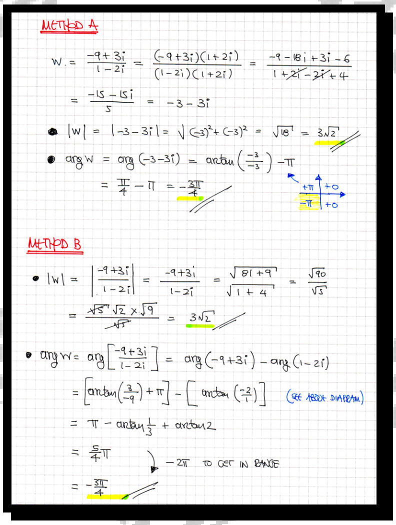
COMPLEX NUMBERS
(Exam Questions I)

Question 1 (**)

$$w = \frac{-9+3\text{i}}{1-2\text{i}} .$$

Find the modulus and the argument of the complex number w.

$\boxed{\text{FP1}}$, $\boxed{|w| = 3\sqrt{2}}$, $\boxed{\arg w = -\frac{3\pi}{4}}$



METHOD A

$w = \frac{-9+3i}{1-2i} = \frac{(-9+3i)(1+2i)}{(1-2i)(1+2i)} = \frac{-9-18i+3i-6}{1+\cancel{2i}-\cancel{2i}+4}$

$= \frac{-15-15i}{5} = -3-3i$

- $|w| = |-3-3i| = \sqrt{(-3)^2+(-3)^2} = \sqrt{18} = 3\sqrt{2}$
- $\arg w = \arg(-3-3i) = \arctan\left(\frac{-3}{-3}\right) - \pi$

$= \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$

METHOD B

- $|w| = \left|\frac{-9+3i}{1-2i}\right| = \frac{|-9+3i|}{|1-2i|} = \frac{\sqrt{81+9}}{\sqrt{1+4}} = \frac{\sqrt{90}}{\sqrt{5}}$

$= \frac{\cancel{\sqrt{5}}\sqrt{2}\times\sqrt{9}}{\cancel{\sqrt{5}}} = 3\sqrt{2}$

- $\arg w = \arg\left[\frac{-9+3i}{1-2i}\right] = \arg(-9+3i) - \arg(1-2i)$

$= \left[\arctan\left(\frac{3}{-9}\right)+\pi\right] - \left[\arctan\left(\frac{-2}{1}\right)\right]$ (SEE ABOVE DIAGRAM)

$= \pi - \arctan\frac{1}{3} + \arctan 2$

$= \frac{5}{4}\pi$ $\quad -2\pi$ TO GET IN RANGE

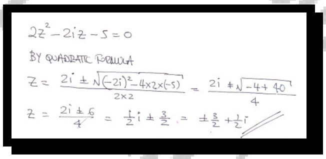
$= -\frac{3\pi}{4}$

Question 2 (**)

Solve the equation

$$2z^2 - 2\text{i}z - 5 = 0, \ z \in \mathbb{C} .$$

$\boxed{z = \pm\frac{3}{2} + \frac{1}{2}\text{i}}$



$2z^2 - 2iz - 5 = 0$

BY QUADRATIC FORMULA

$z = \frac{2i \pm \sqrt{(-2i)^2 - 4\times 2\times(-5)}}{2\times 2} = \frac{2i \pm \sqrt{-4+40}}{4}$

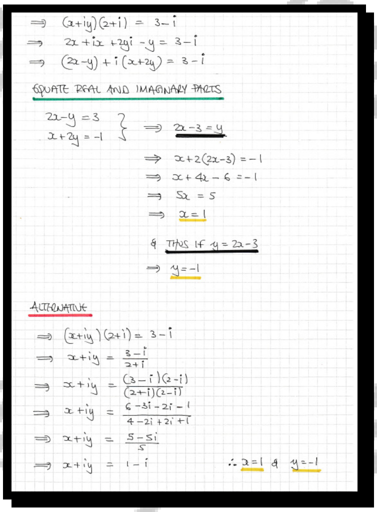
$z = \frac{2i \pm 6}{4} = \frac{1}{2}i \pm \frac{3}{2} = \pm\frac{3}{2} + \frac{1}{2}i$

Question 3 **(**)**

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$(x+\mathrm{i}y)(2+\mathrm{i}) = 3-\mathrm{i}\,.$$

$\boxed{}$, $\boxed{(x,y)=(1,-1)}$



$\Longrightarrow \quad (x+iy)(2+i) = 3-i$

$\Longrightarrow \quad 2x + ix + 2yi - y = 3-i$

$\Longrightarrow \quad (2x-y) + i(x+2y) = 3-i$

EQUATE REAL AND IMAGINARY PARTS

$\left.\begin{array}{l} 2x-y=3 \\ x+2y=-1 \end{array}\right\} \Longrightarrow \underline{2x-3=y}$

$\Longrightarrow \quad x+2(2x-3)=-1$

$\Longrightarrow \quad x+4x-6=-1$

$\Longrightarrow \quad 5x=5$

$\Longrightarrow \quad \underline{x=1}$

& THUS IF $\underline{y=2x-3}$

$\Longrightarrow \quad \underline{y=-1}$

ALTERNATIVE

$\Longrightarrow \quad (x+iy)(2+i) = 3-i$

$\Longrightarrow \quad x+iy = \frac{3-i}{2+i}$

$\Longrightarrow \quad x+iy = \frac{(3-i)(2-i)}{(2+i)(2-i)}$

$\Longrightarrow \quad x+iy = \frac{6-3i-2i-1}{4-2i+2i+1}$

$\Longrightarrow \quad x+iy = \frac{5-5i}{5}$

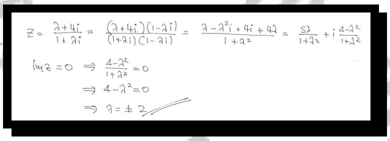
$\Longrightarrow \quad x+iy = 1-i$ $\qquad \therefore \underline{x=1}$ & $\underline{y=-1}$

Question 4 **(**)**

$$z = \frac{\lambda + 4\mathrm{i}}{1+\lambda\mathrm{i}}, \ \lambda \in \mathbb{R}\,.$$

Given that z is a real number, find the possible values of λ.

$\boxed{\lambda = \pm 2}$



$z = \frac{\lambda+4i}{1+\lambda i} = \frac{(\lambda+4i)(1-\lambda i)}{(1+\lambda i)(1-\lambda i)} = \frac{\lambda - \lambda^2 i + 4i + 4\lambda}{1+\lambda^2} = \frac{5\lambda}{1+\lambda^2} + i\,\frac{4-\lambda^2}{1+\lambda^2}$

$\operatorname{Im} z = 0 \Longrightarrow \frac{4-\lambda^2}{1+\lambda^2} = 0$

$\Longrightarrow 4-\lambda^2=0$

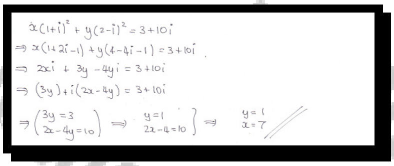
$\Longrightarrow \lambda = \pm 2$

Question 5 **(**)**

Find the values of x and y in the equation

$$x(1+\mathrm{i})^2 + y(2-\mathrm{i})^2 = 3+10\mathrm{i}, \ x \in \mathbb{R}, \ y \in \mathbb{R}.$$

$\boxed{x=7}$, $\boxed{y=1}$



$$x(1+i)^2 + y(2-i)^2 = 3+10i$$

$$\Rightarrow x(1+2i-1) + y(4-4i-1) = 3+10i$$

$$\Rightarrow 2xi + 3y - 4yi = 3+10i$$

$$\Rightarrow (3y) + i(2x-4y) = 3+10i$$

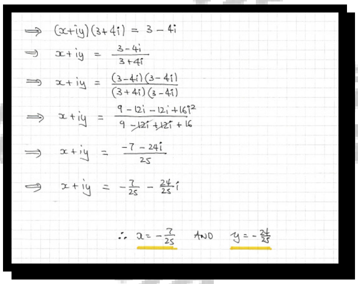
$$\Rightarrow \left(\begin{matrix} 3y = 3 \\ 2x-4y=10 \end{matrix}\right) \Rightarrow \left.\begin{matrix} y=1 \\ 2x-4=10 \end{matrix}\right\} \Rightarrow \begin{matrix} y=1 \\ x=7 \end{matrix} \ //$$

Question 6 **(**)**

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$(x+\mathrm{i}y)(3+4\mathrm{i}) = 3-4\mathrm{i}.$$

$\boxed{\quad}$, $\boxed{(x,y) = \left(-\frac{7}{25}, -\frac{24}{25}\right)}$



$$\Rightarrow (x+iy)(3+4i) = 3-4i$$

$$\Rightarrow x+iy = \frac{3-4i}{3+4i}$$

$$\Rightarrow x+iy = \frac{(3-4i)(3-4i)}{(3+4i)(3-4i)}$$

$$\Rightarrow x+iy = \frac{9-12i-12i+16i^2}{9\cancel{-12i}\cancel{+12i}+16}$$

$$\Rightarrow x+iy = \frac{-7-24i}{25}$$

$$\Rightarrow x+iy = -\frac{7}{25} - \frac{24}{25}i$$

$$\therefore \ \underline{x = -\frac{7}{25}} \quad \text{AND} \quad \underline{y = -\frac{24}{25}}$$

Question 7 (**)

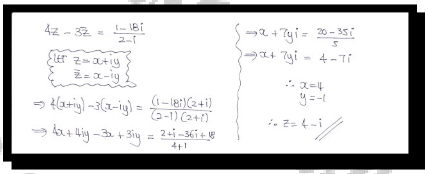
The complex number z satisfies the equation

$$4z-3\overline{z}=\frac{1-18\text{i}}{2-\text{i}},$$

where $\overline{z}$ denotes the complex conjugate of z.

Solve the equation, giving the answer in the form $x+\text{i}y$, where x and y are real numbers.

$\boxed{z=4-\text{i}}$



$4z-3\bar{z}=\frac{1-18i}{2-i}$

let $z=x+iy$
$\bar{z}=x-iy$

$\Rightarrow 4(x+iy)-3(x-iy)=\frac{(1-18i)(2+i)}{(2-1)(2+i)}$

$\Rightarrow 4x+4iy-3x+3iy=\frac{2+i-36i+18}{4+1}$

$\Rightarrow x+7yi=\frac{20-35i}{5}$

$\Rightarrow x+7yi=4-7i$

$\therefore x=4$
$y=-1$

$\therefore z=4-i$

Question 8 (**)

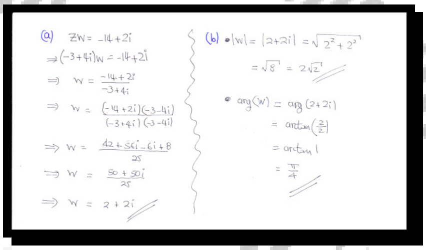
$$z=-3+4\text{i} \quad \text{and} \quad zw=-14+2\text{i}.$$

By showing clear workings, find …

a) … w in the form $a+b\text{i}$, where a and b are real numbers.

b) … the modulus and the argument of w.

$\boxed{w=2+2\text{i}}$, $\boxed{|w|=2\sqrt{2}}$, $\boxed{\arg w=\frac{\pi}{4}}$



(a) $zw=-14+2i$

$\Rightarrow(-3+4i)w=-14+2i$

$\Rightarrow w=\frac{-14+2i}{-3+4i}$

$\Rightarrow w=\frac{(-14+2i)(-3-4i)}{(-3+4i)(-3-4i)}$

$\Rightarrow w=\frac{42+56i-6i+8}{25}$

$\Rightarrow w=\frac{50+50i}{25}$

$\Rightarrow w=2+2i$

(b) • $|w|=|2+2i|=\sqrt{2^2+2^2}$

$=\sqrt{8}=2\sqrt{2}$

• $\arg(w)=\arg(2+2i)$

$=\arctan\left(\frac{2}{2}\right)$

$=\arctan 1$

$=\frac{\pi}{4}$

Question 9 (**)

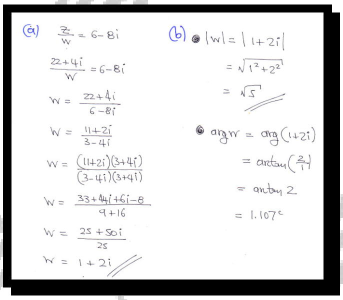
$$z = 22 + 4\mathrm{i} \quad \text{and} \quad \frac{z}{w} = 6 - 8\mathrm{i}.$$

By showing clear workings, find …

a) … w in the form $a + b\mathrm{i}$, where a and b are real numbers .

b) … the modulus and the argument of w.

$\boxed{w = 1 + 2\mathrm{i}}$, $\boxed{|w| = \sqrt{5}}$, $\boxed{\arg w \approx 1.11^{\mathrm{c}}}$



(a) $\frac{z}{w} = 6 - 8i$

$\frac{22+4i}{w} = 6 - 8i$

$w = \frac{22+4i}{6-8i}$

$w = \frac{11+2i}{3-4i}$

$w = \frac{(11+2i)(3+4i)}{(3-4i)(3+4i)}$

$w = \frac{33+44i+6i-8}{9+16}$

$w = \frac{25+50i}{25}$

$w = 1 + 2i$

(b) ● $|w| = |1+2i|$

$= \sqrt{1^2+2^2}$

$= \sqrt{5}$

● $\arg w = \arg(1+2i)$

$= \arctan\left(\frac{2}{1}\right)$

$= \arctan 2$

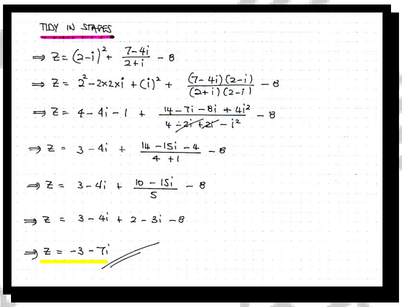
$= 1.107^{c}$

Question 10 (**)

$$z = (2 - \mathrm{i})^2 + \frac{7 - 4\mathrm{i}}{2 + \mathrm{i}} - 8.$$

Express z in the form $x + \mathrm{i}y$, where x and y are real numbers.

$\boxed{\text{FP1-L}}$, $\boxed{-3 - 7\mathrm{i}}$



TIDY IN STAGES

$\Rightarrow z = (2-i)^2 + \frac{7-4i}{2+i} - 8$

$\Rightarrow z = 2^2 - 2\times2\times i + (i)^2 + \frac{(7-4i)(2-i)}{(2+i)(2-i)} - 8$

$\Rightarrow z = 4 - 4i - 1 + \frac{14 - 7i - 8i + 4i^2}{4 \cancel{-2i} \cancel{+2i} - i^2} - 8$

$\Rightarrow z = 3 - 4i + \frac{14 - 15i - 4}{4 + 1} - 8$

$\Rightarrow z = 3 - 4i + \frac{10 - 15i}{5} - 8$

$\Rightarrow z = 3 - 4i + 2 - 3i - 8$

$\Rightarrow z = -3 - 7i$

Question 11 **(**)**

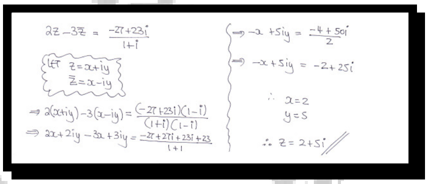
The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$2z - 3\overline{z} = \frac{-27+23\mathrm{i}}{1+\mathrm{i}},$$

giving the answer in the form $x + \mathrm{i}y$, where x and y are real numbers.

$\boxed{z = 2 + 5\mathrm{i}}$



$$2z - 3\overline{z} = \frac{-27+23i}{1+i}$$

LET $z = x+iy$
$\overline{z} = x-iy$

$$\Rightarrow 2(x+iy)-3(x-iy) = \frac{(-27+23i)(1-i)}{(1+i)(1-i)}$$

$$\Rightarrow 2x+2iy-3x+3iy = \frac{-27+27i+23i+23}{1+1}$$

$$\Rightarrow -x+5iy = \frac{-4+50i}{2}$$

$$\Rightarrow -x+5iy = -2+25i$$

$\therefore$ $x=2$
$y=5$

$\therefore z = 2+5i$ //

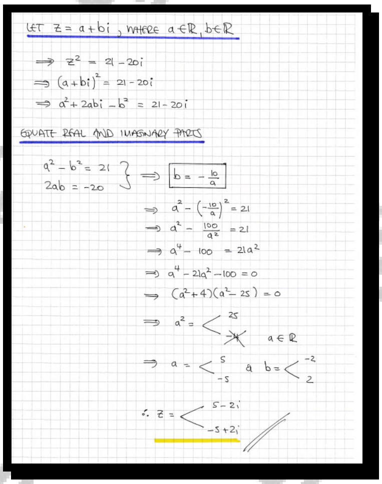
Question 12 **(**+)**

Solve the following equation.

$$z^2 = 21 - 20\mathrm{i}, \quad z \in \mathbb{C}.$$

Give the answers in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$\boxed{\quad}$, $\boxed{z = \pm(5 - 2\mathrm{i})}$



LET $z = a+bi$, WHERE $a \in \mathbb{R}, b \in \mathbb{R}$

$$\Rightarrow z^2 = 21-20i$$
$$\Rightarrow (a+bi)^2 = 21-20i$$
$$\Rightarrow a^2+2abi-b^2 = 21-20i$$

EQUATE REAL AND IMAGINARY PARTS

$$\left.\begin{array}{l} a^2-b^2 = 21 \\ 2ab = -20 \end{array}\right\} \Rightarrow \boxed{b = -\frac{10}{a}}$$

$$\Rightarrow a^2 - \left(-\frac{10}{a}\right)^2 = 21$$
$$\Rightarrow a^2 - \frac{100}{a^2} = 21$$
$$\Rightarrow a^4 - 100 = 21a^2$$
$$\Rightarrow a^4 - 21a^2 - 100 = 0$$
$$\Rightarrow (a^2+4)(a^2-25) = 0$$
$$\Rightarrow a^2 = \begin{cases} 25 \\ \cancel{-4} \end{cases} \quad a \in \mathbb{R}$$
$$\Rightarrow a = \begin{cases} 5 \\ -5 \end{cases} \quad \& \quad b = \begin{cases} -2 \\ 2 \end{cases}$$
$$\therefore z = \begin{cases} 5-2i \\ -5+2i \end{cases}$$ //

Question 13 (+)**

The cubic equation

$$2z^3 - 5z^2 + cz - 5 = 0,\ c \in \mathbb{R},$$

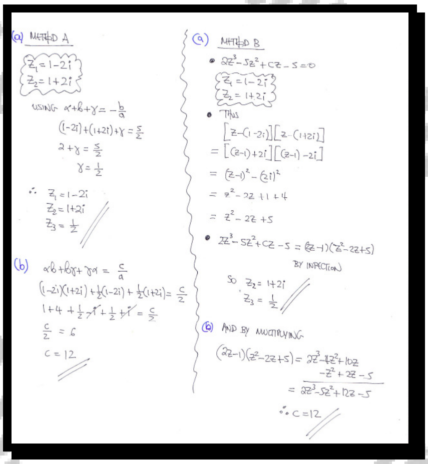
has a solution $z = 1 - 2\mathrm{i}$.

Find in any order …

a) … the other two solutions of the equations.

b) … the value of c.

$\boxed{z_2 = 1 + 2\mathrm{i}}$, $\boxed{z_3 = \frac{1}{2}}$, $\boxed{c = 12}$



(a) METHOD A

$z_1 = 1-2i$

$z_2 = 1+2i$

USING $\alpha + \beta + \gamma = -\frac{b}{a}$

$(1-2i)+(1+2i)+\gamma = \frac{5}{2}$

$2+\gamma = \frac{5}{2}$

$\gamma = \frac{1}{2}$

$\therefore\ z_1 = 1-2i$

$z_2 = 1+2i$

$z_3 = \frac{1}{2}$

(b) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$(1-2i)(1+2i) + \frac{1}{2}(1-2i) + \frac{1}{2}(1+2i) = \frac{c}{2}$

$1+4+\frac{1}{2}\cancel{-i}+\frac{1}{2}\cancel{+i} = \frac{c}{2}$

$\frac{c}{2} = 6$

$c = 12$

(a) METHOD B

- $2z^3 - 5z^2 + cz - 5 = 0$

$z_1 = 1-2i$

$z_2 = 1+2i$

- THUS

$[z-(1-2i)][z-(1+2i)]$

$= [(z-1)+2i][(z-1)-2i]$

$= (z-1)^2 - (2i)^2$

$= z^2 - 2z + 1 + 4$

$= z^2 - 2z + 5$

- $2z^3 - 5z^2 + cz - 5 = (2z-1)(z^2-2z+5)$

BY INSPECTION

SO $z_2 = 1+2i$

$z_3 = \frac{1}{2}$

(b) AND BY MULTIPLYING

$(2z-1)(z^2-2z+5) = 2z^3 - 4z^2 + 10z$

$-z^2 + 2z - 5$

$= 2z^3 - 5z^2 + 12z - 5$

$\therefore c = 12$

Question 14 (+)**

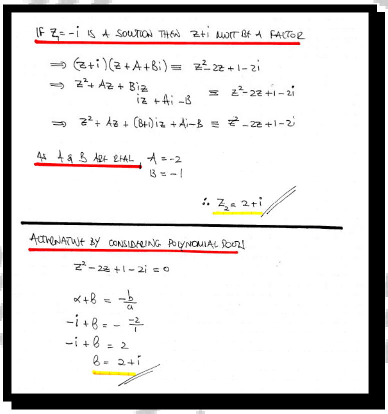
The quadratic equation

$$z^2 - 2z + 1 - 2\mathrm{i} = 0, \; c \in \mathbb{R},$$

has a solution $z = -\mathrm{i}$.

Find the other solution.

$\boxed{\phantom{z_2 = 2+\mathrm{i}}}$, $\boxed{z_2 = 2 + \mathrm{i}}$



IF $Z_1 = -i$ IS A SOLUTION THEN $z+i$ MUST BE A FACTOR

$$\Longrightarrow (z+i)(z+A+Bi) \equiv z^2 - 2z + 1 - 2i$$

$$\Longrightarrow \begin{array}{l} z^2 + Az + Biz \\ \qquad\qquad iz + Ai - B \end{array} \equiv z^2 - 2z + 1 - 2i$$

$$\Longrightarrow z^2 + Az + (B+1)iz + Ai - B \equiv z^2 - 2z + 1 - 2i$$

AS A & B ARE REAL, $A = -2$
$B = -1$

$\therefore Z_2 = 2 + i$

ALTERNATIVE BY CONSIDERING POLYNOMIAL ROOTS

$$z^2 - 2z + 1 - 2i = 0$$

$$\alpha + \beta = -\frac{b}{a}$$

$$-i + \beta = -\frac{-2}{1}$$

$$-i + \beta = 2$$

$$\beta = 2 + i$$

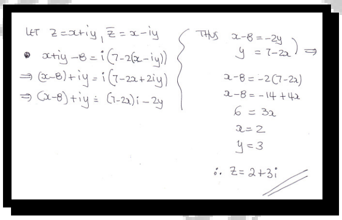
Question 15 **(**+)**

$$z-8=\mathrm{i}\left(7-2\overline{z}\right),\ z\in\mathbb{C}.$$

The complex conjugate of z is denoted by $\overline{z}$.

Determine the value of z in the above equation, giving the answer in the form $x+\mathrm{i}y$, where x and y are real numbers.

$\boxed{z=2+3\mathrm{i}}$



LET $z=x+iy$, $\bar{z}=x-iy$

- $x+iy-8=i(7-2(x-iy))$

$\Rightarrow (x-8)+iy=i(7-2x+2iy)$

$\Rightarrow (x-8)+iy=(7-2x)i-2y$

THUS $x-8=-2y$

$y=7-2x$ $\Rightarrow$

$x-8=-2(7-2x)$

$x-8=-14+4x$

$6=3x$

$x=2$

$y=3$

$\therefore z=2+3i$

Question 16 **(**+)**

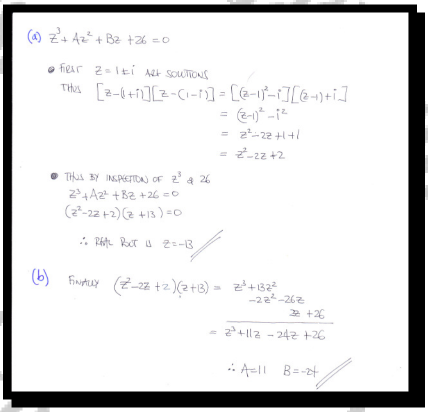
$$z^3+Az^2+Bz+26=0,\ \text{where } A\in\mathbb{R},\ B\in\mathbb{R}$$

One of the roots of the above cubic equation is $1+\mathrm{i}$.

a) Find the real root of the equation.

b) Determine the values of A and B.

$\boxed{z=-13}$, $\boxed{A=11}$, $\boxed{B=-24}$



(a) $z^3+Az^2+Bz+26=0$

- FIRST $z=1\pm i$ ARE SOLUTIONS

THUS $[z-(1+i)][z-(1-i)]=[(z-1)-i][(z-1)+i]$

$=(z-1)^2-i^2$

$=z^2-2z+1+1$

$=z^2-2z+2$

- THUS BY INSPECTION OF z^3 & 26

$z^3+Az^2+Bz+26=0$

$(z^2-2z+2)(z+13)=0$

$\therefore$ REAL ROOT IS $z=-13$

(b) FINALLY $(z^2-2z+2)(z+13)=z^3+13z^2$

$-2z^2-26z$

$2z+26$

$=z^3+11z-24z+26$

$\therefore A=11$ $B=-24$

Question 17 (+)**

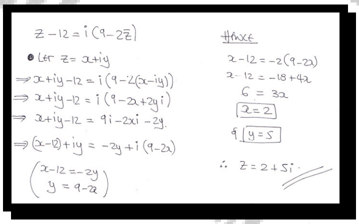
The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$z - 12 = \mathrm{i}\left(9 - 2\overline{z}\right),$$

giving the answer in the form $x + \mathrm{i}y$, where x and y are real numbers.

$\boxed{z = 2 + 5\mathrm{i}}$



$z - 12 = i(9 - 2\bar{z})$

- LET $z = x + iy$

$\Rightarrow x + iy - 12 = i(9 - 2(x - iy))$

$\Rightarrow x + iy - 12 = i(9 - 2x + 2yi)$

$\Rightarrow x + iy - 12 = 9i - 2xi - 2y.$

$\Rightarrow (x - 12) + iy = -2y + i(9 - 2x)$

$\begin{pmatrix} x - 12 = -2y \\ y = 9 - 2x \end{pmatrix}$

HENCE

$x - 12 = -2(9 - 2x)$

$x - 12 = -18 + 4x$

$6 = 3x$

$\boxed{x = 2}$

& $\boxed{y = 5}$

$\therefore \ z = 2 + 5i.$

Question 18 (+)**

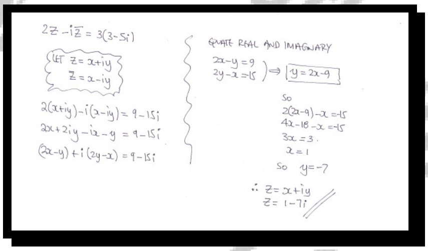
The complex number z satisfies the equation

$$2z - \mathrm{i}\overline{z} = 3\left(3 - 5\mathrm{i}\right),$$

where $\overline{z}$ denotes the complex conjugate of z.

Determine the value of z, giving the answer in the form $x + \mathrm{i}y$, where x and y are real numbers.

$\boxed{z = 1 - 7\mathrm{i}}$



$2z - i\bar{z} = 3(3 - 5i)$

LET $z = x + iy$

$\bar{z} = x - iy$

$2(x + iy) - i(x - iy) = 9 - 15i$

$2x + 2iy - ix - y = 9 - 15i$

$(2x - y) + i(2y - x) = 9 - 15i$

EQUATE REAL AND IMAGINARY

$\left.\begin{matrix} 2x - y = 9 \\ 2y - x = -15 \end{matrix}\right) \Rightarrow \boxed{y = 2x - 9}$

So

$2(2x - 9) - x = -15$

$4x - 18 - x = -15$

$3x = 3.$

$x = 1$

So $y = -7$

$\therefore \ z = x + iy$

$z = 1 - 7i$

Question 19 (**+)

The cubic equation

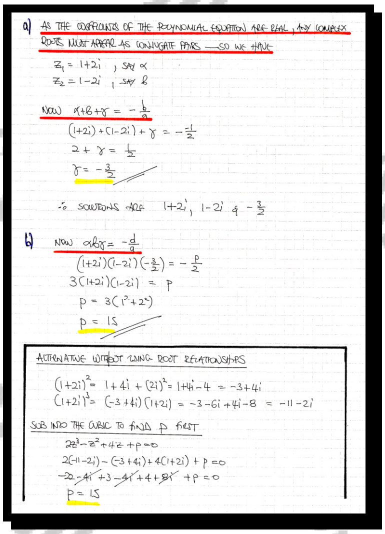
$$2z^3 - z^2 + 4z + p = 0, \quad p \in \mathbb{R},$$

is satisfied by $z = 1 + 2\mathrm{i}$.

a) Find the other two roots of the equation.

b) Determine the value of p.

$\boxed{}$, $\boxed{1 - 2\mathrm{i}, -\frac{3}{2}}$, $\boxed{p = 15}$



a) AS THE COEFFICIENTS OF THE POLYNOMIAL EQUATION ARE REAL, ANY COMPLEX ROOTS MUST APPEAR AS CONJUGATE PAIRS — SO WE HAVE

$z_1 = 1+2i$, SAY α

$z_2 = 1-2i$, SAY β

NOW $\alpha+\beta+\gamma = -\frac{b}{a}$

$(1+2i)+(1-2i)+\gamma = -\frac{-1}{2}$

$2+\gamma = \frac{1}{2}$

$\gamma = -\frac{3}{2}$

$\therefore$ SOLUTIONS ARE $1+2i$, $1-2i$ & $-\frac{3}{2}$

b) NOW $\alpha\beta\gamma = -\frac{d}{a}$

$(1+2i)(1-2i)\left(-\frac{3}{2}\right) = -\frac{p}{2}$

$3(1+2i)(1-2i) = p$

$p = 3(1^2+2^2)$

$p = 15$

ALTERNATIVE WITHOUT USING ROOT RELATIONSHIPS

$(1+2i)^2 = 1+4i+(2i)^2 = 1+4i-4 = -3+4i$

$(1+2i)^3 = (-3+4i)(1+2i) = -3-6i+4i-8 = -11-2i$

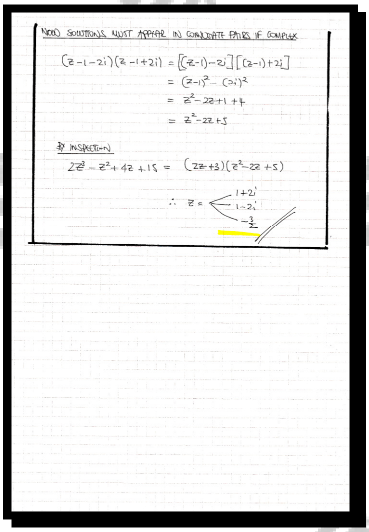
SUB INTO THE CUBIC TO FIND p FIRST

$2z^3 - z^2 + 4z + p = 0$

$2(-11-2i) - (-3+4i) + 4(1+2i) + p = 0$

$-22 \cancel{-4i} + 3 \cancel{-4i} + 4 + \cancel{8i} + p = 0$

$p = 15$



NOW SOLUTIONS MUST APPEAR IN CONJUGATE PAIRS IF COMPLEX

$(z-1-2i)(z-1+2i) = [(z-1)-2i][(z-1)+2i]$

$= (z-1)^2 - (2i)^2$

$= z^2 - 2z + 1 + 4$

$= z^2 - 2z + 5$

BY INSPECTION

$2z^3 - z^2 + 4z + 15 = (2z+3)(z^2-2z+5)$

$\therefore z = \begin{cases} 1+2i \\ 1-2i \\ -\frac{3}{2} \end{cases}$

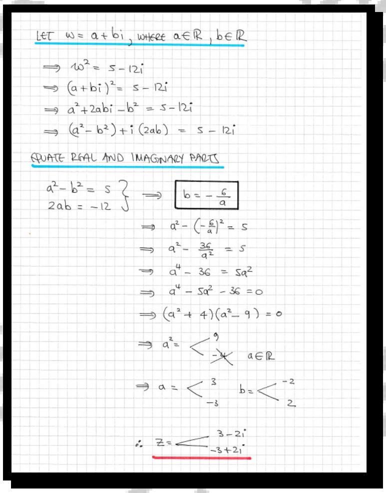
Question 20 (+)**

Solve the following equation.

$$w^2 = 5 - 12\mathrm{i}\,, \quad w \in \mathbb{C}\,.$$

Give the answers in the form $a + b\mathrm{i}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$\boxed{\quad\quad}$, $\boxed{w = \pm(3 - 2\mathrm{i})}$



LET $w = a + bi$, WHERE $a \in \mathbb{R}$, $b \in \mathbb{R}$

$\Rightarrow w^2 = 5 - 12i$

$\Rightarrow (a + bi)^2 = 5 - 12i$

$\Rightarrow a^2 + 2abi - b^2 = 5 - 12i$

$\Rightarrow (a^2 - b^2) + i(2ab) = 5 - 12i$

EQUATE REAL AND IMAGINARY PARTS

$$\left.\begin{array}{l} a^2 - b^2 = 5 \\ 2ab = -12 \end{array}\right\} \Rightarrow \boxed{b = -\frac{6}{a}}$$

$\Rightarrow a^2 - \left(-\frac{6}{a}\right)^2 = 5$

$\Rightarrow a^2 - \frac{36}{a^2} = 5$

$\Rightarrow a^4 - 36 = 5a^2$

$\Rightarrow a^4 - 5a^2 - 36 = 0$

$\Rightarrow (a^2 + 4)(a^2 - 9) = 0$

$\Rightarrow a^2 = \left\langle \begin{array}{l} 9 \\ \cancel{-4} \end{array} \right. \quad a \in \mathbb{R}$

$\Rightarrow a = \left\langle \begin{array}{l} 3 \\ -3 \end{array} \right. \quad b = \left\langle \begin{array}{l} -2 \\ 2 \end{array} \right.$

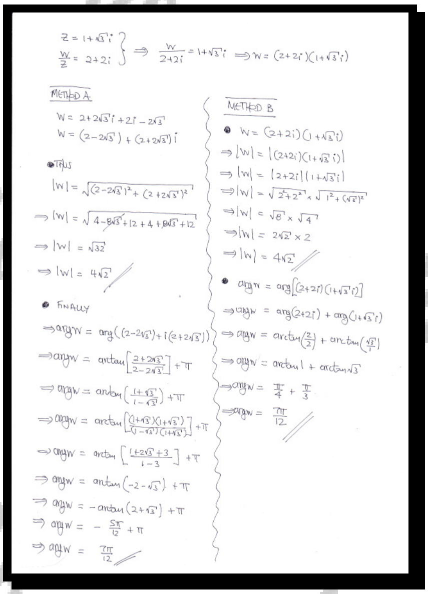
$\therefore z = \left\langle \begin{array}{l} 3 - 2i \\ -3 + 2i \end{array} \right.$

Question 21 (+)**

$$z = 1 + \sqrt{3}\mathrm{i} \qquad \text{and} \qquad \frac{w}{z} = 2 + 2\mathrm{i}\,.$$

Find the exact value of the modulus of w and the exact value of the argument of w.

$\boxed{|w| = 4\sqrt{2}}$, $\boxed{\arg w = \frac{7\pi}{12}}$



$\left.\begin{array}{l} z = 1+\sqrt{3}i \\ \frac{w}{z} = 2+2i \end{array}\right\} \Rightarrow \frac{w}{2+2i} = 1+\sqrt{3}i \Rightarrow w = (2+2i)(1+\sqrt{3}i)$

METHOD A

$w = 2+2\sqrt{3}i+2i-2\sqrt{3}$

$w = (2-2\sqrt{3})+(2+2\sqrt{3})i$

- THUS

$|w| = \sqrt{(2-2\sqrt{3})^2+(2+2\sqrt{3})^2}$

$\Rightarrow |w| = \sqrt{4-\cancel{8\sqrt{3}}+12+4+\cancel{8\sqrt{3}}+12}$

$\Rightarrow |w| = \sqrt{32}$

$\Rightarrow |w| = 4\sqrt{2}$ //

- FINALLY

$\Rightarrow \arg w = \arg((2-2\sqrt{3})+i(2+2\sqrt{3}))$

$\Rightarrow \arg w = \arctan\left[\frac{2+2\sqrt{3}}{2-2\sqrt{3}}\right]+\pi$

$\Rightarrow \arg w = \arctan\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right)+\pi$

$\Rightarrow \arg w = \arctan\left[\frac{(1+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}\right]+\pi$

$\Rightarrow \arg w = \arctan\left[\frac{1+2\sqrt{3}+3}{1-3}\right]+\pi$

$\Rightarrow \arg w = \arctan(-2-\sqrt{3})+\pi$

$\Rightarrow \arg w = -\arctan(2+\sqrt{3})+\pi$

$\Rightarrow \arg w = -\frac{5\pi}{12}+\pi$

$\Rightarrow \arg w = \frac{7\pi}{12}$ //

METHOD B

- $w = (2+2i)(1+\sqrt{3}i)$

$\Rightarrow |w| = |(2+2i)(1+\sqrt{3}i)|$

$\Rightarrow |w| = |2+2i||1+\sqrt{3}i|$

$\Rightarrow |w| = \sqrt{2^2+2^2}\times\sqrt{1^2+(\sqrt{3})^2}$

$\Rightarrow |w| = \sqrt{8}\times\sqrt{4}$

$\Rightarrow |w| = 2\sqrt{2}\times 2$

$\Rightarrow |w| = 4\sqrt{2}$ //

- $\arg w = \arg[(2+2i)(1+\sqrt{3}i)]$

$\Rightarrow \arg w = \arg(2+2i)+\arg(1+\sqrt{3}i)$

$\Rightarrow \arg w = \arctan\left(\frac{2}{2}\right)+\arctan\left(\frac{\sqrt{3}}{1}\right)$

$\Rightarrow \arg w = \arctan 1+\arctan\sqrt{3}$

$\Rightarrow \arg w = \frac{\pi}{4}+\frac{\pi}{3}$

$\Rightarrow \arg w = \frac{7\pi}{12}$ //

Question 22 (**+)

The following cubic equation is given

$$z^3 + az^2 + bz - 5 = 0,$$

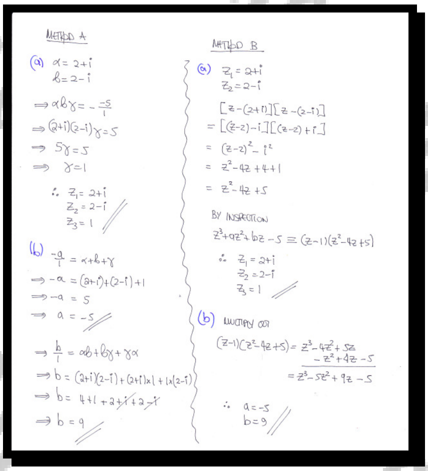
where $a \in \mathbb{R}$, $b \in \mathbb{R}$.

One of the roots of the above cubic equation is $2 + \mathrm{i}$.

a) Find the other two roots.

b) Determine the value of a and the value of b.

$\boxed{z_2 = 2 - \mathrm{i}}$, $\boxed{z_3 = 1}$, $\boxed{a = -5}$, $\boxed{b = 9}$



METHOD A

(a) $\alpha = 2+i$
$\beta = 2-i$

$\Rightarrow \alpha\beta\gamma = -\frac{-5}{1}$

$\Rightarrow (2+i)(2-i)\gamma = 5$

$\Rightarrow 5\gamma = 5$

$\Rightarrow \gamma = 1$

$\therefore z_1 = 2+i$
$z_2 = 2-i$
$z_3 = 1$ //

(b) $\frac{-a}{1} = \alpha + \beta + \gamma$

$\Rightarrow -a = (2+i) + (2-i) + 1$

$\Rightarrow -a = 5$

$\Rightarrow a = -5$ //

$\Rightarrow \frac{b}{1} = \alpha\beta + \beta\gamma + \gamma\alpha$

$\Rightarrow b = (2+i)(2-i) + (2+i)\times 1 + 1\times(2-i)$

$\Rightarrow b = 4 + 1 + 2 + \cancel{i} + 2 - \cancel{i}$

$\Rightarrow b = 9$ //

METHOD B

(a) $z_1 = 2+i$
$z_2 = 2-i$

$[z-(2+i)][z-(2-i)]$

$= [(z-2)-i][(z-2)+i]$

$= (z-2)^2 - i^2$

$= z^2 - 4z + 4 + 1$

$= z^2 - 4z + 5$

BY INSPECTION

$z^3 + az^2 + bz - 5 \equiv (z-1)(z^2-4z+5)$

$\therefore z_1 = 2+i$
$z_2 = 2-i$
$z_3 = 1$ //

(b) MULTIPLY OUT

$(z-1)(z^2-4z+5) = z^3 - 4z^2 + 5z$
$\quad - z^2 + 4z - 5$
$= z^3 - 5z^2 + 9z - 5$

$\therefore a = -5$
$b = 9$ //

Question 23 (**+)

The following cubic equation is given

$$z^3 + pz^2 + 6z + q = 0,$$

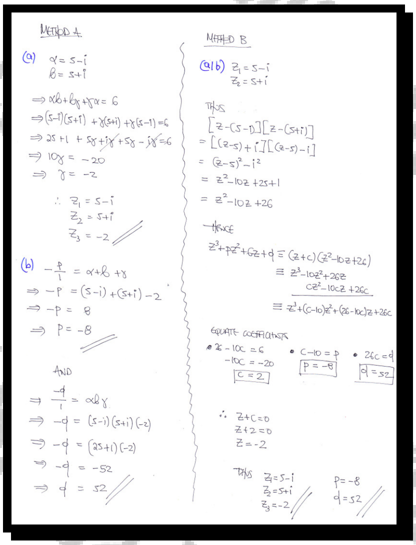
where $p \in \mathbb{R}$, $q \in \mathbb{R}$.

One of the three solutions of the above cubic equation is $5 - \mathrm{i}$.

a) Find the other two solutions of the equation.

b) Determine the value of p and the value of q.

$\boxed{z_2 = 5 + \mathrm{i},\ z_3 = 2}$, $\boxed{p = -8}$, $\boxed{q = 52}$



METHOD A

(a) $\alpha = 5 - i$
$\beta = 5 + i$

$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = 6$
$\Rightarrow (5-i)(5+i) + \gamma(5+i) + \gamma(5-1) = 6$
$\Rightarrow 25 + 1 + 5\gamma + \cancel{i\gamma} + 5\gamma - \cancel{i\gamma} = 6$
$\Rightarrow 10\gamma = -20$
$\Rightarrow \gamma = -2$

$\therefore z_1 = 5 - i$
$z_2 = 5 + i$
$z_3 = -2$ //

(b) $-\frac{p}{1} = \alpha + \beta + \gamma$
$\Rightarrow -p = (5-i) + (5+i) - 2$
$\Rightarrow -p = 8$
$\Rightarrow p = -8$ //

AND

$\Rightarrow \frac{-q}{1} = \alpha\beta\gamma$
$\Rightarrow -q = (5-i)(5+i)(-2)$
$\Rightarrow -q = (25+1)(-2)$
$\Rightarrow -q = -52$
$\Rightarrow q = 52$ //

METHOD B

(a|b) $z_1 = 5 - i$
$z_2 = 5 + i$

THUS

$[z - (5-i)][z - (5+i)]$
$= [(z-5) + i][(z-5) - i]$
$= (z-5)^2 - i^2$
$= z^2 - 10z + 25 + 1$
$= z^2 - 10z + 26$

HENCE

$z^3 + pz^2 + 6z + q \equiv (z+c)(z^2 - 10z + 26)$
$\equiv z^3 - 10z^2 + 26z$
$\quad cz^2 - 10cz + 26c$
$\equiv z^3 + (c-10)z^2 + (26-10c)z + 26c$

EQUATE COEFFICIENTS

- $26 - 10c = 6$
  $-10c = -20$
  $\boxed{c = 2}$
- $c - 10 = p$
  $\boxed{p = -8}$
- $26c = q$
  $\boxed{q = 52}$

$\therefore z + c = 0$
$z + 2 = 0$
$z = -2$

THUS $z_1 = 5 - i$
$z_2 = 5 + i$
$z_3 = -2$ //

$p = -8$
$q = 52$ //

Question 24 **(**+)**

The complex number z is defined as

$$z = \mathrm{i}(1+\mathrm{i})(1-2\mathrm{i})^2 .$$

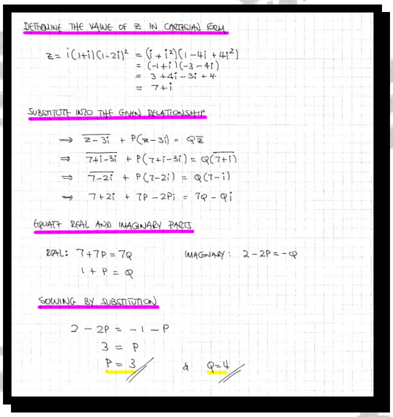
It is further given that

$$\overline{z-3\mathrm{i}} \; + \; P(z-3\mathrm{i}) \; = \; Q\,\overline{z}$$

where P and Q are **real** constants.

Find the value of P and the value of Q.

$\boxed{}$, $\boxed{P=3}$, $\boxed{Q=4}$



DETERMINE THE VALUE OF z IN CARTESIAN FORM

$$\begin{aligned} z = i(1+i)(1-2i)^2 &= (i+i^2)(1-4i+4i^2) \\ &= (-1+i)(-3-4i) \\ &= 3+4i-3i+4 \\ &= 7+i \end{aligned}$$

SUBSTITUTE INTO THE GIVEN RELATIONSHIP

$$\Rightarrow \overline{z-3i} + P(z-3i) = Q\overline{z}$$

$$\Rightarrow \overline{7+i-3i} + P(7+i-3i) = Q(\overline{7+i})$$

$$\Rightarrow \overline{7-2i} + P(7-2i) = Q(7-i)$$

$$\Rightarrow 7+2i + 7P - 2Pi = 7Q - Qi$$

EQUATE REAL AND IMAGINARY PARTS

REAL: $7+7P = 7Q$

$1+P = Q$

IMAGINARY: $2-2P = -Q$

SOLVING BY SUBSTITUTION

$$2-2P = -1-P$$

$$3 = P$$

$$P = 3 \quad \& \quad Q = 4$$

Question 25 (***)

$$z = \sqrt{3} + \mathrm{i} \quad \text{and} \quad w = 3\mathrm{i}.$$

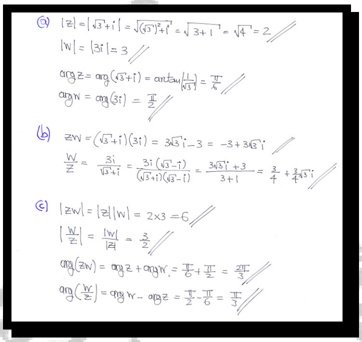
a) Find, in exact form where appropriate, the modulus and argument of z and the modulus and argument of w.

b) Determine simplified expressions for zw and $\frac{w}{z}$, giving the answers in the form $x + \mathrm{i}y$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$.

c) Find, in exact form where appropriate, the modulus and argument of zw and the modulus and argument of $\frac{w}{z}$.

$\boxed{|z| = 2,\ |w| = 3}$, $\boxed{\arg z = \frac{\pi}{6},\ \arg w = \frac{\pi}{2}}$, $\boxed{zw = -3 + 3\sqrt{3}\mathrm{i}}$, $\boxed{\frac{w}{z} = \frac{3}{4} + \frac{3}{4}\sqrt{3}\mathrm{i}}$,

$\boxed{|zw| = 6,\ \left|\frac{w}{z}\right| = \frac{3}{2}}$, $\boxed{\arg(zw) = \frac{2\pi}{3}, \arg\left(\frac{w}{z}\right) = \frac{\pi}{3}}$



(a) $|z| = |\sqrt{3}+i| = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{3+1} = \sqrt{4} = 2$ //

$|w| = |3i| = 3$ //

$\arg z = \arg(\sqrt{3}+i) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ //

$\arg w = \arg(3i) = \frac{\pi}{2}$ //

(b) $zw = (\sqrt{3}+i)(3i) = 3\sqrt{3}i - 3 = -3 + 3\sqrt{3}i$ //

$\frac{w}{z} = \frac{3i}{\sqrt{3}+i} = \frac{3i(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)} = \frac{3\sqrt{3}i+3}{3+1} = \frac{3}{4} + \frac{3}{4}\sqrt{3}i$ //

(c) $|zw| = |z||w| = 2\times3 = 6$ //

$\left|\frac{w}{z}\right| = \frac{|w|}{|z|} = \frac{3}{2}$ //

$\arg(zw) = \arg z + \arg w = \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3}$ //

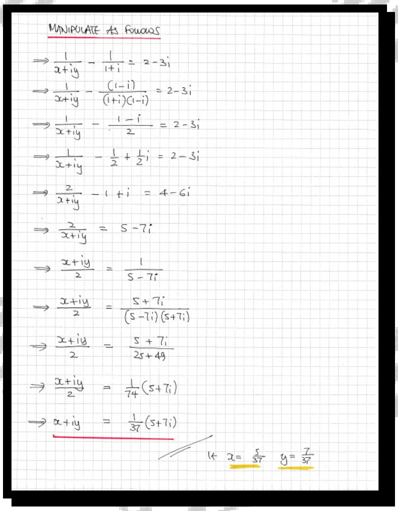
$\arg\left(\frac{w}{z}\right) = \arg w - \arg z = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$ //

Question 26 (***)

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$\frac{1}{x+\mathrm{i}y} - \frac{1}{1+\mathrm{i}} = 2 - 3\mathrm{i}.$$

$\boxed{}$, $\boxed{(x,y) = \left(\frac{5}{37}, \frac{7}{37}\right)}$



MANIPULATE AS FOLLOWS

$$\Rightarrow \frac{1}{x+iy} - \frac{1}{1+i} = 2-3i$$

$$\Rightarrow \frac{1}{x+iy} - \frac{(1-i)}{(1+i)(1-i)} = 2-3i$$

$$\Rightarrow \frac{1}{x+iy} - \frac{1-i}{2} = 2-3i$$

$$\Rightarrow \frac{1}{x+iy} - \frac{1}{2} + \frac{1}{2}i = 2-3i$$

$$\Rightarrow \frac{2}{x+iy} - 1 + i = 4-6i$$

$$\Rightarrow \frac{2}{x+iy} = 5-7i$$

$$\Rightarrow \frac{x+iy}{2} = \frac{1}{5-7i}$$

$$\Rightarrow \frac{x+iy}{2} = \frac{5+7i}{(5-7i)(5+7i)}$$

$$\Rightarrow \frac{x+iy}{2} = \frac{5+7i}{25+49}$$

$$\Rightarrow \frac{x+iy}{2} = \frac{1}{74}(5+7i)$$

$$\Rightarrow x+iy = \frac{1}{37}(5+7i)$$

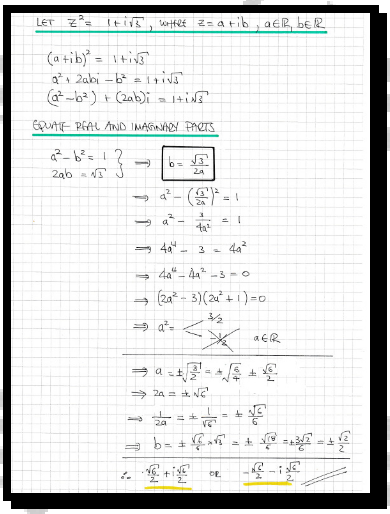
ie $x = \frac{5}{37}$ $y = \frac{7}{37}$

Question 27 (***)

Find the square roots of $1+i\sqrt{3}$.

Give the answers in the form $a+bi$, where $a\in\mathbb{R}$ and $b\in\mathbb{R}$.

$\boxed{}$, $\boxed{\pm\frac{1}{2}\left(\sqrt{6}+i\sqrt{2}\right)}$



LET $z^2 = 1+i\sqrt{3}$, WHERE $z=a+ib$, $a\in\mathbb{R}$, $b\in\mathbb{R}$

$$(a+ib)^2 = 1+i\sqrt{3}$$
$$a^2+2abi-b^2 = 1+i\sqrt{3}$$
$$(a^2-b^2)+(2ab)i = 1+i\sqrt{3}$$

EQUATE REAL AND IMAGINARY PARTS

$$\left.\begin{array}{l} a^2-b^2=1 \\ 2ab=\sqrt{3} \end{array}\right\} \Longrightarrow \boxed{b=\frac{\sqrt{3}}{2a}}$$
$$\Longrightarrow a^2-\left(\frac{\sqrt{3}}{2a}\right)^2=1$$
$$\Longrightarrow a^2-\frac{3}{4a^2}=1$$
$$\Longrightarrow 4a^4-3=4a^2$$
$$\Longrightarrow 4a^4-4a^2-3=0$$
$$\Longrightarrow (2a^2-3)(2a^2+1)=0$$
$$\Longrightarrow a^2 = \begin{cases} 3/2 \\ \cancel{-1/2} \end{cases} \quad a\in\mathbb{R}$$

$$\Longrightarrow a=\pm\sqrt{\frac{3}{2}}=\pm\sqrt{\frac{6}{4}}\pm\frac{\sqrt{6}}{2}$$
$$\Longrightarrow 2a=\pm\sqrt{6}$$
$$\Longrightarrow \frac{1}{2a}=\pm\frac{1}{\sqrt{6}}=\pm\frac{\sqrt{6}}{6}$$
$$\Longrightarrow b=\pm\frac{\sqrt{6}}{6}\times\sqrt{3}=\pm\frac{\sqrt{18}}{6}=\pm\frac{3\sqrt{2}}{6}=\pm\frac{\sqrt{2}}{2}$$

$$\therefore \quad \frac{\sqrt{6}}{2}+i\frac{\sqrt{6}}{2} \quad \text{OR} \quad -\frac{\sqrt{6}}{2}-i\frac{\sqrt{6}}{2} \quad //$$

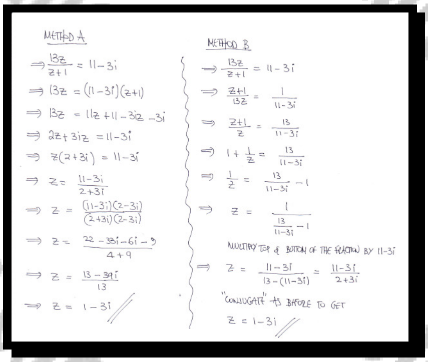
Question 28 **(***)**

Solve the equation

$$\frac{13z}{z+1} = 11 - 3\text{i}, \quad z \in \mathbb{C},$$

giving the answer in the form $x + \text{i}y$, where x and y are real numbers.

$\boxed{z = 1 - 3\text{i}}$



METHOD A

$\Rightarrow \frac{13z}{z+1} = 11-3i$

$\Rightarrow 13z = (11-3i)(z+1)$

$\Rightarrow 13z = 11z + 11 - 3iz - 3i$

$\Rightarrow 2z + 3iz = 11-3i$

$\Rightarrow z(2+3i) = 11-3i$

$\Rightarrow z = \frac{11-3i}{2+3i}$

$\Rightarrow z = \frac{(11-3i)(2-3i)}{(2+3i)(2-3i)}$

$\Rightarrow z = \frac{22-33i-6i-9}{4+9}$

$\Rightarrow z = \frac{13-39i}{13}$

$\Rightarrow z = 1-3i$ //

METHOD B

$\Rightarrow \frac{13z}{z+1} = 11-3i$

$\Rightarrow \frac{z+1}{13z} = \frac{1}{11-3i}$

$\Rightarrow \frac{z+1}{z} = \frac{13}{11-3i}$

$\Rightarrow 1 + \frac{1}{z} = \frac{13}{11-3i}$

$\Rightarrow \frac{1}{z} = \frac{13}{11-3i} - 1$

$\Rightarrow z = \frac{1}{\frac{13}{11-3i} - 1}$

MULTIPLY TOP & BOTTOM OF THE FRACTION BY 11-3i

$\Rightarrow z = \frac{11-3i}{13-(11-3i)} = \frac{11-3i}{2+3i}$

"CONJUGATE" AS BEFORE TO GET

$z = 1-3i$ //

Question 29 **(***)**

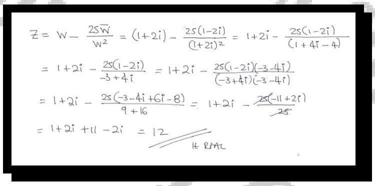
The complex conjugate of w is denoted by $\overline{w}$.

Given further that

$$w = 1 + 2\text{i} \quad \text{and} \quad z = w - \frac{25\overline{w}}{w^2},$$

show clearly that z is a real number, stating its value.

$\boxed{12}$



$z = w - \frac{25\overline{w}}{w^2} = (1+2i) - \frac{25(1-2i)}{(1+2i)^2} = 1+2i - \frac{25(1-2i)}{(1+4i-4)}$

$= 1+2i - \frac{25(1-2i)}{-3+4i} = 1+2i - \frac{25(1-2i)(-3-4i)}{(-3+4i)(-3-4i)}$

$= 1+2i - \frac{25(-3-4i+6i-8)}{9+16} = 1+2i - \frac{\cancel{25}(-11+2i)}{\cancel{25}}$

$= 1+2i+11-2i = 12$ //

IE REAL

Question 30 **(***)**

The following cubic equation is given

$$z^3 + 2z^2 + az + b = 0 ,$$

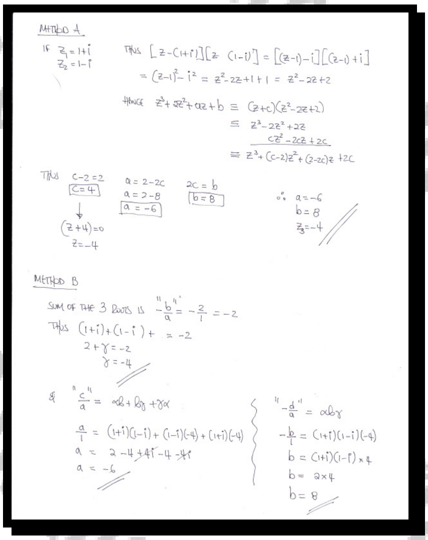
where $a \in \mathbb{R}$, $b \in \mathbb{R}$.

One of the roots of the above cubic equation is $1 + \mathrm{i}$.

a) Find the real root of the equation.

b) Find the value of a and the value of b .

$\boxed{z = -4}$, $\boxed{a = -6}$, $\boxed{b = 8}$



METHOD A

IF $z_1 = 1+i$
$z_2 = 1-i$

THUS $[z-(1+i)][z\ (1-i)] = [(z-1)-i][(z-1)+i]$
$= (z-1)^2 - i^2 = z^2-2z+1+1 = z^2-2z+2$

HENCE $z^3+2z^2+az+b \equiv (z+c)(z^2-2z+2)$
$\equiv z^3-2z^2+2z$
$\quad cz^2-2cz+2c$
$\equiv z^3+(c-2)z^2+(2-2c)z+2c$

THUS $c-2=2$
$\boxed{c=4}$
$\downarrow$
$(z+4)=0$
$z=-4$

$a = 2-2c$
$a = 2-8$
$\boxed{a=-6}$

$2c = b$
$\boxed{b=8}$

$\therefore a=-6$
$b=8$
$z_3=-4$

METHOD B

SUM OF THE 3 ROOTS IS "$-\frac{b}{a}$" $= -\frac{2}{1} = -2$

THUS $(1+i)+(1-i)+\quad = -2$
$2+\gamma=-2$
$\gamma=-4$

& "$\frac{c}{a}$" $= \alpha\beta+\beta\gamma+\gamma\alpha$

$\frac{a}{1} = (1+i)(1-i)+(1-i)(-4)+(1+i)(-4)$
$a = 2-4\cancel{+4i}-4\cancel{-4i}$
$a = -6$

"$-\frac{d}{a}$" $= \alpha\beta\gamma$
$-\frac{b}{1} = (1+i)(1-i)(-4)$
$b = (1+i)(1-i)\times 4$
$b = 2\times 4$
$b = 8$

Question 31 **(***)**

The following complex numbers are given.

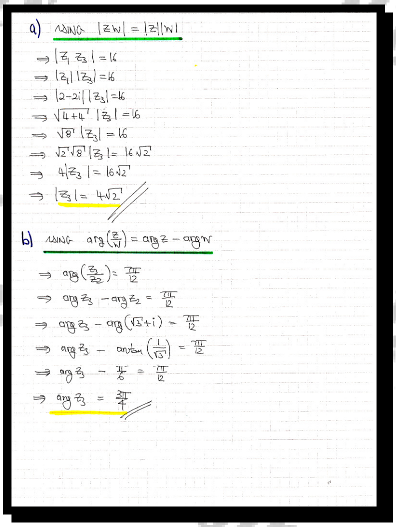
$$z_1 = 2 - 2\mathrm{i}, \quad z_2 = \sqrt{3} + \mathrm{i} \quad \text{and} \quad z_3 = a + b\mathrm{i} \quad \text{where} \quad a \in \mathbb{R},\ b \in \mathbb{R}.$$

a) If $|z_1 z_3| = 16$, find the modulus z_3.

b) Given further that $\arg\left(\dfrac{z_3}{z_2}\right) = \dfrac{7\pi}{12}$, determine the argument of z_3.

c) Find the values of a and b, and hence show $\dfrac{z_3}{z_1} = -2$.

$\boxed{\quad}$, $\boxed{|z_3| = 4\sqrt{2}}$, $\boxed{\arg z_3 = \dfrac{3\pi}{4}}$, $\boxed{a = -4}$, $\boxed{b = 4}$



a) USING $|zw| = |z||w|$

$\Rightarrow |z_1 z_3| = 16$

$\Rightarrow |z_1||z_3| = 16$

$\Rightarrow |2-2i||z_3| = 16$

$\Rightarrow \sqrt{4+4}\,|z_3| = 16$

$\Rightarrow \sqrt{8}\,|z_3| = 16$

$\Rightarrow \sqrt{2}\sqrt{8}\,|z_3| = 16\sqrt{2}$

$\Rightarrow 4|z_3| = 16\sqrt{2}$

$\Rightarrow |z_3| = 4\sqrt{2}$ //

b) USING $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$

$\Rightarrow \arg\left(\frac{z_3}{z_2}\right) = \frac{7\pi}{12}$

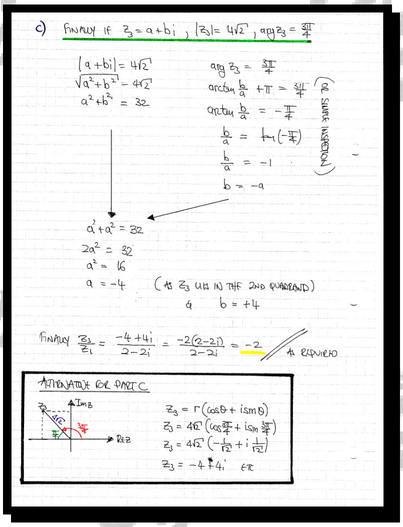
$\Rightarrow \arg z_3 - \arg z_2 = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 - \arg(\sqrt{3}+i) = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 - \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 - \frac{\pi}{6} = \frac{7\pi}{12}$

$\Rightarrow \arg z_3 = \frac{3\pi}{4}$ //



c) FINALLY IF $z_3 = a+bi$, $|z_3| = 4\sqrt{2}$, $\arg z_3 = \frac{3\pi}{4}$

$|a+bi| = 4\sqrt{2}$

$\sqrt{a^2+b^2} = 4\sqrt{2}$

$a^2+b^2 = 32$

$\arg z_3 = \frac{3\pi}{4}$

$\arctan\frac{b}{a} + \pi = \frac{3\pi}{4}$

$\arctan\frac{b}{a} = -\frac{\pi}{4}$

$\frac{b}{a} = \tan\left(-\frac{\pi}{4}\right)$

$\frac{b}{a} = -1$

$b = -a$

(OR SIMPLE INSPECTION)

$a^2 + a^2 = 32$

$2a^2 = 32$

$a^2 = 16$

$a = -4$ (AS z_3 LIES IN THE 2ND QUADRANT)

& $b = +4$

FINALLY $\frac{z_3}{z_1} = \frac{-4+4i}{2-2i} = \frac{-2(2-2i)}{2-2i} = -2$ // AS REQUIRED

ALTERNATIVE FOR PART C

$z_3 = r(\cos\theta + i\sin\theta)$

$z_3 = 4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$z_3 = 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$z_3 = -4+4i$ ETC

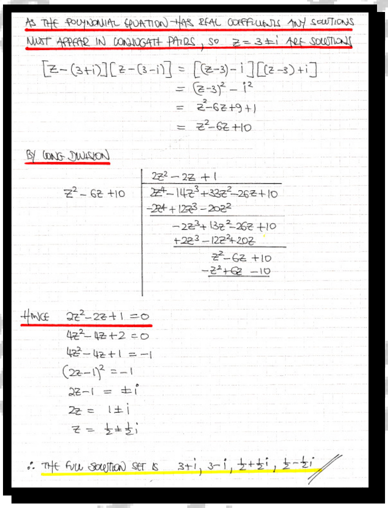
Question 32 (***)

Solve the equation

$$2z^4 - 14z^3 + 33z^2 - 26z + 10 = 0,\ z \in \mathbb{C}$$

given that one of its roots is $3+\mathrm{i}$.

$\boxed{}$, $\boxed{z = 3+\mathrm{i},\quad z = 3-\mathrm{i},\quad z = \frac{1}{2}+\frac{1}{2}\mathrm{i},\quad z = \frac{1}{2}-\frac{1}{2}\mathrm{i}}$



AS THE POLYNOMIAL EQUATION HAS REAL COEFFICIENTS ANY SOLUTIONS MUST APPEAR IN CONJUGATE PAIRS, SO $z = 3 \pm i$ ARE SOLUTIONS

$$[z-(3+i)][z-(3-i)] = [(z-3)-i][(z-3)+i]$$
$$= (z-3)^2 - i^2$$
$$= z^2 - 6z + 9 + 1$$
$$= z^2 - 6z + 10$$

BY LONG DIVISION

$$\begin{array}{r|l} & 2z^2 - 2z + 1 \\ \hline z^2 - 6z + 10 & 2z^4 - 14z^3 + 33z^2 - 26z + 10 \\ & -2z^4 + 12z^3 - 20z^2 \\ \hline & \quad -2z^3 + 13z^2 - 26z + 10 \\ & \quad +2z^3 - 12z^2 + 20z \\ \hline & \qquad z^2 - 6z + 10 \\ & \qquad -z^2 + 6z - 10 \end{array}$$

HENCE $2z^2 - 2z + 1 = 0$

$$4z^2 - 4z + 2 = 0$$
$$4z^2 - 4z + 1 = -1$$
$$(2z-1)^2 = -1$$
$$2z - 1 = \pm i$$
$$2z = 1 \pm i$$
$$z = \tfrac{1}{2} \pm \tfrac{1}{2}i$$

$\therefore$ THE FULL SOLUTION SET IS $3+i,\ 3-i,\ \frac{1}{2}+\frac{1}{2}i,\ \frac{1}{2}-\frac{1}{2}i$

Question 33 (***)

$$2z^3 + pz^2 + qz + 16 = 0,\ p \in \mathbb{R},\ q \in \mathbb{R}.$$

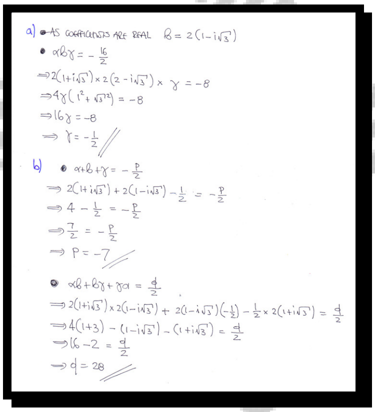
The above cubic equation has roots α, β and γ, where γ is real.

It is given that $\alpha = 2\left(1 + \mathrm{i}\sqrt{3}\right)$.

a) Find the other two roots, β and γ.

b) Determine the values of p and q.

$\boxed{\beta = 2\left(1 - \mathrm{i}\sqrt{3}\right)}$, $\boxed{\gamma = -\frac{1}{2}}$, $\boxed{p = -7}$, $\boxed{q = 28}$



a) • AS COEFFICIENTS ARE REAL $\beta = 2(1 - i\sqrt{3})$

• $\alpha\beta\gamma = -\frac{16}{2}$

$\Rightarrow 2(1 + i\sqrt{3}) \times 2(2 - i\sqrt{3}) \times \gamma = -8$

$\Rightarrow 4\gamma(1^2 + \sqrt{3}^2) = -8$

$\Rightarrow 16\gamma = -8$

$\Rightarrow \gamma = -\frac{1}{2}$ //

b) • $\alpha + \beta + \gamma = -\frac{p}{2}$

$\Rightarrow 2(1 + i\sqrt{3}) + 2(1 - i\sqrt{3}) - \frac{1}{2} = -\frac{p}{2}$

$\Rightarrow 4 - \frac{1}{2} = -\frac{p}{2}$

$\Rightarrow \frac{7}{2} = -\frac{p}{2}$

$\Rightarrow p = -7$ //

• $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{q}{2}$

$\Rightarrow 2(1 + i\sqrt{3}) \times 2(1 - i\sqrt{3}) + 2(1 - i\sqrt{3})(-\frac{1}{2}) - \frac{1}{2} \times 2(1 + i\sqrt{3}) = \frac{q}{2}$

$\Rightarrow 4(1 + 3) - (1 - i\sqrt{3}) - (1 + i\sqrt{3}) = \frac{q}{2}$

$\Rightarrow 16 - 2 = \frac{q}{2}$

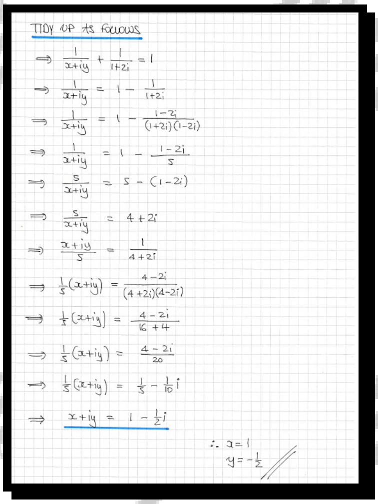
$\Rightarrow q = 28$ //

Question 34 (***)

Find the value of x and the value of y in the following equation, given that $x, y \in \mathbb{R}$.

$$\frac{1}{x+\mathrm{i}y}+\frac{1}{1+2\mathrm{i}}=1.$$

$\boxed{}$, $\boxed{(x,y)=\left(1,-\tfrac{1}{2}\right)}$



TIDY UP AS FOLLOWS

$$\Rightarrow \frac{1}{x+iy}+\frac{1}{1+2i}=1$$

$$\Rightarrow \frac{1}{x+iy}=1-\frac{1}{1+2i}$$

$$\Rightarrow \frac{1}{x+iy}=1-\frac{1-2i}{(1+2i)(1-2i)}$$

$$\Rightarrow \frac{1}{x+iy}=1-\frac{1-2i}{5}$$

$$\Rightarrow \frac{5}{x+iy}=5-(1-2i)$$

$$\Rightarrow \frac{5}{x+iy}=4+2i$$

$$\Rightarrow \frac{x+iy}{5}=\frac{1}{4+2i}$$

$$\Rightarrow \frac{1}{5}(x+iy)=\frac{4-2i}{(4+2i)(4-2i)}$$

$$\Rightarrow \frac{1}{5}(x+iy)=\frac{4-2i}{16+4}$$

$$\Rightarrow \frac{1}{5}(x+iy)=\frac{4-2i}{20}$$

$$\Rightarrow \frac{1}{5}(x+iy)=\frac{1}{5}-\frac{1}{10}i$$

$$\Rightarrow x+iy=1-\frac{1}{2}i$$

$\therefore x=1$

$y=-\frac{1}{2}$ //

Question 35 (***)

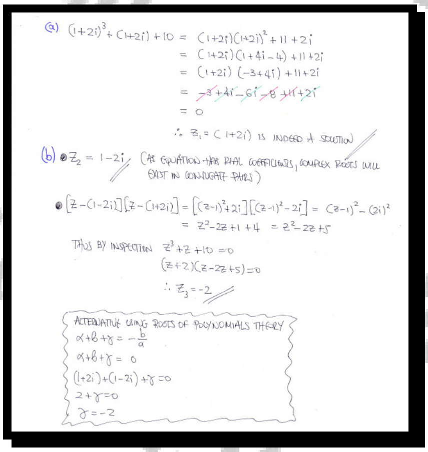
Consider the cubic equation

$$z^3 + z + 10 = 0, \quad z \in \mathbb{C}.$$

a) Verify that $1+2\mathrm{i}$ is a root of this equation.

b) Find the other two roots.

$$\boxed{z_1 = 1-2\mathrm{i}, \quad z_2 = -2}$$



(a) $(1+2i)^3 + (1+2i) + 10 = (1+2i)(1+2i)^2 + 11 + 2i$

$= (1+2i)(1+4i-4) + 11 + 2i$

$= (1+2i)(-3+4i) + 11 + 2i$

$= \cancel{-3} \cancel{+4i} \cancel{-6i} \cancel{-8} \cancel{+11} \cancel{+2i}$

$= 0$

$\therefore z_1 = (1+2i)$ IS INDEED A SOLUTION

(b) ● $z_2 = 1-2i$ (AS EQUATION HAS REAL COEFFICIENTS, COMPLEX ROOTS WILL EXIST IN CONJUGATE PAIRS)

● $[z-(1-2i)][z-(1+2i)] = [(z-1)+2i][(z-1)-2i] = (z-1)^2 - (2i)^2$

$= z^2 - 2z + 1 + 4 = z^2 - 2z + 5$

THUS BY INSPECTION $z^3 + z + 10 = 0$

$(z+2)(z-2z+5) = 0$

$\therefore z_3 = -2$

ALTERNATIVE USING ROOTS OF POLYNOMIALS THEORY

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha + \beta + \gamma = 0$

$(1+2i) + (1-2i) + \gamma = 0$

$2 + \gamma = 0$

$\gamma = -2$

Question 36 (***)

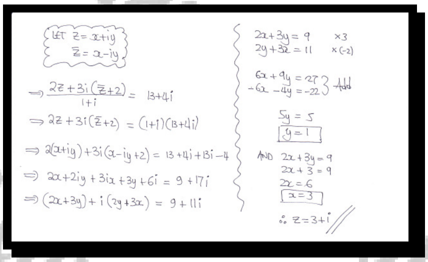
The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$\frac{2z+3\text{i}\left(\overline{z}+2\right)}{1+\text{i}}=13+4\text{i},$$

giving the answer in the form $x+\text{i}y$, where x and y are real numbers.

$$\boxed{z=3+\text{i}}$$



$$\boxed{\begin{aligned}\text{LET } z&=x+iy\\ \bar{z}&=x-iy\end{aligned}}$$

$$\Rightarrow \frac{2z+3i(\bar{z}+2)}{1+i}=13+4i$$

$$\Rightarrow 2z+3i(\bar{z}+2)=(1+i)(13+4i)$$

$$\Rightarrow 2(x+iy)+3i(x-iy+2)=13+4i+13i-4$$

$$\Rightarrow 2x+2iy+3ix+3y+6i=9+17i$$

$$\Rightarrow (2x+3y)+i(2y+3x)=9+11i$$

$$\begin{aligned}2x+3y&=9 & \times 3\\ 2y+3x&=11 & \times(-2)\end{aligned}$$

$$\left.\begin{aligned}6x+9y&=27\\ -6x-4y&=-22\end{aligned}\right\}\text{Add}$$

$$5y=5$$

$$\boxed{y=1}$$

$$\text{AND } \begin{aligned}2x+3y&=9\\ 2x+3&=9\\ 2x&=6\end{aligned}$$

$$\boxed{x=3}$$

$$\therefore z=3+i \,//$$

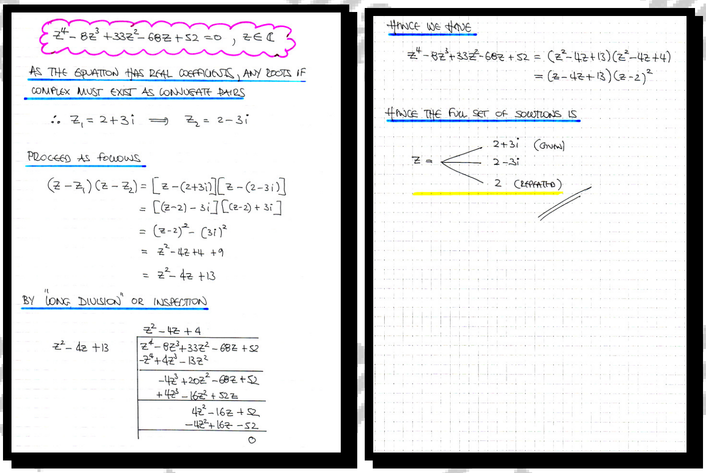
Question 37 (***)

$$z^4 - 8z^3 + 33z^2 - 68z + 52 = 0,\ z \in \mathbb{C}.$$

One of the roots of the above quartic equation is $2 + 3\text{i}$.

Find the other roots of the equation.

$\boxed{\quad}$, $\boxed{z = 2 - 3\text{i},\ z = 2}$



$z^4 - 8z^3 + 33z^2 - 68z + 52 = 0$, $z \in \mathbb{C}$

AS THE EQUATION HAS REAL COEFFICIENTS, ANY ROOTS IF COMPLEX MUST EXIST AS CONJUGATE PAIRS

$\therefore\ z_1 = 2 + 3i \implies z_2 = 2 - 3i$

PROCEED AS FOLLOWS

$$\begin{aligned}(z - z_1)(z - z_2) &= [z - (2+3i)][z - (2-3i)] \\ &= [(z-2) - 3i][(z-2) + 3i] \\ &= (z-2)^2 - (3i)^2 \\ &= z^2 - 4z + 4 + 9 \\ &= z^2 - 4z + 13\end{aligned}$$

BY "LONG DIVISION" OR INSPECTION

$$\begin{array}{r|l} & z^2 - 4z + 4 \\ \hline z^2 - 4z + 13 & z^4 - 8z^3 + 33z^2 - 68z + 52 \\ & -z^4 + 4z^3 - 13z^2 \\ \hline & -4z^3 + 20z^2 - 68z + 52 \\ & +4z^3 - 16z^2 + 52z \\ \hline & 4z^2 - 16z + 52 \\ & -4z^2 + 16z - 52 \\ \hline & 0 \end{array}$$

HENCE WE HAVE

$$\begin{aligned}z^4 - 8z^3 + 33z^2 - 68z + 52 &= (z^2 - 4z + 13)(z^2 - 4z + 4) \\ &= (z - 4z + 13)(z - 2)^2\end{aligned}$$

HENCE THE FULL SET OF SOLUTIONS IS

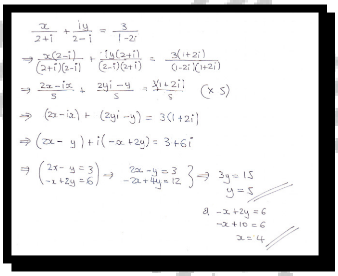
$$z = \begin{cases} 2 + 3i & \text{(GIVEN)} \\ 2 - 3i & \\ 2 & \text{(REPEATED)} \end{cases}$$

Question 38 (***)

Find the values of x and y in the equation

$$\frac{x}{2+\text{i}} + \frac{\text{i}y}{2-\text{i}} = \frac{3}{1-2\text{i}},\ x \in \mathbb{R},\ y \in \mathbb{R}.$$

$\boxed{x = 4}$, $\boxed{y = 5}$



$$\frac{x}{2+i} + \frac{iy}{2-i} = \frac{3}{1-2i}$$

$$\Rightarrow \frac{x(2-i)}{(2+i)(2-i)} + \frac{iy(2+i)}{(2-i)(2+i)} = \frac{3(1+2i)}{(1-2i)(1+2i)}$$

$$\Rightarrow \frac{2x - ix}{5} + \frac{2yi - y}{5} = \frac{3(1+2i)}{5} \quad (\times 5)$$

$$\Rightarrow (2x - ix) + (2yi - y) = 3(1 + 2i)$$

$$\Rightarrow (2x - y) + i(-x + 2y) = 3 + 6i$$

$$\Rightarrow \begin{pmatrix} 2x - y = 3 \\ -x + 2y = 6 \end{pmatrix} \Rightarrow \left.\begin{array}{l} 2x - y = 3 \\ -2x + 4y = 12 \end{array}\right\} \Rightarrow \begin{array}{l} 3y = 15 \\ y = 5 \end{array}$$

& $-x + 2y = 6$

$-x + 10 = 6$

$x = 4$

Question 39 **(***)**

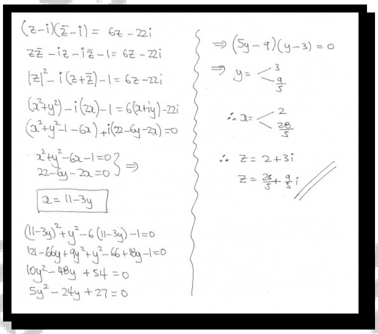
The complex conjugate of z is denoted by $\overline{z}$.

Find the two solutions of the equation

$$(z-\mathrm{i})(\overline{z}-\mathrm{i})=6z-22\mathrm{i},\ z\in\mathbb{C},$$

giving the answers in the form $x+\mathrm{i}y$, where x and y are real numbers.

$\boxed{z_1=2+3\mathrm{i}}$, $\boxed{z_2=\frac{28}{5}+\frac{9}{5}\mathrm{i}}$



$(z-i)(\bar{z}-i) = 6z-22i$

$z\bar{z}-iz-i\bar{z}-1 = 6z-22i$

$|z|^2-i(z+\bar{z})-1 = 6z-22i$

$(x^2+y^2)-i(2x)-1 = 6(x+iy)-22i$

$(x^2+y^2-1-6x)+i(22-6y-2x) = 0$

$\left.\begin{array}{l} x^2+y^2-6x-1=0 \\ 22-6y-2x=0 \end{array}\right\} \Rightarrow$

$\boxed{x = 11-3y}$

$(11-3y)^2+y^2-6(11-3y)-1=0$

$121-66y+9y^2+y^2-66+18y-1=0$

$10y^2-48y+54=0$

$5y^2-24y+27=0$

$\Rightarrow (5y-9)(y-3)=0$

$\Rightarrow y = \begin{cases} 3 \\ \frac{9}{5} \end{cases}$

$\therefore x = \begin{cases} 2 \\ \frac{28}{5} \end{cases}$

$\therefore z = 2+3i$

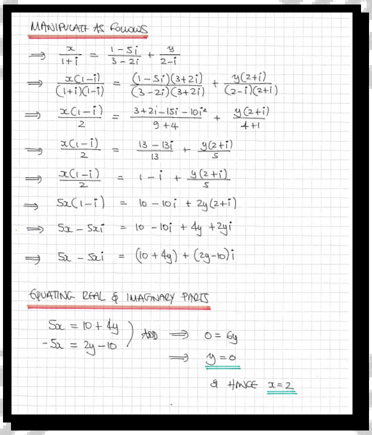
$z = \frac{28}{5}+\frac{9}{5}i$

Question 40 (***)

Find the value of x and the value of y in the following equation, given further that $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$$\frac{x}{1+\mathrm{i}} = \frac{1-5\mathrm{i}}{3-2\mathrm{i}} + \frac{y}{2-\mathrm{i}}.$$

$\boxed{}$, $\boxed{(x, y) = (2, 0)}$



MANIPULATE AS FOLLOWS

$$\Longrightarrow \frac{x}{1+i} = \frac{1-5i}{3-2i} + \frac{y}{2-i}$$

$$\Longrightarrow \frac{x(1-i)}{(1+i)(1-i)} = \frac{(1-5i)(3+2i)}{(3-2i)(3+2i)} + \frac{y(2+i)}{(2-i)(2+i)}$$

$$\Longrightarrow \frac{x(1-i)}{2} = \frac{3+2i-15i-10i^2}{9+4} + \frac{y(2+i)}{4+1}$$

$$\Longrightarrow \frac{x(1-i)}{2} = \frac{13-13i}{13} + \frac{y(2+i)}{5}$$

$$\Longrightarrow \frac{x(1-i)}{2} = 1-i + \frac{y(2+i)}{5}$$

$$\Longrightarrow 5x(1-i) = 10-10i + 2y(2+i)$$

$$\Longrightarrow 5x - 5xi = 10 - 10i + 4y + 2yi$$

$$\Longrightarrow 5x - 5xi = (10+4y) + (2y-10)i$$

EQUATING REAL & IMAGINARY PARTS

$$\left.\begin{array}{r} 5x = 10+4y \\ -5x = 2y-10 \end{array}\right) \text{ADD} \Longrightarrow 0 = 6y$$

$$\Longrightarrow \underline{\underline{y=0}}$$

& HENCE $\underline{\underline{x=2}}$

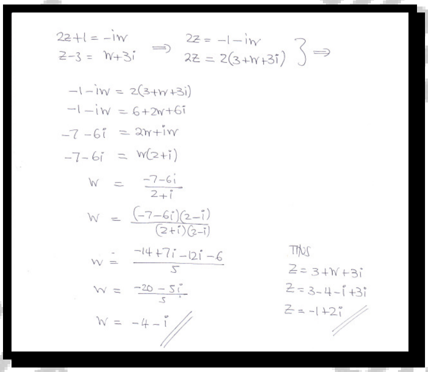
Question 41 **(***)**

Find the value of z and the value of w in the following simultaneous equations

$$2z+1=-\mathrm{i}w$$

$$z-3=w+3\mathrm{i}\,.$$

$$\boxed{z=-1+2\mathrm{i},\ w=-4-\mathrm{i}}$$



$$\left.\begin{aligned}2z+1&=-iw\\ z-3&=w+3i\end{aligned}\right. \implies \left.\begin{aligned}2z&=-1-iw\\ 2z&=2(3+w+3i)\end{aligned}\right\}\implies$$

$$-1-iw=2(3+w+3i)$$
$$-1-iw=6+2w+6i$$
$$-7-6i=2w+iw$$
$$-7-6i=w(2+i)$$
$$w=\frac{-7-6i}{2+i}$$
$$w=\frac{(-7-6i)(2-i)}{(2+i)(2-i)}$$
$$w=\frac{-14+7i-12i-6}{5}$$
$$w=\frac{-20-5i}{5}$$
$$w=-4-i$$

THUS

$$z=3+w+3i$$
$$z=3-4-i+3i$$
$$z=-1+2i$$

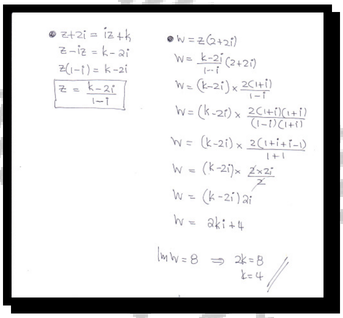
Question 42 **(***)**

It is given that

$$z+2\mathrm{i}=\mathrm{i}z+k\,,\ k\in\mathbb{R} \qquad \text{and} \qquad \frac{w}{z}=2+2\mathrm{i}\,,\ \operatorname{Im}w=8\,.$$

Determine the value of k.

$$\boxed{k=4}$$



- $z+2i=iz+k$

$$z-iz=k-2i$$
$$z(1-i)=k-2i$$
$$\boxed{z=\frac{k-2i}{1-i}}$$

- $w=z(2+2i)$

$$w=\frac{k-2i}{1-i}(2+2i)$$
$$w=(k-2i)\times\frac{2(1+i)}{1-i}$$
$$w=(k-2i)\times\frac{2(1+i)(1+i)}{(1-i)(1+i)}$$
$$w=(k-2i)\times\frac{2(1+i+i-1)}{1+1}$$
$$w=(k-2i)\times\frac{\cancel{2}\times2i}{\cancel{2}}$$
$$w=(k-2i)2i$$
$$w=2ki+4$$

$$\operatorname{Im}w=8 \implies 2k=8$$
$$k=4$$

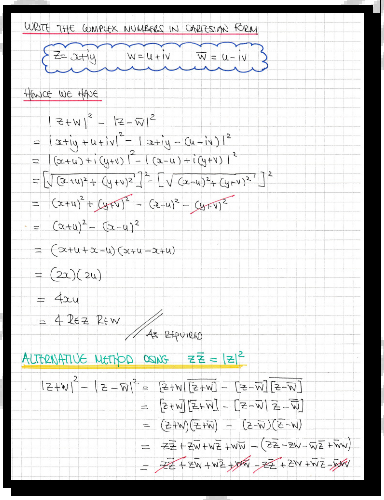
Question 43 (***+)

Given that z and w are complex numbers prove that

$$|z+w|^2 - |z-\overline{w}|^2 = 4\,\text{Re}\,z\;\text{Re}\,w\,,$$

where $\overline{w}$ denotes the complex conjugate of w.

$\boxed{\text{proof}}$



WRITE THE COMPLEX NUMBERS IN CARTESIAN FORM

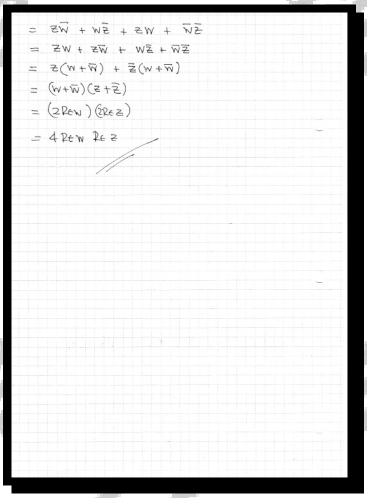
$z = x+iy \qquad w = u+iv \qquad \overline{w} = u-iv$

HENCE WE HAVE

$$\begin{aligned}
&|z+w|^2 - |z-\overline{w}|^2 \\
&= |x+iy+u+iv|^2 - |x+iy-(u-iv)|^2 \\
&= |(x+u)+i(y+v)|^2 - |(x-u)+i(y+v)|^2 \\
&= \left[\sqrt{(x+u)^2+(y+v)^2}\right]^2 - \left[\sqrt{(x-u)^2+(y+v)^2}\right]^2 \\
&= (x+u)^2 + \cancel{(y+v)^2} - (x-u)^2 - \cancel{(y+v)^2} \\
&= (x+u)^2 - (x-u)^2 \\
&= (x+u+x-u)(x+u-x+u) \\
&= (2x)(2u) \\
&= 4xu \\
&= 4\,\text{Re}\,z\;\text{Re}\,w \quad // \text{ AS REQUIRED}
\end{aligned}$$

ALTERNATIVE METHOD USING $z\overline{z} = |z|^2$

$$\begin{aligned}
|z+w|^2 - |z-\overline{w}|^2 &= [z+w]\overline{[z+w]} - [z-\overline{w}]\overline{[z-\overline{w}]} \\
&= [z+w][\overline{z}+\overline{w}] - [z-\overline{w}][\overline{z}-\overline{\overline{w}}] \\
&= (z+w)(\overline{z}+\overline{w}) - (z-\overline{w})(\overline{z}-w) \\
&= z\overline{z}+z\overline{w}+w\overline{z}+w\overline{w} - (z\overline{z}-zw-\overline{w}\overline{z}+\overline{w}w) \\
&= \cancel{z\overline{z}}+z\overline{w}+w\overline{z}+\cancel{w\overline{w}}-\cancel{z\overline{z}}+zw+\overline{w}\overline{z}-\cancel{\overline{w}w}
\end{aligned}$$



$$\begin{aligned}
&= z\overline{w} + w\overline{z} + zw + \overline{w}\overline{z} \\
&= zw + z\overline{w} + w\overline{z} + \overline{w}\overline{z} \\
&= z(w+\overline{w}) + \overline{z}(w+\overline{w}) \\
&= (w+\overline{w})(z+\overline{z}) \\
&= (2\,\text{Re}\,w)(2\,\text{Re}\,z) \\
&= 4\,\text{Re}\,w\;\text{Re}\,z \quad //
\end{aligned}$$

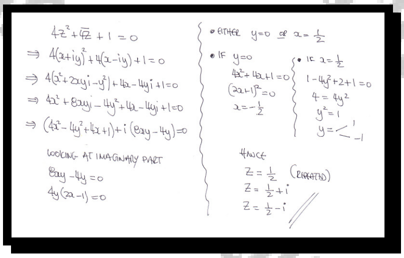
Question 44 (***+)

Find the three solutions of the equation

$$4z^2 + 4\overline{z} + 1 = 0, \quad z \in \mathbb{C},$$

where $\overline{z}$ denotes the complex conjugate of z.

$$\boxed{z = \frac{1}{2}, \ \frac{1}{2} + \mathrm{i}, \ \frac{1}{2} - \mathrm{i}}$$



$4z^2 + 4\overline{z} + 1 = 0$

$\Rightarrow 4(x+iy)^2 + 4(x-iy) + 1 = 0$

$\Rightarrow 4(x^2 + 2xyi - y^2) + 4x - 4yi + 1 = 0$

$\Rightarrow 4x^2 + 8xyi - 4y^2 + 4x - 4yi + 1 = 0$

$\Rightarrow (4x^2 - 4y^2 + 4x + 1) + i(8xy - 4y) = 0$

LOOKING AT IMAGINARY PART

$8xy - 4y = 0$

$4y(2x-1) = 0$

- EITHER $y=0$ or $x = \frac{1}{2}$
- IF $y=0$

  $4x^2 + 4x + 1 = 0$

  $(2x+1)^2 = 0$

  $x = -\frac{1}{2}$
- IF $x = \frac{1}{2}$

  $1 - 4y^2 + 2 + 1 = 0$

  $4 = 4y^2$

  $y^2 = 1$

  $y = \begin{cases} 1 \\ -1 \end{cases}$

HENCE

$z = \frac{1}{2}$ (REPEATED)

$z = \frac{1}{2} + i$

$z = \frac{1}{2} - i$ //

Question 45 (***+)

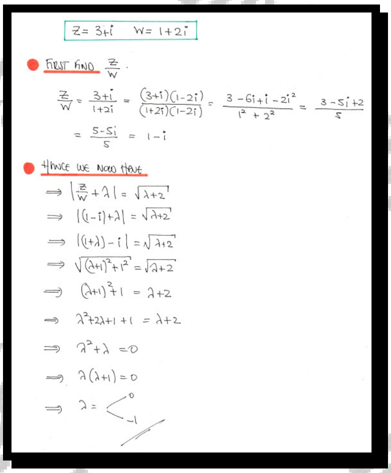
The complex numbers z and w are defined as

$$z = 3 + \mathrm{i} \quad \text{and} \quad w = 1 + 2\mathrm{i}.$$

Determine the possible values of the real constant λ if

$$\left| \frac{z}{w} + \lambda \right| = \sqrt{\lambda + 2}.$$

, $\boxed{\lambda = 0, \quad \lambda = -1}$



$\boxed{z = 3+i \quad w = 1+2i}$

- FIRST FIND $\frac{z}{w}$.

$\frac{z}{w} = \frac{3+i}{1+2i} = \frac{(3+i)(1-2i)}{(1+2i)(1-2i)} = \frac{3 - 6i + i - 2i^2}{1^2 + 2^2} = \frac{3 - 5i + 2}{5}$

$= \frac{5-5i}{5} = 1 - i$

- HENCE WE NOW HAVE

$\Longrightarrow \left|\frac{z}{w} + \lambda\right| = \sqrt{\lambda+2}$

$\Longrightarrow |(1-i) + \lambda| = \sqrt{\lambda+2}$

$\Longrightarrow |(1+\lambda) - i| = \sqrt{\lambda+2}$

$\Longrightarrow \sqrt{(\lambda+1)^2 + 1^2} = \sqrt{\lambda+2}$

$\Longrightarrow (\lambda+1)^2 + 1 = \lambda + 2$

$\Longrightarrow \lambda^2 + 2\lambda + 1 + 1 = \lambda + 2$

$\Longrightarrow \lambda^2 + \lambda = 0$

$\Longrightarrow \lambda(\lambda+1) = 0$

$\Longrightarrow \lambda = \begin{cases} 0 \\ -1 \end{cases}$ //

Question 46 (***+)

The complex number z satisfies the equation

$$z^2 = 3 + 4\text{i}\,.$$

a) Find the possible values of …

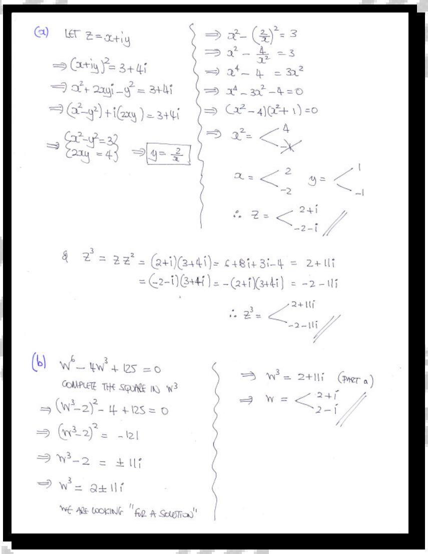
i. … z.

ii. … z^3.

b) Hence, by showing detailed workings, find a solution of the equation

$$w^6 - 4w^3 + 125 = 0,\ w \in \mathbb{C},$$

$\boxed{z = \pm(2+\text{i})}$, $\boxed{z^3 = 2 \pm 11\text{i}}$, $\boxed{w = \pm(2+\text{i})}$



(a) LET $z = x + iy$

$\Rightarrow (x+iy)^2 = 3+4i$

$\Rightarrow x^2 + 2xyi - y^2 = 3+4i$

$\Rightarrow (x^2-y^2) + i(2xy) = 3+4i$

$\Rightarrow \left\{ \begin{array}{l} x^2-y^2=3 \\ 2xy = 4 \end{array} \right\} \Rightarrow \boxed{y = \frac{2}{x}}$

$\Rightarrow x^2 - \left(\frac{2}{x}\right)^2 = 3$

$\Rightarrow x^2 - \frac{4}{x^2} = 3$

$\Rightarrow x^4 - 4 = 3x^2$

$\Rightarrow x^4 - 3x^2 - 4 = 0$

$\Rightarrow (x^2-4)(x^2+1) = 0$

$\Rightarrow x^2 = < \begin{array}{l} 4 \\ \times \end{array}$

$x = < \begin{array}{l} 2 \\ -2 \end{array} \quad y = < \begin{array}{l} 1 \\ -1 \end{array}$

$\therefore z = < \begin{array}{l} 2+i \\ -2-i \end{array}$ //

& $z^3 = zz^2 = (2+i)(3+4i) = 6+8i+3i-4 = 2+11i$

$= (-2-i)(3+4i) = -(2+i)(3+4i) = -2-11i$

$\therefore z^3 = < \begin{array}{l} 2+11i \\ -2-11i \end{array}$ //

(b) $w^6 - 4w^3 + 125 = 0$

COMPLETE THE SQUARE IN w^3

$\Rightarrow (w^3-2)^2 - 4 + 125 = 0$

$\Rightarrow (w^3-2)^2 = -121$

$\Rightarrow w^3 - 2 = \pm 11i$

$\Rightarrow w^3 = 2 \pm 11i$

WE ARE LOOKING "FOR A SOLUTION"

$\Rightarrow w^3 = 2+11i$ (PART a)

$\Rightarrow w = < \begin{array}{l} 2+i \\ 2-i \end{array}$ //

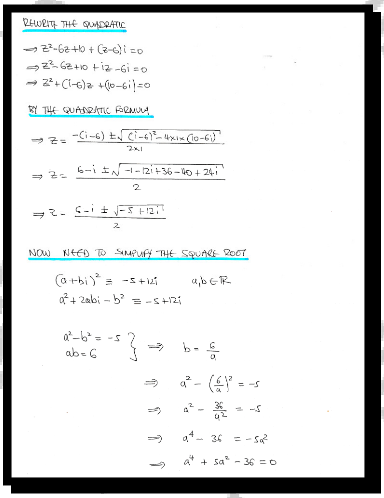
Question 47 (***+)

Solve the following quadratic equation

$$z^2 - 6z + 10 + (z-6)\mathrm{i} = 0, \quad z \in \mathbb{C}.$$

Give the answers in the form $a + b\mathrm{i}$, $a \in \mathbb{R}$, $b \in \mathbb{R}$.

$\boxed{}$, $\boxed{z_1 = 4 + \mathrm{i}}$, $\boxed{z_2 = 2 - 2\mathrm{i}}$



REWRITE THE QUADRATIC

$\Rightarrow z^2 - 6z + 10 + (z-6)i = 0$

$\Rightarrow z^2 - 6z + 10 + iz - 6i = 0$

$\Rightarrow z^2 + (i-6)z + (10-6i) = 0$

BY THE QUADRATIC FORMULA

$$\Rightarrow z = \frac{-(i-6) \pm \sqrt{(i-6)^2 - 4 \times 1 \times (10-6i)}}{2 \times 1}$$

$$\Rightarrow z = \frac{6 - i \pm \sqrt{-1 - 12i + 36 - 40 + 24i}}{2}$$

$$\Rightarrow z = \frac{6 - i \pm \sqrt{-5 + 12i}}{2}$$

NOW NEED TO SIMPLIFY THE SQUARE ROOT

$(a+bi)^2 \equiv -5 + 12i \qquad a, b \in \mathbb{R}$

$a^2 + 2abi - b^2 \equiv -5 + 12i$

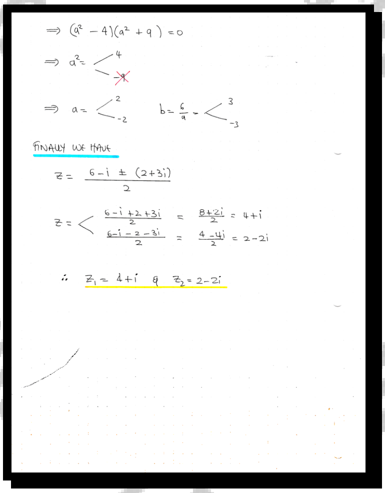
$$\left.\begin{array}{l} a^2 - b^2 = -5 \\ ab = 6 \end{array}\right\} \Rightarrow b = \frac{6}{a}$$

$\Rightarrow a^2 - \left(\frac{6}{a}\right)^2 = -5$

$\Rightarrow a^2 - \frac{36}{a^2} = -5$

$\Rightarrow a^4 - 36 = -5a^2$

$\Rightarrow a^4 + 5a^2 - 36 = 0$



$\Rightarrow (a^2 - 4)(a^2 + 9) = 0$

$$\Rightarrow a^2 = \left\langle \begin{array}{l} 4 \\ \cancel{-9} \end{array} \right.$$

$$\Rightarrow a = \left\langle \begin{array}{l} 2 \\ -2 \end{array} \right. \qquad b = \frac{6}{a} = \left\langle \begin{array}{l} 3 \\ -3 \end{array} \right.$$

FINALLY WE HAVE

$$z = \frac{6 - i \pm (2+3i)}{2}$$

$$z = \left\langle \begin{array}{l} \frac{6 - i + 2 + 3i}{2} = \frac{8 + 2i}{2} = 4 + i \\ \frac{6 - i - 2 - 3i}{2} = \frac{4 - 4i}{2} = 2 - 2i \end{array} \right.$$

$\therefore \; z_1 = 4 + i \;\; \& \;\; z_2 = 2 - 2i$

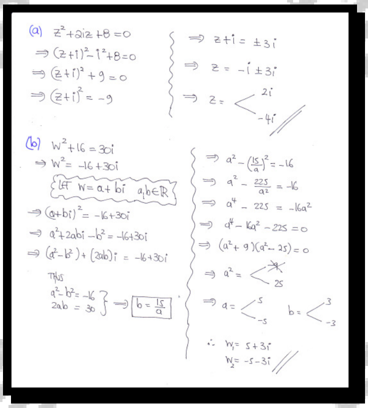
Question 48 **(***+)**

Solve teach of the following equations.

a) $z^2 + 2iz + 8 = 0$, $z \in \mathbb{C}$.

b) $w^2 + 16 = 30i$, $w \in \mathbb{C}$.

$\boxed{z_1 = 2i,\ z_2 = -4i}$, $\boxed{w = \pm(3+5i)}$



(a) $z^2+2iz+8=0$

$\Rightarrow (z+i)^2-i^2+8=0$

$\Rightarrow (z+i)^2+9=0$

$\Rightarrow (z+i)^2=-9$

$\Rightarrow z+i=\pm 3i$

$\Rightarrow z=-i\pm 3i$

$\Rightarrow z = \left\langle \begin{array}{l} 2i \\ -4i \end{array} \right.$ //

(b) $w^2+16=30i$

$\Rightarrow w^2=-16+30i$

LET $w=a+bi \quad a,b\in\mathbb{R}$

$\Rightarrow (a+bi)^2=-16+30i$

$\Rightarrow a^2+2abi-b^2=-16+30i$

$\Rightarrow (a^2-b^2)+(2ab)i=-16+30i$

Thus

$\left.\begin{array}{l} a^2-b^2=-16 \\ 2ab=30 \end{array}\right\} \Rightarrow \boxed{b=\frac{15}{a}}$

$\Rightarrow a^2-\left(\frac{15}{a}\right)^2=-16$

$\Rightarrow a^2-\frac{225}{a^2}=-16$

$\Rightarrow a^4-225=-16a^2$

$\Rightarrow a^4-16a^2-225=0$

$\Rightarrow (a^2+9)(a^2-25)=0$

$\Rightarrow a^2 = \left\langle \begin{array}{l} \cancel{-9} \\ 25 \end{array} \right.$

$\Rightarrow a = \left\langle \begin{array}{l} 5 \\ -5 \end{array} \right. \quad b = \left\langle \begin{array}{l} 3 \\ -3 \end{array} \right.$

$\therefore\ w_1=5+3i$

$w_2=-5-3i$ //

Question 49 **(***+)**

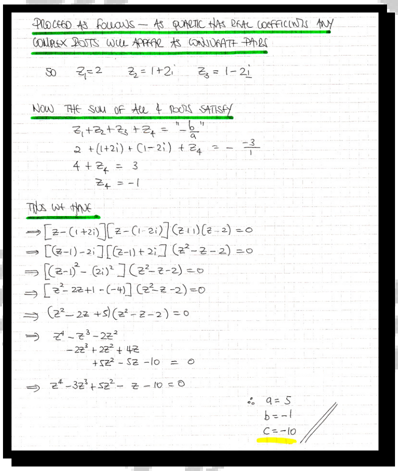
It is given that $z = 2$ and $z = 1 + 2\text{i}$ are solutions of the equation

$$z^4 - 3z^3 + az^2 + bz + c = 0 .$$

where a, b and c are real constants.

Determine the values of a, b and c.

$\boxed{\quad}$, $\boxed{a = 5}$, $\boxed{b = -1}$, $\boxed{c = -10}$



PROCEED AS FOLLOWS – AS QUARTIC HAS REAL COEFFICIENTS ANY COMPLEX ROOTS WILL APPEAR AS CONJUGATE PAIRS

SO $z_1 = 2 \qquad z_2 = 1+2i \qquad z_3 = 1-2i$

NOW THE SUM OF ALL 4 ROOTS SATISFY

$$z_1 + z_2 + z_3 + z_4 = \text{"}-\frac{b}{a}\text{"}$$
$$2 + (1+2i) + (1-2i) + z_4 = -\frac{-3}{1}$$
$$4 + z_4 = 3$$
$$z_4 = -1$$

THUS WE HAVE

$$\Rightarrow [z-(1+2i)][z-(1-2i)](z+1)(z-2) = 0$$
$$\Rightarrow [(z-1)-2i][(z-1)+2i](z^2-z-2) = 0$$
$$\Rightarrow [(z-1)^2 - (2i)^2](z^2-z-2) = 0$$
$$\Rightarrow [z^2-2z+1-(-4)](z^2-z-2) = 0$$
$$\Rightarrow (z^2-2z+5)(z^2-z-2) = 0$$
$$\Rightarrow z^4 - z^3 - 2z^2 - 2z^3 + 2z^2 + 4z + 5z^2 - 5z - 10 = 0$$
$$\Rightarrow z^4 - 3z^3 + 5z^2 - z - 10 = 0$$

$\therefore a = 5$
$b = -1$
$c = -10$ //

Question 50 **(***+)**

The following complex numbers are given

$$z = \frac{1+\mathrm{i}}{1-\mathrm{i}} \quad \text{and} \quad w = \frac{\sqrt{2}}{1-\mathrm{i}}.$$

a) Calculate the modulus of z and the modulus of w.

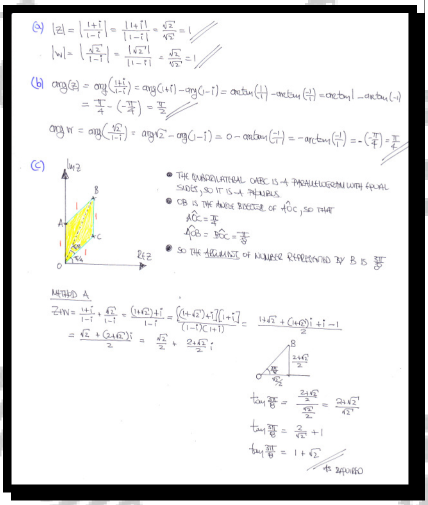
b) Find the argument of z and the argument of w.

In a standard Argand diagram, the points A, B and C represent the numbers z, $z+w$ and w respectively. The origin of the Argand diagram is denoted by O.

c) By considering the quadrilateral $OABC$ and the argument of $z+w$, show that

$$\tan\left(\frac{3\pi}{8}\right) = 1+\sqrt{2}.$$

$\boxed{|z|=1}$, $\boxed{|w|=1}$, $\boxed{\arg z = \frac{\pi}{2}}$, $\boxed{\arg w = \frac{\pi}{4}}$



(a) $|z| = \left|\frac{1+i}{1-i}\right| = \frac{|1+i|}{|1-i|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$ //

$|w| = \left|\frac{\sqrt{2}}{1-i}\right| = \frac{|\sqrt{2}|}{|1-i|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$ //

(b) $\arg(z) = \arg\left(\frac{1+i}{1-i}\right) = \arg(1+i) - \arg(1-i) = \arctan\left(\frac{1}{1}\right) - \arctan\left(\frac{-1}{1}\right) = \arctan 1 - \arctan(-1)$

$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$ //

$\arg w = \arg\left(\frac{\sqrt{2}}{1-i}\right) = \arg\sqrt{2} - \arg(1-i) = 0 - \arctan\left(\frac{-1}{1}\right) = -\arctan\left(\frac{-1}{1}\right) = -\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$ //

(c)

- THE QUADRILATERAL OABC IS A PARALLELOGRAM WITH EQUAL SIDES, SO IT IS A RHOMBUS.
- OB IS THE ANGLE BISECTOR OF $A\hat{O}C$, SO THAT
  $A\hat{O}C = \frac{\pi}{4}$
  $A\hat{O}B = B\hat{O}C = \frac{\pi}{8}$
- SO THE ARGUMENT OF NUMBER REPRESENTED BY B IS $\frac{3\pi}{8}$

METHOD A

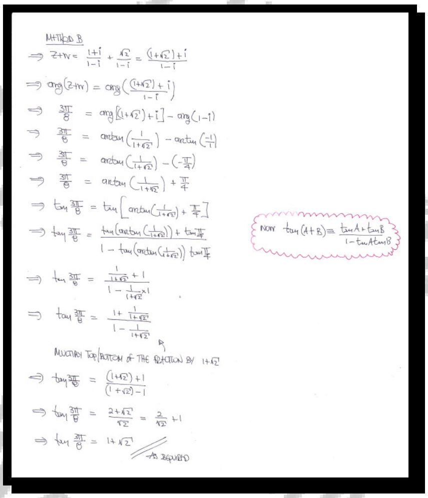
$z+w = \frac{1+i}{1-i} + \frac{\sqrt{2}}{1-i} = \frac{(1+\sqrt{2})+i}{1-i} = \frac{[(1+\sqrt{2})+i][1+i]}{(1-i)(1+i)} = \frac{1+\sqrt{2} + (1+\sqrt{2})i + i - 1}{2}$

$= \frac{\sqrt{2} + (2+\sqrt{2})i}{2} = \frac{\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}i$

$\tan\frac{3\pi}{8} = \frac{\frac{2+\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2+\sqrt{2}}{\sqrt{2}}$

$\tan\frac{3\pi}{8} = \frac{2}{\sqrt{2}} + 1$

$\tan\frac{3\pi}{8} = 1 + \sqrt{2}$ // AS REQUIRED



METHOD B

$\Rightarrow z+w = \frac{1+i}{1-i} + \frac{\sqrt{2}}{1-i} = \frac{(1+\sqrt{2})+i}{1-i}$

$\Rightarrow \arg(z+w) = \arg\left(\frac{(1+\sqrt{2})+i}{1-i}\right)$

$\Rightarrow \frac{3\pi}{8} = \arg[(1+\sqrt{2})+i] - \arg(1-i)$

$\Rightarrow \frac{3\pi}{8} = \arctan\left(\frac{1}{1+\sqrt{2}}\right) - \arctan\left(\frac{-1}{1}\right)$

$\Rightarrow \frac{3\pi}{8} = \arctan\left(\frac{1}{1+\sqrt{2}}\right) - \left(-\frac{\pi}{4}\right)$

$\Rightarrow \frac{3\pi}{8} = \arctan\left(\frac{1}{1+\sqrt{2}}\right) + \frac{\pi}{4}$

$\Rightarrow \tan\frac{3\pi}{8} = \tan\left[\arctan\left(\frac{1}{1+\sqrt{2}}\right) + \frac{\pi}{4}\right]$

$\Rightarrow \tan\frac{3\pi}{8} = \frac{\tan\left(\arctan\left(\frac{1}{1+\sqrt{2}}\right)\right) + \tan\frac{\pi}{4}}{1 - \tan\left(\arctan\left(\frac{1}{1+\sqrt{2}}\right)\right)\tan\frac{\pi}{4}}$

NOW $\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\Rightarrow \tan\frac{3\pi}{8} = \frac{\frac{1}{1+\sqrt{2}} + 1}{1 - \frac{1}{1+\sqrt{2}} \times 1}$

$\Rightarrow \tan\frac{3\pi}{8} = \frac{1 + \frac{1}{1+\sqrt{2}}}{1 - \frac{1}{1+\sqrt{2}}}$

MULTIPLY TOP/BOTTOM OF THE FRACTION BY $1+\sqrt{2}$

$\Rightarrow \tan\frac{3\pi}{8} = \frac{(1+\sqrt{2})+1}{(1+\sqrt{2})-1}$

$\Rightarrow \tan\frac{3\pi}{8} = \frac{2+\sqrt{2}}{\sqrt{2}} = \frac{2}{\sqrt{2}} + 1$

$\Rightarrow \tan\frac{3\pi}{8} = 1+\sqrt{2}$ // AS REQUIRED

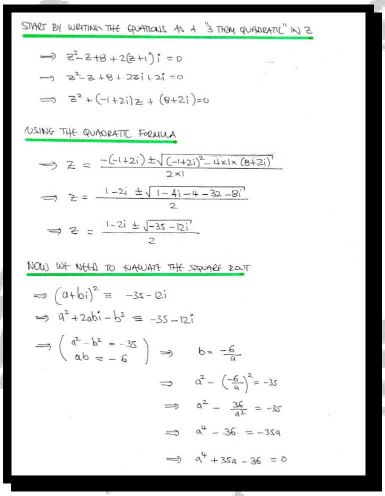
Question 51 (***+)

Solve the following quadratic equation

$$z^2 - z + 8 + 2(z+1)\mathrm{i} = 0, \quad z \in \mathbb{C}.$$

Give the answers in the form $a + b\mathrm{i}$, $a \in \mathbb{R}$, $b \in \mathbb{R}$.

$\boxed{}$, $\boxed{z_1 = 2\mathrm{i}}$, $\boxed{z_2 = 1 - 4\mathrm{i}}$



START BY WRITING THE EQUATIONS AS A "3 TERM QUADRATIC" IN z

$\Rightarrow z^2 - z + 8 + 2(z+1)i = 0$

$\Rightarrow z^2 - z + 8 + 2zi + 2i = 0$

$\Rightarrow z^2 + (-1+2i)z + (8+2i) = 0$

USING THE QUADRATIC FORMULA

$\Rightarrow z = \frac{-(-1+2i) \pm \sqrt{(-1+2i)^2 - 4 \times 1 \times (8+2i)}}{2 \times 1}$

$\Rightarrow z = \frac{1-2i \pm \sqrt{1-4i-4-32-8i}}{2}$

$\Rightarrow z = \frac{1-2i \pm \sqrt{-35-12i}}{2}$

NOW WE NEED TO EVALUATE THE SQUARE ROOT

$\Rightarrow (a+bi)^2 \equiv -35-12i$

$\Rightarrow a^2 + 2abi - b^2 \equiv -35-12i$

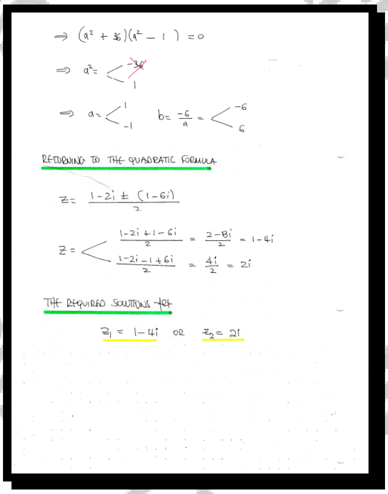
$\Rightarrow \begin{pmatrix} a^2 - b^2 = -35 \\ ab = -6 \end{pmatrix} \Rightarrow b = \frac{-6}{a}$

$\Rightarrow a^2 - \left(\frac{-6}{a}\right)^2 = -35$

$\Rightarrow a^2 - \frac{36}{a^2} = -35$

$\Rightarrow a^4 - 36 = -35a$

$\Rightarrow a^4 + 35a - 36 = 0$



$\Rightarrow (a^2 + 36)(a^2 - 1) = 0$

$\Rightarrow a^2 = \begin{cases} \cancel{-36} \\ 1 \end{cases}$

$\Rightarrow a = \begin{cases} 1 \\ -1 \end{cases} \quad b = \frac{-6}{a} = \begin{cases} -6 \\ 6 \end{cases}$

RETURNING TO THE QUADRATIC FORMULA

$z = \frac{1-2i \pm (1-6i)}{2}$

$z = \begin{cases} \frac{1-2i+1-6i}{2} = \frac{2-8i}{2} = 1-4i \\ \frac{1-2i-1+6i}{2} = \frac{4i}{2} = 2i \end{cases}$

THE REQUIRED SOLUTIONS ARE

$z_1 = 1-4i$ OR $z_2 = 2i$

Question 52 (***+)

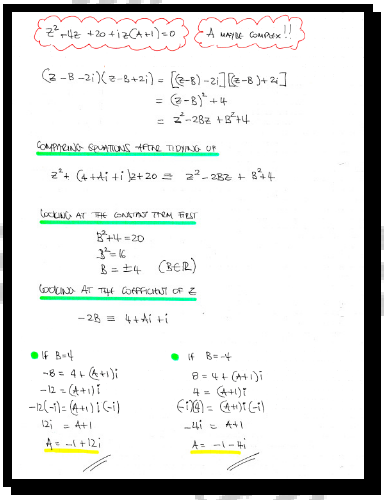
The quadratic equation

$$z^2 + 4z + 20 + \mathrm{i}\,z(A+1) = 0 \ ,$$

where A is a constant, has complex conjugate roots.

If one of the roots of this quadratic is $z = B + 2\mathrm{i}$, where B is a **real** constant, find the possible values of A.

$\boxed{}$, $\boxed{A = -1 + 12\mathrm{i} \ \cup \ A = -1 - 4\mathrm{i}}$



$z^2+4z+20+iz(A+1)=0$ A MAYBE COMPLEX!!

$(z-B-2i)(z-B+2i) = [(z-B)-2i][(z-B)+2i]$
$= (z-B)^2+4$
$= z^2-2Bz+B^2+4$

COMPARING EQUATIONS AFTER TIDYING UP

$z^2+(4+Ai+i)z+20 \equiv z^2-2Bz+B^2+4$

LOOKING AT THE CONSTANT TERM FIRST

$B^2+4=20$
$B^2=16$
$B=\pm 4 \quad (B \in \mathbb{R})$

LOOKING AT THE COEFFICIENT OF z

$-2B \equiv 4+Ai+i$

- If $B=4$
  $-8 = 4+(A+1)i$
  $-12 = (A+1)i$
  $-12(-i) = (A+1)i(-i)$
  $12i = A+1$
  $A = -1+12i$ //
- If $B=-4$
  $8 = 4+(A+1)i$
  $4 = (A+1)i$
  $(-i)(4) = (A+1)i(-i)$
  $-4i = A+1$
  $A = -1-4i$ //

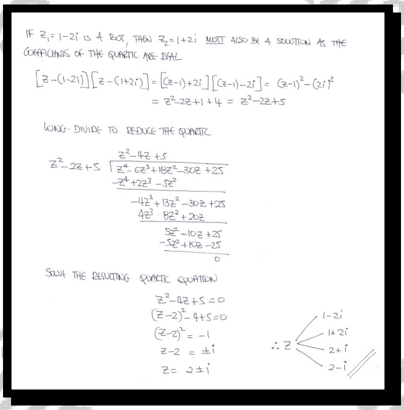
Question 53 **(***+)**

If $1-2\mathrm{i}$ is a root of the quartic equation

$$z^4-6z^3+18z^2-30z+25=0$$

find the other three roots.

$$\boxed{z_2=1+2\mathrm{i},\quad z_3=2-\mathrm{i},\quad z_4=2+\mathrm{i}}$$



IF $z_1=1-2i$ IS A ROOT, THEN $z_2=1+2i$ MUST ALSO BE A SOLUTION AS THE COEFFICIENTS OF THE QUARTIC ARE REAL

$$[z-(1-2i)][z-(1+2i)]=[(z-1)+2i][(z-1)-2i]=(z-1)^2-(2i)^2$$
$$=z^2-2z+1+4=z^2-2z+5$$

LONG·DIVIDE TO REDUCE THE QUARTIC

$$\begin{array}{r}
z^2-4z+5 \\
z^2-2z+5 \,\big)\, z^4-6z^3+18z^2-30z+25 \\
-z^4+2z^3-5z^2 \\
\hline
-4z^3+13z^2-30z+25 \\
4z^3-8z^2+20z \\
\hline
5z^2-10z+25 \\
-5z^2+10z-25 \\
\hline
0
\end{array}$$

SOLVE THE RESULTING QUADRATIC EQUATION

$$z^2-4z+5=0$$
$$(z-2)^2-4+5=0$$
$$(z-2)^2=-1$$
$$z-2=\pm i$$
$$z=2\pm i$$

$$\therefore z \begin{cases} 1-2i \\ 1+2i \\ 2+i \\ 2-i \end{cases}$$

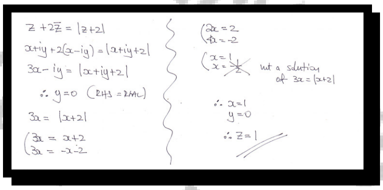
Question 54 **(****)**

The complex conjugate of z is denoted by $\overline{z}$.

Solve the equation

$$z+2\overline{z}=|z+2|,\ z\in\mathbb{C}.$$

$$\boxed{z=1}$$



$$z+2\overline{z}=|z+2|$$
$$x+iy+2(x-iy)=|x+iy+2|$$
$$3x-iy=|x+iy+2|$$
$$\therefore y=0 \quad (\text{RHS} = \text{REAL})$$
$$3x=|x+2|$$
$$\begin{cases} 3x=x+2 \\ 3x=-x-2 \end{cases}$$
$$\begin{cases} 2x=2 \\ 4x=-2 \end{cases}$$
$$\begin{cases} x=1 \\ \cancel{x=-\tfrac{1}{2}} \end{cases}$$ not a solution of $3x=|x+2|$

$$\therefore x=1$$
$$y=0$$
$$\therefore z=1$$

Question 55 (****)

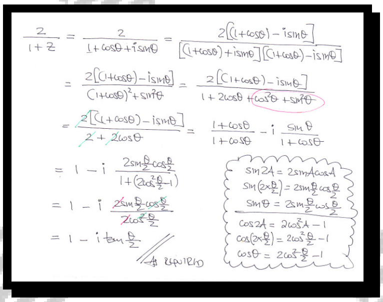
It is given that

$$z = \cos\theta + \mathrm{i}\sin\theta,\ 0 \le z < 2\pi .$$

Show clearly that

$$\frac{2}{1+z} = 1 - \mathrm{i}\tan\left(\frac{\theta}{2}\right).$$

$\boxed{\text{proof}}$



$$\frac{2}{1+z} = \frac{2}{1+\cos\theta+\mathrm{i}\sin\theta} = \frac{2\left[(1+\cos\theta)-\mathrm{i}\sin\theta\right]}{\left[(1+\cos\theta)+\mathrm{i}\sin\theta\right]\left[(1+\cos\theta)-\mathrm{i}\sin\theta\right]}$$

$$= \frac{2\left[(1+\cos\theta)-\mathrm{i}\sin\theta\right]}{(1+\cos\theta)^2+\sin^2\theta} = \frac{2\left[(1+\cos\theta)-\mathrm{i}\sin\theta\right]}{1+2\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$= \frac{\cancel{2}\left[(1+\cos\theta)-\mathrm{i}\sin\theta\right]}{\cancel{2}+\cancel{2}\cos\theta} = \frac{1+\cos\theta}{1+\cos\theta} - \mathrm{i}\,\frac{\sin\theta}{1+\cos\theta}$$

$$= 1 - \mathrm{i}\,\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+\left(2\cos^2\frac{\theta}{2}-1\right)}$$

$$= 1 - \mathrm{i}\,\frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}$$

$$= 1 - \mathrm{i}\tan\frac{\theta}{2}$$

// AS REQUIRED

$$\sin 2A = 2\sin A\cos A$$
$$\sin\left(2\times\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\cos 2A = 2\cos^2 A - 1$$
$$\cos\left(2\times\frac{\theta}{2}\right) = 2\cos^2\frac{\theta}{2} - 1$$
$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

Question 56 (****)

$$\frac{(3+4\mathrm{i})(1+2\mathrm{i})}{1+3\mathrm{i}} = q(1+\mathrm{i}) \ , \quad q \in \mathbb{R} \, .$$

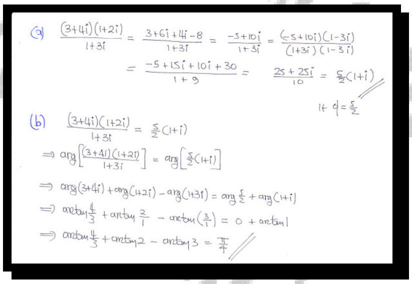
a) Find the value of q.

b) Hence simplify

$$\arctan\frac{4}{3} + \arctan 2 - \arctan 3 \, ,$$

giving the answer in terms of π.

$\boxed{q = \frac{5}{2}}$, $\boxed{\frac{1}{4}\pi}$



(a) $\frac{(3+4i)(1+2i)}{1+3i} = \frac{3+6i+4i-8}{1+3i} = \frac{-5+10i}{1+3i} = \frac{(-5+10i)(1-3i)}{(1+3i)(1-3i)}$

$= \frac{-5+15i+10i+30}{1+9} = \frac{25+25i}{10} = \frac{5}{2}(1+i)$

$\therefore q = \frac{5}{2}$

(b) $\frac{(3+4i)(1+2i)}{1+3i} = \frac{5}{2}(1+i)$

$\Longrightarrow \arg\left[\frac{(3+4i)(1+2i)}{1+3i}\right] = \arg\left[\frac{5}{2}(1+i)\right]$

$\Longrightarrow \arg(3+4i) + \arg(1+2i) - \arg(1+3i) = \arg\frac{5}{2} + \arg(1+i)$

$\Longrightarrow \arctan\frac{4}{3} + \arctan\frac{2}{1} - \arctan\left(\frac{3}{1}\right) = 0 + \arctan 1$

$\Longrightarrow \arctan\frac{4}{3} + \arctan 2 - \arctan 3 = \frac{\pi}{4}$

Question 57 (****)

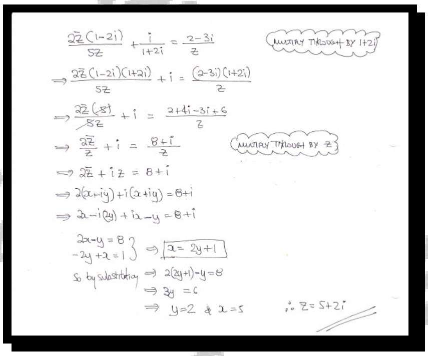
The complex conjugate of the complex number z is denoted by $\overline{z}$.

Solve the equation

$$\frac{2\overline{z}(1-2\mathrm{i})}{5z}+\frac{\mathrm{i}}{1+2\mathrm{i}}=\frac{2-3\mathrm{i}}{z},$$

giving the answer in the form $x+\mathrm{i}y$.

$\boxed{z=5+2\mathrm{i}}$



$\frac{2\bar{z}(1-2i)}{5z}+\frac{i}{1+2i}=\frac{2-3i}{z}$ MULTIPLY THROUGH BY 1+2i

$\Rightarrow \frac{2\bar{z}(1-2i)(1+2i)}{5z}+i=\frac{(2-3i)(1+2i)}{z}$

$\Rightarrow \frac{2\bar{z}(\cancel{5})}{\cancel{5}z}+i=\frac{2+4i-3i+6}{z}$

$\Rightarrow \frac{2\bar{z}}{z}+i=\frac{8+i}{z}$ MULTIPLY THROUGH BY z

$\Rightarrow 2\bar{z}+iz=8+i$

$\Rightarrow 2(x-iy)+i(x+iy)=8+i$

$\Rightarrow 2x-i(2y)+ix-y=8+i$

$\left.\begin{array}{l}2x-y=8\\-2y+x=1\end{array}\right\}\Rightarrow \boxed{x=2y+1}$

So by substitution $\Rightarrow 2(2y+1)-y=8$

$\Rightarrow 3y=6$

$\Rightarrow y=2$ & $x=5$ $\therefore z=5+2i$

Question 58 **(****)**

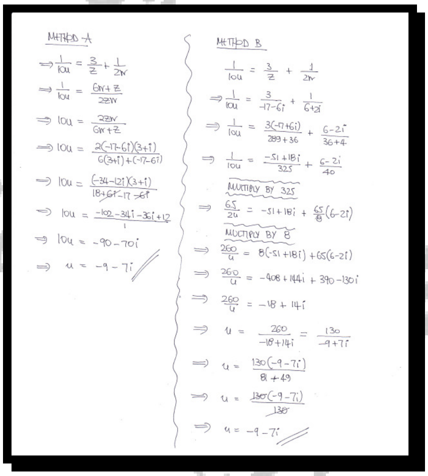
It is given that

$$z = -17 - 6\text{i} \qquad \text{and} \qquad w = 3 + \text{i}\,.$$

Find the value of u given further that

$$\frac{1}{10u} = \frac{3}{z} + \frac{1}{2w}\,.$$

$$\boxed{u = -9 - 7\text{i}}$$



METHOD A

$\Rightarrow \frac{1}{10u} = \frac{3}{z} + \frac{1}{2w}$

$\Rightarrow \frac{1}{10u} = \frac{6w+z}{2zw}$

$\Rightarrow 10u = \frac{2zw}{6w+z}$

$\Rightarrow 10u = \frac{2(-17-6i)(3+i)}{6(3+i)+(-17-6i)}$

$\Rightarrow 10u = \frac{(-34-12i)(3+i)}{18+\cancel{6i}-17-\cancel{6i}}$

$\Rightarrow 10u = \frac{-102-34i-36i+12}{1}$

$\Rightarrow 10u = -90-70i$

$\Rightarrow u = -9-7i$ //

METHOD B

$\frac{1}{10u} = \frac{3}{z} + \frac{1}{2w}$

$\Rightarrow \frac{1}{10u} = \frac{3}{-17-6i} + \frac{1}{6+2i}$

$\Rightarrow \frac{1}{10u} = \frac{3(-17+6i)}{289+36} + \frac{6-2i}{36+4}$

$\Rightarrow \frac{1}{10u} = \frac{-51+18i}{325} + \frac{6-2i}{40}$

MULTIPLY BY 325

$\Rightarrow \frac{65}{2u} = -51+18i + \frac{65}{8}(6-2i)$

MULTIPLY BY 8

$\Rightarrow \frac{260}{u} = 8(-51+18i) + 65(6-2i)$

$\Rightarrow \frac{260}{u} = -408+144i+390-130i$

$\Rightarrow \frac{260}{u} = -18+14i$

$\Rightarrow u = \frac{260}{-18+14i} = \frac{130}{-9+7i}$

$\Rightarrow u = \frac{130(-9-7i)}{81+49}$

$\Rightarrow u = \frac{\cancel{130}(-9-7i)}{\cancel{130}}$

$\Rightarrow u = -9-7i$ //

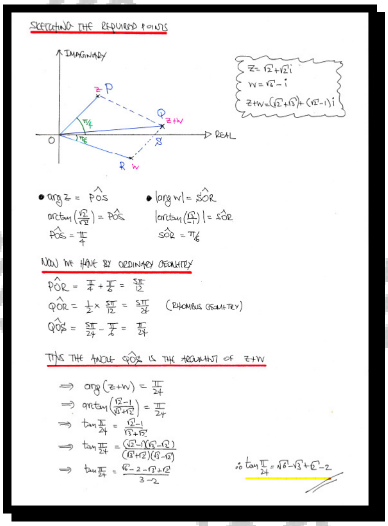
Question 59 (****)

Sketch on a standard Argand diagram the locus of the points $z = \sqrt{2}(1+\mathrm{i})$, $w = \sqrt{3} - \mathrm{i}$ and $z+w$, and use geometry to prove that

$$\tan\left(\frac{\pi}{24}\right) = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2 .$$

You must justify all the steps in this proof.

$\boxed{}$, $\boxed{\text{proof}}$



SKETCHING THE REQUIRED POINTS

$z = \sqrt{2} + \sqrt{2}\,\mathrm{i}$
$w = \sqrt{3} - \mathrm{i}$
$z+w = (\sqrt{2}+\sqrt{3}) + (\sqrt{2}-1)\mathrm{i}$

- $\arg z = \hat{POS}$
  $\arctan\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \hat{POS}$
  $\hat{POS} = \frac{\pi}{4}$
- $|\arg w| = \hat{SOR}$
  $\left|\arctan\left(\frac{\sqrt{3}}{-1}\right)\right| = \hat{SOR}$
  $\hat{SOR} = \pi/6$

NOW WE HAVE BY ORDINARY GEOMETRY

$\hat{POR} = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$

$\hat{QOR} = \frac{1}{2} \times \frac{5\pi}{12} = \frac{5\pi}{24}$ (RHOMBUS GEOMETRY)

$\hat{QOS} = \frac{5\pi}{24} - \frac{\pi}{6} = \frac{\pi}{24}$

THUS THE ANGLE $\hat{QOS}$ IS THE ARGUMENT OF $z+w$

$\Rightarrow \arg(z+w) = \frac{\pi}{24}$

$\Rightarrow \arctan\left(\frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}\right) = \frac{\pi}{24}$

$\Rightarrow \tan\frac{\pi}{24} = \frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}$

$\Rightarrow \tan\frac{\pi}{24} = \frac{(\sqrt{2}-1)(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$

$\Rightarrow \tan\frac{\pi}{24} = \frac{\sqrt{6}-2-\sqrt{3}+\sqrt{2}}{3-2}$

$\therefore \tan\frac{\pi}{24} = \sqrt{6}-\sqrt{3}+\sqrt{2}-2$

Question 60 **(****)**

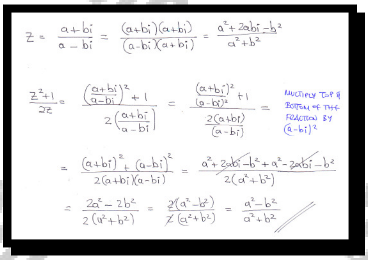
The complex number z is given by

$$z = \frac{a+b\mathrm{i}}{a-b\mathrm{i}}, \ a \in \mathbb{R}, \ b \in \mathbb{R}.$$

Show clearly that

$$\frac{z^2+1}{2z} = \frac{a^2-b^2}{a^2+b^2}.$$

$\boxed{\text{proof}}$



$$z = \frac{a+bi}{a-bi} = \frac{(a+bi)(a+bi)}{(a-bi)(a+bi)} = \frac{a^2+2abi-b^2}{a^2+b^2}$$

$$\frac{z^2+1}{2z} = \frac{\left(\frac{a+bi}{a-bi}\right)^2+1}{2\left(\frac{a+bi}{a-bi}\right)} = \frac{\frac{(a+bi)^2}{(a-bi)^2}+1}{\frac{2(a+bi)}{(a-bi)}} =$$

MULTIPLY TOP & BOTTOM OF THE FRACTION BY $(a-bi)^2$

$$= \frac{(a+bi)^2+(a-bi)^2}{2(a+bi)(a-bi)} = \frac{a^2+\cancel{2abi}-b^2+a^2-\cancel{2abi}-b^2}{2(a^2+b^2)}$$

$$= \frac{2a^2-2b^2}{2(a^2+b^2)} = \frac{\cancel{2}(a^2-b^2)}{\cancel{2}(a^2+b^2)} = \frac{a^2-b^2}{a^2+b^2}$$

Question 61 (****)

It is given that

$$z = \frac{1+8\text{i}}{1-2\text{i}} \ .$$

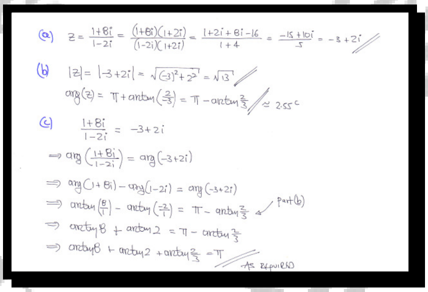
a) Express z in the form $x+\text{i}y$, where x and y are real numbers.

b) Find the modulus and argument of z.

c) Show clearly that

$$\arctan 8 + \arctan 2 + \arctan\frac{2}{3} = \pi \ .$$

$\boxed{z = -3+2\text{i}}$, $\boxed{|z| = \sqrt{13}}$, $\boxed{\arg z \approx 2.55^{\text{c}}}$



(a) $z = \frac{1+8i}{1-2i} = \frac{(1+8i)(1+2i)}{(1-2i)(1+2i)} = \frac{1+2i+8i-16}{1+4} = \frac{-15+10i}{5} = -3+2i$ //

(b) $|z| = |-3+2i| = \sqrt{(-3)^2+2^2} = \sqrt{13}$ //

$\arg(z) = \pi + \arctan\left(\frac{2}{-3}\right) = \pi - \arctan\frac{2}{3}$ // $\approx 2.55^c$

(c) $\frac{1+8i}{1-2i} = -3+2i$

$\Rightarrow \arg\left(\frac{1+8i}{1-2i}\right) = \arg(-3+2i)$

$\Rightarrow \arg(1+8i) - \arg(1-2i) = \arg(-3+2i)$

$\Rightarrow \arctan\left(\frac{8}{1}\right) - \arctan\left(\frac{-2}{1}\right) = \pi - \arctan\frac{2}{3}$ ← part (b)

$\Rightarrow \arctan 8 + \arctan 2 = \pi - \arctan\frac{2}{3}$

$\Rightarrow \arctan 8 + \arctan 2 + \arctan\frac{2}{3} = \pi$ //

As required

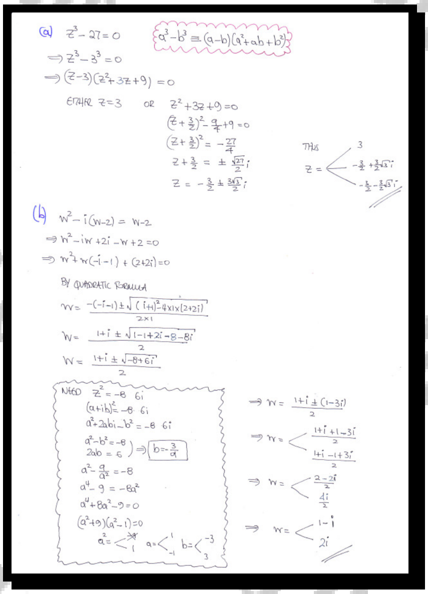
Question 62 (****+)

Solve each of the following equations.

a) $z^3 - 27 = 0$.

b) $w^2 - \mathrm{i}(w-2) = (w-2)$.

$\boxed{z_1 = 3, \quad z_2 = \frac{3}{2}\left(-1 \pm \sqrt{3}\right)}$, $\boxed{w_1 = 2\mathrm{i}, \quad w_2 = 1 - \mathrm{i}}$



(a) $z^3 - 27 = 0$

$\boxed{a^3 - b^3 \equiv (a-b)(a^2+ab+b^2)}$

$\Rightarrow z^3 - 3^3 = 0$

$\Rightarrow (z-3)(z^2+3z+9) = 0$

EITHER $z = 3$ OR $z^2+3z+9=0$

$\left(z+\frac{3}{2}\right)^2 - \frac{9}{4} + 9 = 0$

$\left(z+\frac{3}{2}\right)^2 = -\frac{27}{4}$

$z + \frac{3}{2} = \pm \frac{\sqrt{27}}{2} i$

$z = -\frac{3}{2} \pm \frac{3\sqrt{3}}{2} i$

THUS $z = \left\langle \begin{array}{l} 3 \\ -\frac{3}{2} + \frac{3}{2}\sqrt{3}\,i \\ -\frac{3}{2} - \frac{3}{2}\sqrt{3}\,i \end{array} \right.$

(b) $w^2 - i(w-2) = w-2$

$\Rightarrow w^2 - iw + 2i - w + 2 = 0$

$\Rightarrow w^2 + w(-i-1) + (2+2i) = 0$

BY QUADRATIC FORMULA

$w = \frac{-(-i-1) \pm \sqrt{(i+1)^2 - 4\times1\times(2+2i)}}{2\times1}$

$w = \frac{1+i \pm \sqrt{1-1+2i-8-8i}}{2}$

$w = \frac{1+i \pm \sqrt{-8+6i}}{2}$

NEED $z^2 = -8 \quad 6i$

$(a+ib)^2 = -8 \quad 6i$

$a^2 + 2abi - b^2 = -8 \quad 6i$

$\left.\begin{array}{l} a^2 - b^2 = -8 \\ 2ab = 6 \end{array}\right) \Rightarrow \boxed{b = -\frac{3}{a}}$

$a^2 - \frac{9}{a^2} = -8$

$a^4 - 9 = -8a^2$

$a^4 + 8a^2 - 9 = 0$

$(a^2+9)(a^2-1) = 0$

$a^2 = \left\langle \begin{array}{l} \cancel{-9} \\ 1 \end{array} \right. \quad a = \left\langle \begin{array}{l} 1 \\ -1 \end{array} \right. \quad b = \left\langle \begin{array}{l} -3 \\ 3 \end{array} \right.$

$\Rightarrow w = \frac{1+i \pm (1-3i)}{2}$

$\Rightarrow w = \left\langle \begin{array}{l} \frac{1+i+1-3i}{2} \\ \frac{1+i-1+3i}{2} \end{array} \right.$

$\Rightarrow w = \left\langle \begin{array}{l} \frac{2-2i}{2} \\ \frac{4i}{2} \end{array} \right.$

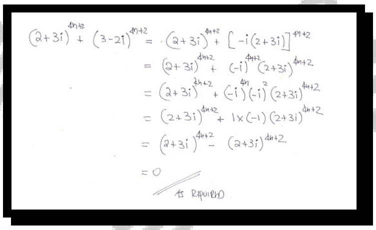
$\Rightarrow w = \left\langle \begin{array}{l} 1-i \\ 2i \end{array} \right.$

Question 63 (****+)

$$z = (2+3\mathrm{i})^{4n+2} + (3-2\mathrm{i})^{4n+2}, \; n \in \mathbb{N}.$$

Show clearly that $z=0$ for all $n \in \mathbb{N}$.

proof



$$(2+3i)^{4n+2} + (3-2i)^{4n+2} = (2+3i)^{4n+2} + \left[-i(2+3i)\right]^{4n+2}$$
$$= (2+3i)^{4n+2} + (-i)^{4n+2}(2+3i)^{4n+2}$$
$$= (2+3i)^{4n+2} + (-i)^{4n}(-i)^{2}(2+3i)^{4n+2}$$
$$= (2+3i)^{4n+2} + 1\times(-1)(2+3i)^{4n+2}$$
$$= (2+3i)^{4n+2} - (2+3i)^{4n+2}$$
$$= 0$$

AS REQUIRED

Question 64 (****+)

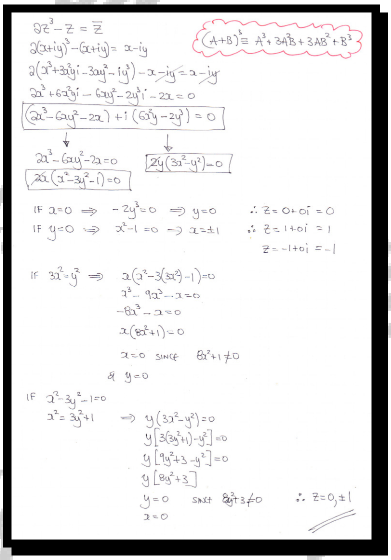
The complex conjugate of z is denoted by $\overline{z}$.

Show clearly that the equation

$$2z^3 - z = \overline{z},$$

is satisfied either by $z=0$ or $z=\pm1$.

proof



$$2z^3 - z = \overline{z}$$

$$(A+B)^3 \equiv A^3 + 3A^2B + 3AB^2 + B^3$$

$$2(x+iy)^3 - (x+iy) = x - iy$$
$$2(x^3+3x^2yi-3xy^2-iy^3) - x - \cancel{iy} = x - \cancel{iy}$$
$$2x^3+6x^2yi-6xy^2-2y^3i-2x=0$$
$$\boxed{(2x^3-6xy^2-2x)+i(6x^2y-2y^3)=0}$$

$$2x^3-6xy^2-2x=0 \qquad \boxed{\cancel{2}y(3x^2-y^2)=0}$$
$$\boxed{\cancel{2}x(x^2-3y^2-1)=0}$$

$$\text{IF } x=0 \Rightarrow -2y^3=0 \Rightarrow y=0 \qquad \therefore z=0+0i=0$$
$$\text{IF } y=0 \Rightarrow x^2-1=0 \Rightarrow x=\pm1 \qquad \therefore z=1+0i=1$$
$$z=-1+0i=-1$$

$$\text{IF } 3x^2=y^2 \Rightarrow x(x^2-3(3x^2)-1)=0$$
$$x^3-9x^3-x=0$$
$$-8x^3-x=0$$
$$x(8x^2+1)=0$$
$$x=0 \text{ SINCE } 8x^2+1\neq0$$
$$\text{\& } y=0$$

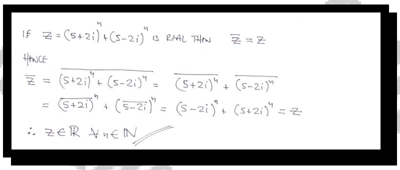
$$\text{IF } x^2-3y^2-1=0$$
$$x^2=3y^2+1 \Rightarrow y(3x^2-y^2)=0$$
$$y\left[3(3y^2+1)-y^2\right]=0$$
$$y\left[9y^2+3-y^2\right]=0$$
$$y\left[8y^2+3\right]$$
$$y=0 \text{ SINCE } 8y^2+3\neq0 \qquad \therefore z=0,\pm1$$
$$x=0$$

Question 65 **(****+)**

$$z = (5+2\mathrm{i})^n + (5-2\mathrm{i})^n, \ n \in \mathbb{N}.$$

Show clearly that z is a real number.

proof



IF $z = (5+2i)^n + (5-2i)^n$ IS REAL THEN $\bar{z} = z$

HENCE

$\bar{z} = \overline{(5+2i)^n + (5-2i)^n} = \overline{(5+2i)^n} + \overline{(5-2i)^n}$

$= \left(\overline{5+2i}\right)^n + \left(\overline{5-2i}\right)^n = (5-2i)^n + (5+2i)^n = z$

$\therefore z \in \mathbb{R} \ \forall n \in \mathbb{N}$

Question 66 (****+)

The complex number z satisfies the relationship

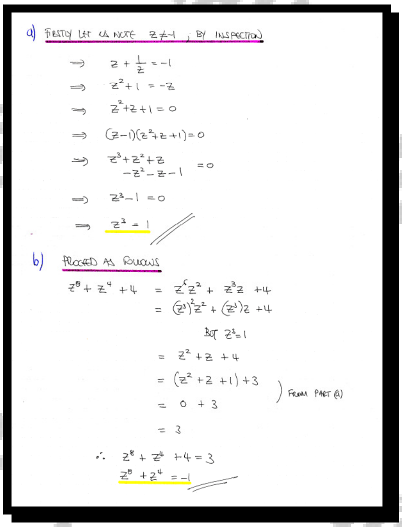
$$z + \frac{1}{z} = -1, \; z \neq 0 .$$

Show clearly that ….

a) … $z^3 = 1$.

b) … $z^8 + z^4 = -1$.

$\boxed{}$, $\boxed{\text{proof}}$



a) FIRSTLY LET US NOTE $z \neq -1$, BY INSPECTION

$$\Rightarrow \quad z + \frac{1}{z} = -1$$
$$\Rightarrow \quad z^2 + 1 = -z$$
$$\Rightarrow \quad z^2 + z + 1 = 0$$
$$\Rightarrow \quad (z-1)(z^2+z+1) = 0$$
$$\Rightarrow \quad \begin{matrix} z^3 + z^2 + z \\ -z^2 - z - 1 \end{matrix} = 0$$
$$\Rightarrow \quad z^3 - 1 = 0$$
$$\Rightarrow \quad z^3 = 1$$

b) PROCEED AS FOLLOWS

$$z^8 + z^4 + 4 = z^6 z^2 + z^3 z + 4$$
$$= (z^3)^2 z^2 + (z^3) z + 4$$

BUT $z^3 = 1$

$$= z^2 + z + 4$$
$$= (z^2 + z + 1) + 3$$
$$= 0 + 3$$

FROM PART (a)

$$= 3$$

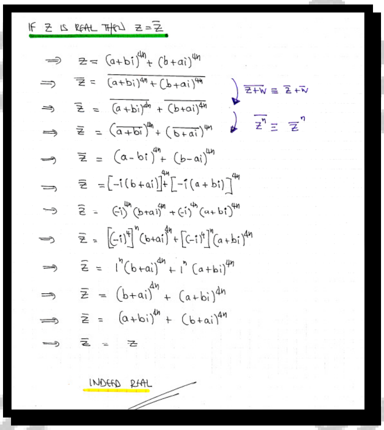
$$\therefore \quad z^8 + z^4 + 4 = 3$$
$$z^8 + z^4 = -1$$

Question 67 (**+)**

$$z = (a+b\mathrm{i})^{4n} + (b+a\mathrm{i})^{4n}, \ a \in \mathbb{R}, \ b \in \mathbb{R}, \ n \in \mathbb{N}.$$

Show that z is a real number.

$\boxed{\text{FP1-V}}$, $\boxed{\text{proof}}$



IF Z IS REAL THEN $z = \bar{z}$

$\Rightarrow z = (a+bi)^{4n} + (b+ai)^{4n}$

$\Rightarrow \bar{z} = \overline{(a+bi)^{4n} + (b+ai)^{4n}}$ $\quad \overline{z+w} \equiv \bar{z} + \bar{w}$

$\Rightarrow \bar{z} = \overline{(a+bi)^{4n}} + \overline{(b+ai)^{4n}}$ $\quad \overline{z^n} \equiv \bar{z}^n$

$\Rightarrow \bar{z} = (\overline{a+bi})^{4n} + (\overline{b+ai})^{4n}$

$\Rightarrow \bar{z} = (a-bi)^{4n} + (b-ai)^{4n}$

$\Rightarrow \bar{z} = [-i(b+ai)]^{4n} + [-i(a+bi)]^{4n}$

$\Rightarrow \bar{z} = (-i)^{4n}(b+ai)^{4n} + (-i)^{4n}(a+bi)^{4n}$

$\Rightarrow \bar{z} = [(-i)^4]^n(b+ai)^{4n} + [(-i)^4]^n(a+bi)^{4n}$

$\Rightarrow \bar{z} = 1^n(b+ai)^{4n} + 1^n(a+bi)^{4n}$

$\Rightarrow \bar{z} = (b+ai)^{4n} + (a+bi)^{4n}$

$\Rightarrow \bar{z} = (a+bi)^{4n} + (b+ai)^{4n}$

$\Rightarrow \bar{z} = z$

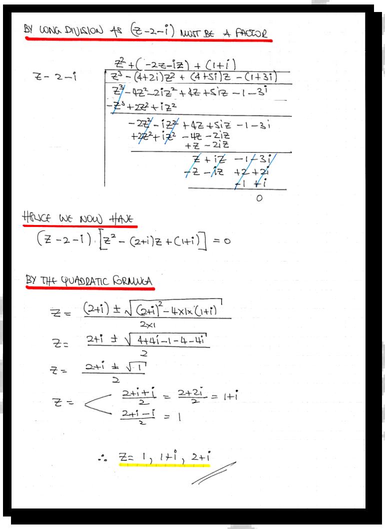
INDEED REAL

Question 68 (****+)

$$z^3-(4+2\mathrm{i})z^2+(4+5\mathrm{i})z-(1+3\mathrm{i})=0\,,\ z\in\mathbb{C}\,.$$

Given that one of the solutions of the above cubic equation is $z=2+\mathrm{i}$, find the other two solutions.

$\boxed{\phantom{z=2+\mathrm{i}}}$, $\boxed{z=1}$, $\boxed{z=1+\mathrm{i}}$



BY LONG DIVISION AS $(z-2-i)$ MUST BE A FACTOR

$$\begin{array}{r|l} & z^2+(-2z-iz)+(1+i) \\ \hline z-2-i & z^3-(4+2i)z^2+(4+5i)z-(1+3i) \\ \hline & \cancel{z^3}-4z^2-2iz^2+4z+5iz-1-3i \\ & \cancel{-z^3}+2z^2+iz^2 \\ \hline & -\cancel{2z^2}-\cancel{iz^2}+4z+5iz-1-3i \\ & +\cancel{2z^2}+\cancel{iz^2}-4z-2iz \\ & +z-2iz \\ \hline & \cancel{z}+\cancel{iz}-\cancel{1}-\cancel{3i} \\ & \cancel{-z}-\cancel{iz}+\cancel{2}+\cancel{2i} \\ & \cancel{-1}+\cancel{i} \\ \hline & 0 \end{array}$$

HENCE WE NOW HAVE

$$(z-2-i)\cdot\left[z^2-(2+i)z+(1+i)\right]=0$$

BY THE QUADRATIC FORMULA

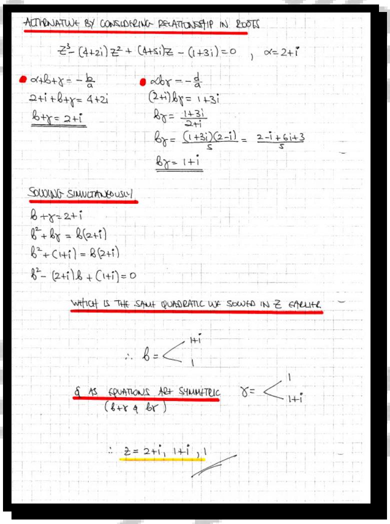
$$z=\frac{(2+i)\pm\sqrt{(2+i)^2-4\times1\times(1+i)}}{2\times1}$$

$$z=\frac{2+i\pm\sqrt{4+4i-1-4-4i}}{2}$$

$$z=\frac{2+i\pm\sqrt{-1}}{2}$$

$$z=\left\langle\begin{array}{l}\frac{2+i+i}{2}=\frac{2+2i}{2}=1+i\\ \frac{2+i-i}{2}=1\end{array}\right.$$

$$\therefore\ z=1,\ 1+i,\ 2+i$$



ALTERNATIVE BY CONSIDERING RELATIONSHIP IN ROOTS

$$z^3-(4+2i)z^2+(4+5i)z-(1+3i)=0\ ,\quad \alpha=2+i$$

- $\alpha+\beta+\gamma=-\frac{b}{a}$

$2+i+\beta+\gamma=4+2i$

$\beta+\gamma=2+i$

- $\alpha\beta\gamma=-\frac{d}{a}$

$(2+i)\beta\gamma=1+3i$

$\beta\gamma=\frac{1+3i}{2+i}$

$\beta\gamma=\frac{(1+3i)(2-i)}{5}=\frac{2-i+6i+3}{5}$

$\beta\gamma=1+i$

SOLVING SIMULTANEOUSLY

$\beta+\gamma=2+i$

$\beta^2+\beta\gamma=\beta(2+i)$

$\beta^2+(1+i)=\beta(2+i)$

$\beta^2-(2+i)\beta+(1+i)=0$

WHICH IS THE SAME QUADRATIC WE SOLVED IN z EARLIER

$$\therefore\ \beta=\left\langle\begin{array}{l}1+i\\1\end{array}\right.$$

& AS EQUATIONS ARE SYMMETRIC $(\beta+\gamma\ \&\ \beta\gamma)$ $\gamma=\left\langle\begin{array}{l}1\\1+i\end{array}\right.$

$$\therefore\ z=2+i,\ 1+i,\ 1$$

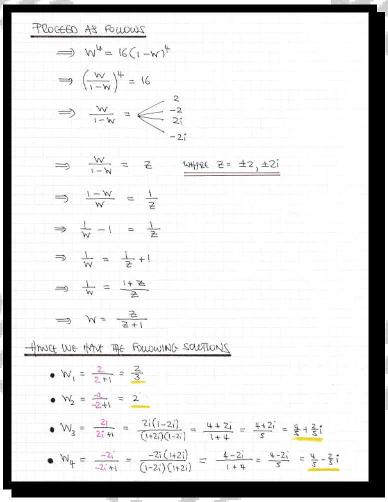
Question 69 (*****)

Find the solutions of the equation

$$w^4 = 16(1-w)^4,$$

giving the answers in the form $x+\mathrm{i}y$, where $x \in \mathbb{R}$, $y \in \mathbb{R}$.

$\boxed{}$, $\boxed{z_1 = 2, \quad z_2 = \frac{2}{3}, \quad z_3 = \frac{4}{5} + \mathrm{i}\frac{2}{5}, \quad z_4 = \frac{4}{5} - \mathrm{i}\frac{2}{5}}$



PROCEED AS FOLLOWS

$\Rightarrow w^4 = 16(1-w)^4$

$\Rightarrow \left(\frac{w}{1-w}\right)^4 = 16$

$\Rightarrow \frac{w}{1-w} = \begin{cases} 2 \\ -2 \\ 2i \\ -2i \end{cases}$

$\Rightarrow \frac{w}{1-w} = z$ WHERE $z = \pm 2, \pm 2i$

$\Rightarrow \frac{1-w}{w} = \frac{1}{z}$

$\Rightarrow \frac{1}{w} - 1 = \frac{1}{z}$

$\Rightarrow \frac{1}{w} = \frac{1}{z} + 1$

$\Rightarrow \frac{1}{w} = \frac{1+z}{z}$

$\Rightarrow w = \frac{z}{z+1}$

HENCE WE HAVE THE FOLLOWING SOLUTIONS

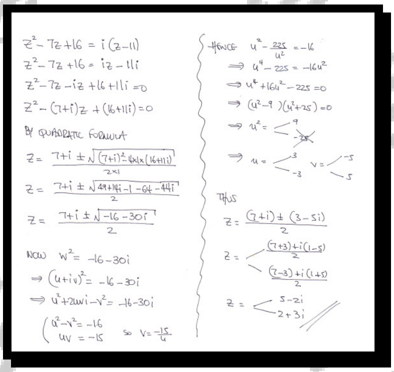
- $w_1 = \frac{2}{2+1} = \frac{2}{3}$
- $w_2 = \frac{-2}{-2+1} = 2$
- $w_3 = \frac{2i}{2i+1} = \frac{2i(1-2i)}{(1+2i)(1-2i)} = \frac{4+2i}{1+4} = \frac{4+2i}{5} = \frac{4}{5} + \frac{2}{5}i$
- $w_4 = \frac{-2i}{-2i+1} = \frac{-2i(1+2i)}{(1-2i)(1+2i)} = \frac{4-2i}{1+4} = \frac{4-2i}{5} = \frac{4}{5} - \frac{2}{5}i$

Question 70 (*****)

Solve the quadratic equation

$$z^2 - 7z + 16 = \mathrm{i}(z-11), \quad z \in \mathbb{C}.$$

$\boxed{}$, $\boxed{z = 2+3\mathrm{i}, \ z = 5-2\mathrm{i}}$



$z^2 - 7z + 16 = i(z-11)$

$z^2 - 7z + 16 = iz - 11i$

$z^2 - 7z - iz + 16 + 11i = 0$

$z^2 - (7+i)z + (16+11i) = 0$

BY QUADRATIC FORMULA

$z = \frac{7+i \pm \sqrt{(7+i)^2 - 4\times1\times(16+11i)}}{2\times1}$

$z = \frac{7+i \pm \sqrt{49+14i-1-64-44i}}{2}$

$z = \frac{7+i \pm \sqrt{-16-30i}}{2}$

NOW $w^2 = -16-30i$

$\Rightarrow (u+iv)^2 = -16-30i$

$\Rightarrow u^2 + 2uvi - v^2 = -16-30i$

$\left(\begin{array}{l} u^2 - v^2 = -16 \\ uv = -15 \end{array}\right.$ so $v = \frac{-15}{u}$

HENCE $u^2 - \frac{225}{u^2} = -16$

$\Rightarrow u^4 - 225 = -16u^2$

$\Rightarrow u^4 + 16u^2 - 225 = 0$

$\Rightarrow (u^2-9)(u^2+25) = 0$

$\Rightarrow u^2 = \begin{cases} 9 \\ \cancel{-25} \end{cases}$

$\Rightarrow u = \begin{cases} 3 \\ -3 \end{cases} \quad v = \begin{cases} -5 \\ 5 \end{cases}$

THUS

$z = \frac{(7+i) \pm (3-5i)}{2}$

$z = \begin{cases} \frac{(7+3)+i(1-5)}{2} \\ \frac{(7-3)+i(1+5)}{2} \end{cases}$

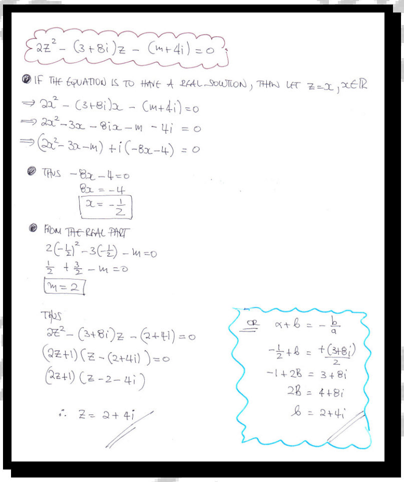
$z = \begin{cases} 5-2i \\ 2+3i \end{cases}$ //

Question 71 (*****)

$$2z^2-(3+8\mathrm{i})z-(m+4\mathrm{i})=0,\ z\in\mathbb{C}.$$

Given that m is a real constant, find the two solutions of the above equation given further that one of these solutions is real.

$\boxed{\quad\quad}$, $\boxed{z=\frac{1}{2}}$, $\boxed{z=2+4\mathrm{i}}$



$2z^2-(3+8i)z-(m+4i)=0$

- IF THE EQUATION IS TO HAVE A REAL SOLUTION, THEN LET $z=x,\ x\in\mathbb{R}$

$\Rightarrow 2x^2-(3+8i)x-(m+4i)=0$

$\Rightarrow 2x^2-3x-8ix-m-4i=0$

$\Rightarrow (2x^2-3x-m)+i(-8x-4)=0$

- THUS $-8x-4=0$

$8x=-4$

$\boxed{x=-\frac{1}{2}}$

- FROM THE REAL PART

$2(-\frac{1}{2})^2-3(-\frac{1}{2})-m=0$

$\frac{1}{2}+\frac{3}{2}-m=0$

$\boxed{m=2}$

THUS

$2z^2-(3+8i)z-(2+4i)=0$

$(2z+1)(z-(2+4i))=0$

$(2z+1)(z-2-4i)$

$\therefore\ z=2+4i$

OR $\alpha+\beta=-\frac{b}{a}$

$-\frac{1}{2}+\beta=+\frac{(3+8i)}{2}$

$-1+2\beta=3+8i$

$2\beta=4+8i$

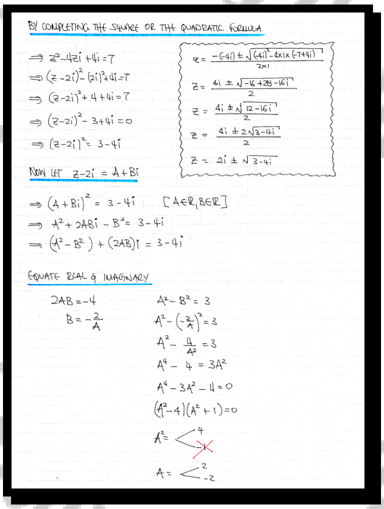
$\beta=2+4i$

Question 72 (*****)

Solve the quadratic equation

$$z^2 - 4zi + 4i = 7 \,, \; z \in \mathbb{C} \,.$$

$\boxed{}$, $\boxed{z = -2 + 3i, \; z = 2 + i}$



BY COMPLETING THE SQUARE OR THE QUADRATIC FORMULA

$\Rightarrow z^2 - 4zi + 4i = 7$

$\Rightarrow (z-2i)^2 - (2i)^2 + 4i = 7$

$\Rightarrow (z-2i)^2 + 4 + 4i = 7$

$\Rightarrow (z-2i)^2 - 3 + 4i = 0$

$\Rightarrow (z-2i)^2 = 3 - 4i$

$z = \frac{-(-4i) \pm \sqrt{(-4i)^2 - 4 \times 1 \times (-7+4i)}}{2 \times 1}$

$z = \frac{4i \pm \sqrt{-16 + 28 - 16i}}{2}$

$z = \frac{4i \pm \sqrt{12 - 16i}}{2}$

$z = \frac{4i \pm 2\sqrt{3-4i}}{2}$

$z = 2i \pm \sqrt{3-4i}$

NOW LET $z - 2i = A + Bi$

$\Rightarrow (A + Bi)^2 = 3 - 4i \qquad [A \in \mathbb{R}, B \in \mathbb{R}]$

$\Rightarrow A^2 + 2ABi - B^2 = 3 - 4i$

$\Rightarrow (A^2 - B^2) + (2AB)i = 3 - 4i$

EQUATE REAL & IMAGINARY

$2AB = -4$

$B = -\frac{2}{A}$

$A^2 - B^2 = 3$

$A^2 - \left(-\frac{2}{A}\right)^2 = 3$

$A^2 - \frac{4}{A^2} = 3$

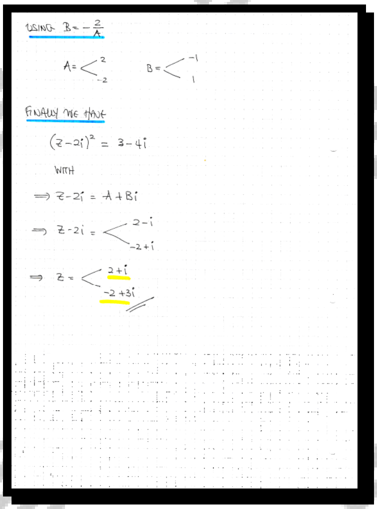
$A^4 - 4 = 3A^2$

$A^4 - 3A^2 - 4 = 0$

$(A^2 - 4)(A^2 + 1) = 0$

$A^2 = \begin{cases} 4 \\ \cancel{-1} \end{cases}$

$A = \begin{cases} 2 \\ -2 \end{cases}$



USING $B = -\frac{2}{A}$

$A = \begin{cases} 2 \\ -2 \end{cases} \qquad B = \begin{cases} -1 \\ 1 \end{cases}$

FINALLY WE HAVE

$(z - 2i)^2 = 3 - 4i$

WITH

$\Rightarrow z - 2i = A + Bi$

$\Rightarrow z - 2i = \begin{cases} 2 - i \\ -2 + i \end{cases}$

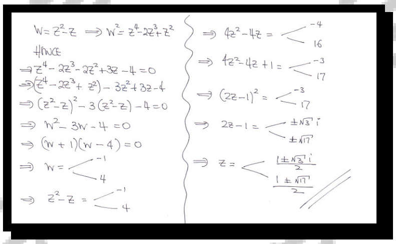
$\Rightarrow z = \begin{cases} 2 + i \\ -2 + 3i \end{cases}$

Question 73 (*****)

$$z^4 - 2z^3 - 2z^2 + 3z - 4 = 0 \,, \; z \in \mathbb{C} \,.$$

By using the substitution $w = z^2 - z$, or otherwise, find in exact form the four solutions of the above equation.

$\boxed{}$, $\boxed{z = \frac{1 \pm \sqrt{17}}{2}, \; \frac{1 \pm i\sqrt{3}}{2}}$



$w = z^2 - z \Rightarrow w^2 = z^4 - 2z^3 + z^2$

HENCE

$\Rightarrow z^4 - 2z^3 - 2z^2 + 3z - 4 = 0$

$\Rightarrow (z^4 - 2z^3 + z^2) - 3z^2 + 3z - 4$

$\Rightarrow (z^2 - z)^2 - 3(z^2 - z) - 4 = 0$

$\Rightarrow w^2 - 3w - 4 = 0$

$\Rightarrow (w + 1)(w - 4) = 0$

$\Rightarrow w = \begin{cases} -1 \\ 4 \end{cases}$

$\Rightarrow z^2 - z = \begin{cases} -1 \\ 4 \end{cases}$

$\Rightarrow 4z^2 - 4z = \begin{cases} -4 \\ 16 \end{cases}$

$\Rightarrow 4z^2 - 4z + 1 = \begin{cases} -3 \\ 17 \end{cases}$

$\Rightarrow (2z - 1)^2 = \begin{cases} -3 \\ 17 \end{cases}$

$\Rightarrow 2z - 1 = \begin{cases} \pm\sqrt{3}\,i \\ \pm\sqrt{17} \end{cases}$

$\Rightarrow z = \begin{cases} \frac{1 \pm \sqrt{3}\,i}{2} \\ \frac{1 \pm \sqrt{17}}{2} \end{cases}$

Question 74 (*****)

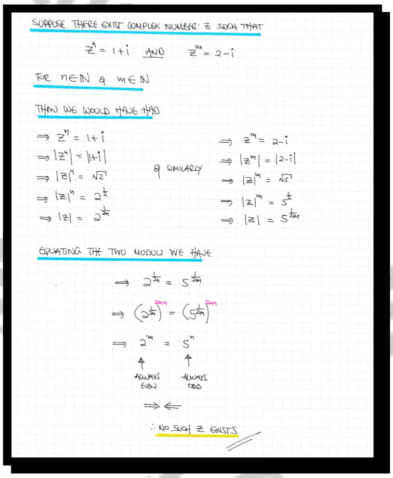
Show that if n and m are natural numbers, then the equations

$$z^n = 1 + \mathrm{i}$$

$$z^m = 2 - \mathrm{i},$$

have no common solution for $z \in \mathbb{C}$.

$\boxed{}$, $\boxed{\text{proof}}$



SUPPOSE THERE EXIST COMPLEX NUMBER z SUCH THAT

$z^n = 1+i$ AND $z^m = 2-i$

FOR $n \in \mathbb{N}$ & $m \in \mathbb{N}$

THEN WE WOULD HAVE HAD

$\Rightarrow z^n = 1+i$
$\Rightarrow |z^n| = |1+i|$
$\Rightarrow |z|^n = \sqrt{2}$
$\Rightarrow |z|^n = 2^{\frac{1}{2}}$
$\Rightarrow |z| = 2^{\frac{1}{2n}}$

& SIMILARLY

$\Rightarrow z^m = 2-i$
$\Rightarrow |z^m| = |2-i|$
$\Rightarrow |z|^m = \sqrt{5}$
$\Rightarrow |z|^m = 5^{\frac{1}{2}}$
$\Rightarrow |z| = 5^{\frac{1}{2m}}$

EQUATING THE TWO MODULI WE HAVE

$$\Rightarrow 2^{\frac{1}{2n}} = 5^{\frac{1}{2m}}$$

$$\Rightarrow \left(2^{\frac{1}{2n}}\right)^{2mn} = \left(5^{\frac{1}{2m}}\right)^{2mn}$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{ALWAYS} \\ \text{EVEN}}}{2^m} = \underset{\substack{\uparrow \\ \text{ALWAYS} \\ \text{ODD}}}{5^n}$$

$$\Rightarrow\Leftarrow$$

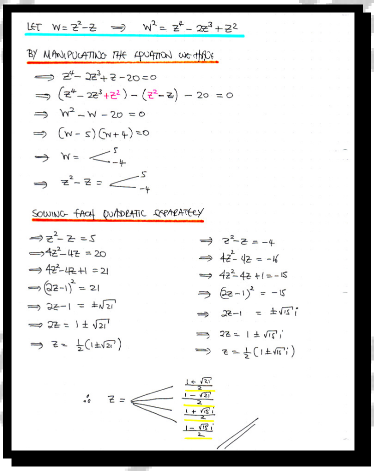
$\therefore$ NO SUCH z EXISTS

Question 75 (*****)

$$z^4 - 2z^3 + z - 20 = 0, \; z \in \mathbb{C}.$$

By using the substitution $w = z^2 - z$, or otherwise, find in exact form the four solutions of the above equation.

$\boxed{}$, $\boxed{z = \frac{1 \pm \sqrt{21}}{2}, \frac{1 \pm i\sqrt{15}}{2}}$



LET $w = z^2 - z \implies w^2 = z^4 - 2z^3 + z^2$

BY MANIPULATING THE EQUATION WE HAVE

$\implies z^4 - 2z^3 + z - 20 = 0$

$\implies (z^4 - 2z^3 + z^2) - (z^2 - z) - 20 = 0$

$\implies w^2 - w - 20 = 0$

$\implies (w-5)(w+4) = 0$

$\implies w = \begin{cases} 5 \\ -4 \end{cases}$

$\implies z^2 - z = \begin{cases} 5 \\ -4 \end{cases}$

SOLVING EACH QUADRATIC SEPARATELY

$\implies z^2 - z = 5$

$\implies 4z^2 - 4z = 20$

$\implies 4z^2 - 4z + 1 = 21$

$\implies (2z-1)^2 = 21$

$\implies 2z - 1 = \pm\sqrt{21}$

$\implies 2z = 1 \pm \sqrt{21}$

$\implies z = \frac{1}{2}(1 \pm \sqrt{21})$

$\implies z^2 - z = -4$

$\implies 4z^2 - 4z = -16$

$\implies 4z^2 - 4z + 1 = -15$

$\implies (2z-1)^2 = -15$

$\implies 2z - 1 = \pm\sqrt{15}\,i$

$\implies 2z = 1 \pm \sqrt{15}\,i$

$\implies z = \frac{1}{2}(1 \pm \sqrt{15}\,i)$

$\therefore \; z = \begin{cases} \frac{1+\sqrt{21}}{2} \\ \frac{1-\sqrt{21}}{2} \\ \frac{1+\sqrt{15}\,i}{2} \\ \frac{1-\sqrt{15}\,i}{2} \end{cases}$ //

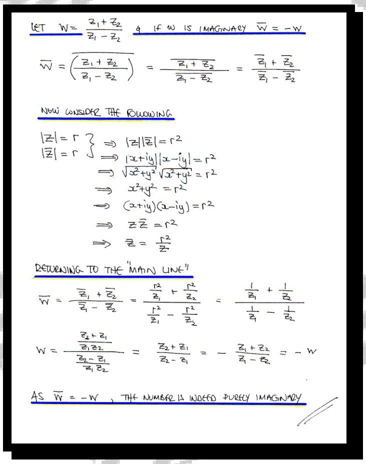
Question 76 (*****)

Two distinct complex numbers z_1 and z_2 are such so that $|z_1| = |z_2| = r \neq 0$.

Show clearly that $\frac{z_1 + z_2}{z_1 - z_2}$ is purely imaginary.

You may find the result $z\overline{z} = |z|^2 = r^2$ useful.

$\boxed{}$, $\boxed{\text{proof}}$



LET $w = \frac{z_1 + z_2}{z_1 - z_2}$ & IF w IS IMAGINARY $\overline{w} = -w$

$$\overline{w} = \overline{\left(\frac{z_1 + z_2}{z_1 - z_2}\right)} = \frac{\overline{z_1 + z_2}}{\overline{z_1 - z_2}} = \frac{\overline{z_1} + \overline{z_2}}{\overline{z_1} - \overline{z_2}}$$

NOW CONSIDER THE FOLLOWING

$$\left.\begin{array}{l} |z| = r \\ |\overline{z}| = r \end{array}\right\} \Rightarrow |z||\overline{z}| = r^2$$
$$\Rightarrow |x+iy||x-iy| = r^2$$
$$\Rightarrow \sqrt{x^2+y^2}\sqrt{x^2+y^2} = r^2$$
$$\Rightarrow x^2+y^2 = r^2$$
$$\Rightarrow (x+iy)(x-iy) = r^2$$
$$\Rightarrow z\overline{z} = r^2$$
$$\Rightarrow \overline{z} = \frac{r^2}{z}$$

RETURNING TO THE "MAIN LINE"

$$\overline{w} = \frac{\overline{z_1} + \overline{z_2}}{\overline{z_1} - \overline{z_2}} = \frac{\frac{r^2}{z_1} + \frac{r^2}{z_2}}{\frac{r^2}{z_1} - \frac{r^2}{z_2}} = \frac{\frac{1}{z_1} + \frac{1}{z_2}}{\frac{1}{z_1} - \frac{1}{z_2}}$$

$$w = \frac{\frac{z_2 + z_1}{z_1 z_2}}{\frac{z_2 - z_1}{z_1 z_2}} = \frac{z_2 + z_1}{z_2 - z_1} = -\frac{z_1 + z_2}{z_1 - z_2} = -w$$

AS $\overline{w} = -w$, THE NUMBER IS INDEED PURELY IMAGINARY

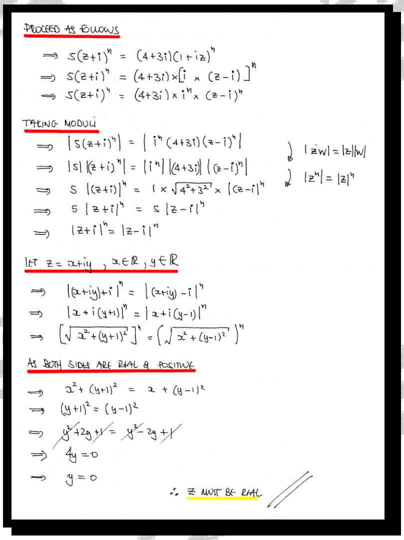
Question 77 (*****)

The complex number z satisfies the relationship

$$5(z+\mathrm{i})^n = (4+3\mathrm{i})(1+\mathrm{i}z)^n, \; n \in \mathbb{R}.$$

Show that z is a real number.

$\boxed{}$, $\boxed{\text{proof}}$



PROCEED AS FOLLOWS

$$\Rightarrow 5(z+i)^n = (4+3i)(1+iz)^n$$
$$\Rightarrow 5(z+i)^n = (4+3i)\times[i \times (z-i)]^n$$
$$\Rightarrow 5(z+i)^n = (4+3i)\times i^n \times (z-i)^n$$

TAKING MODULI

$$\Rightarrow |5(z+i)^n| = |i^n(4+3i)(z-i)^n| \qquad |zw| = |z||w|$$
$$\Rightarrow |5|\,|(z+i)^n| = |i^n|\,|(4+3i)|\,|(z-i)^n| \qquad |z^n| = |z|^n$$
$$\Rightarrow 5\,|(z+i)|^n = 1 \times \sqrt{4^2+3^2} \times |(z-i|^n$$
$$\Rightarrow 5\,|z+i|^n = 5\,|z-i|^n$$
$$\Rightarrow |z+i|^n = |z-i|^n$$

LET $z = x+iy$, $x \in \mathbb{R}$, $y \in \mathbb{R}$

$$\Rightarrow |(x+iy)+i|^n = |(x+iy)-i|^n$$
$$\Rightarrow |x+i(y+1)|^n = |x+i(y-1)|^n$$
$$\Rightarrow \left[\sqrt{x^2+(y+1)^2}\right]^n = \left(\sqrt{x^2+(y-1)^2}\right)^n$$

AS BOTH SIDES ARE REAL & POSITIVE

$$\Rightarrow x^2+(y+1)^2 = x+(y-1)^2$$
$$\Rightarrow (y+1)^2 = (y-1)^2$$
$$\Rightarrow \cancel{y^2}+2y+\cancel{1} = \cancel{y^2}-2y+\cancel{1}$$
$$\Rightarrow 4y = 0$$
$$\Rightarrow y = 0$$

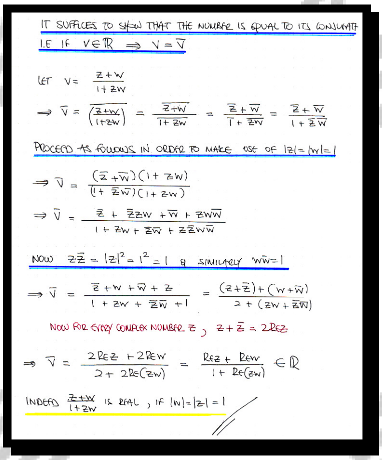
$\therefore$ z MUST BE REAL //

Question 78 (*****)

The complex numbers z and w are such so that $|z| = |w| = 1$.

Show clearly that $\dfrac{z+w}{1+zw}$ is real.

SPX-D, proof



IT SUFFICES TO SHOW THAT THE NUMBER IS EQUAL TO ITS CONJUGATE

I.E IF $v \in \mathbb{R} \Rightarrow v = \bar{v}$

LET $v = \dfrac{z+w}{1+zw}$

$\Rightarrow \bar{v} = \overline{\left(\dfrac{z+w}{1+zw}\right)} = \dfrac{\overline{z+w}}{\overline{1+zw}} = \dfrac{\bar{z}+\bar{w}}{\bar{1}+\overline{zw}} = \dfrac{\bar{z}+\bar{w}}{1+\bar{z}\bar{w}}$

PROCEED AS FOLLOWS IN ORDER TO MAKE USE OF $|z| = |w| = 1$

$\Rightarrow \bar{v} = \dfrac{(\bar{z}+\bar{w})(1+zw)}{(1+\bar{z}\bar{w})(1+zw)}$

$\Rightarrow \bar{v} = \dfrac{\bar{z} + \bar{z}zw + \bar{w} + zw\bar{w}}{1 + zw + \bar{z}\bar{w} + z\bar{z}w\bar{w}}$

NOW $z\bar{z} = |z|^2 = 1^2 = 1$ & SIMILARLY $w\bar{w} = 1$

$\Rightarrow \bar{v} = \dfrac{\bar{z} + w + \bar{w} + z}{1 + zw + \bar{z}\bar{w} + 1} = \dfrac{(z+\bar{z}) + (w+\bar{w})}{2 + (zw + \bar{z}\bar{w})}$

NOW FOR EVERY COMPLEX NUMBER z, $z + \bar{z} = 2\text{Re}z$

$\Rightarrow \bar{v} = \dfrac{2\text{Re}z + 2\text{Re}w}{2 + 2\text{Re}(zw)} = \dfrac{\text{Re}z + \text{Re}w}{1 + \text{Re}(zw)} \in \mathbb{R}$

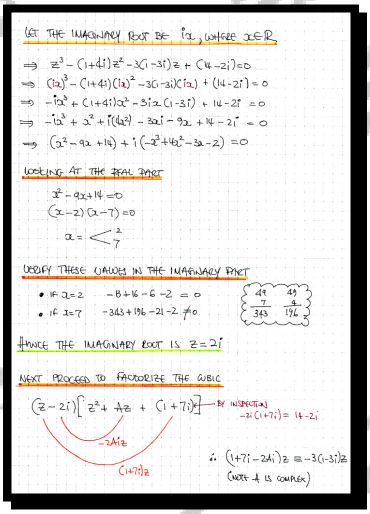
INDEED $\dfrac{z+w}{1+zw}$ IS REAL, IF $|w| = |z| = 1$

Question 79 (*****)

$$z^3 - 2(2-\mathrm{i})z^2 + (8-3\mathrm{i})z - 5 + \mathrm{i} = 0, \quad z \in \mathbb{C}.$$

Find the three solutions of the above equation given that one of these solutions is real.

$\boxed{\quad}$, $\boxed{z=1}$, $\boxed{z=2-3\mathrm{i}}$, $\boxed{z=2+\mathrm{i}}$



LET THE IMAGINARY ROOT BE ix, WHERE $x \in \mathbb{R}$

$\Rightarrow z^3 - (1+4i)z^2 - 3(1-3i)z + (4-2i) = 0$

$\Rightarrow (ix)^3 - (1+4i)(ix)^2 - 3(1-3i)(ix) + (14-2i) = 0$

$\Rightarrow -ix^3 + (1+4i)x^2 - 3ix(1-3i) + 14 - 2i = 0$

$\Rightarrow -ix^3 + x^2 + i(4x^2) - 3xi - 9x + 14 - 2i = 0$

$\Rightarrow (x^2 - 9x + 14) + i(-x^3 + 4x^2 - 3x - 2) = 0$

LOOKING AT THE REAL PART

$x^2 - 9x + 14 = 0$

$(x-2)(x-7) = 0$

$x = \left\langle \begin{matrix} 2 \\ 7 \end{matrix} \right.$

VERIFY THESE VALUES IN THE IMAGINARY PART

- IF $x = 2$ $\quad -8 + 16 - 6 - 2 = 0$
- IF $x = 7$ $\quad -343 + 196 - 21 - 2 \neq 0$

$\begin{array}{r} 49 \\ 7 \\ \hline 343 \end{array} \quad \begin{array}{r} 49 \\ 4 \\ \hline 196 \end{array}$

HENCE THE IMAGINARY ROOT IS $z = 2i$

NEXT PROCEED TO FACTORIZE THE CUBIC

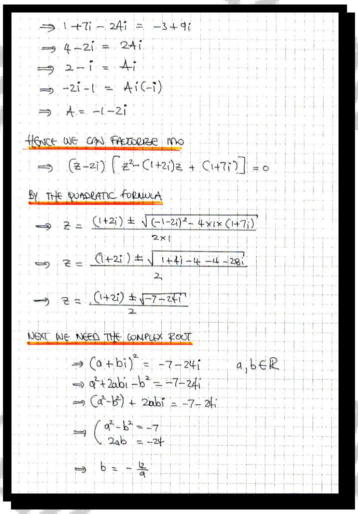
$(z - 2i)\left[z^2 + Az + (1+7i)\right] \leftarrow$ BY INSPECTION $-2i(1+7i) = 14 - 2i$

$-2Aiz$

$(1+7i)z$

$\therefore (1 + 7i - 2Ai)z \equiv -3(1-3i)z$

(NOTE A IS COMPLEX)



$\Rightarrow 1 + 7i - 2Ai = -3 + 9i$

$\Rightarrow 4 - 2i = 2Ai$

$\Rightarrow 2 - i = Ai$

$\Rightarrow -2i - 1 = Ai(-i)$

$\Rightarrow A = -1 - 2i$

HENCE WE CAN FACTORIZE INTO

$\Rightarrow (z - 2i)\left[z^2 - (1+2i)z + (1+7i)\right] = 0$

BY THE QUADRATIC FORMULA

$\Rightarrow z = \dfrac{(1+2i) \pm \sqrt{(-1-2i)^2 - 4 \times 1 \times (1+7i)}}{2 \times 1}$

$\Rightarrow z = \dfrac{(1+2i) \pm \sqrt{1 + 4i - 4 - 4 - 28i}}{2}$

$\Rightarrow z = \dfrac{(1+2i) \pm \sqrt{-7-24i}}{2}$

NEXT WE NEED THE COMPLEX ROOT

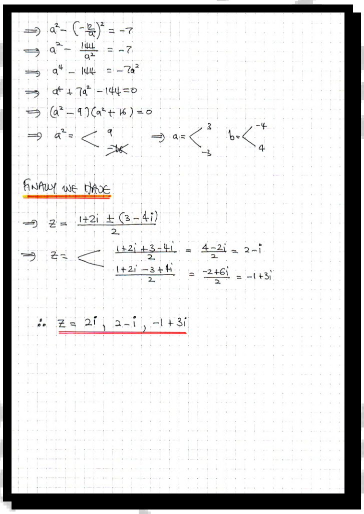
$\Rightarrow (a + bi)^2 = -7 - 24i \qquad a, b \in \mathbb{R}$

$\Rightarrow a^2 + 2abi - b^2 = -7 - 24i$

$\Rightarrow (a^2 - b^2) + 2abi = -7 - 24i$

$\Rightarrow \left(\begin{matrix} a^2 - b^2 = -7 \\ 2ab = -24 \end{matrix} \right.$

$\Rightarrow b = -\dfrac{12}{a}$



$\Rightarrow a^2 - \left(-\dfrac{12}{a}\right)^2 = -7$

$\Rightarrow a^2 - \dfrac{144}{a^2} = -7$

$\Rightarrow a^4 - 144 = -7a^2$

$\Rightarrow a^4 + 7a^2 - 144 = 0$

$\Rightarrow (a^2 - 9)(a^2 + 16) = 0$

$\Rightarrow a^2 = \left\langle \begin{matrix} 9 \\ \cancel{-16} \end{matrix} \right. \Rightarrow a = \left\langle \begin{matrix} 3 \\ -3 \end{matrix} \right. \quad b = \left\langle \begin{matrix} -4 \\ 4 \end{matrix} \right.$

FINALLY WE HAVE

$\Rightarrow z = \dfrac{1 + 2i \pm (3 - 4i)}{2}$

$\Rightarrow z = \left\langle \begin{matrix} \dfrac{1 + 2i + 3 - 4i}{2} = \dfrac{4 - 2i}{2} = 2 - i \\ \dfrac{1 + 2i - 3 + 4i}{2} = \dfrac{-2 + 6i}{2} = -1 + 3i \end{matrix} \right.$

$\therefore z = 2i, \ 2 - i, \ -1 + 3i$

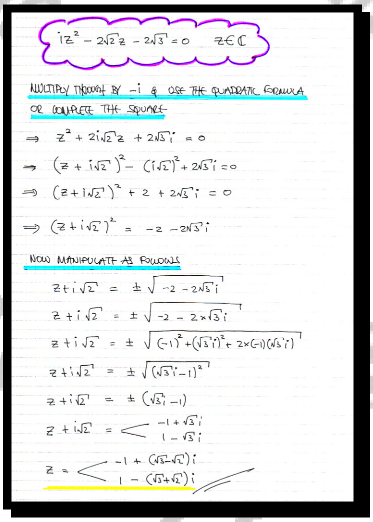
Question 80 (*****)

Solve the quadratic equation

$$\mathrm{i}z^2 - 2\sqrt{2}\,z - 2\sqrt{3} = 0\,,\quad z \in \mathbb{C}\,.$$

Give the answers in the form $x + \mathrm{i}y$, where x and y are exact real constants.

$\boxed{}$, $\boxed{z = -1 + \mathrm{i}\left(\sqrt{3} - \sqrt{2}\right),\quad z = 1 - \mathrm{i}\left(\sqrt{3} + \sqrt{2}\right)}$



$$\boxed{iz^2 - 2\sqrt{2}z - 2\sqrt{3} = 0 \qquad z \in \mathbb{C}}$$

MULTIPLY THROUGH BY $-i$ & USE THE QUADRATIC FORMULA OR COMPLETE THE SQUARE

$\Rightarrow \quad z^2 + 2i\sqrt{2}z + 2\sqrt{3}i = 0$

$\Rightarrow \quad (z + i\sqrt{2})^2 - (i\sqrt{2})^2 + 2\sqrt{3}i = 0$

$\Rightarrow \quad (z + i\sqrt{2})^2 + 2 + 2\sqrt{3}i = 0$

$\Rightarrow \quad (z + i\sqrt{2})^2 = -2 - 2\sqrt{3}i$

NOW MANIPULATE AS FOLLOWS

$z + i\sqrt{2} = \pm\sqrt{-2 - 2\sqrt{3}i}$

$z + i\sqrt{2} = \pm\sqrt{-2 - 2\times\sqrt{3}i}$

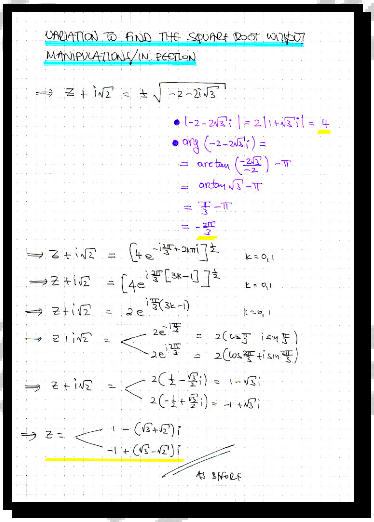
$z + i\sqrt{2} = \pm\sqrt{(-1)^2 + (\sqrt{3}i)^2 + 2\times(-1)(\sqrt{3}i)}$

$z + i\sqrt{2} = \pm\sqrt{(\sqrt{3}i - 1)^2}$

$z + i\sqrt{2} = \pm(\sqrt{3}i - 1)$

$z + i\sqrt{2} = \left< \begin{matrix} -1 + \sqrt{3}i \\ 1 - \sqrt{3}i \end{matrix} \right.$

$z = \left< \begin{matrix} -1 + (\sqrt{3} - \sqrt{2})i \\ 1 - (\sqrt{3} + \sqrt{2})i \end{matrix} \right.$



VARIATION TO FIND THE SQUARE ROOT WITHOUT MANIPULATIONS/IN SPECTION

$\Rightarrow \quad z + i\sqrt{2} = \pm\sqrt{-2 - 2i\sqrt{3}}$

- $|-2 - 2\sqrt{3}i| = 2|1 + \sqrt{3}i| = 4$
- $\arg(-2 - 2\sqrt{3}i) =$

$= \arctan\left(\frac{-2\sqrt{3}}{-2}\right) - \pi$

$= \arctan\sqrt{3} - \pi$

$= \frac{\pi}{3} - \pi$

$= -\frac{2\pi}{3}$

$\Rightarrow z + i\sqrt{2} = \left[4e^{-i\frac{2\pi}{3} + 2k\pi i}\right]^{\frac{1}{2}} \qquad k = 0, 1$

$\Rightarrow z + i\sqrt{2} = \left[4e^{i\frac{2\pi}{3}[3k - 1]}\right]^{\frac{1}{2}} \qquad k = 0, 1$

$\Rightarrow z + i\sqrt{2} = 2e^{i\frac{\pi}{3}(3k - 1)} \qquad k = 0, 1$

$\rightarrow z + i\sqrt{2} = \left< \begin{matrix} 2e^{-i\frac{\pi}{3}} = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) \\ 2e^{i\frac{2\pi}{3}} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \end{matrix} \right.$

$\Rightarrow z + i\sqrt{2} = \left< \begin{matrix} 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}i \\ 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i \end{matrix} \right.$

$\Rightarrow z = \left< \begin{matrix} 1 - (\sqrt{3} + \sqrt{2})i \\ -1 + (\sqrt{3} - \sqrt{2})i \end{matrix} \right.$

AS BEFORE

Question 81 (*****)

The complex number z satisfies the equation

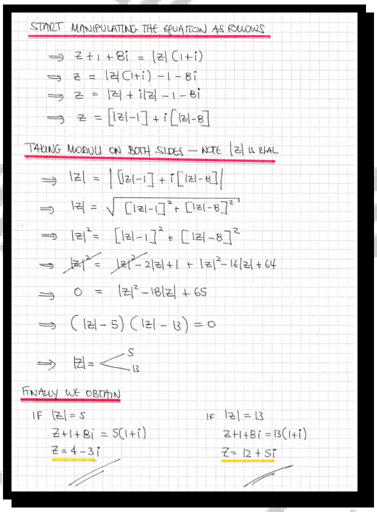
$$z+1+8\text{i}=|z|(1+\text{i}).$$

Show clearly that

$$|z|^2-18|z|+65=0,$$

and hence find the possible values of z.

$\boxed{}$, $\boxed{z=4-3\text{i}}$, $\boxed{z=12+5\text{i}}$



START MANIPULATING THE EQUATION AS FOLLOWS

$\Rightarrow z+1+8i = |z|(1+i)$

$\Rightarrow z = |z|(1+i)-1-8i$

$\Rightarrow z = |z|+i|z|-1-8i$

$\Rightarrow z = [|z|-1]+i[|z|-8]$

TAKING MODULI ON BOTH SIDES — NOTE $|z|$ IS REAL

$\Rightarrow |z| = \big|[|z|-1]+i[|z|-8]\big|$

$\Rightarrow |z| = \sqrt{[|z|-1]^2+[|z|-8]^2}$

$\Rightarrow |z|^2 = [|z|-1]^2+[|z|-8]^2$

$\Rightarrow \cancel{|z|^2} = \cancel{|z|^2}-2|z|+1+|z|^2-16|z|+64$

$\Rightarrow 0 = |z|^2-18|z|+65$

$\Rightarrow (|z|-5)(|z|-13)=0$

$\Rightarrow |z| = \begin{cases} 5 \\ 13 \end{cases}$

FINALLY WE OBTAIN

IF $|z|=5$

$z+1+8i = 5(1+i)$

$z = 4-3i$

IF $|z|=13$

$z+1+8i = 13(1+i)$

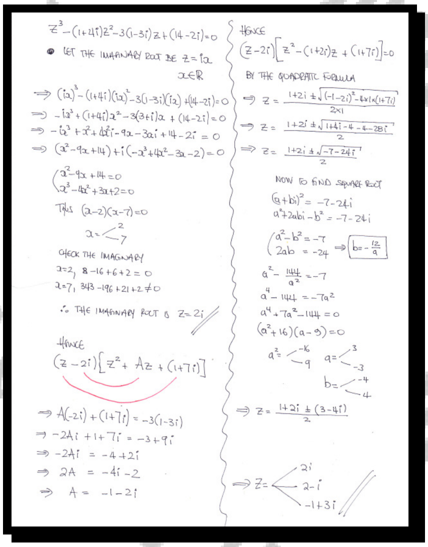
$z = 12+5i$

Question 82 (*****)

$$z^3 - (2+4\mathrm{i})z^2 - 3(1-3\mathrm{i})z + 14 - 2\mathrm{i} = 0, \ z \in \mathbb{C}.$$

Find the three solutions of the above equation given that one of these solutions is purely imaginary.

$\boxed{\text{SP - F}}$, $\boxed{z = 2\mathrm{i}}$, $\boxed{z = 2 - \mathrm{i}}$, $\boxed{z = -1 + 3\mathrm{i}}$



$z^3 - (1+4i)z^2 - 3(1-3i)z + (14-2i) = 0$

● LET THE IMAGINARY ROOT BE $z = ix$
$x \in \mathbb{R}$

$\Rightarrow (ix)^3 - (1+4i)(ix)^2 - 3(1-3i)(ix) + (14-2i) = 0$

$\Rightarrow -ix^3 + (1+4i)x^2 - 3(3+i)x + (14-2i) = 0$

$\Rightarrow -ix^3 + x^2 + 4x^2i - 9x - 3xi + 14 - 2i = 0$

$\Rightarrow (x^2 - 9x + 14) + i(-x^3 + 4x^2 - 3x - 2) = 0$

$\begin{cases} x^2 - 9x + 14 = 0 \\ x^3 - 4x^2 + 3x + 2 = 0 \end{cases}$

THUS $(x-2)(x-7) = 0$

$x = \begin{cases} 2 \\ 7 \end{cases}$

CHECK THE IMAGINARY

$x = 2, \ 8 - 16 + 6 + 2 = 0$

$x = 7, \ 343 - 196 + 21 + 2 \neq 0$

$\therefore$ THE IMAGINARY ROOT IS $z = 2i$ //

HENCE

$(z - 2i)\left[z^2 + Az + (1+7i)\right]$

$\Rightarrow A(-2i) + (1+7i) = -3(1-3i)$

$\Rightarrow -2Ai + 1 + 7i = -3 + 9i$

$\Rightarrow -2Ai = -4 + 2i$

$\Rightarrow 2A = -4i - 2$

$\Rightarrow A = -1 - 2i$

HENCE

$(z - 2i)\left[z^2 - (1+2i)z + (1+7i)\right] = 0$

BY THE QUADRATIC FORMULA

$\Rightarrow z = \dfrac{1+2i \pm \sqrt{(-1-2i)^2 - 4 \times 1 \times (1+7i)}}{2 \times 1}$

$\Rightarrow z = \dfrac{1+2i \pm \sqrt{1+4i-4-4-28i}}{2}$

$\Rightarrow z = \dfrac{1+2i \pm \sqrt{-7-24i}}{2}$

NOW TO FIND SQUARE ROOT

$(a+bi)^2 = -7 - 24i$

$a^2 + 2abi - b^2 = -7 - 24i$

$\begin{cases} a^2 - b^2 = -7 \\ 2ab = -24 \end{cases} \Rightarrow \boxed{b = -\frac{12}{a}}$

$a^2 - \dfrac{144}{a^2} = -7$

$a^4 - 144 = -7a^2$

$a^4 + 7a^2 - 144 = 0$

$(a^2 + 16)(a - 9) = 0$

$a^2 = \begin{cases} -16 \\ 9 \end{cases} \quad a = \begin{cases} 3 \\ -3 \end{cases}$

$b = \begin{cases} -4 \\ 4 \end{cases}$

$\Rightarrow z = \dfrac{1+2i \pm (3-4i)}{2}$

$\Rightarrow z = \begin{cases} 2i \\ 2-i \\ -1+3i \end{cases}$ //

Question 83 (*****)

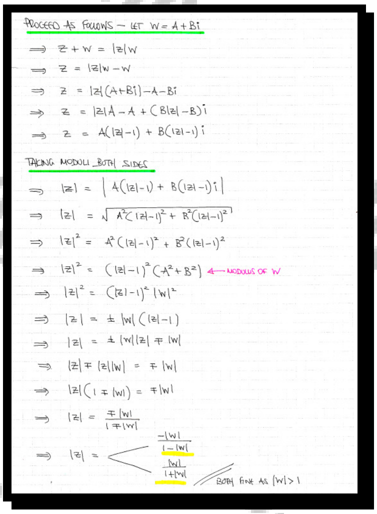
It is given that

$$z + w = |z| w ,$$

where $z \in \mathbb{C}$, $w \in \mathbb{C}$, and $|w| > 1$.

Determine an exact simplified expression for $|z|$, in terms of $|w|$.

$\boxed{\text{IPx-P}}$, $\boxed{|z| = \frac{\mp |w|}{1 \mp |w|} = \frac{\pm |w|}{\pm |w| - 1}}$



PROCEED AS FOLLOWS – LET $W = A + Bi$

$$\Rightarrow z + w = |z| w$$

$$\Rightarrow z = |z| w - w$$

$$\Rightarrow z = |z|(A + Bi) - A - Bi$$

$$\Rightarrow z = |z| A - A + (B|z| - B) i$$

$$\Rightarrow z = A(|z| - 1) + B(|z| - 1) i$$

TAKING MODULI BOTH SIDES

$$\Rightarrow |z| = |A(|z| - 1) + B(|z| - 1) i|$$

$$\Rightarrow |z| = \sqrt{A^2 (|z| - 1)^2 + B^2 (|z| - 1)^2}$$

$$\Rightarrow |z|^2 = A^2 (|z| - 1)^2 + B^2 (|z| - 1)^2$$

$$\Rightarrow |z|^2 = (|z| - 1)^2 (A^2 + B^2) \leftarrow \text{MODULUS OF } W$$

$$\Rightarrow |z|^2 = (|z| - 1)^2 |w|^2$$

$$\Rightarrow |z| = \pm |w| (|z| - 1)$$

$$\Rightarrow |z| = \pm |w| |z| \mp |w|$$

$$\Rightarrow |z| \mp |z| |w| = \mp |w|$$

$$\Rightarrow |z| (1 \mp |w|) = \mp |w|$$

$$\Rightarrow |z| = \frac{\mp |w|}{1 \mp |w|}$$

$$\Rightarrow |z| = \left< \begin{array}{l} \dfrac{-|w|}{1 - |w|} \\ \dfrac{|w|}{1 + |w|} \end{array} \right.$$

// BOTH FINE AS $|W| > 1$