COMPLEX NUMBERS
(part 1)

BASIC COMPLEX ALGEBRA

Question 1

Simplify the following complex number expressions, giving the final answer in the form $a+b\text{i}$, where $a \in \mathbb{R}$, $b \in \mathbb{R}$.

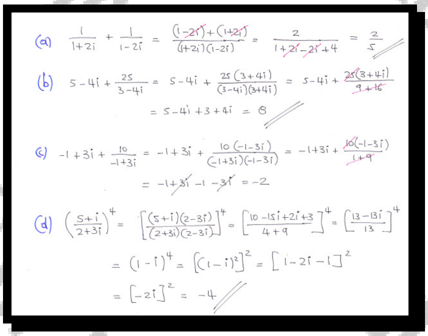
a) $\dfrac{1}{1+2\text{i}}+\dfrac{1}{1-2\text{i}}$

b) $5-4\text{i}+\dfrac{25}{3-4\text{i}}$

c) $-1+3\text{i}+\dfrac{10}{-1+3\text{i}}$

d) $\left(\dfrac{5+\text{i}}{2+3\text{i}}\right)^4$

$\boxed{\frac{2}{5}}$, $\boxed{8}$, $\boxed{-2}$, $\boxed{-4}$


(a) $\frac{1}{1+2i}+\frac{1}{1-2i}=\frac{(1-2i)+(1+2i)}{(1+2i)(1-2i)}=\frac{2}{1+2i-2i+4}=\frac{2}{5}$

(b) $5-4i+\frac{25}{3-4i}=5-4i+\frac{25(3+4i)}{(3-4i)(3+4i)}=5-4i+\frac{25(3+4i)}{9+16}$
$=5-4i+3+4i=8$

(c) $-1+3i+\frac{10}{-1+3i}=-1+3i+\frac{10(-1-3i)}{(-1+3i)(-1-3i)}=-1+3i+\frac{10(-1-3i)}{1+9}$
$=-1+3i-1-3i=-2$

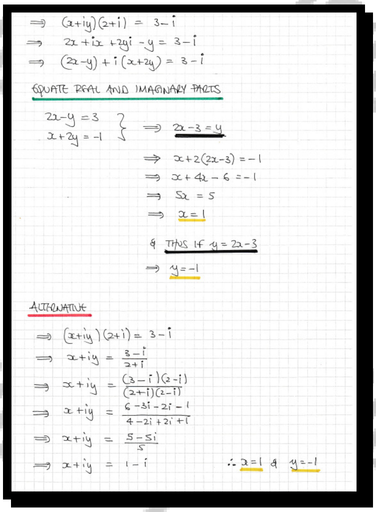
(d) $\left(\frac{5+i}{2+3i}\right)^4=\left[\frac{(5+i)(2-3i)}{(2+3i)(2-3i)}\right]^4=\left[\frac{10-15i+2i+3}{4+9}\right]^4=\left(\frac{13-13i}{13}\right)^4$
$=(1-i)^4=\left[(1-i)^2\right]^2=\left[1-2i-1\right]^2$
$=\left[-2i\right]^2=-4$

Question 2

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$(x + iy)(2 + i) = 3 - i.$$

$$\boxed{(x, y) = (1, -1)}$$

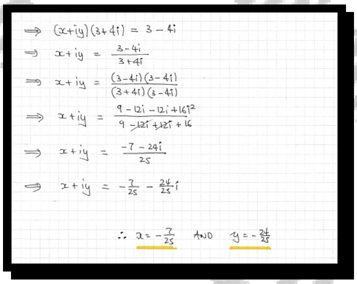


Question 3

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$(x + iy)(3 + 4i) = 3 - 4i.$$

$$\boxed{(x, y) = \left(-\frac{7}{25}, -\frac{24}{25}\right)}$$

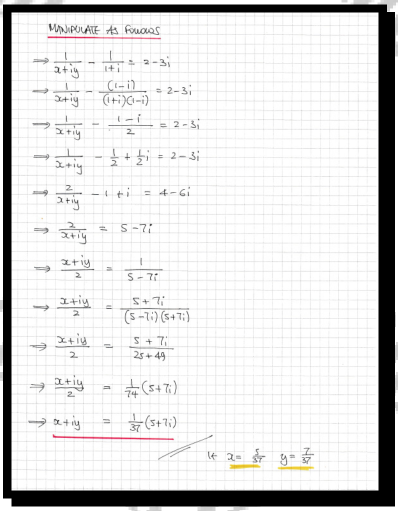


Question 4

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$\frac{1}{x+\mathrm{i}y} - \frac{1}{1+\mathrm{i}} = 2 - 3\mathrm{i}.$$

$$\boxed{(x, y) = \left(\frac{5}{37}, \frac{7}{37}\right)}$$



MANIPULATE AS FOLLOWS

$$\Rightarrow \frac{1}{x+iy} - \frac{1}{1+i} = 2-3i$$

$$\Rightarrow \frac{1}{x+iy} - \frac{(1-i)}{(1+i)(1-i)} = 2-3i$$

$$\Rightarrow \frac{1}{x+iy} - \frac{1-i}{2} = 2-3i$$

$$\Rightarrow \frac{1}{x+iy} - \frac{1}{2} + \frac{1}{2}i = 2-3i$$

$$\Rightarrow \frac{2}{x+iy} - 1 + i = 4-6i$$

$$\Rightarrow \frac{2}{x+iy} = 5-7i$$

$$\Rightarrow \frac{x+iy}{2} = \frac{1}{5-7i}$$

$$\Rightarrow \frac{x+iy}{2} = \frac{5+7i}{(5-7i)(5+7i)}$$

$$\Rightarrow \frac{x+iy}{2} = \frac{5+7i}{25+49}$$

$$\Rightarrow \frac{x+iy}{2} = \frac{1}{74}(5+7i)$$

$$\Rightarrow x+iy = \frac{1}{37}(5+7i)$$

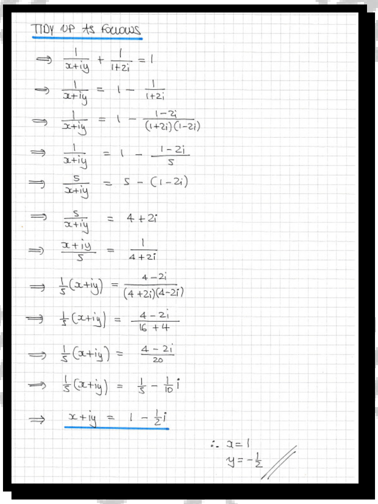
i.e. $x = \frac{5}{37}$ $y = \frac{7}{37}$

Question 5

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$\frac{1}{x+\mathrm{i}y}+\frac{1}{1+2\mathrm{i}}=1.$$

$$\boxed{(x,y)=\left(1,-\tfrac{1}{2}\right)}$$


TIDY UP AS FOLLOWS

$\Rightarrow \frac{1}{x+iy}+\frac{1}{1+2i}=1$

$\Rightarrow \frac{1}{x+iy}=1-\frac{1}{1+2i}$

$\Rightarrow \frac{1}{x+iy}=1-\frac{1-2i}{(1+2i)(1-2i)}$

$\Rightarrow \frac{1}{x+iy}=1-\frac{1-2i}{5}$

$\Rightarrow \frac{5}{x+iy}=5-(1-2i)$

$\Rightarrow \frac{5}{x+iy}=4+2i$

$\Rightarrow \frac{x+iy}{5}=\frac{1}{4+2i}$

$\Rightarrow \frac{1}{5}(x+iy)=\frac{4-2i}{(4+2i)(4-2i)}$

$\Rightarrow \frac{1}{5}(x+iy)=\frac{4-2i}{16+4}$

$\Rightarrow \frac{1}{5}(x+iy)=\frac{4-2i}{20}$

$\Rightarrow \frac{1}{5}(x+iy)=\frac{1}{5}-\frac{1}{10}i$

$\Rightarrow x+iy=1-\frac{1}{2}i$

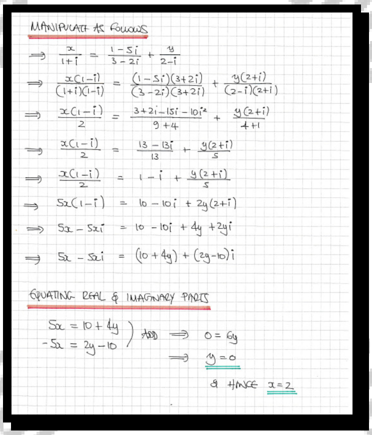
$\therefore x=1$, $y=-\frac{1}{2}$

Question 6

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$\frac{x}{1+\mathrm{i}} = \frac{1-5\mathrm{i}}{3-2\mathrm{i}} + \frac{y}{2-\mathrm{i}}.$$

$\boxed{(x, y) = (2,0)}$



MANIPULATE AS FOLLOWS

$$\Rightarrow \frac{x}{1+i} = \frac{1-5i}{3-2i} + \frac{y}{2-i}$$

$$\Rightarrow \frac{x(1-i)}{(1+i)(1-i)} = \frac{(1-5i)(3+2i)}{(3-2i)(3+2i)} + \frac{y(2+i)}{(2-i)(2+i)}$$

$$\Rightarrow \frac{x(1-i)}{2} = \frac{3+2i-15i-10i^2}{9+4} + \frac{y(2+i)}{4+1}$$

$$\Rightarrow \frac{x(1-i)}{2} = \frac{13-13i}{13} + \frac{y(2+i)}{5}$$

$$\Rightarrow \frac{x(1-i)}{2} = 1-i + \frac{y(2+i)}{5}$$

$$\Rightarrow 5x(1-i) = 10-10i + 2y(2+i)$$

$$\Rightarrow 5x - 5xi = 10 - 10i + 4y + 2yi$$

$$\Rightarrow 5x - 5xi = (10+4y) + (2y-10)i$$

EQUATING REAL & IMAGINARY PARTS

$5x = 10 + 4y$
$-5x = 2y - 10$) ADD $\Rightarrow 0 = 6y$

$\Rightarrow y = 0$

& HENCE $x = 2$

Question 7

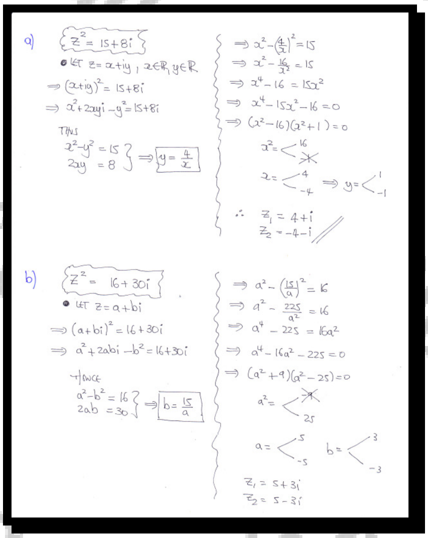
Find the square roots of the following complex numbers.

a) $15+8\mathrm{i}$

b) $16+30\mathrm{i}$

Give the answers in the form $a+b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$\boxed{\pm(4+\mathrm{i})}$, $\boxed{\pm(5+3\mathrm{i})}$



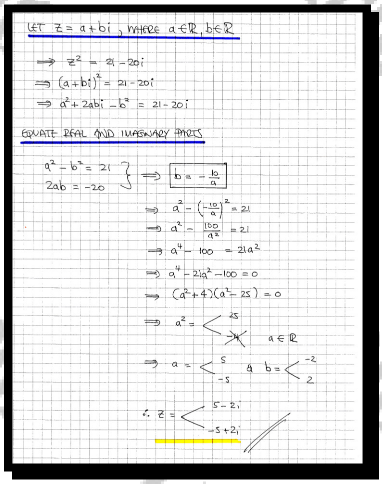
Question 8

Solve the following equation.

$$z^2 = 21 - 20\mathrm{i}, \quad z \in \mathbb{C}.$$

Give the answers in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$$\boxed{z = \pm(5 - 2\mathrm{i})}$$



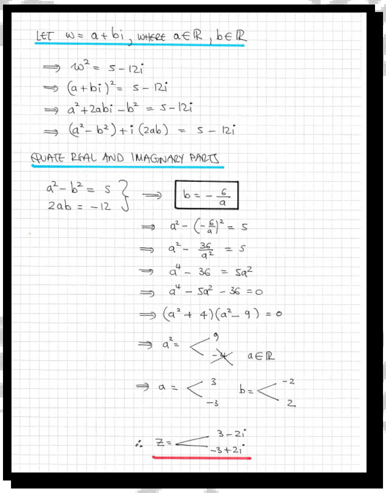
Question 9

Solve the following equation.

$$w^2 = 5 - 12\text{i}, \quad w \in \mathbb{C}.$$

Give the answers in the form $a + b\text{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$$\boxed{w = \pm(3 - 2\text{i})}$$

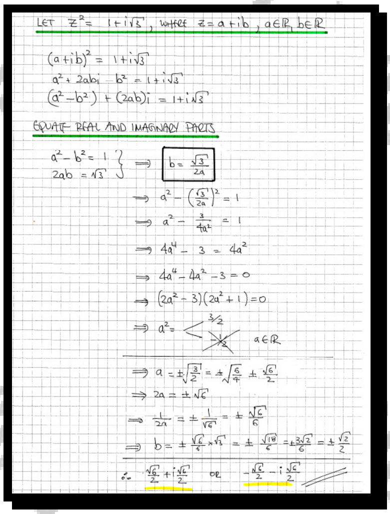


Question 10

Find the square roots of $1+i\sqrt{3}$.

Give the answers in the form $a+bi$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$$\boxed{\pm\frac{1}{2}\left(\sqrt{6}+i\sqrt{2}\right)}$$



LET $z^2 = 1+i\sqrt{3}$, WHERE $z = a+ib$, $a \in \mathbb{R}$, $b \in \mathbb{R}$

$(a+ib)^2 = 1+i\sqrt{3}$

$a^2 + 2abi - b^2 = 1+i\sqrt{3}$

$(a^2-b^2) + (2ab)i = 1+i\sqrt{3}$

EQUATE REAL AND IMAGINARY PARTS

$a^2 - b^2 = 1$, $2ab = \sqrt{3}$ $\Rightarrow$ $b = \frac{\sqrt{3}}{2a}$

$\Rightarrow a^2 - \left(\frac{\sqrt{3}}{2a}\right)^2 = 1$

$\Rightarrow a^2 - \frac{3}{4a^2} = 1$

$\Rightarrow 4a^4 - 3 = 4a^2$

$\Rightarrow 4a^4 - 4a^2 - 3 = 0$

$\Rightarrow (2a^2-3)(2a^2+1) = 0$

$\Rightarrow a^2 = \frac{3}{2}$ (or $-\frac{1}{2}$, crossed out), $a \in \mathbb{R}$

$\Rightarrow a = \pm\sqrt{\frac{3}{2}} = \pm\sqrt{\frac{6}{4}} \pm \frac{\sqrt{6}}{2}$

$\Rightarrow 2a = \pm\sqrt{6}$

$\Rightarrow \frac{1}{2a} = \pm\frac{1}{\sqrt{6}} = \pm\frac{\sqrt{6}}{6}$

$\Rightarrow b = \pm\frac{\sqrt{6}}{6}\times\sqrt{3} = \pm\frac{\sqrt{18}}{6} = \pm\frac{3\sqrt{2}}{6} = \pm\frac{\sqrt{2}}{2}$

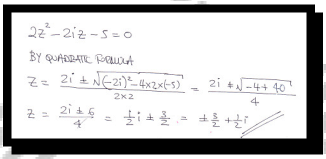
$\therefore \frac{\sqrt{6}}{2} + i\frac{\sqrt{6}}{2}$ OR $-\frac{\sqrt{6}}{2} - i\frac{\sqrt{6}}{2}$

Question 11

Solve the equation

$$2z^2 - 2\mathrm{i}z - 5 = 0, \ z \in \mathbb{C}.$$

$$\boxed{z = \pm\frac{3}{2} + \frac{1}{2}\mathrm{i}}$$



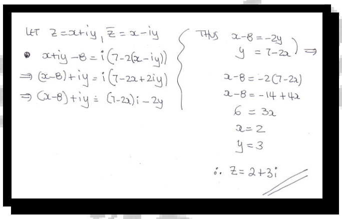
Question 12

$$z - 8 = \mathrm{i}\left(7 - 2\bar{z}\right), \ z \in \mathbb{C}.$$

The complex conjugate of z is denoted by $\bar{z}$.

Determine the value of z in the above equation, giving the answer in the form $x + \mathrm{i}y$, where x and y are real numbers.

$$\boxed{z = 2 + 3\mathrm{i}}$$



Question 13

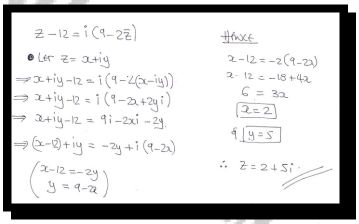
The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$z - 12 = \mathrm{i}\left(9 - 2\overline{z}\right),$$

giving the answer in the form $x + \mathrm{i}y$, where x and y are real numbers.

$\boxed{z = 2 + 5\mathrm{i}}$



Question 14

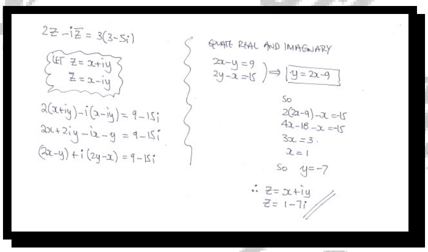
The complex number z satisfies the equation

$$2z - \mathrm{i}\overline{z} = 3\left(3 - 5\mathrm{i}\right),$$

where $\overline{z}$ denotes the complex conjugate of z.

Determine the value of z, giving the answer in the form $x + \mathrm{i}y$, where x and y are real numbers.

$\boxed{z = 1 - 7\mathrm{i}}$



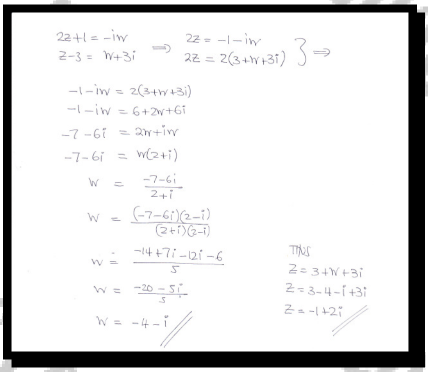
Question 15

Find the value of z and the value of w in the following simultaneous equations

$$2z+1=-\mathrm{i}w$$

$$z-3=w+3\mathrm{i}\ .$$

$$\boxed{z=-1+2\mathrm{i},\ w=-4-\mathrm{i}}$$



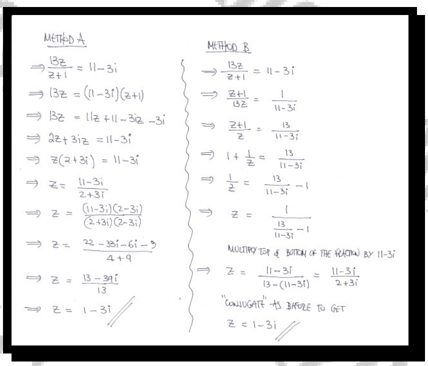
Question 16

Solve the equation

$$\frac{13z}{z+1}=11-3\mathrm{i}\ ,\quad z\in\mathbb{C}\ ,$$

giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$$\boxed{z=1-3\mathrm{i}}$$



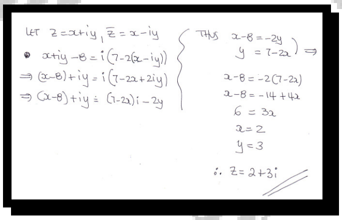
Question 17

$$z - 8 = \mathrm{i}\left(7 - 2\overline{z}\right), \ z \in \mathbb{C}.$$

The complex conjugate of z is denoted by $\overline{z}$.

Determine the value of z in the above equation, giving the answer in the form $x + \mathrm{i}y$, where x and y are real numbers.

$\boxed{z = 2 + 3\mathrm{i}}$


LET $z = x+iy$, $\bar{z} = x - iy$

• $x+iy-8 = i(7-2(x-iy))$

$\Rightarrow (x-8)+iy = i(7-2x+2iy)$

$\Rightarrow (x-8)+iy = (7-2x)i - 2y$

THUS $x-8 = -2y$, $y = 7-2x$ $\Rightarrow$

$x-8 = -2(7-2x)$

$x-8 = -14+4x$

$6 = 3x$

$x = 2$

$y = 3$

$\therefore z = 2+3i$

Question 18

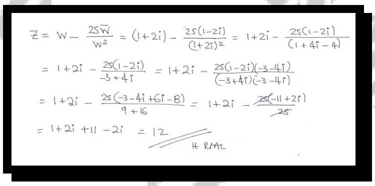
The complex conjugate of w is denoted by $\overline{w}$.

Given further that

$$w = 1 + 2\mathrm{i} \quad \text{and} \quad z = w - \frac{25\overline{w}}{w^2},$$

show clearly that z is a real number, stating its value.

$\boxed{12}$


$z = w - \frac{25\bar{w}}{w^2} = (1+2i) - \frac{25(1-2i)}{(1+2i)^2} = 1+2i - \frac{25(1-2i)}{(1+4i-4)}$

$= 1+2i - \frac{25(1-2i)}{-3+4i} = 1+2i - \frac{25(1-2i)(-3-4i)}{(-3+4i)(-3-4i)}$

$= 1+2i - \frac{25(-3-4i+6i-8)}{9+16} = 1+2i - \frac{25(-11+2i)}{25}$

$= 1+2i+11-2i = 12$

IT REAL

Question 19

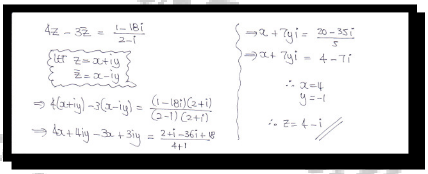
The complex number z satisfies the equation

$$4z - 3\overline{z} = \frac{1-18\text{i}}{2-\text{i}},$$

where $\overline{z}$ denotes the complex conjugate of z.

Solve the equation, giving the answer in the form $x+\text{i}y$, where x and y are real numbers.

$\boxed{z = 4-\text{i}}$



Question 20

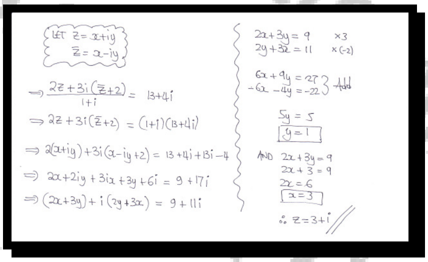
The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$\frac{2z + 3\text{i}\left(\overline{z}+2\right)}{1+\text{i}} = 13+4\text{i},$$

giving the answer in the form $x+\text{i}y$, where x and y are real numbers.

$\boxed{z = 3+\text{i}}$



Question 21

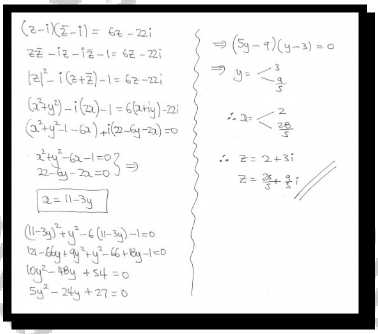
The complex conjugate of z is denoted by $\overline{z}$.

Find the two solutions of the equation

$$(z-\mathrm{i})(\overline{z}-\mathrm{i}) = 6z - 22\mathrm{i}, \quad z \in \mathbb{C},$$

giving the answers in the form $x + \mathrm{i}y$, where x and y are real numbers.

$\boxed{z_1 = 2+3\mathrm{i}}$, $\boxed{z_2 = \frac{28}{5} + \frac{9}{5}\mathrm{i}}$



Question 22

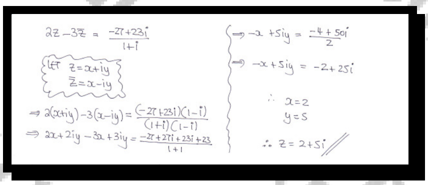
The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$2z - 3\overline{z} = \frac{-27+23\mathrm{i}}{1+\mathrm{i}},$$

giving the answer in the form $x + \mathrm{i}y$, where x and y are real numbers.

$\boxed{z = 2+5\mathrm{i}}$



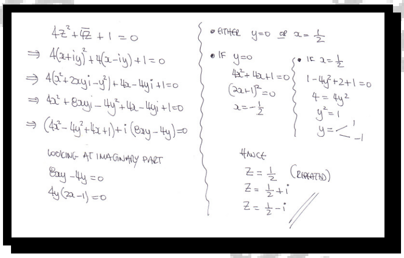
Question 23

Find the three solutions of the equation

$$4z^2 + 4\overline{z} + 1 = 0, \ z \in \mathbb{C},$$

where $\overline{z}$ denotes the complex conjugate of z.

$$\boxed{z = \frac{1}{2}, \ \frac{1}{2} + \mathrm{i}, \ \frac{1}{2} - \mathrm{i}}$$


$4z^2+4\overline{z}+1=0$
$\Rightarrow 4(x+iy)^2+4(x-iy)+1=0$
$\Rightarrow 4(x^2+2xyi-y^2)+4x-4yi+1=0$
$\Rightarrow 4x^2+8xyi-4y^2+4x-4yi+1=0$
$\Rightarrow (4x^2-4y^2+4x+1)+i(8xy-4y)=0$
LOOKING AT IMAGINARY PART
$8xy-4y=0$
$4y(2x-1)=0$
• EITHER $y=0$ or $x=\frac{1}{2}$
• IF $y=0$: $4x^2+4x+1=0$, $(2x+1)^2=0$, $x=-\frac{1}{2}$
• IF $x=\frac{1}{2}$: $1-4y^2+2+1=0$, $4=4y^2$, $y^2=1$, $y=1$ or -1
HENCE
$z=\frac{1}{2}$ (REPEATED)
$z=\frac{1}{2}+i$
$z=\frac{1}{2}-i$

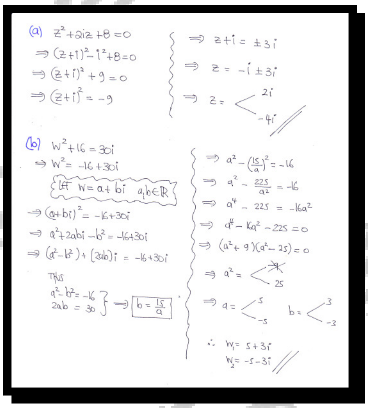
Question 24

Solve the following equations.

a) $z^2 + 2\mathrm{i}z + 8 = 0, \ z \in \mathbb{C}$.

b) $w^2 + 16 = 30\mathrm{i}, \ w \in \mathbb{C}$.

$$\boxed{z_1 = 2\mathrm{i}, \ z_2 = -4\mathrm{i}}, \ \boxed{w = \pm(3+5\mathrm{i})}$$


(a) $z^2+2iz+8=0$
$\Rightarrow (z+i)^2-i^2+8=0$
$\Rightarrow (z+i)^2+9=0$
$\Rightarrow (z+i)^2=-9$
$\Rightarrow z+i=\pm 3i$
$\Rightarrow z=-i\pm 3i$
$\Rightarrow z = 2i$ or $-4i$
(b) $w^2+16=30i$
$\Rightarrow w^2=-16+30i$
LET $w=a+bi$, $a,b\in\mathbb{R}$
$\Rightarrow (a+bi)^2=-16+30i$
$\Rightarrow a^2+2abi-b^2=-16+30i$
$\Rightarrow (a^2-b^2)+(2ab)i=-16+30i$
THUS $a^2-b^2=-16$, $2ab=30$ $\Rightarrow b=\frac{15}{a}$
$\Rightarrow a^2-\left(\frac{15}{a}\right)^2=-16$
$\Rightarrow a^2-\frac{225}{a^2}=-16$
$\Rightarrow a^4-225=-16a^2$
$\Rightarrow a^4+16a^2-225=0$
$\Rightarrow (a^2+9)(a^2-25)=0$
$\Rightarrow a^2 = 25$ (or -9, rejected)
$\Rightarrow a = 5$ or -5, $b = 3$ or -3
$\therefore w_1=5+3i$, $w_2=-5-3i$

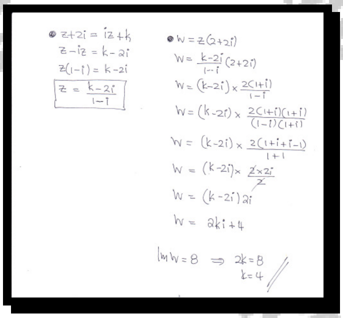
Question 25

It is given that

$$z + 2\mathrm{i} = \mathrm{i}z + k\,,\ k \in \mathbb{R} \qquad \text{and} \qquad \frac{w}{z} = 2 + 2\mathrm{i}\,,\ \operatorname{Im} w = 8\,.$$

Determine the value of k.

$\boxed{k = 4}$



Question 26

The complex number z satisfies the equation

$$z^2 = 3 + 4\text{i}\,.$$

a) Find the possible values of …

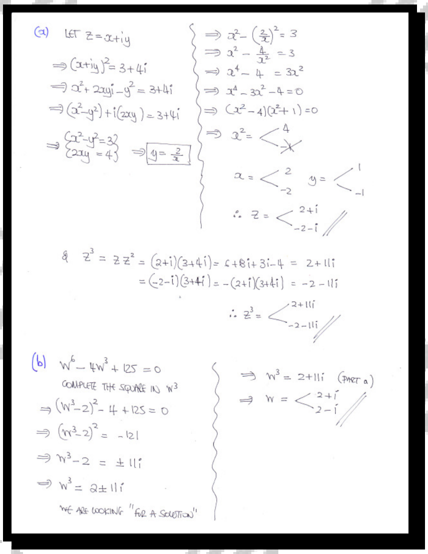
a.… z.

b.… z^3.

b) Hence, by showing detailed workings, find a solution of the equation

$$w^6 - 4w^3 + 125 = 0,\ w \in \mathbb{C},$$

$\boxed{z = \pm(2+\text{i})}$, $\boxed{z^3 = 2 \pm 11\text{i}}$, $\boxed{w = \pm(2+\text{i})}$



Question 27

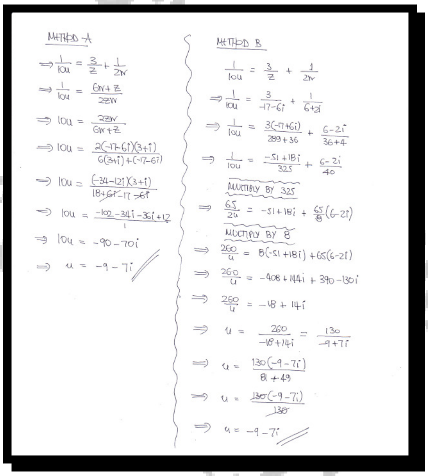
It is given that

$$z = -17 - 6\text{i} \text{ and } w = 3 + \text{i}.$$

Find the value of u given further that

$$\frac{1}{10u} = \frac{3}{z} + \frac{1}{2w}.$$

$\boxed{u = -9 - 7\text{i}}$



METHOD A

$$\Rightarrow \frac{1}{10u} = \frac{3}{z} + \frac{1}{2w}$$
$$\Rightarrow \frac{1}{10u} = \frac{6w + z}{2zw}$$
$$\Rightarrow 10u = \frac{2zw}{6w + z}$$
$$\Rightarrow 10u = \frac{2(-17-6i)(3+i)}{6(3+i)+(-17-6i)}$$
$$\Rightarrow 10u = \frac{(-34-12i)(3+i)}{18+6i-17-6i}$$
$$\Rightarrow 10u = \frac{-102-34i-36i+12}{1}$$
$$\Rightarrow 10u = -90-70i$$
$$\Rightarrow u = -9-7i$$

METHOD B

$$\frac{1}{10u} = \frac{3}{z} + \frac{1}{2w}$$
$$\Rightarrow \frac{1}{10u} = \frac{3}{-17-6i} + \frac{1}{6+2i}$$
$$\Rightarrow \frac{1}{10u} = \frac{3(-17+6i)}{289+36} + \frac{6-2i}{36+4}$$
$$\Rightarrow \frac{1}{10u} = \frac{-51+18i}{325} + \frac{6-2i}{40}$$

MULTIPLY BY 325

$$\Rightarrow \frac{65}{2u} = -51+18i + \frac{65}{8}(6-2i)$$

MULTIPLY BY 8

$$\Rightarrow \frac{260}{u} = 8(-51+18i) + 65(6-2i)$$
$$\Rightarrow \frac{260}{u} = -408+144i+390-130i$$
$$\Rightarrow \frac{260}{u} = -18+14i$$
$$\Rightarrow u = \frac{260}{-18+14i} = \frac{130}{-9+7i}$$
$$\Rightarrow u = \frac{130(-9-7i)}{81+49}$$
$$\Rightarrow u = \frac{130(-9-7i)}{130}$$
$$\Rightarrow u = -9-7i$$

Question 28

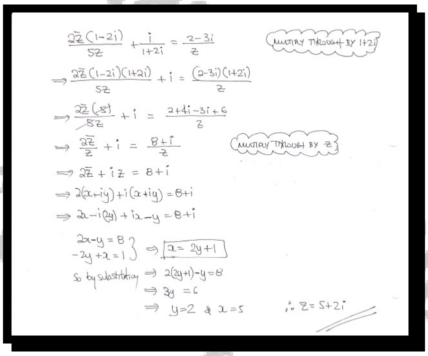
The complex conjugate of the complex number z is denoted by $\overline{z}$.

Solve the equation

$$\frac{2\overline{z}(1-2\mathrm{i})}{5z}+\frac{\mathrm{i}}{1+2\mathrm{i}}=\frac{2-3\mathrm{i}}{z},$$

giving the answer in the form $x+\mathrm{i}y$.

$\boxed{z=5+2\mathrm{i}}$



Question 29

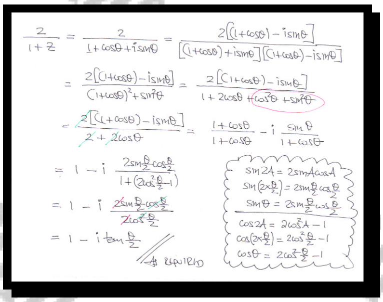
It is given that

$$z = \cos\theta + \mathrm{i}\sin\theta\,,\ 0 \le z < 2\pi\,.$$

Show clearly that

$$\frac{2}{1+z} = 1 - \mathrm{i}\tan\left(\frac{\theta}{2}\right).$$

proof



Question 30

By considering the solutions of the equation

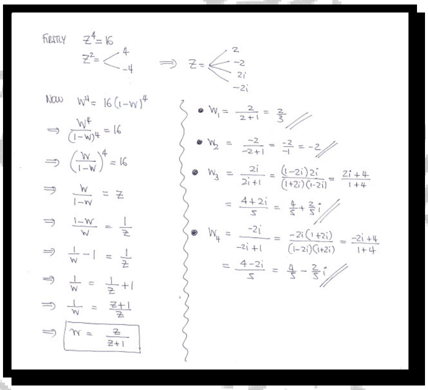
$$z^4 = 16,$$

find the solutions of the equation

$$w^4 = 16(1-w)^4,$$

giving the answers in the form $x + \mathrm{i}y$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$\boxed{z_1 = 2, \quad z_2 = \frac{2}{3}, \quad z_3 = \frac{4}{5} + \mathrm{i}\frac{2}{5}, \quad z_4 = \frac{4}{5} - \mathrm{i}\frac{2}{5}}$$



Question 31

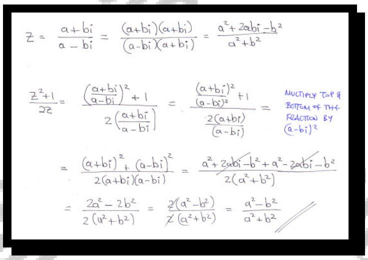
The complex number z is given by

$$z = \frac{a+bi}{a-bi}, \; a \in \mathbb{R}, \; b \in \mathbb{R}.$$

Show clearly that

$$\frac{z^2+1}{2z} = \frac{a^2-b^2}{a^2+b^2}.$$

proof



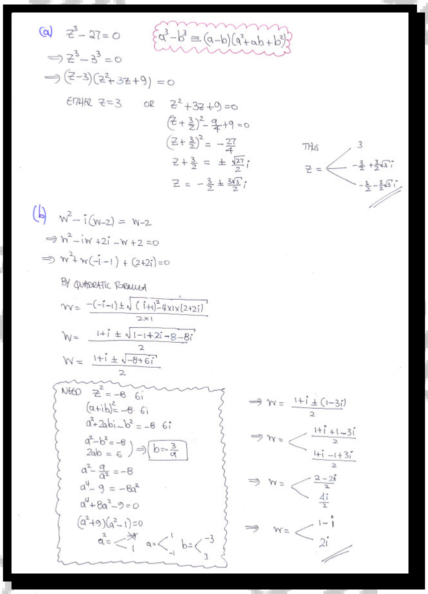
Question 32

Solve the following equations.

a) $z^3 - 27 = 0$.

b) $w^2 - \mathrm{i}(w-2) = (w-2)$.

$\boxed{z_1 = 3, \quad z_2 = \frac{3}{2}\left(-1 \pm \sqrt{3}\right)}$, $\boxed{w_1 = 2\mathrm{i}, \quad w_2 = 1 - \mathrm{i}}$



(a) $z^3 - 27 = 0$ $\quad a^3 - b^3 \equiv (a-b)(a^2+ab+b^2)$

$\Rightarrow z^3 - 3^3 = 0$

$\Rightarrow (z-3)(z^2+3z+9) = 0$

EITHER $z = 3$ OR $z^2+3z+9=0$

$\left(z+\frac{3}{2}\right)^2 - \frac{9}{4} + 9 = 0$

$\left(z+\frac{3}{2}\right)^2 = -\frac{27}{4}$

$z + \frac{3}{2} = \pm \frac{\sqrt{27}}{2}i$

$z = -\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$

THUS $z = 3, \; -\frac{3}{2}+\frac{3}{2}\sqrt{3}i, \; -\frac{3}{2}-\frac{3}{2}\sqrt{3}i$

(b) $w^2 - i(w-2) = w-2$

$\Rightarrow w^2 - iw + 2i - w + 2 = 0$

$\Rightarrow w^2 + w(-i-1) + (2+2i) = 0$

BY QUADRATIC FORMULA

$w = \frac{-(-i-1) \pm \sqrt{(i+1)^2 - 4\times1\times(2+2i)}}{2\times1}$

$w = \frac{1+i \pm \sqrt{1-1+2i-8-8i}}{2}$

$w = \frac{1+i \pm \sqrt{-8+6i}}{2}$

NEED $z^2 = -8\ 6i$

$(a+ib)^2 = -8\ 6i$

$a^2 + 2abi - b^2 = -8\ 6i$

$a^2 - b^2 = -8$, $2ab = 6$ $\Rightarrow b = -\frac{3}{a}$

$a^2 - \frac{9}{a^2} = -8$

$a^4 - 9 = -8a^2$

$a^4 + 8a^2 - 9 = 0$

$(a^2+9)(a^2-1) = 0$

$a^2 = 1$, $a = 1, -1$, $b = -3, 3$

$\Rightarrow w = \frac{1+i \pm (1-3i)}{2}$

$\Rightarrow w = \frac{1+i+1-3i}{2}, \; \frac{1+i-1+3i}{2}$

$\Rightarrow w = \frac{2-2i}{2}, \; \frac{4i}{2}$

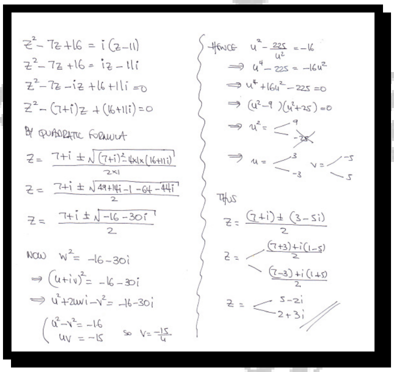
$\Rightarrow w = 1-i, \; 2i$

Question 33

Solve the quadratic equation

$$z^2 - 7z + 16 = \mathrm{i}(z - 11), \; z \in \mathbb{C}.$$

$$\boxed{z = 2 + 3\mathrm{i}, \; z = 5 - 2\mathrm{i}}$$



$z^2 - 7z + 16 = i(z-11)$
$z^2 - 7z + 16 = iz - 11i$
$z^2 - 7z - iz + 16 + 11i = 0$
$z^2 - (7+i)z + (16+11i) = 0$

BY QUADRATIC FORMULA

$z = \dfrac{7+i \pm \sqrt{(7+i)^2 - 4\times1\times(16+11i)}}{2\times1}$

$z = \dfrac{7+i \pm \sqrt{49+14i-1-64-44i}}{2}$

$z = \dfrac{7+i \pm \sqrt{-16-30i}}{2}$

NOW $w^2 = -16-30i$
$\Rightarrow (u+iv)^2 = -16-30i$
$\Rightarrow u^2 + 2uvi - v^2 = -16-30i$
$u^2 - v^2 = -16$
$uv = -15$ so $v = -\frac{15}{u}$

HENCE $u^2 - \frac{225}{u^2} = -16$
$\Rightarrow u^4 - 225 = -16u^2$
$\Rightarrow u^4 + 16u^2 - 225 = 0$
$\Rightarrow (u^2-9)(u^2+25) = 0$
$\Rightarrow u^2 = 9, \; -25$ (rejected)
$\Rightarrow u = 3, -3 \quad v = -5, 5$

THUS

$z = \dfrac{(7+i) \pm (3-5i)}{2}$

$z = \dfrac{(7+3)+i(1-5)}{2}, \; \dfrac{(7-3)+i(1+5)}{2}$

$z = 5-2i, \; 2+3i$

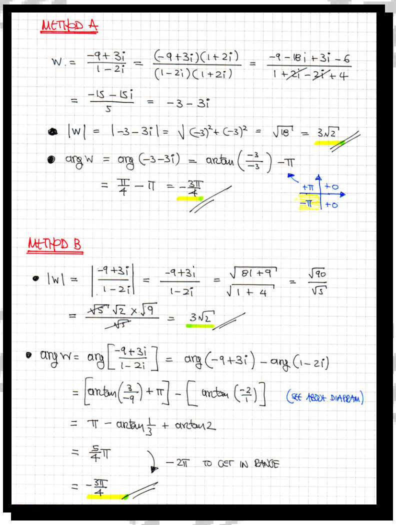
MODULUS AND ARGUMENT

Question 1

$$w = \frac{-9+3\text{i}}{1-2\text{i}}.$$

Find the modulus and the argument of the complex number w.

$$\boxed{|w| = 3\sqrt{2}}, \quad \boxed{\arg w = -\frac{3\pi}{4}}$$


METHOD A

$w = \frac{-9+3i}{1-2i} = \frac{(-9+3i)(1+2i)}{(1-2i)(1+2i)} = \frac{-9-18i+3i-6}{1+2i-2i+4}$

$= \frac{-15-15i}{5} = -3-3i$

• $|w| = |-3-3i| = \sqrt{(-3)^2+(-3)^2} = \sqrt{18} = 3\sqrt{2}$

• $\arg w = \arg(-3-3i) = \arctan\left(\frac{-3}{-3}\right) - \pi$

$= \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$

$+\pi$, $+0$, $-\pi$, $+0$

METHOD B

• $|w| = \left|\frac{-9+3i}{1-2i}\right| = \frac{|-9+3i|}{|1-2i|} = \frac{\sqrt{81+9}}{\sqrt{1+4}} = \frac{\sqrt{90}}{\sqrt{5}}$

$= \frac{\sqrt{5}\sqrt{2}\times\sqrt{9}}{\sqrt{5}} = 3\sqrt{2}$

• $\arg w = \arg\left[\frac{-9+3i}{1-2i}\right] = \arg(-9+3i) - \arg(1-2i)$

$= \left[\arctan\left(\frac{3}{-9}\right)+\pi\right] - \left[\arctan\left(\frac{-2}{1}\right)\right]$ (SEE ABOVE DIAGRAM)

$= \pi - \arctan\frac{1}{3} + \arctan 2$

$= \frac{5}{4}\pi$ → -2π TO GET IN RANGE

$= -\frac{3\pi}{4}$

Question 2

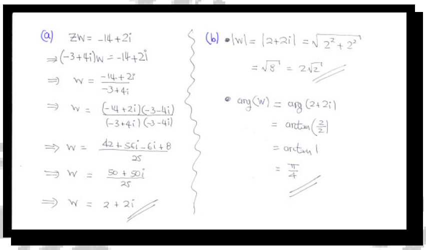
$$z = -3 + 4\mathrm{i} \qquad \text{and} \qquad zw = -14 + 2\mathrm{i}.$$

By showing clear workings, find …

a) … w in the form $a + b\mathrm{i}$, where a and b are real numbers.

b) … the modulus and the argument of w.

$\boxed{w = 2 + 2\mathrm{i}}$, $\boxed{|w| = 2\sqrt{2}}$, $\boxed{\arg w = \frac{\pi}{4}}$



Question 3

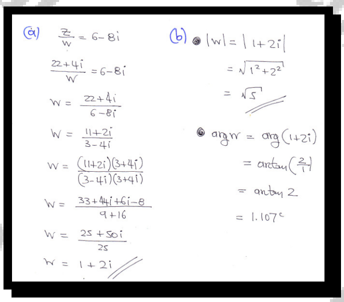
$$z = 22 + 4\mathrm{i} \qquad \text{and} \qquad \frac{z}{w} = 6 - 8\mathrm{i}.$$

By showing clear workings, find …

a) … w in the form $a + b\mathrm{i}$, where a and b are real numbers .

b) … the modulus and the argument of w.

$\boxed{w = 1 + 2\mathrm{i}}$, $\boxed{|w| = \sqrt{5}}$, $\boxed{\arg w \approx 1.11^{\mathrm{c}}}$

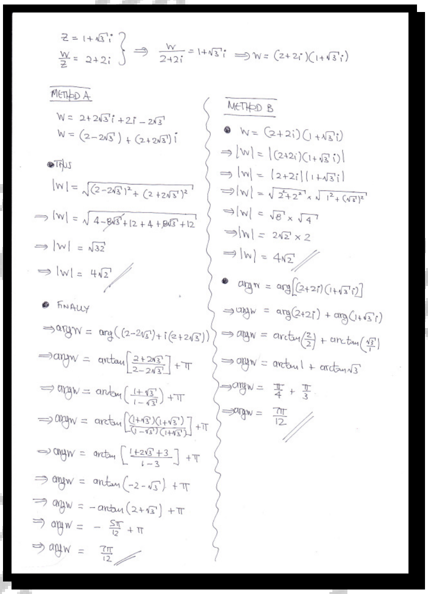


Question 4

$$z = 1 + \sqrt{3}\mathrm{i} \qquad \text{and} \qquad \frac{w}{z} = 2 + 2\mathrm{i}\,.$$

Find the exact value of the modulus of w and the exact value of the argument of w.

$\boxed{|w| = 4\sqrt{2}}$, $\boxed{\arg w = \frac{7\pi}{12}}$



$z = 1+\sqrt{3}i$, $\frac{w}{z} = 2+2i$ $\Rightarrow \frac{w}{2+2i} = 1+\sqrt{3}i \Rightarrow w = (2+2i)(1+\sqrt{3}i)$

METHOD A

$w = 2+2\sqrt{3}i+2i-2\sqrt{3}$

$w = (2-2\sqrt{3}) + (2+2\sqrt{3})i$

• THUS

$|w| = \sqrt{(2-2\sqrt{3})^2 + (2+2\sqrt{3})^2}$

$\Rightarrow |w| = \sqrt{4-8\sqrt{3}+12+4+8\sqrt{3}+12}$

$\Rightarrow |w| = \sqrt{32}$

$\Rightarrow |w| = 4\sqrt{2}$

• FINALLY

$\Rightarrow \arg w = \arg\left((2-2\sqrt{3}) + i(2+2\sqrt{3})\right)$

$\Rightarrow \arg w = \arctan\left[\frac{2+2\sqrt{3}}{2-2\sqrt{3}}\right] + \pi$

$\Rightarrow \arg w = \arctan\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right) + \pi$

$\Rightarrow \arg w = \arctan\left[\frac{(1+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}\right] + \pi$

$\Rightarrow \arg w = \arctan\left[\frac{1+2\sqrt{3}+3}{1-3}\right] + \pi$

$\Rightarrow \arg w = \arctan(-2-\sqrt{3}) + \pi$

$\Rightarrow \arg w = -\arctan(2+\sqrt{3}) + \pi$

$\Rightarrow \arg w = -\frac{5\pi}{12} + \pi$

$\Rightarrow \arg w = \frac{7\pi}{12}$

METHOD B

• $w = (2+2i)(1+\sqrt{3}i)$

$\Rightarrow |w| = |(2+2i)(1+\sqrt{3}i)|$

$\Rightarrow |w| = |2+2i||1+\sqrt{3}i|$

$\Rightarrow |w| = \sqrt{2^2+2^2} \times \sqrt{1^2+(\sqrt{3})^2}$

$\Rightarrow |w| = \sqrt{8} \times \sqrt{4}$

$\Rightarrow |w| = 2\sqrt{2} \times 2$

$\Rightarrow |w| = 4\sqrt{2}$

• $\arg w = \arg\left[(2+2i)(1+\sqrt{3}i)\right]$

$\Rightarrow \arg w = \arg(2+2i) + \arg(1+\sqrt{3}i)$

$\Rightarrow \arg w = \arctan\left(\frac{2}{2}\right) + \arctan\left(\frac{\sqrt{3}}{1}\right)$

$\Rightarrow \arg w = \arctan 1 + \arctan\sqrt{3}$

$\Rightarrow \arg w = \frac{\pi}{4} + \frac{\pi}{3}$

$\Rightarrow \arg w = \frac{7\pi}{12}$

Question 5

The following complex numbers are given.

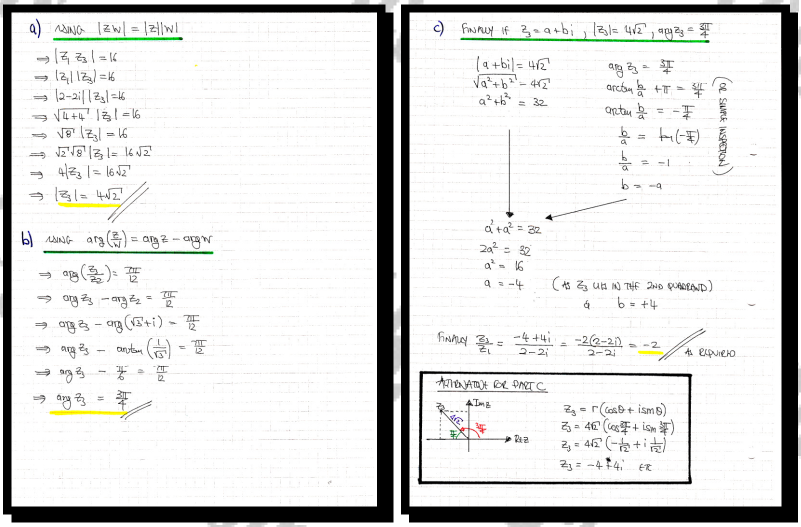
$$z_1 = 2 - 2\mathrm{i}, \quad z_2 = \sqrt{3} + \mathrm{i} \quad \text{and} \quad z_3 = a + b\mathrm{i} \quad \text{where } a \in \mathbb{R},\ b \in \mathbb{R}.$$

a) If $|z_1 z_3| = 16$, find the modulus z_3.

b) Given further that $\arg\left(\dfrac{z_3}{z_2}\right) = \dfrac{7\pi}{12}$, determine the argument of z_3.

c) Find the values of a and b, and hence show $\dfrac{z_3}{z_1} = -2$.

$$\boxed{|z_3| = 4\sqrt{2}}, \quad \boxed{\arg z_3 = \frac{3\pi}{4}}, \quad \boxed{a = -4}, \quad \boxed{b = 4}$$


a) USING $|zw| = |z||w|$

$\Rightarrow |z_1 z_3| = 16$

$\Rightarrow |z_1||z_3| = 16$

$\Rightarrow |2-2i||z_3| = 16$

$\Rightarrow \sqrt{4+4}\,|z_3| = 16$

$\Rightarrow \sqrt{8}\,|z_3| = 16$

$\Rightarrow \sqrt{2}\sqrt{8}\,|z_3| = 16\sqrt{2}$

$\Rightarrow 4|z_3| = 16\sqrt{2}$

$\Rightarrow |z_3| = 4\sqrt{2}$

b) USING $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$

$\Rightarrow \arg\left(\frac{z_3}{z_2}\right) = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 - \arg z_2 = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 - \arg(\sqrt{3}+i) = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 - \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 - \frac{\pi}{6} = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 = \frac{3\pi}{4}$

c) FINALLY IF $z_3 = a+bi$, $|z_3| = 4\sqrt{2}$, $\arg z_3 = \frac{3\pi}{4}$

$|a+bi| = 4\sqrt{2}$

$\sqrt{a^2+b^2} = 4\sqrt{2}$

$a^2+b^2 = 32$

$\arg z_3 = \frac{3\pi}{4}$

$\arctan\frac{b}{a} + \pi = \frac{3\pi}{4}$ (OR SIMPLE INSPECTION)

$\arctan\frac{b}{a} = -\frac{\pi}{4}$

$\frac{b}{a} = \tan\left(-\frac{\pi}{4}\right)$

$\frac{b}{a} = -1$

$b = -a$

$a^2 + a^2 = 32$

$2a^2 = 32$

$a^2 = 16$

$a = -4$ (AS z_3 LIES IN THE 2ND QUADRANT)

& $b = +4$

FINALLY $\frac{z_3}{z_1} = \frac{-4+4i}{2-2i} = \frac{-2(2-2i)}{2-2i} = -2$ AS REQUIRED

ALTERNATIVE FOR PART C

$z_3 = r(\cos\theta + i\sin\theta)$

$z_3 = 4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$z_3 = 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$z_3 = -4 + 4i$ ETC

Question 6

$$z = \sqrt{3} + \mathrm{i} \quad \text{and} \quad w = 3\mathrm{i}.$$

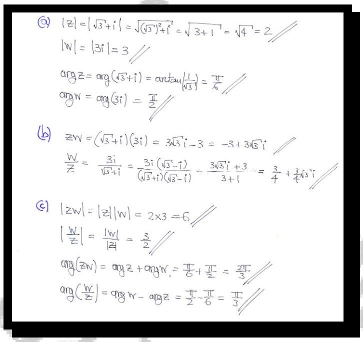
a) Find, in exact form where appropriate, the modulus and argument of z and the modulus and argument of w.

b) Determine simplified expressions for zw and $\frac{w}{z}$, giving the answers in the form $x + \mathrm{i}y$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$.

c) Find, in exact form where appropriate, the modulus and argument of zw and the modulus and argument of $\frac{w}{z}$.

$\boxed{|z| = 2,\ |w| = 3}$, $\boxed{\arg z = \frac{\pi}{6},\ \arg w = \frac{\pi}{2}}$, $\boxed{zw = -3 + 3\sqrt{3}\mathrm{i}}$, $\boxed{\frac{w}{z} = \frac{3}{4} + \frac{3}{4}\sqrt{3}\mathrm{i}}$,

$\boxed{|zw| = 6,\ \left|\frac{w}{z}\right| = \frac{3}{2}}$, $\boxed{\arg(zw) = \frac{2\pi}{3}, \arg\left(\frac{w}{z}\right) = \frac{\pi}{3}}$


(a) $|z| = |\sqrt{3}+i| = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{3+1} = \sqrt{4} = 2$

$|w| = |3i| = 3$

$\arg z = \arg(\sqrt{3}+i) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\arg w = \arg(3i) = \frac{\pi}{2}$

(b) $zw = (\sqrt{3}+i)(3i) = 3\sqrt{3}i - 3 = -3+3\sqrt{3}i$

$\frac{w}{z} = \frac{3i}{\sqrt{3}+i} = \frac{3i(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)} = \frac{3\sqrt{3}i+3}{3+1} = \frac{3}{4} + \frac{3}{4}\sqrt{3}i$

(c) $|zw| = |z||w| = 2\times3 = 6$

$\left|\frac{w}{z}\right| = \frac{|w|}{|z|} = \frac{3}{2}$

$\arg(zw) = \arg z + \arg w = \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3}$

$\arg\left(\frac{w}{z}\right) = \arg w - \arg z = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

Question 7

The following complex numbers are given

$$z = \frac{1+\mathrm{i}}{1-\mathrm{i}} \quad \text{and} \quad w = \frac{\sqrt{2}}{1-\mathrm{i}}.$$

a) Calculate the modulus of z and the modulus of w.

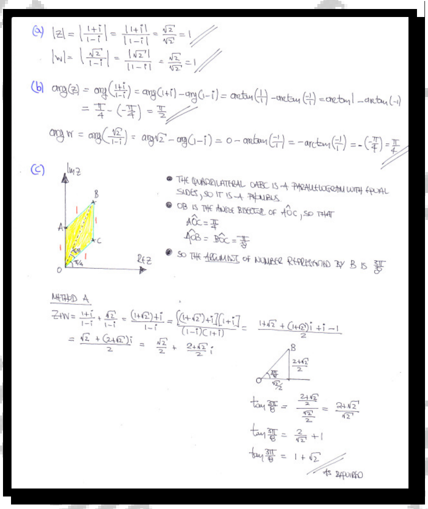
b) Find the argument of z and the argument of w.

In a standard Argand diagram, the points A, B and C represent the numbers z, $z+w$ and w respectively. The origin of the Argand diagram is denoted by O.

c) By considering the quadrilateral $OABC$ and the argument of $z+w$, show that

$$\tan\left(\frac{3\pi}{8}\right) = 1+\sqrt{2}.$$

$\boxed{|z|=1}$, $\boxed{|w|=1}$ $\boxed{\arg z = \frac{\pi}{2}}$, $\boxed{\arg w = \frac{\pi}{4}}$



(a) $|z| = \left|\frac{1+i}{1-i}\right| = \frac{|1+i|}{|1-i|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$

$|w| = \left|\frac{\sqrt{2}}{1-i}\right| = \frac{|\sqrt{2}|}{|1-i|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$

(b) $\arg(z) = \arg\left(\frac{1+i}{1-i}\right) = \arg(1+i) - \arg(1-i) = \arctan\left(\frac{1}{1}\right) - \arctan\left(\frac{-1}{1}\right) = \arctan 1 - \arctan(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

$\arg w = \arg\left(\frac{\sqrt{2}}{1-i}\right) = \arg\sqrt{2} - \arg(1-i) = 0 - \arctan\left(\frac{-1}{1}\right) = -\arctan\left(\frac{-1}{1}\right) = -\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$

(c)

- THE QUADRILATERAL OABC IS A PARALLELOGRAM WITH EQUAL SIDES, SO IT IS A RHOMBUS
- OB IS THE ANGLE BISECTOR OF $A\hat{O}C$, SO THAT $A\hat{O}C = \frac{\pi}{4}$, $A\hat{O}B = B\hat{O}C = \frac{\pi}{8}$
- SO THE ARGUMENT OF NUMBER REPRESENTED BY B IS $\frac{3\pi}{8}$

METHOD A

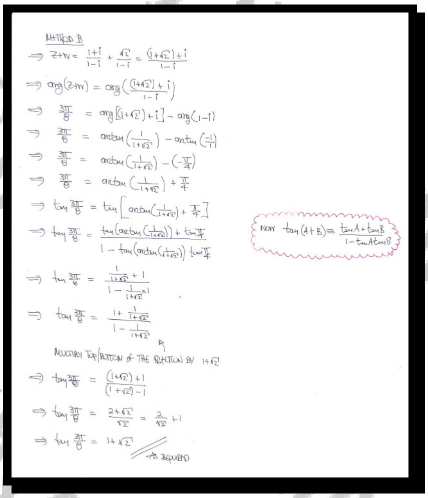
$z+w = \frac{1+i}{1-i} + \frac{\sqrt{2}}{1-i} = \frac{(1+\sqrt{2})+i}{1-i} = \frac{[(1+\sqrt{2})+i][1+i]}{(1-i)(1+i)} = \frac{1+\sqrt{2}+(1+\sqrt{2})i+i-1}{2}$

$= \frac{\sqrt{2}+(2+\sqrt{2})i}{2} = \frac{\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}i$

$\tan\frac{3\pi}{8} = \frac{\frac{2+\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2+\sqrt{2}}{\sqrt{2}}$

$\tan\frac{3\pi}{8} = \frac{2}{\sqrt{2}} + 1$

$\tan\frac{3\pi}{8} = 1+\sqrt{2}$ AS REQUIRED



METHOD B

$\Rightarrow z+w = \frac{1+i}{1-i} + \frac{\sqrt{2}}{1-i} = \frac{(1+\sqrt{2})+i}{1-i}$

$\Rightarrow \arg(z+w) = \arg\left(\frac{(1+\sqrt{2})+i}{1-i}\right)$

$\Rightarrow \frac{3\pi}{8} = \arg[(1+\sqrt{2})+i] - \arg(1-i)$

$\Rightarrow \frac{3\pi}{8} = \arctan\left(\frac{1}{1+\sqrt{2}}\right) - \arctan\left(\frac{-1}{1}\right)$

$\Rightarrow \frac{3\pi}{8} = \arctan\left(\frac{1}{1+\sqrt{2}}\right) - \left(-\frac{\pi}{4}\right)$

$\Rightarrow \frac{3\pi}{8} = \arctan\left(\frac{1}{1+\sqrt{2}}\right) + \frac{\pi}{4}$

$\Rightarrow \tan\frac{3\pi}{8} = \tan\left[\arctan\left(\frac{1}{1+\sqrt{2}}\right) + \frac{\pi}{4}\right]$

$\Rightarrow \tan\frac{3\pi}{8} = \frac{\tan\left(\arctan\left(\frac{1}{1+\sqrt{2}}\right)\right) + \tan\frac{\pi}{4}}{1 - \tan\left(\arctan\left(\frac{1}{1+\sqrt{2}}\right)\right)\tan\frac{\pi}{4}}$

NOW $\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\Rightarrow \tan\frac{3\pi}{8} = \frac{\frac{1}{1+\sqrt{2}} + 1}{1 - \frac{1}{1+\sqrt{2}} \times 1}$

$\Rightarrow \tan\frac{3\pi}{8} = \frac{1 + \frac{1}{1+\sqrt{2}}}{1 - \frac{1}{1+\sqrt{2}}}$

MULTIPLY TOP/BOTTOM OF THE FRACTION BY $1+\sqrt{2}$

$\Rightarrow \tan\frac{3\pi}{8} = \frac{(1+\sqrt{2})+1}{(1+\sqrt{2})-1}$

$\Rightarrow \tan\frac{3\pi}{8} = \frac{2+\sqrt{2}}{\sqrt{2}} = \frac{2}{\sqrt{2}} + 1$

$\Rightarrow \tan\frac{3\pi}{8} = 1+\sqrt{2}$ AS REQUIRED

Question 8

$$\frac{(3+4\mathrm{i})(1+2\mathrm{i})}{1+3\mathrm{i}} = q(1+\mathrm{i}) \ , \quad q \in \mathbb{R} \ .$$

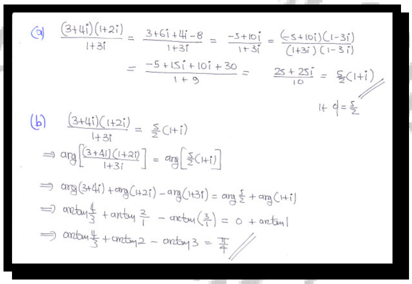
a) Find the value of q.

b) Hence simplify

$$\arctan\frac{4}{3} + \arctan 2 - \arctan 3 \, ,$$

giving the answer in terms of π.

$\boxed{q = \frac{5}{2}}$, $\boxed{\frac{1}{4}\pi}$



(a) $\frac{(3+4i)(1+2i)}{1+3i} = \frac{3+6i+4i-8}{1+3i} = \frac{-5+10i}{1+3i} = \frac{(-5+10i)(1-3i)}{(1+3i)(1-3i)}$

$= \frac{-5+15i+10i+30}{1+9} = \frac{25+25i}{10} = \frac{5}{2}(1+i)$

i.e. $q = \frac{5}{2}$

(b) $\frac{(3+4i)(1+2i)}{1+3i} = \frac{5}{2}(1+i)$

$\Rightarrow \arg\left[\frac{(3+4i)(1+2i)}{1+3i}\right] = \arg\left[\frac{5}{2}(1+i)\right]$

$\Rightarrow \arg(3+4i) + \arg(1+2i) - \arg(1+3i) = \arg\frac{5}{2} + \arg(1+i)$

$\Rightarrow \arctan\frac{4}{3} + \arctan\frac{2}{1} - \arctan\left(\frac{3}{1}\right) = 0 + \arctan 1$

$\Rightarrow \arctan\frac{4}{3} + \arctan 2 - \arctan 3 = \frac{\pi}{4}$

Question 9

It is given that

$$z = \frac{1+8\text{i}}{1-2\text{i}} \ .$$

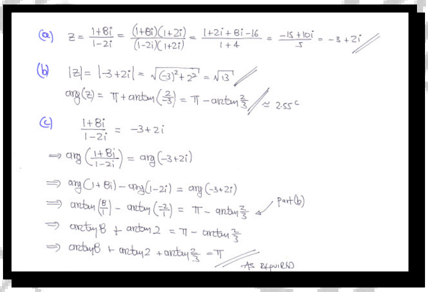
a) Express z in the form $x + \text{i}y$.

b) Find the modulus and argument of z.

c) Show clearly that

$$\arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \pi \ .$$

$\boxed{z = -3 + 2\text{i}}$, $\boxed{|z| = \sqrt{13}}$, $\boxed{\arg z \approx 2.55^{\text{c}}}$



COMPLEX POLYNOMIAL QUESTIONS

Question 1

The cubic equation

$$2z^3 - 5z^2 + cz - 5 = 0, \; c \in \mathbb{R},$$

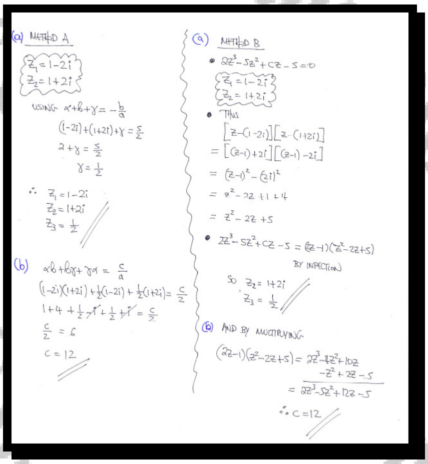
has a solution $z = 1 - 2\text{i}$.

Find in any order …

a) … the other two solutions of the equations.

b) … the value of c.

$\boxed{z_2 = 1 + 2\text{i}}$, $\boxed{z_3 = \frac{1}{2}}$, $\boxed{c = 12}$



Question 2

The following cubic equation is given

$$z^3 + az^2 + bz - 5 = 0,$$

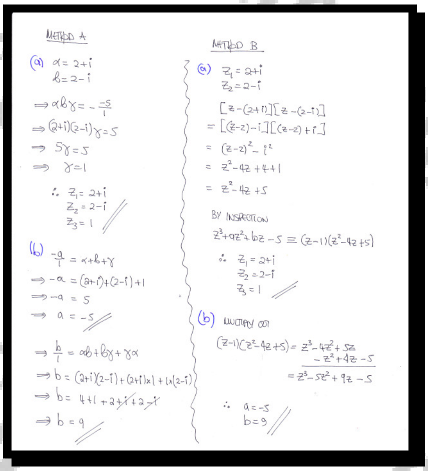
where $a \in \mathbb{R}$, $b \in \mathbb{R}$.

One of the roots of the above cubic equation is $2 + \mathrm{i}$.

a) Find the other two roots.

b) Determine the value of a and the value of b.

$\boxed{z_2 = 2 - \mathrm{i}}$, $\boxed{z_3 = 1}$, $\boxed{a = -5}$, $\boxed{b = 9}$



Question 3

The following cubic equation is given

$$z^3 + pz^2 + 6z + q = 0,$$

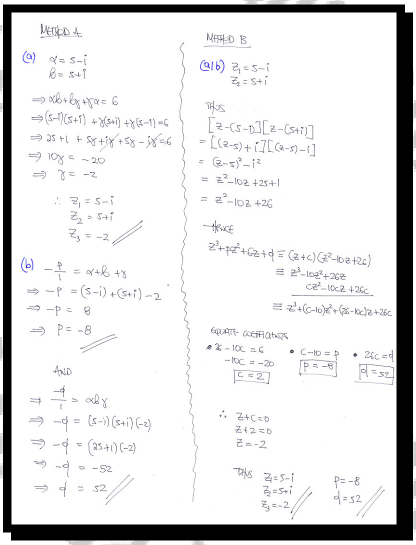
where $p \in \mathbb{R}$, $q \in \mathbb{R}$.

One of the three solutions of the above cubic equation is $5 - \mathrm{i}$.

a) Find the other two solutions of the equation.

b) Determine the value of p and the value of q.

$\boxed{z_2 = 5 + \mathrm{i},\ z_3 = 2}$, $\boxed{p = -8}$, $\boxed{q = 52}$



METHOD A

(a) $\alpha = 5 - i$
$\beta = 5 + i$

$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = 6$
$\Rightarrow (5-i)(5+i) + \gamma(5+i) + \gamma(5-i) = 6$
$\Rightarrow 25 + 1 + 5\gamma + i\gamma + 5\gamma - i\gamma = 6$
$\Rightarrow 10\gamma = -20$
$\Rightarrow \gamma = -2$

$\therefore z_1 = 5 - i$
$z_2 = 5 + i$
$z_3 = -2$

(b) $-\frac{p}{1} = \alpha + \beta + \gamma$
$\Rightarrow -p = (5-i) + (5+i) - 2$
$\Rightarrow -p = 8$
$\Rightarrow p = -8$

AND

$\Rightarrow \frac{-q}{1} = \alpha\beta\gamma$
$\Rightarrow -q = (5-i)(5+i)(-2)$
$\Rightarrow -q = (25+1)(-2)$
$\Rightarrow -q = -52$
$\Rightarrow q = 52$

METHOD B

(a|b) $z_1 = 5 - i$
$z_2 = 5 + i$

THUS
$[z - (5-i)][z - (5+i)]$
$= [(z-5) + i][(z-5) - i]$
$= (z-5)^2 - i^2$
$= z^2 - 10z + 25 + 1$
$= z^2 - 10z + 26$

HENCE
$z^3 + pz^2 + 6z + q \equiv (z+c)(z^2 - 10z + 26)$
$\equiv z^3 - 10z^2 + 26z$
$\quad cz^2 - 10cz + 26c$
$\equiv z^3 + (c-10)z^2 + (26-10c)z + 26c$

EQUATE COEFFICIENTS
- $26 - 10c = 6$, $-10c = -20$, $\boxed{c = 2}$
- $c - 10 = p$, $\boxed{p = -8}$
- $26c = q$, $\boxed{q = 52}$

$\therefore z + c = 0$
$z + 2 = 0$
$z = -2$

THUS $z_1 = 5 - i$, $z_2 = 5 + i$, $z_3 = -2$; $p = -8$, $q = 52$

Question 4

The following cubic equation is given

$$z^3 + 2z^2 + az + b = 0,$$

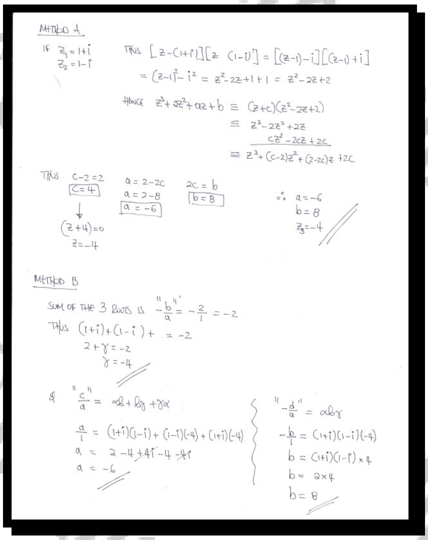
where $a \in \mathbb{R}$, $b \in \mathbb{R}$.

One of the roots of the above cubic equation is $1 + \mathrm{i}$.

a) Find the real root of the equation.

b) Find the value of a and the value of b.

$\boxed{z = -4}$, $\boxed{a = -6}$, $\boxed{b = 8}$



Question 5

The following cubic equation is given

$$z^3 + Az^2 + Bz + 26 = 0,$$

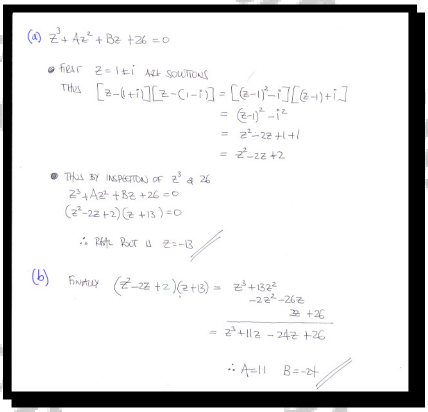
where $A \in \mathbb{R}$, $B \in \mathbb{R}$

One of the roots of the above cubic equation is $1+\mathrm{i}$.

a) Find the real root of the equation.

b) Determine the value of A and the value of B.

$\boxed{z=-13}$, $\boxed{A=11}$, $\boxed{B=-24}$


(a) $z^3 + Az^2 + Bz + 26 = 0$

- FIRST $z = 1 \pm i$ ARE SOLUTIONS

THUS $[z-(1+i)][z-(1-i)] = [(z-1)-i][(z-1)+i]$

$= (z-1)^2 - i^2$

$= z^2 - 2z + 1 + 1$

$= z^2 - 2z + 2$

- THUS BY INSPECTION OF z^3 & 26

$z^3 + Az^2 + Bz + 26 = 0$

$(z^2 - 2z + 2)(z + 13) = 0$

$\therefore$ REAL ROOT IS $z = -13$

(b) FINALLY $(z^2 - 2z + 2)(z + 13) = z^3 + 13z^2 - 2z^2 - 26z + 2z + 26$

$= z^3 + 11z - 24z + 26$

$\therefore A = 11 \quad B = -24$

Question 6

The cubic equation

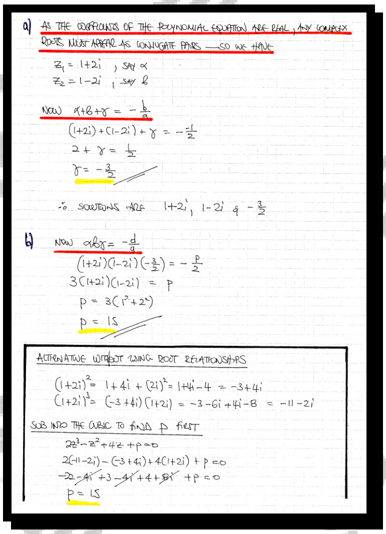
$$2z^3 - z^2 + 4z + p = 0, \ p \in \mathbb{R},$$

is satisfied by $z = 1 + 2\text{i}$.

a) Find the other two roots of the equation.

b) Determine the value of p.

$\boxed{1 - 2\text{i}, -\frac{3}{2}}$, $\boxed{p = 15}$



a) AS THE COEFFICIENTS OF THE POLYNOMIAL EQUATION ARE REAL, ANY COMPLEX ROOTS MUST APPEAR AS CONJUGATE PAIRS — SO WE HAVE

$z_1 = 1+2i$, SAY α

$z_2 = 1-2i$, SAY β

NOW $\alpha+\beta+\gamma = -\frac{b}{a}$

$(1+2i)+(1-2i)+\gamma = -\frac{-1}{2}$

$2+\gamma = \frac{1}{2}$

$\gamma = -\frac{3}{2}$

$\therefore$ SOLUTIONS ARE $1+2i$, $1-2i$ & $-\frac{3}{2}$

b) NOW $\alpha\beta\gamma = -\frac{d}{a}$

$(1+2i)(1-2i)\left(-\frac{3}{2}\right) = -\frac{p}{2}$

$3(1+2i)(1-2i) = p$

$p = 3(1^2+2^2)$

$p = 15$

ALTERNATIVE WITHOUT USING ROOT RELATIONSHIPS

$(1+2i)^2 = 1+4i+(2i)^2 = 1+4i-4 = -3+4i$

$(1+2i)^3 = (-3+4i)(1+2i) = -3-6i+4i-8 = -11-2i$

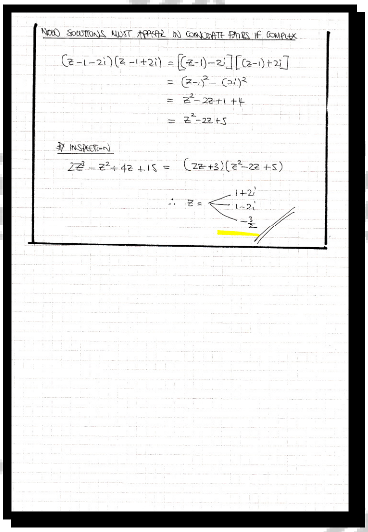
SUB INTO THE CUBIC TO FIND p FIRST

$2z^3 - z^2 + 4z + p = 0$

$2(-11-2i) - (-3+4i) + 4(1+2i) + p = 0$

$-22 - 4i + 3 - 4i + 4 + 8i + p = 0$

$p = 15$



NOW SOLUTIONS MUST APPEAR IN CONJUGATE PAIRS IF COMPLEX

$(z-1-2i)(z-1+2i) = [(z-1)-2i][(z-1)+2i]$

$= (z-1)^2 - (2i)^2$

$= z^2 - 2z + 1 + 4$

$= z^2 - 2z + 5$

BY INSPECTION

$2z^3 - z^2 + 4z + 15 = (2z+3)(z^2-2z+5)$

$\therefore z = 1+2i, \ 1-2i, \ -\frac{3}{2}$

Question 7

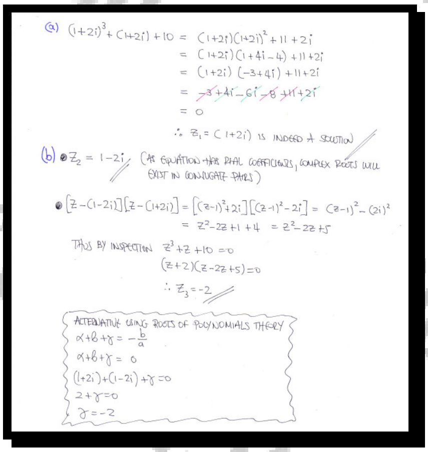
Consider the cubic equation

$$z^3 + z + 10 = 0, \ z \in \mathbb{C}.$$

a) Verify that $1+2i$ is a root of this equation.

b) Find the other two roots.

$$\boxed{z_1 = 1-2i, \quad z_2 = -2}$$



(a) $(1+2i)^3 + (1+2i) + 10 = (1+2i)(1+2i)^2 + 11 + 2i$
$= (1+2i)(1+4i-4) + 11 + 2i$
$= (1+2i)(-3+4i) + 11 + 2i$
$= -3 + 4i - 6i - 8 + 11 + 2i$
$= 0$

$\therefore z_1 = (1+2i)$ IS INDEED A SOLUTION

(b) • $z_2 = 1-2i$ (AS EQUATION HAS REAL COEFFICIENTS, COMPLEX ROOTS WILL EXIST IN CONJUGATE PAIRS)

• $[z-(1-2i)][z-(1+2i)] = [(z-1)^2+2i][(z-1)^2-2i] = (z-1)^2-(2i)^2$
$= z^2-2z+1+4 = z^2-2z+5$

THUS BY INSPECTION $z^3+z+10=0$
$(z+2)(z-2z+5)=0$
$\therefore z_3 = -2$

ALTERNATIVE USING ROOTS OF POLYNOMIALS THEORY
$\alpha+\beta+\gamma = -\frac{b}{a}$
$\alpha+\beta+\gamma = 0$
$(1+2i)+(1-2i)+\gamma = 0$
$2+\gamma=0$
$\gamma=-2$

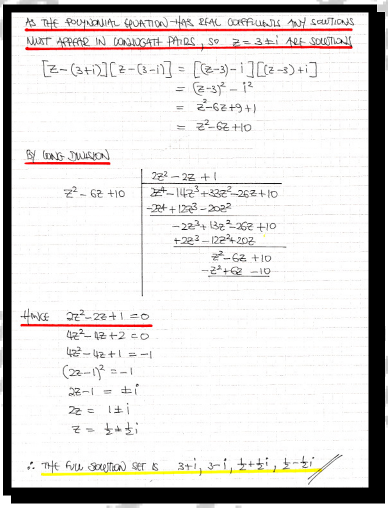
Question 8

Solve the equation

$$2z^4 - 14z^3 + 33z^2 - 26z + 10 = 0, \; z \in \mathbb{C}$$

given that one of its roots is $3+\mathrm{i}$.

$$\boxed{z = 3+\mathrm{i}, \quad z = 3-\mathrm{i}, \quad z = \frac{1}{2}+\frac{1}{2}\mathrm{i}, \quad z = \frac{1}{2}-\frac{1}{2}\mathrm{i}}$$


AS THE POLYNOMIAL EQUATION HAS REAL COEFFICIENTS ANY SOLUTIONS MUST APPEAR IN CONJUGATE PAIRS, SO $z = 3 \pm i$ ARE SOLUTIONS

$[z-(3+i)][z-(3-i)] = [(z-3)-i][(z-3)+i]$
$= (z-3)^2 - i^2$
$= z^2 - 6z + 9 + 1$
$= z^2 - 6z + 10$

BY LONG DIVISION

$z^2 - 6z + 10 \,\big|\, 2z^4 - 14z^3 + 33z^2 - 26z + 10$, quotient $2z^2 - 2z + 1$
$-2z^4 + 12z^3 - 20z^2$
$-2z^3 + 13z^2 - 26z + 10$
$+2z^3 - 12z^2 + 20z$
$z^2 - 6z + 10$
$-z^2 + 6z - 10$

HENCE $2z^2 - 2z + 1 = 0$
$4z^2 - 4z + 2 = 0$
$4z^2 - 4z + 1 = -1$
$(2z-1)^2 = -1$
$2z - 1 = \pm i$
$2z = 1 \pm i$
$z = \frac{1}{2} \pm \frac{1}{2} i$

∴ THE FULL SOLUTION SET IS $3+i,\ 3-i,\ \frac{1}{2}+\frac{1}{2}i,\ \frac{1}{2}-\frac{1}{2}i$

Question 9

$$2z^3 + pz^2 + qz + 16 = 0, \ p \in \mathbb{R}, \ q \in \mathbb{R}.$$

The above cubic equation has roots α, β and γ, where γ is real.

It is given that $\alpha = 2\left(1 + \mathrm{i}\sqrt{3}\right)$.

a) Find the other two roots, β and γ.

b) Determine the values of p and q.

$\boxed{\beta = 2\left(1 - \mathrm{i}\sqrt{3}\right)}$, $\boxed{\gamma = -\frac{1}{2}}$, $\boxed{p = -7}$, $\boxed{q = 28}$

a) • AS COEFFICIENTS ARE REAL $\beta = 2(1 - i\sqrt{3})$

• $\alpha\beta\gamma = -\frac{16}{2}$

$\Rightarrow 2(1+i\sqrt{3}) \times 2(2 - i\sqrt{3}) \times \gamma = -8$

$\Rightarrow 4\gamma(1^2 + \sqrt{3}^2) = -8$

$\Rightarrow 16\gamma = -8$

$\Rightarrow \gamma = -\frac{1}{2}$

b) • $\alpha + \beta + \gamma = -\frac{p}{2}$

$\Rightarrow 2(1+i\sqrt{3}) + 2(1-i\sqrt{3}) - \frac{1}{2} = -\frac{p}{2}$

$\Rightarrow 4 - \frac{1}{2} = -\frac{p}{2}$

$\Rightarrow \frac{7}{2} = -\frac{p}{2}$

$\Rightarrow p = -7$

• $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{q}{2}$

$\Rightarrow 2(1+i\sqrt{3}) \times 2(1-i\sqrt{3}) + 2(1-i\sqrt{3})(-\frac{1}{2}) - \frac{1}{2} \times 2(1+i\sqrt{3}) = \frac{q}{2}$

$\Rightarrow 4(1+3) - (1-i\sqrt{3}) - (1+i\sqrt{3}) = \frac{q}{2}$

$\Rightarrow 16 - 2 = \frac{q}{2}$

$\Rightarrow q = 28$

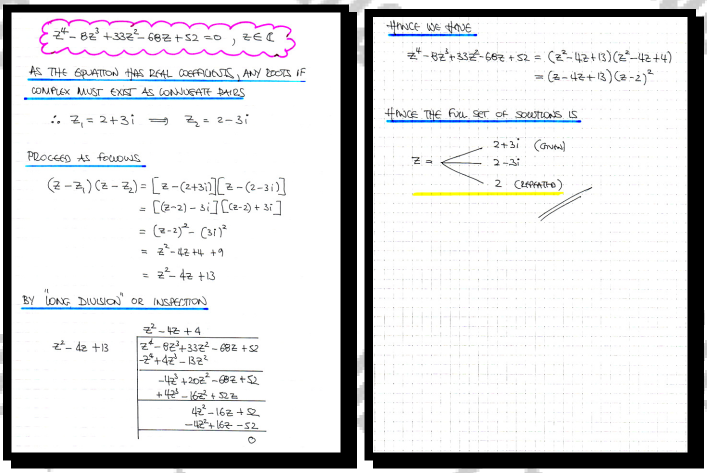
Question 10

$$z^4 - 8z^3 + 33z^2 - 68z + 52 = 0, \ z \in \mathbb{C} .$$

One of the roots of the above quartic equation, is $2 + 3\text{i}$.

Find the other roots of the equation.

$\boxed{z = 2 - 3\text{i}, \ z = 2}$



$z^4 - 8z^3 + 33z^2 - 68z + 52 = 0 \ , \ z \in \mathbb{C}$

AS THE EQUATION HAS REAL COEFFICIENTS, ANY ROOTS IF COMPLEX MUST EXIST AS CONJUGATE PAIRS

$\therefore \ z_1 = 2 + 3i \implies z_2 = 2 - 3i$

PROCEED AS FOLLOWS

$$\begin{aligned}(z - z_1)(z - z_2) &= [z - (2+3i)][z - (2-3i)] \\ &= [(z-2) - 3i][(z-2) + 3i] \\ &= (z-2)^2 - (3i)^2 \\ &= z^2 - 4z + 4 + 9 \\ &= z^2 - 4z + 13\end{aligned}$$

BY "LONG DIVISION" OR INSPECTION

$$\begin{array}{r|l} & z^2 - 4z + 4 \\ \hline z^2 - 4z + 13 & z^4 - 8z^3 + 33z^2 - 68z + 52 \\ & -z^4 + 4z^3 - 13z^2 \\ \hline & -4z^3 + 20z^2 - 68z + 52 \\ & +4z^3 - 16z^2 + 52z \\ \hline & 4z^2 - 16z + 52 \\ & -4z^2 + 16z - 52 \\ \hline & 0 \end{array}$$

HENCE WE HAVE

$$\begin{aligned}z^4 - 8z^3 + 33z^2 - 68z + 52 &= (z^2 - 4z + 13)(z^2 - 4z + 4) \\ &= (z - 4z + 13)(z-2)^2\end{aligned}$$

HENCE THE FULL SET OF SOLUTIONS IS

$z = 2 + 3i$ (GIVEN)

$z = 2 - 3i$

$z = 2$ (REPEATED)

Question 11

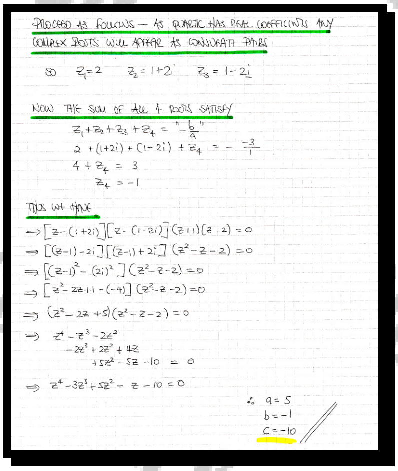
It is given that $z=2$ and $z=1+2\mathrm{i}$ are solutions of the equation

$$z^4-3z^3+az^2+bz+c=0.$$

where a, b and c are real constants.

Determine the values of a, b and c.

$\boxed{a=5}$, $\boxed{b=-1}$, $\boxed{c=-10}$


PROCEED AS FOLLOWS — AS QUARTIC HAS REAL COEFFICIENTS ANY COMPLEX ROOTS WILL APPEAR AS CONJUGATE PAIRS

SO $z_1=2$ $\quad z_2=1+2i$ $\quad z_3=1-2i$

NOW THE SUM OF ALL 4 ROOTS SATISFY

$z_1+z_2+z_3+z_4 = \text{"}-\frac{b}{a}\text{"}$

$2+(1+2i)+(1-2i)+z_4 = -\frac{-3}{1}$

$4+z_4=3$

$z_4=-1$

THUS WE HAVE

$\Rightarrow [z-(1+2i)][z-(1-2i)](z+1)(z-2)=0$

$\Rightarrow [(z-1)-2i][(z-1)+2i](z^2-z-2)=0$

$\Rightarrow [(z-1)^2-(2i)^2](z^2-z-2)=0$

$\Rightarrow [z^2-2z+1-(-4)](z^2-z-2)=0$

$\Rightarrow (z^2-2z+5)(z^2-z-2)=0$

$\Rightarrow z^4-z^3-2z^2$
$\quad -2z^3+2z^2+4z$
$\quad +5z^2-5z-10=0$

$\Rightarrow z^4-3z^3+5z^2-z-10=0$

$\therefore a=5$
$b=-1$
$c=-10$

Question 12

If $1-2\mathrm{i}$ is a root of the quartic equation

$$z^4 - 6z^3 + 18z^2 - 30z + 25 = 0$$

find the other three roots.

$$\boxed{z_2 = 1+2\mathrm{i}, \quad z_3 = 2-\mathrm{i}, \quad z_4 = 2+\mathrm{i}}$$