VECTOR PRACTICE

Part B

THE CROSS PRODUCT

Question 1

Find in each of the following cases $\mathbf{a} \wedge \mathbf{b}$, where the vectors $\mathbf{a}$ and $\mathbf{b}$ are

a) $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$

b) $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}$

c) $\mathbf{a} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

d) $\mathbf{a} = 7\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

e) $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$$\boxed{\mathbf{i} + 3\mathbf{j} - 17\mathbf{k}}, \boxed{-\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}}, \boxed{5\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}}, \boxed{-10\mathbf{i} - 18\mathbf{j} + 22\mathbf{k}}, \boxed{-11\mathbf{i} + 18\mathbf{j} + 17\mathbf{k}}$$

(a) $(2,5,1)\wedge(3,-1,0) = \begin{vmatrix} i & j & k \\ 2 & 5 & 1 \\ 3 & -1 & 0 \end{vmatrix} = [0-(-1), 3-0, -2-15] = (1,3,-17)$

(b) $(1,2,1)\wedge(3,-1,-1) = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 3 & -1 & -1 \end{vmatrix} = [-2-(-1), 3-(-1), -1-6] = (-1,4,-7)$

(c) $(3,-1,-2)\wedge(1,3,1) = \begin{vmatrix} i & j & k \\ 3 & -1 & -2 \\ 1 & 3 & 1 \end{vmatrix} = [-1-(-6), -2-3, 9-(-1)] = (5,-5,10)$

(d) $(7,1,4)\wedge(-1,3,2) = \begin{vmatrix} i & j & k \\ 7 & 1 & 4 \\ -1 & 3 & 2 \end{vmatrix} = [2-12, -4-14, 21-(-1)] = (-10,-18,22)$

(e) $(2,5,-4)\wedge(-3,1,-3) = \begin{vmatrix} i & j & k \\ 2 & 5 & -4 \\ -3 & 1 & -3 \end{vmatrix} = [-15-(-4), 12-(-6), 2-(-15)] = (-11,18,17)$

Question 2

Find a **unit** vector perpendicular to both

$$\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{b} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

$$\boxed{\frac{1}{\sqrt{330}}(-4\mathbf{i} + 5\mathbf{j} + 17\mathbf{k})}$$

$\underline{a} \wedge \underline{b} = \begin{vmatrix} i & j & k \\ 2 & 5 & 1 \\ 3 & -1 & -1 \end{vmatrix} = [-5-(-1), 3-(-2), -2-15] = (-4,5,-17)$

$|-4,5,-17| = \sqrt{16+25+289} = \sqrt{330}$

$\therefore$ REQUIRED VECTOR IS $\frac{1}{\sqrt{330}}(-4,5,-17)$

Question 3

The vectors $\mathbf{a}$ and $\mathbf{b}$, are not parallel.

Simplify fully

$$(2\mathbf{a}+\mathbf{b})\wedge(\mathbf{a}-2\mathbf{b}).$$

$\boxed{5\mathbf{b}\wedge\mathbf{a}=-5\mathbf{a}\wedge\mathbf{b}}$

USING THE FACT THAT THE "CROSS PRODUCT" IS DISTRIBUTIVE OVER ADDITION & SUBTRACTION, WE OBTAIN

$$(2\underline{a}+\underline{b})\wedge(\underline{a}-2\underline{b}) = 2\underline{a}\wedge\underline{a} - 4\underline{a}\wedge\underline{b} + \underline{b}\wedge\underline{a} - 2\underline{b}\wedge\underline{b}$$

NEXT WE USE THE PROPERTIES

- $\underline{u}\wedge\underline{u}=\underline{0}$ FOR ALL $\underline{u}$
- $\underline{u}\wedge\underline{v} = -\underline{v}\wedge\underline{u}$ FOR ALL $\underline{u}$ & $\underline{v}$

$$\ldots = \underline{0} + 4\underline{b}\wedge\underline{a} + \underline{b}\wedge\underline{a} - \underline{0}$$

$$= 5\underline{b}\wedge\underline{a}$$

[OR INDEED $-5\underline{a}\wedge\underline{b}$]

Question 4

The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are not parallel.

Simplify fully

$$\mathbf{a}\cdot\left[\mathbf{b}\wedge(\mathbf{c}+\mathbf{a})\right].$$

$\boxed{\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})}$

APPLY THE CROSS PRODUCT FIRST

$$\Rightarrow \underline{a}\cdot[\underline{b}\wedge(\underline{c}+\underline{a})] = \underline{a}\cdot[\underline{b}\wedge\underline{c} + \underline{b}\wedge\underline{a}]$$

$$= \underline{a}\cdot\underline{b}\wedge\underline{c} + \underline{a}\cdot\underline{b}\wedge\underline{a}$$

NOW $\underline{b}\wedge\underline{a}$ IS PERPENDICULAR TO $\underline{a}$, SO $\underline{a}\cdot(\underline{b}\wedge\underline{a})=0$

$$\therefore\ \underline{a}\cdot[\underline{b}\wedge(\underline{c}+\underline{a})] = \underline{a}\cdot\underline{b}\wedge\underline{c}$$

Question 5

The following vectors are given

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$
$$\mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{c} = \mathbf{j} + 3\mathbf{k}$$

a) Show that the three vectors are coplanar.

b) Express $\mathbf{a}$ in terms of $\mathbf{b}$ and $\mathbf{c}$.

$$\boxed{\mathbf{a} = 2\mathbf{b} - \mathbf{c}}$$

(a) $\underline{a} = (2,3,-1)$ If COPLANAR $\underline{a} \wedge \underline{b} \cdot \underline{c} = 0$

$\underline{b} = (1,2,1)$

$\underline{c} = (0,1,3)$

$$\begin{vmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -3 + 3 = 0$$

(b) $\lambda \underline{b} + \mu \underline{c} = \underline{a}$

$\lambda(1,2,1) + \mu(0,1,3) = (2,3,-1)$

$(\lambda, 2\lambda + \mu, \lambda + 3\mu) = (2,3,-1)$

$\therefore \lambda = 2$

$2\lambda + \mu = 3$

$4 + \mu = 3$

$\mu = -1$

$\therefore \underline{a} = 2\underline{b} - \underline{c}$

Question 6

The following three vectors are given

$$\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$
$$\mathbf{c} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$$

where λ is a scalar constant.

a) If the three vectors given above are coplanar, find the value of λ.

b) Express $\mathbf{a}$ in terms of $\mathbf{b}$ and $\mathbf{c}$.

$\boxed{\lambda = 1}$, $\boxed{\mathbf{a} = 3\mathbf{c} - \mathbf{b}}$

a) IF THE VECTORS ARE COPLANAR, THE CROSS PRODUCT OF ANY TWO WILL BE PERPENDICULAR TO THE THIRD

$$\Rightarrow (\underline{a} \wedge \underline{b}) \cdot \underline{c} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 2 & \lambda \\ 1 & 3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 1\begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \lambda\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow (3-6) - 2(1-4) + \lambda(3-6) = 0$$

$$\Rightarrow -3 + 6 - 3\lambda = 0$$

$$\Rightarrow 3 = 3\lambda$$

$$\Rightarrow \lambda = 1$$

b) SETTING UP AN EQUATION

$$\underline{a} = p\underline{b} + q\underline{c}$$

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = p\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + q\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

EQUATE SAY $\underline{i}$ & $\underline{k}$ (THE $\underline{j}$ SHOULD BALANCE)

$$\left.\begin{aligned} 2p + q &= 1 \\ p + q &= 2 \end{aligned}\right\} \Rightarrow p = -1 \quad \& \quad q = 3$$

$$\therefore \underline{a} = 3\underline{c} - \underline{b}$$

Question 7

The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such so that

$$\mathbf{c} \wedge \mathbf{a} = \mathbf{i} \quad \text{and} \quad \mathbf{b} \wedge \mathbf{c} = 2\mathbf{k} .$$

Express $(\mathbf{a}+\mathbf{b}) \wedge (\mathbf{a}+\mathbf{b}+2\mathbf{c})$ in terms of $\mathbf{i}$ and $\mathbf{k}$.

$\boxed{-2\mathbf{i}+4\mathbf{k}}$

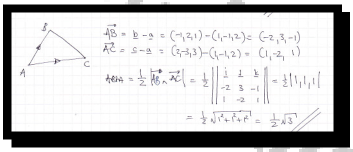
$$\begin{aligned}(\underline{a}+\underline{b})\wedge(\underline{a}+\underline{b}+2\underline{c}) &= \cancel{(\underline{a}+\underline{b})\wedge(\underline{a}+\underline{b})} + (\underline{a}+\underline{b})\wedge 2\underline{c} \\ &= 2\underline{a}\wedge\underline{c} + 2\underline{b}\wedge\underline{c} \\ &= -2\underline{i} + 2(2\underline{k}) \\ &= -2\underline{i} + 4\underline{k}\end{aligned}$$

CROSS PRODUCT GEOMETRIC APPLICATIONS

Question 1

Find the area of the triangle with vertices at $A(1,-1,2)$, $B(-1,2,1)$ and $C(2,-3,3)$.

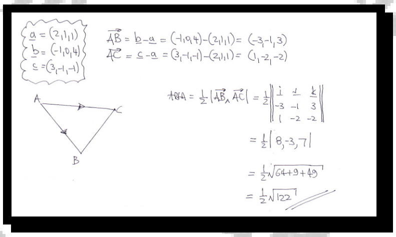
$$\boxed{\frac{1}{2}\sqrt{3}}$$



Question 2

Find the area of the triangle with vertices at $A(2,1,1)$, $B(-1,0,4)$ and $C(3,-1,-1)$.

$$\boxed{\frac{1}{2}\sqrt{122}}$$



Question 3

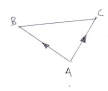
A triangle has vertices at $A(-2,-2,0)$, $B(6,8,6)$ and $C(-6,8,12)$.

Find the area of the triangle ABC.

$\boxed{90}$

$\underline{a} = (-2,-2,0)$
$\underline{b} = (6,8,6)$
$\underline{c} = (-6,8,12)$

$\overrightarrow{AB} = \underline{b} - \underline{a} = (6,8,6) - (-2,-2,0) = (8,10,6)$
$\overrightarrow{AC} = \underline{c} - \underline{a} = (-6,8,12) - (-2,-2,0) = (-4,10,12)$



$$\text{AREA} = \frac{1}{2}\left|\overrightarrow{AB} \wedge \overrightarrow{AC}\right| = \frac{1}{2}\left\|\begin{matrix} \underline{i} & \underline{j} & \underline{k} \\ 8 & 10 & 6 \\ -4 & 10 & 12 \end{matrix}\right\|$$

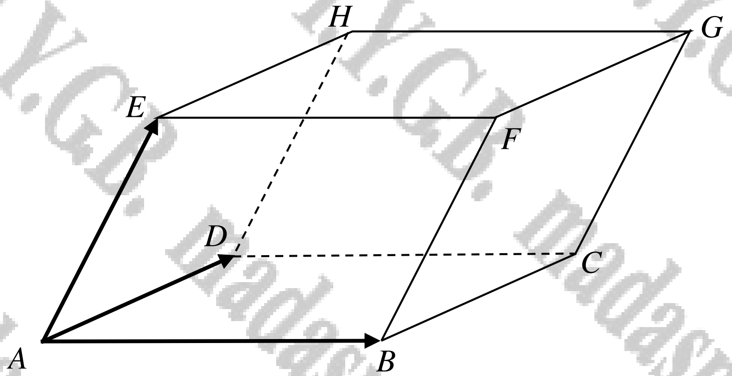
$$= \frac{1}{2}\left|60, -120, -120\right|$$

$$= \frac{1}{2}\sqrt{3600 + 14400 + 14400}$$

$$= 90$$

Question 4

A parallelepiped has vertices at the points $A(2,1,t)$, $B(3,3,2)$, $D(4,0,5)$ and $E(1,-2,7)$, where t is a scalar constant.



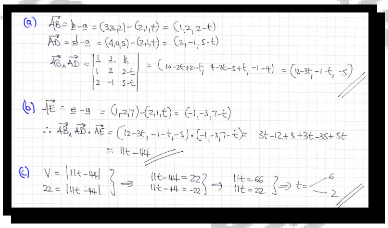
a) Calculate $\overrightarrow{AB} \wedge \overrightarrow{AD}$, in terms of t.

b) Find the value of $\overrightarrow{AB} \wedge \overrightarrow{AD} \cdot \overrightarrow{AE}$

The volume of the parallelepiped is 22 cubic units.

c) Determine the possible values of t.

$\boxed{(12-3t)\mathbf{i}+(-t-1)\mathbf{j}-5\mathbf{k}}$, $\boxed{11t-44}$, $\boxed{t=2,6}$



(a) $\overrightarrow{AB} = \underline{b}-\underline{a} = (3,3,2)-(2,1,t) = (1,2,2-t)$

$\overrightarrow{AD} = \underline{d}-\underline{a} = (4,0,5)-(2,1,t) = (2,-1,5-t)$

$\overrightarrow{AB} \wedge \overrightarrow{AD} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 2-t \\ 2 & -1 & 5-t \end{vmatrix} = (10-2t+2-t,\ 4-2t-5+t,\ -1-4) = (12-3t, -1-t, -5)$

(b) $\overrightarrow{AE} = \underline{e}-\underline{a} = (1,-2,7)-(2,1,t) = (-1,-3,7-t)$

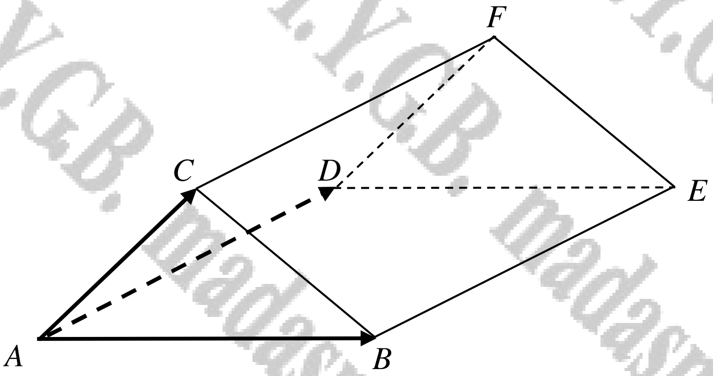
$\therefore \overrightarrow{AB} \wedge \overrightarrow{AD} \cdot \overrightarrow{AE} = (12-3t,-1-t,-5)\cdot(-1,-3,7-t) = 3t-12+3+3t-35+5t$

$= 11t-44$

(c) $\left.\begin{array}{l} V = |11t-44| \\ 22 = |11t-44| \end{array}\right\} \Rightarrow \left.\begin{array}{l} 11t-44 = 22 \\ 11t-44 = -22 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 11t = 66 \\ 11t = 22 \end{array}\right\} \Rightarrow t = \begin{cases} 6 \\ 2 \end{cases}$

Question 5

A triangular prism has vertices at the points $A(3,3,3)$, $B(1,3,t)$, $C(5,1,5)$ and $F(8,0,10)$, where t is a scalar constant.



The face ABC is parallel to the face DEF and the lines AD, BE and CF are parallel to each other.

a) Calculate $\overrightarrow{AB} \wedge \overrightarrow{AC}$, in terms of t.

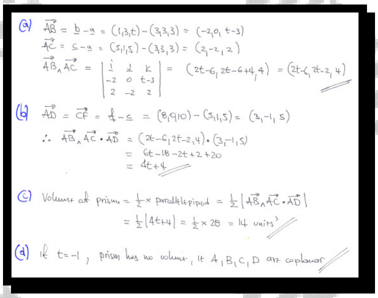
b) Find the value of $\overrightarrow{AB} \wedge \overrightarrow{AC} \cdot \overrightarrow{AD}$, in terms of t.

The value of t is taken to be 6.

c) Determine the volume of the prism for this value of t.

d) Explain the geometrical significance if $t = -1$.

$\boxed{(2t-6)\mathbf{i} + (2t-2)\mathbf{j} + 4\mathbf{k}}$, $\boxed{4t+4}$, $\boxed{V = 14 \text{ cubic units}}$, $\boxed{A, B, C, D \text{ are coplanar, so no volume}}$



(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (1,3,t) - (3,3,3) = (-2,0,t-3)$

$\overrightarrow{AC} = \underline{c} - \underline{a} = (5,1,5) - (3,3,3) = (2,-2,2)$

$\overrightarrow{AB} \wedge \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & t-3 \\ 2 & -2 & 2 \end{vmatrix} = (2t-6, 2t-6+4, 4) = (2t-6, 2t-2, 4)$

(b) $\overrightarrow{AD} = \overrightarrow{CF} = \underline{f} - \underline{c} = (8,0,10) - (5,1,5) = (3,-1,5)$

$\therefore \overrightarrow{AB} \wedge \overrightarrow{AC} \cdot \overrightarrow{AD} = (2t-6, 2t-2, 4) \cdot (3,-1,5)$

$= 6t - 18 - 2t + 2 + 20$

$= 4t + 4$

(c) Volume of prism $= \frac{1}{2} \times$ parallelepiped $= \frac{1}{2} \left| \overrightarrow{AB} \wedge \overrightarrow{AC} \cdot \overrightarrow{AD} \right|$

$= \frac{1}{2} |4t+4| = \frac{1}{2} \times 28 = 14$ units^3

(d) if $t = -1$, prism has no volume, ie A, B, C, D are coplanar

Question 6

A tetrahedron has vertices at the points $A(-3,6,4)$, $B(0,11,0)$, $C(4,1,28)$ and $D(7,k,24)$, where k is a scalar constant.

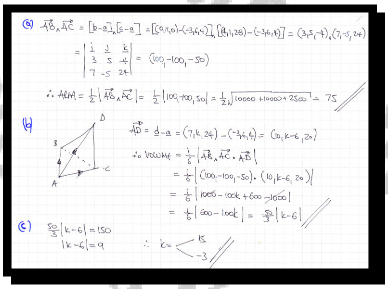
a) Calculate the area of the triangle ABC.

b) Find the volume of the tetrahedron $ABCD$, in terms of k.

The volume of the tetrahedron is 150 cubic units.

c) Determine the possible values of k.

$\boxed{\text{area} = 75}$, $\boxed{\text{volume} = \frac{50}{3}|k-6|}$, $\boxed{k = -3, 15}$



(a) $\vec{AB} \wedge \vec{AC} = [\underline{b}-\underline{a}] \wedge [\underline{c}-\underline{a}] = [(0,11,0)-(-3,6,4)] \wedge [(4,1,28)-(-3,6,4)] = (3,5,-4) \wedge (7,-5,24)$

$= \begin{vmatrix} i & j & k \\ 3 & 5 & -4 \\ 7 & -5 & 24 \end{vmatrix} = (100,-100,-50)$

$\therefore$ AREA $= \frac{1}{2}|\vec{AB} \wedge \vec{AC}| = \frac{1}{2}|100,-100,50| = \frac{1}{2}\sqrt{10000+10000+2500} = 75$

(b) $\vec{AD} = \underline{d}-\underline{a} = (7,k,24)-(-3,6,4) = (10,k-6,20)$

$\therefore$ VOLUME $= \frac{1}{6}|\vec{AB} \wedge \vec{AC} \cdot \vec{AD}|$

$= \frac{1}{6}|(100,-100,-50)\cdot(10,k-6,20)|$

$= \frac{1}{6}|\cancel{1000} - 100k + 600 - \cancel{1000}|$

$= \frac{1}{6}|600-100k| = \frac{50}{3}|k-6|$

(c) $\frac{50}{3}|k-6| = 150$

$|k-6| = 9$

$\therefore k = 15, -3$

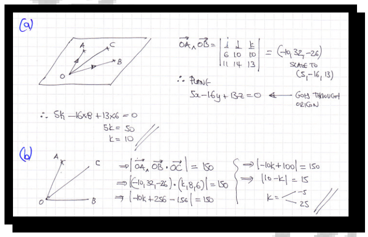
Question 7

With respect to a fixed origin O the points A, B and C, have respective coordinates $(6,10,10)$, $(11,14,13)$ and $(k,8,6)$, where k is a constant.

a) Given that all the three points lie on a plane which contains the origin, find the value of k.

b) Given instead that OA, OB, OC are edges of a parallelepiped of volume 150 cubic units determine the possible values of k.

$\boxed{k=10}$, $\boxed{k=-5,25}$



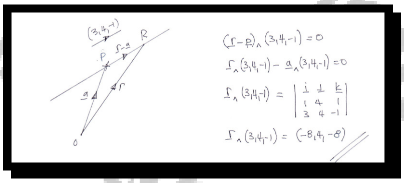
LINES

Question 1

Find an equation of the straight line that passes through the point $P(1,4,1)$ and is parallel to the vector $3\mathbf{i}+4\mathbf{j}-\mathbf{k}$.

Give the answer in the form $\mathbf{r}\wedge\mathbf{a}=\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

$$\boxed{\mathbf{r}\wedge(3\mathbf{i}+4\mathbf{j}-\mathbf{k})=-8\mathbf{i}+4\mathbf{j}-8\mathbf{k}}$$

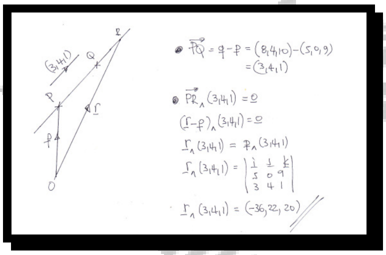


Question 2

Find an equation of the straight line that passes through the points $P(5,0,9)$ and $Q(8,4,10)$.

Give the answer in the form $\mathbf{r}\wedge\mathbf{a}=\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

$$\boxed{\mathbf{r}\wedge(3\mathbf{i}+4\mathbf{j}+\mathbf{k})=-36\mathbf{i}+22\mathbf{j}+20\mathbf{k}}$$



Question 3

A straight line has equation

$$\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}),$$

where λ is a scalar constant.

Convert the above equation into Cartesian form.

$$\boxed{\frac{x-4}{1} = \frac{y-2}{8} = \frac{z-5}{-3}}$$

$\underline{r} = (4,2,5) + \lambda(1,8,-3)$

$(x,y,z) = (\lambda+4, 8\lambda+2, 5-3\lambda)$

$$\left.\begin{array}{l} x = \lambda+4 \\ y = 8\lambda+2 \\ z = 5-3\lambda \end{array}\right\} \Rightarrow \begin{array}{l} \lambda = x-4 \\ \lambda = \frac{y-2}{8} \\ \lambda = \frac{z-5}{-3} \end{array}$$

$\therefore \frac{x-4}{1} = \frac{y-2}{8} = \frac{z-5}{-3}$

OR

$x-4 = \frac{y-2}{8} = \frac{5-z}{3}$

Question 4

A straight line has equation

$$\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \lambda(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

where λ is a scalar parameter.

Convert the above equation into Cartesian form.

$$\boxed{\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z}{2}}$$

$\underline{r} = (2,-3,0) + \lambda(1,-2,2)$

$\underline{r} = \lambda+2, -3-2\lambda, 2\lambda$

$$\left.\begin{array}{l} x = \lambda+2 \\ y = -3-2\lambda \\ z = 2\lambda \end{array}\right\} \Rightarrow \begin{array}{l} \lambda = x-2 \\ \lambda = \frac{-3-y}{2} \\ \lambda = \frac{z}{2} \end{array}$$

$\therefore x-2 = \frac{-3-y}{2} = \frac{z}{2}$

$x-2 = \frac{y+3}{-2} = \frac{z}{2}$

Question 5

Convert the equation of the straight line

$$\frac{x-3}{2}=\frac{y+2}{3}=\frac{5-z}{7}$$

into a vector equation of the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and λ is a scalar parameter.

$$\boxed{\mathbf{r}=3\mathbf{i}-2\mathbf{j}+5\mathbf{k}+\lambda(2\mathbf{i}+3\mathbf{j}-7\mathbf{k})}$$

$$\frac{x-3}{2}=\frac{y+2}{3}=\frac{5-z}{7}=\lambda$$

$$\left.\begin{aligned}\frac{x-3}{2}&=\lambda\\ \frac{y+2}{3}&=\lambda\\ \frac{5-z}{7}&=\lambda\end{aligned}\right\}\Rightarrow\begin{aligned}x&=2\lambda+3\\ y&=3\lambda-2\\ z&=-7\lambda+5\end{aligned}\qquad\therefore\ \underline{r}=(3,-2,5)+\lambda(2,3,-7)$$

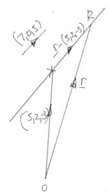
Question 6

A straight line has equation

$$\left[\mathbf{r}-(5\mathbf{i}+2\mathbf{j}-3\mathbf{k})\right]\wedge(7\mathbf{i}+5\mathbf{k})=\mathbf{0}$$

Convert the above equation into the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and λ is a scalar parameter.

$$\boxed{\frac{x-5}{7}=\frac{z+3}{5},\ y=2}$$



- REWRITE LINE IN PARAMETRIC

$$\underline{r}=(5,2,-3)+\lambda(7,0,5)$$

- WRITE IN CARTESIAN BY INSPECTION

$$\begin{aligned}x&=7\lambda+5\\ y&=2\\ z&=5\lambda-3\end{aligned}\quad\Rightarrow\quad\begin{aligned}\lambda&=\frac{x-5}{7}\\ \lambda&=\frac{z+3}{5}\end{aligned}$$

$\therefore\ \frac{x-5}{7}=\frac{z+3}{5}$ AND $y=2$

OR $5x-25=7z+21$

$5x=7z+46$ & $y=2$

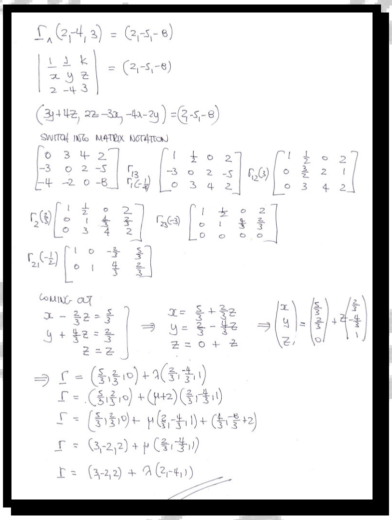
Question 7

A straight line has equation

$$\mathbf{r} \wedge (2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) = (2\mathbf{i} - 5\mathbf{j} - 8\mathbf{k})$$

Convert the above equation into the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and λ is a scalar parameter.

$$\boxed{\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) \quad \text{or} \quad \mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k})}$$



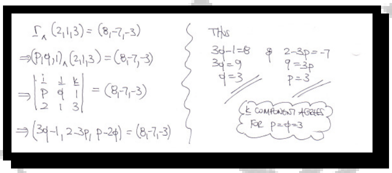
Question 8

If the point $A(p, q, 1)$ lies on the straight line with vector equation

$$\mathbf{r} \wedge (2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = (8\mathbf{i} - 7\mathbf{j} - 3\mathbf{k}),$$

find the value of each of the scalar constants p and q.

$$\boxed{p = q = 3}$$



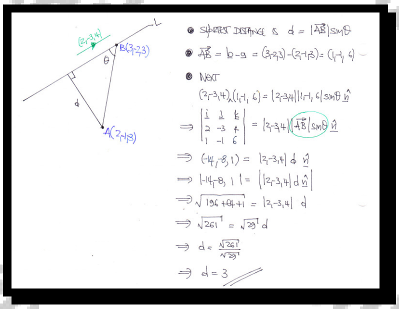
Question 9

The straight line L has equation

$$\left[\mathbf{r}-(3\mathbf{i}-2\mathbf{j}+3\mathbf{k})\right]\wedge(2\mathbf{i}-3\mathbf{k}+4\mathbf{k})=0\,.$$

Use a method involving the cross product to show that the shortest distance of the point $(2,-1,-3)$ from L is 3 units.

proof

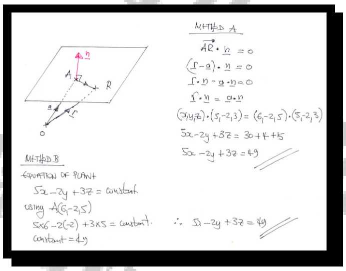


PLANES

Question 1

Find a Cartesian equation of the plane that passes through the point $A(6,-2,5)$, and its normal is in the direction $5\mathbf{i}-2\mathbf{j}+3\mathbf{k}$.

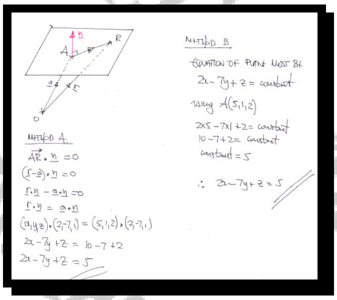
$$\boxed{5x-2y+3z=49}$$



Question 2

Find a Cartesian equation of the plane that passes through the point $A(5,1,2)$, and its normal is in the direction $2\mathbf{i}-7\mathbf{j}+\mathbf{k}$.

$$\boxed{2x-7y+z=5}$$

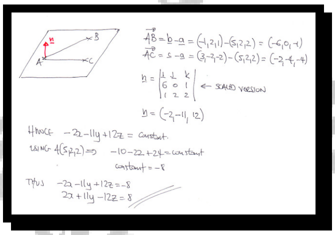


Question 3

Find a Cartesian equation of the plane that passes through the points

$$A(5,2,2),\quad B(-1,2,1) \quad \text{and} \quad C(3,-2,-2).$$

$$\boxed{2x+11y-12z=8}$$



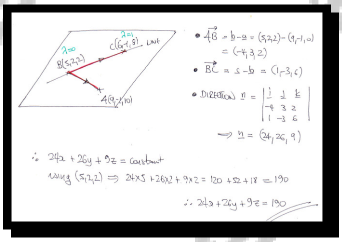
Question 4

Determine a Cartesian equation of the plane that contains the point $A(9,-1,0)$ and the straight line with vector equation

$$\mathbf{r} = 5\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}),$$

where λ is a scalar parameter.

$$\boxed{24x+26y+9z=190}$$



Question 5

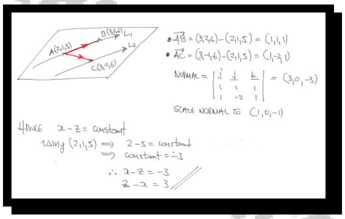
Find a Cartesian equation of the plane that contains the parallel straight lines with vector equations

$$\mathbf{r}_1 = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = 3\mathbf{i} - \mathbf{j} + 6\mathbf{k} + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k}),$$

where λ and μ are scalar parameters.

.

$$\boxed{z - x = 3}$$



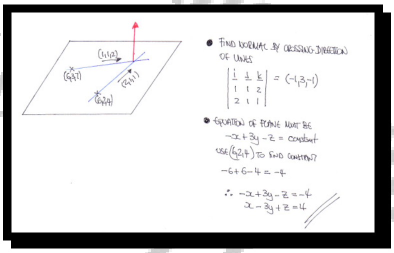
Question 6

Find a Cartesian equation of the plane that contains the intersecting straight lines with vector equations

$$\mathbf{r}_1 = 6\mathbf{i} + 3\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = 6\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k}),$$

where λ and μ are scalar parameters.

$$\boxed{x - 3y + z = 4}$$

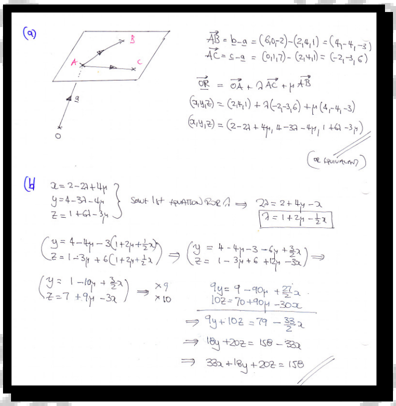


Question 7

a) Find a set of parametric equations for the plane that passes through the points $A(2,4,1)$, $B(6,0,-2)$ and $C(0,1,7)$.

b) Eliminate the parameters to obtain a Cartesian equation of the plane.

$$\boxed{(x,y,z)=(2-2\lambda+4\mu,4-3\lambda-4\mu,1+6\lambda-3\mu)},\ \boxed{33x+18y+20z=158}$$



GEOMETRIC PROBLEMS

(WITH PLANES AND LINES)

Question 1

Find the coordinates of the point of intersection of the plane with equation

$$x + 2y + 3z = 4$$

and the straight line with equation

$$\mathbf{r} = -\mathbf{i} + \mathbf{j} - 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k}),$$

where λ is a scalar parameter.

$\boxed{(1,3,-1)}$

$x + 2y + 3z = 4$
$(x,y,z) = (\lambda-1, \lambda+1, 2\lambda-5)$

$\Rightarrow (\lambda-1) + 2(\lambda+1) + 3(2\lambda-5) = 4$
$\Rightarrow \lambda - 1 + 2\lambda + 2 + 6\lambda - 15 = 4$
$\Rightarrow 9\lambda = 18$
$\Rightarrow \lambda = 2$

$\therefore P(1,3,-1)$

Question 2

Find the coordinates of the point of intersection of the plane with equation

$$3x + 2y - 7z = 2$$

and the straight line with equation

$$\mathbf{r} = 9\mathbf{i} + 2\mathbf{j} + 7\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}),$$

where λ is a scalar parameter.

$\boxed{(5,-10,-1)}$

$3x + 2y - 7z = 2$

$\underline{r} = (9,2,7) + \lambda(1,3,2)$

$(x,y,z) = (\lambda+9, 3\lambda+2, 2\lambda+7)$

SOLVING SIMULTANEOUSLY

$3(\lambda+9) + 2(3\lambda+2) - 7(2\lambda+7) = 2$
$3\lambda + 27 + 6\lambda + 4 - 14\lambda - 49 = 2$
$-20 = 5\lambda$
$\lambda = -4$

$\therefore$ REQUIRED POINT
$x = -4 + 9 = 5$
$y = 3(-4) + 2 = -10$
$z = 2(-4) + 7 = -1$

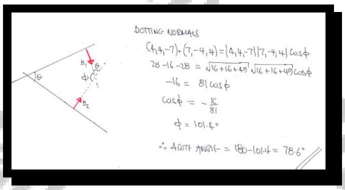
$\therefore (5,-10,-1)$

Question 3

Find the size of the acute angle formed by the planes with Cartesian equations

$$4x + 4y - 7z = 13 \quad \text{and} \quad 7x - 4y + 4z = 6.$$

$\boxed{78.6°}$

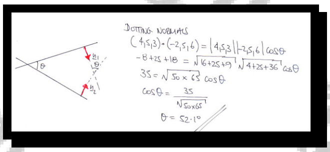


Question 4

Find the size of the acute angle between the planes with Cartesian equations

$$4x + 5y + 3z = 82 \quad \text{and} \quad -2x + 5y + 6z = 124.$$

$\boxed{52.1°}$



Question 5

Find the size of the acute angle between the plane with equation

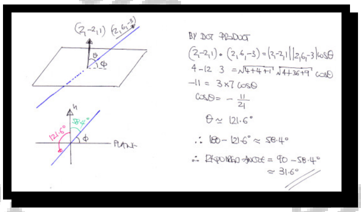
$$2x-2y+z=12$$

and the straight line with equation

$$\mathbf{r}=7\mathbf{i}-\mathbf{j}+2\mathbf{k}+\lambda(2\mathbf{i}+6\mathbf{j}-3\mathbf{k}),$$

where λ is a scalar parameter.

$\boxed{31.6°}$



Question 6

Find the size of the acute angle formed between the plane with Cartesian equation

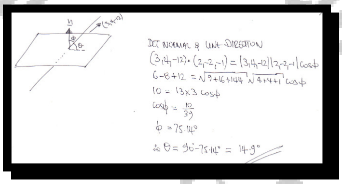
$$2x-2y-z=2$$

and the straight line with vector equation

$$\mathbf{r}=2\mathbf{i}+\mathbf{j}-5\mathbf{k}+\lambda(3\mathbf{i}+4\mathbf{j}-12\mathbf{k}),$$

where λ is a scalar parameter.

$\boxed{14.9°}$



Question 7

Find the size of the acute angle between the plane with equation

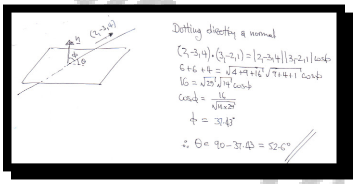
$$3x-2y+z=5$$

and the straight line with equation

$$\mathbf{r}=-3\mathbf{i}-2\mathbf{j}+3\mathbf{k}+\lambda(2\mathbf{i}-3\mathbf{j}+4\mathbf{k}),$$

where λ is a scalar parameter.

$\boxed{52.6°}$

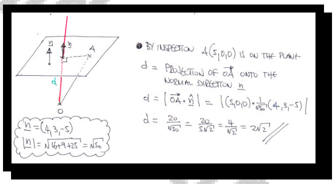


Question 8

Find shortest distance of the origin O from the plane with equation

$$4x+3y-5z=20.$$

$\boxed{2\sqrt{2}}$

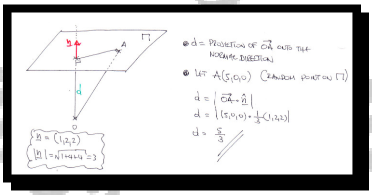


Question 9

Find shortest distance of the origin O from the plane with equation

$$x+2y+2z=5.$$

$$\boxed{\frac{5}{3}}$$

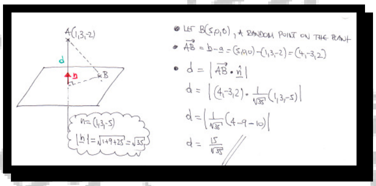


Question 10

Find shortest distance from the point $A(1,3,-2)$ to the plane with Cartesian equation

$$x+3y-5z=5.$$

$$\boxed{\frac{15}{\sqrt{35}}}$$

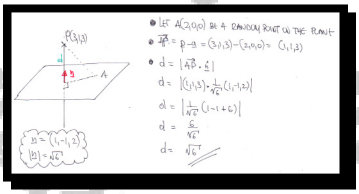


Question 11

Find shortest distance from the point $P(3,1,3)$ to the plane with Cartesian equation

$$x - y + 2z = 2.$$

$\boxed{\sqrt{6}}$

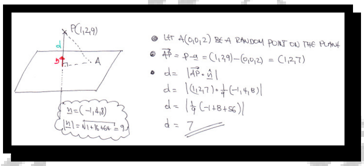


Question 12

Find shortest distance of the point $P(1,2,9)$ from the plane with Cartesian equation

$$-x + 4y + 8z = 16.$$

$\boxed{7}$

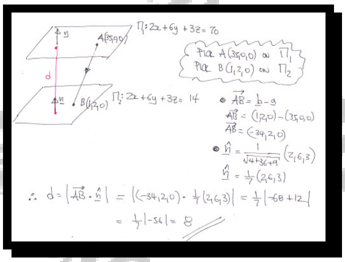


Question 13

Find the distance between the parallel planes with Cartesian equations

$$2x + 6y + 3z = 70 \quad \text{and} \quad 2x + 6y + 3z = 14.$$

$\boxed{8}$



Question 14

The straight line with vector equation

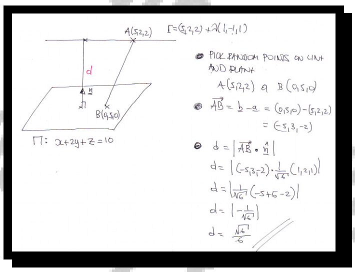
$$\mathbf{r} = (\lambda + 5)\mathbf{i} + (2 - \lambda)\mathbf{j} + (\lambda + 2)\mathbf{k},$$

where λ is a scalar parameter, is parallel to the plane with Cartesian equation

$$x + 2y + z = 10.$$

Find the distance between the plane and the straight line.

$\boxed{\frac{\sqrt{6}}{6}}$

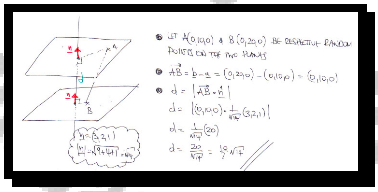


Question 15

Find the distance between the parallel planes with Cartesian equations

$$3x+2y+z=20 \quad \text{and} \quad 3x+2y+z=40.$$

$$\boxed{\frac{10}{7}\sqrt{14}}$$



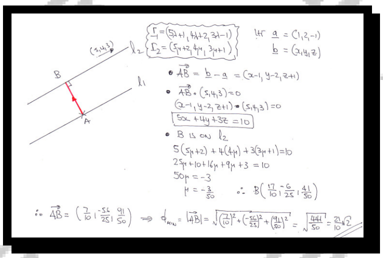
Question 16

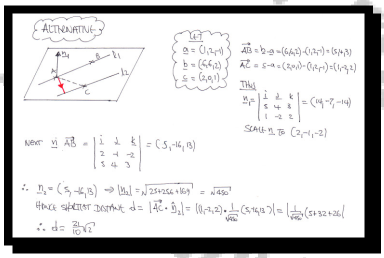
Find the distance between the parallel straight lines with vector equations

$$\mathbf{r}_1=\mathbf{i}+2\mathbf{j}-\mathbf{k}+\lambda(5\mathbf{i}+4\mathbf{j}+3\mathbf{k}) \quad \text{and} \quad \mathbf{r}_2=2\mathbf{i}+\mathbf{k}+\mu(5\mathbf{i}+4\mathbf{j}+3\mathbf{k}),$$

where λ and μ are a scalar parameters.

$$\boxed{\frac{21}{10}\sqrt{2}}$$



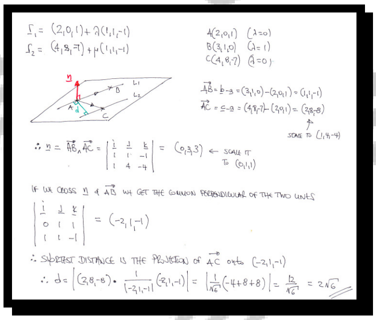


Question 17

Find the distance between the parallel straight lines with vector equations

$$\left[\mathbf{r}-(2\mathbf{i}+\mathbf{k})\right]\wedge(\mathbf{i}+\mathbf{j}-\mathbf{k}) = \mathbf{0} \quad \text{and} \quad x-4=y-8=-z-7.$$

$\boxed{2\sqrt{6}}$



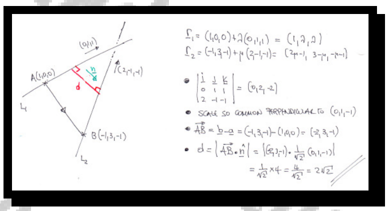
Question 18

Find the shortest distance between the skew straight lines with vector equations

$$\mathbf{r}_1 = \mathbf{i}+\lambda(\mathbf{j}+\mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = -\mathbf{i}+3\mathbf{j}-\mathbf{k}+\mu(2\mathbf{i}-\mathbf{j}-\mathbf{k}),$$

where λ and μ are a scalar parameters.

$\boxed{2\sqrt{2}}$



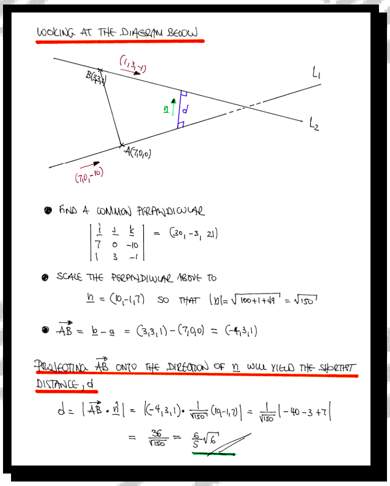
Question 19

Find the shortest distance between the skew straight lines with vector equations

$$\mathbf{r}_1 = 7\mathbf{i} + \lambda(7\mathbf{i} - 10\mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = 3\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \mu(\mathbf{i} + 3\mathbf{j} - \mathbf{k}),$$

where λ and μ are a scalar parameters.

$\boxed{\frac{6}{5}\sqrt{6}}$



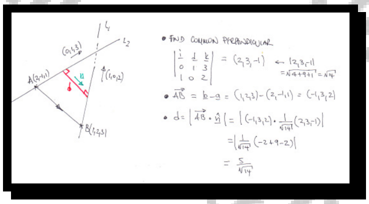
Question 20

Find the shortest distance between the skew straight lines with vector equations

$$\mathbf{r}_1 = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(\mathbf{j} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r}_2 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{k}),$$

where λ and μ are a scalar parameters.

$\boxed{\frac{5}{\sqrt{14}}}$



Question 21

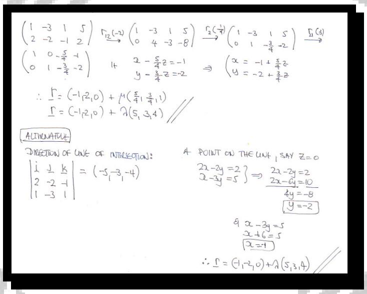
Find the intersection of the planes with Cartesian equations

$$2x-2y-z=2 \quad \text{and} \quad x-3y+z=5,$$

giving the answer in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$,

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and λ is a scalar parameter.

$\boxed{\mathbf{r}=-\mathbf{i}-2\mathbf{j}+\lambda(5\mathbf{i}+3\mathbf{j}+4\mathbf{k})}$



$\begin{pmatrix}1&-3&1&5\\2&-2&-1&2\end{pmatrix} \xrightarrow{r_{12}(-2)} \begin{pmatrix}1&-3&1&5\\0&4&-3&-8\end{pmatrix} \xrightarrow{r_2(\frac{1}{4})} \begin{pmatrix}1&-3&1&5\\0&1&-\frac{3}{4}&-2\end{pmatrix} \xrightarrow{r_{21}(3)}$

$\begin{pmatrix}1&0&-\frac{5}{4}&-1\\0&1&-\frac{3}{4}&-2\end{pmatrix}$ i.e. $x-\frac{5}{4}z=-1$, $y-\frac{3}{4}z=-2$ $\Rightarrow$ $x=-1+\frac{5}{4}z$, $y=-2+\frac{3}{4}z$

$\therefore \underline{r}=(-1,-2,0)+\mu(\frac{5}{4},\frac{3}{4},1)$

$\underline{r}=(-1,-2,0)+\lambda(5,3,4)$

ALTERNATIVE

DIRECTION OF LINE OF INTERSECTION: $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-2&-1\\1&-3&1\end{vmatrix}=(-5,-3,-4)$

A POINT ON THE LINE, SAY $z=0$

$2x-2y=2$, $x-3y=5$ $\Rightarrow$ $2x-2y=2$, $2x-6y=10$; $4y=-8$; $\boxed{y=-2}$

& $x-3y=5$; $x+6=5$; $\boxed{x=-1}$

$\therefore \underline{r}=(-1,-2,0)+\lambda(5,3,4)$

Question 22

Show that the planes with Cartesian equations

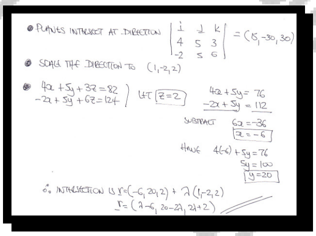
$$4x+5y+3z=82 \quad \text{and} \quad -2x+5y+6z=124$$

intersect along the straight line with equation

$$\mathbf{r}=(\lambda-6)\mathbf{i}+(20-2\lambda)\mathbf{j}+(2\lambda+2)\mathbf{k},$$

where λ is a scalar parameter.

$\boxed{\text{proof}}$



• PLANES INTERSECT AT DIRECTION $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&5&3\\-2&5&6\end{vmatrix}=(15,-30,30)$

• SCALE THE DIRECTION TO $(1,-2,2)$

• $4x+5y+3z=82$, $-2x+5y+6z=124$ LET $\boxed{z=2}$ $4x+5y=76$, $-2x+5y=112$

SUBTRACT $6x=-36$ $\boxed{x=-6}$

HENCE $4(-6)+5y=76$; $5y=100$; $\boxed{y=20}$

$\therefore$ INTERSECTION IS $\underline{r}=(-6,20,2)+\lambda(1,-2,2)$

$\underline{r}=(\lambda-6,20-2\lambda,2\lambda+2)$

Question 23

The planes Π_1 and Π_2 have Cartesian equations:

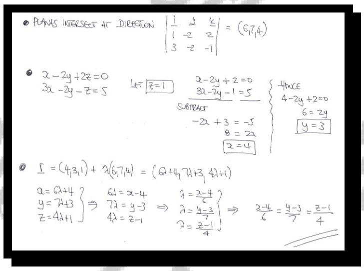
$$\Pi_1 : x - 2y + 2z = 0$$

$$\Pi_2 : 3x - 2y - z = 5$$

Show that the two planes intersect along the straight line with Cartesian equation

$$\frac{x-4}{6} = \frac{y-3}{7} = \frac{z-1}{4}.$$

$\boxed{\text{proof}}$



- PLANES INTERSECT AT DIRECTION $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 2 \\ 3 & -2 & -1 \end{vmatrix} = (6,7,4)$

- $x - 2y + 2z = 0$
  $3x - 2y - z = 5$

  LET $\boxed{z=1}$

  $x - 2y + 2 = 0$
  $3x - 2y - 1 = 5$

  SUBTRACT

  $-2x + 3 = -5$
  $8 = 2x$
  $\boxed{x = 4}$

  HENCE
  $4 - 2y + 2 = 0$
  $6 = 2y$
  $\boxed{y = 3}$

- $\underline{r} = (4,3,1) + \lambda(6,7,4) = (6\lambda+4, 7\lambda+3, 4\lambda+1)$

  $\left.\begin{array}{l} x = 6\lambda+4 \\ y = 7\lambda+3 \\ z = 4\lambda+1 \end{array}\right\} \Rightarrow \begin{array}{l} 6\lambda = x-4 \\ 7\lambda = y-3 \\ 4\lambda = z-1 \end{array} \Rightarrow \left.\begin{array}{l} \lambda = \frac{x-4}{6} \\ \lambda = \frac{y-3}{7} \\ \lambda = \frac{z-1}{4} \end{array}\right\} \Rightarrow \frac{x-4}{6} = \frac{y-3}{7} = \frac{z-1}{4}$