HYPERBOLIC FUNCTIONS

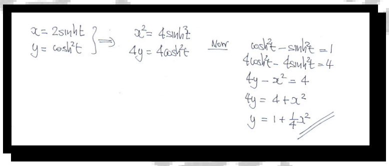
Question 1 **(**)**

A curve is given parametrically by the equations

$$x = 2\sinh t\,,\ \ y = \cosh^2 t\,,\ t \in \mathbb{R}\,.$$

Find a Cartesian equation of the curve, in the form $y = f(x)$.

$$\boxed{y = 1 + \frac{1}{4}x^2}$$



$$\left.\begin{aligned} x &= 2\sinh t \\ y &= \cosh^2 t \end{aligned}\right\} \Longrightarrow \begin{aligned} x^2 &= 4\sinh^2 t \\ 4y &= 4\cosh^2 t \end{aligned}$$

<u>Now</u>

$$\cosh^2 t - \sinh^2 t = 1$$
$$4\cosh^2 t - 4\sinh^2 t = 4$$
$$4y - x^2 = 4$$
$$4y = 4 + x^2$$
$$y = 1 + \tfrac{1}{4}x^2$$

Question 2 **(**)**

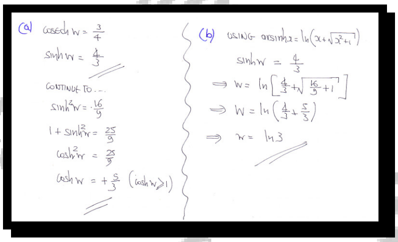
It is given that

$$\operatorname{cosech} w = \frac{3}{4}\,.$$

a) Use hyperbolic identities to find the exact values of $\sinh w$ and $\cosh w$.

b) Hence find the exact value of w, in terms of natural logarithms.

$$\boxed{\sinh w = \frac{4}{3}}\,,\ \boxed{\cosh w = \frac{5}{3}}\,,\ \boxed{w = \ln 3}$$



(a) $\operatorname{cosech} w = \frac{3}{4}$

$\sinh w = \frac{4}{3}$

CONTINUE TO...

$\sinh^2 w = \frac{16}{9}$

$1 + \sinh^2 w = \frac{25}{9}$

$\cosh^2 w = \frac{25}{9}$

$\cosh w = +\frac{5}{3}$ $(\cosh w \geqslant 1)$

(b) USING $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$

$\sinh w = \frac{4}{3}$

$\Rightarrow w = \ln\left[\frac{4}{3} + \sqrt{\frac{16}{9}+1}\right]$

$\Rightarrow w = \ln\left(\frac{4}{3} + \frac{5}{3}\right)$

$\Rightarrow w = \ln 3$

Question 3 (**)

$$f(x) = \operatorname{artanh} x\,,\ x \in \mathbb{R}\,,\ -1 < x < 1\,.$$

a) Show clearly that

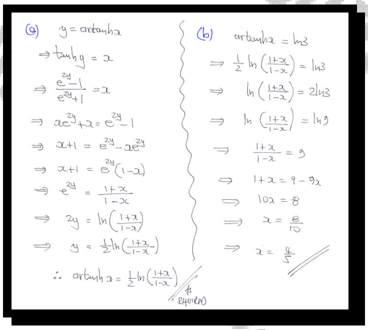
$$f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\ x \in \mathbb{R}\,,\ -1 < x < 1\,.$$

b) Without the use of any calculating aid solve the equation

$$\operatorname{artanh} x = \ln 3\,,$$

showing clearly all the relevant steps in the calculation.

$$\boxed{x = \frac{4}{5}}$$



(a) $y = \operatorname{artanh} x$

$\Rightarrow \tanh y = x$

$\Rightarrow \frac{e^{2y}-1}{e^{2y}+1} = x$

$\Rightarrow xe^{2y} + x = e^{2y} - 1$

$\Rightarrow x + 1 = e^{2y} - xe^{2y}$

$\Rightarrow x + 1 = e^{2y}(1-x)$

$\Rightarrow e^{2y} = \frac{1+x}{1-x}$

$\Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right)$

$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$\therefore \operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ // As REQUIRED

(b) $\operatorname{artanh} x = \ln 3$

$\Rightarrow \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \ln 3$

$\Rightarrow \ln\left(\frac{1+x}{1-x}\right) = 2\ln 3$

$\Rightarrow \ln\left(\frac{1+x}{1-x}\right) = \ln 9$

$\Rightarrow \frac{1+x}{1-x} = 9$

$\Rightarrow 1 + x = 9 - 9x$

$\Rightarrow 10x = 8$

$\Rightarrow x = \frac{8}{10}$

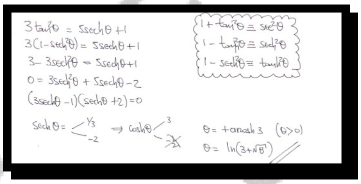
$\Rightarrow x = \frac{4}{5}$ //

Question 4 (**+)

Find, in exact logarithmic form, the positive root of the equation

$$3\tanh^2\theta = 5\operatorname{sech}\theta + 1\,,\ \theta \in \mathbb{R}\,.$$

$$\boxed{\theta = \ln\left(3+\sqrt{8}\right)}$$



$3\tanh^2\theta = 5\operatorname{sech}\theta + 1$

$3(1-\operatorname{sech}^2\theta) = 5\operatorname{sech}\theta + 1$

$3 - 3\operatorname{sech}^2\theta = 5\operatorname{sech}\theta + 1$

$0 = 3\operatorname{sech}^2\theta + 5\operatorname{sech}\theta - 2$

$(3\operatorname{sech}\theta - 1)(\operatorname{sech}\theta + 2) = 0$

$\operatorname{sech}\theta = \frac{1}{3}, -2 \Rightarrow \cosh\theta = 3, \cancel{-\frac{1}{2}}$

$1 + \tan^2\theta \equiv \sec^2\theta$

$1 - \tanh^2\theta \equiv \operatorname{sech}^2\theta$

$1 - \operatorname{sech}^2\theta \equiv \tanh^2\theta$

$\theta = +\operatorname{arcosh} 3 \quad (\theta > 0)$

$\theta = \ln(3+\sqrt{8})$ //

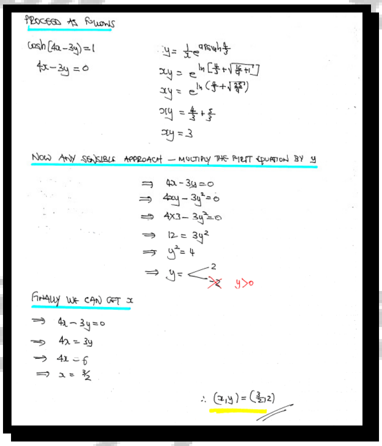
Question 5 (**+)

Given that $x > 0$ and $y > 0$, solve the simultaneous equations

$$\cosh(4x - 3y) = 1$$

$$y = \frac{1}{x}\mathrm{e}^{\operatorname{arsinh}\frac{4}{3}}.$$

$\boxed{}$, $\boxed{x = \frac{3}{2}, \ y = 2}$



PROCEED AS FOLLOWS

$\cosh(4x-3y)=1$

$4x-3y=0$

$y=\frac{1}{x}e^{\text{arsinh}\frac{4}{3}}$

$xy=e^{\ln\left[\frac{4}{3}+\sqrt{\frac{16}{9}+1}\right]}$

$xy=e^{\ln\left(\frac{4}{3}+\sqrt{\frac{25}{9}}\right)}$

$xy=\frac{4}{3}+\frac{5}{3}$

$xy=3$

NOW ANY SENSIBLE APPROACH – MULTIPLY THE FIRST EQUATION BY y

$\Rightarrow 4x-3y=0$

$\Rightarrow 4xy-3y^2=0$

$\Rightarrow 4\times3-3y^2=0$

$\Rightarrow 12=3y^2$

$\Rightarrow y^2=4$

$\Rightarrow y=\begin{cases}2\\ \cancel{-2}\end{cases}$ $y>0$

FINALLY WE CAN GET x

$\Rightarrow 4x-3y=0$

$\Rightarrow 4x=3y$

$\Rightarrow 4x=6$

$\Rightarrow x=\frac{3}{2}$

$\therefore (x,y)=\left(\frac{3}{2},2\right)$

Question 6 (**+)

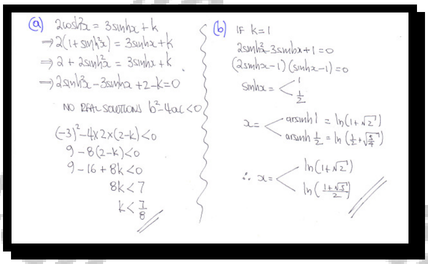
Consider the following hyperbolic equation, given in terms of a constant k.

$$2\cosh^2 x = 3\sinh x + k.$$

a) Find the range of values of k for which the above equation has no real solutions.

b) Given further that $k = 1$, find in exact logarithmic form, the solutions of the above equation.

$\boxed{k < \frac{7}{8}}$, $\boxed{x = \ln\left(1+\sqrt{2}\right),\ \ln\left(\frac{1+\sqrt{5}}{2}\right)}$



(a) $2\cosh^2 x = 3\sinh x + k$

$\Rightarrow 2(1+\sinh^2 x) = 3\sinh x + k$

$\Rightarrow 2 + 2\sinh^2 x = 3\sinh x + k$

$\Rightarrow 2\sinh^2 x - 3\sinh x + 2 - k = 0$

NO REAL SOLUTIONS $b^2 - 4ac < 0$

$(-3)^2 - 4 \times 2 \times (2-k) < 0$

$9 - 8(2-k) < 0$

$9 - 16 + 8k < 0$

$8k < 7$

$k < \frac{7}{8}$

(b) IF $k=1$

$2\sinh^2 x - 3\sinh x + 1 = 0$

$(2\sinh x - 1)(\sinh x - 1) = 0$

$\sinh x = \left< \begin{matrix} 1 \\ \frac{1}{2} \end{matrix} \right.$

$x = \left< \begin{matrix} \operatorname{arsinh} 1 = \ln(1+\sqrt{2}) \\ \operatorname{arsinh} \frac{1}{2} = \ln\left(\frac{1}{2}+\sqrt{\frac{5}{4}}\right) \end{matrix} \right.$

$\therefore x = \left< \begin{matrix} \ln(1+\sqrt{2}) \\ \ln\left(\frac{1+\sqrt{5}}{2}\right) \end{matrix} \right.$

Question 7 (+)**

$$f(x) = \operatorname{artanh} x\,,\ x \in \mathbb{R}\,,\ -1 < x < 1\,.$$

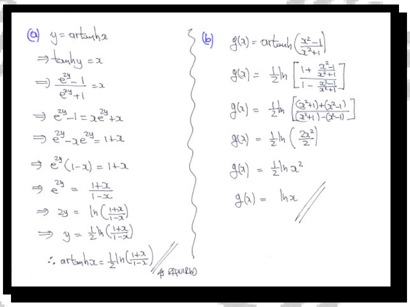
a) Show clearly that

$$f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\ x \in \mathbb{R}\,,\ -1 < x < 1\,.$$

b) Hence simplify fully

$$g(x) = \operatorname{artanh}\left(\frac{x^2-1}{x^2+1}\right),\ x > 0\,.$$

$$\boxed{g(x) = \ln x}$$



(a) $y = \operatorname{artanh} x$

$\Rightarrow \tanh y = x$

$\Rightarrow \frac{e^{2y}-1}{e^{2y}+1} = x$

$\Rightarrow e^{2y}-1 = xe^{2y}+x$

$\Rightarrow e^{2y}-xe^{2y} = 1+x$

$\Rightarrow e^{2y}(1-x) = 1+x$

$\Rightarrow e^{2y} = \frac{1+x}{1-x}$

$\Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right)$

$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$\therefore \operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ // AS REQUIRED

(b) $g(x) = \operatorname{artanh}\left(\frac{x^2-1}{x^2+1}\right)$

$g(x) = \frac{1}{2}\ln\left[\frac{1+\frac{x^2-1}{x^2+1}}{1-\frac{x^2-1}{x^2+1}}\right]$

$g(x) = \frac{1}{2}\ln\left[\frac{(x^2+1)+(x^2-1)}{(x^2+1)-(x^2-1)}\right]$

$g(x) = \frac{1}{2}\ln\left(\frac{2x^2}{2}\right)$

$g(x) = \frac{1}{2}\ln x^2$

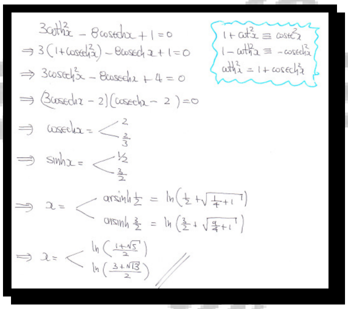
$g(x) = \ln x$ //

Question 8 **(**+)**

Solve the following equation, giving each of the answers in exact simplified form, in terms of natural logarithms.

$$3\coth^2 x - 8\operatorname{cosech} x + 1 = 0 .$$

$$\boxed{x = \ln\left[\tfrac{1}{2}\left(1+\sqrt{5}\right)\right], \quad x = \ln\left[\tfrac{1}{2}\left(3+\sqrt{13}\right)\right]}$$



$3\coth^2 x - 8\operatorname{cosech} x + 1 = 0$

$\Rightarrow 3\left(1+\operatorname{cosech}^2 x\right) - 8\operatorname{cosech} x + 1 = 0$

$\Rightarrow 3\operatorname{cosech}^2 x - 8\operatorname{cosech} x + 4 = 0$

$\Rightarrow \left(3\operatorname{cosech} x - 2\right)\left(\operatorname{cosech} x - 2\right) = 0$

$\Rightarrow \operatorname{cosech} x = < \begin{matrix} 2 \\ \frac{2}{3} \end{matrix}$

$\Rightarrow \sinh x = < \begin{matrix} \frac{1}{2} \\ \frac{3}{2} \end{matrix}$

$\Rightarrow x = < \begin{matrix} \operatorname{arsinh}\frac{1}{2} = \ln\left(\frac{1}{2}+\sqrt{\frac{1}{4}+1}\right) \\ \operatorname{arsinh}\frac{3}{2} = \ln\left(\frac{3}{2}+\sqrt{\frac{9}{4}+1}\right) \end{matrix}$

$\Rightarrow x = < \begin{matrix} \ln\left(\frac{1+\sqrt{5}}{2}\right) \\ \ln\left(\frac{3+\sqrt{13}}{2}\right) \end{matrix}$ //

$1+\cot^2 x \equiv \operatorname{cosec}^2 x$

$1-\coth^2 x \equiv -\operatorname{cosech}^2 x$

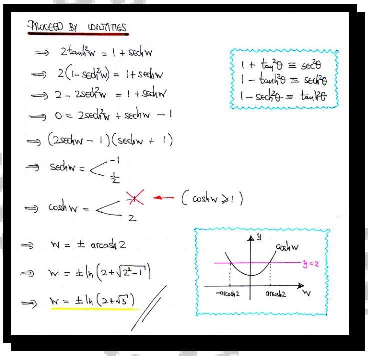
$\coth^2 x = 1+\operatorname{cosech}^2 x$

Question 9 **(**+)**

Solve the following equation, giving the solutions as exact simplified natural logarithms.

$$2\tanh^2 w = 1 + \operatorname{sech} w , \quad w \in \mathbb{R} .$$

$$\boxed{\quad\quad} , \quad \boxed{w = \pm\ln\left(2+\sqrt{3}\right)}$$



PROCEED BY IDENTITIES

$\Rightarrow 2\tanh^2 w = 1 + \operatorname{sech} w$

$\Rightarrow 2\left(1-\operatorname{sech}^2 w\right) = 1+\operatorname{sech} w$

$\Rightarrow 2 - 2\operatorname{sech}^2 w = 1 + \operatorname{sech} w$

$\Rightarrow 0 = 2\operatorname{sech}^2 w + \operatorname{sech} w - 1$

$\Rightarrow \left(2\operatorname{sech} w - 1\right)\left(\operatorname{sech} w + 1\right)$

$\Rightarrow \operatorname{sech} w = < \begin{matrix} -1 \\ \frac{1}{2} \end{matrix}$

$\Rightarrow \cosh w = < \begin{matrix} \cancel{-1} \longleftarrow (\cosh w \geqslant 1) \\ 2 \end{matrix}$

$\Rightarrow w = \pm \operatorname{arcosh} 2$

$\Rightarrow w = \pm\ln\left(2+\sqrt{2^2-1}\right)$

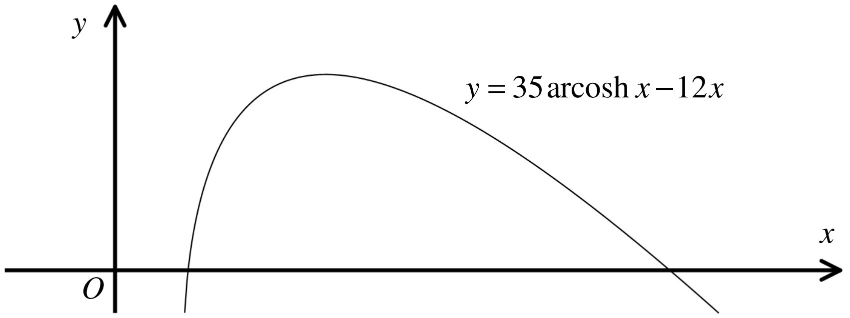
$\Rightarrow w = \pm\ln\left(2+\sqrt{3}\right)$ //

$1+\tan^2\theta \equiv \sec^2\theta$

$1-\tanh^2\theta \equiv \operatorname{sech}^2\theta$

$1-\operatorname{sech}^2\theta \equiv \tanh^2\theta$

Question 10 (**+)



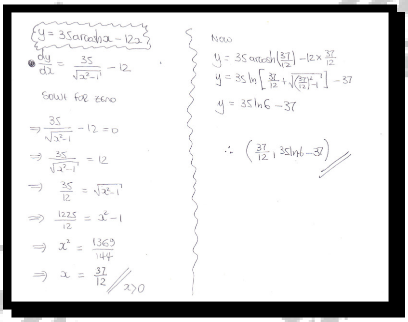
The figure above shows the graph of the curve with equation

$$y = 35\operatorname{arcosh} x - 12x\,,\ x \in \mathbb{R}\,,\ x \geq 1.$$

The curve has a single stationary point with coordinates $\left(\frac{a}{b}, c\ln 6 - d\right)$, where a, b, c and d are positive integers.

Determine the values of a, b, c and d.

$\boxed{a = 37}$, $\boxed{b = 12}$, $\boxed{c = 35}$, $\boxed{d = 37}$



$y = 35\operatorname{arcosh} x - 12x$

$\frac{dy}{dx} = \frac{35}{\sqrt{x^2-1}} - 12$

SOLVE FOR ZERO

$\Rightarrow \frac{35}{\sqrt{x^2-1}} - 12 = 0$

$\Rightarrow \frac{35}{\sqrt{x^2-1}} = 12$

$\Rightarrow \frac{35}{12} = \sqrt{x^2-1}$

$\Rightarrow \frac{1225}{12} = x^2 - 1$

$\Rightarrow x^2 = \frac{1369}{144}$

$\Rightarrow x = \frac{37}{12}$ // $x>0$

Now

$y = 35\operatorname{arcosh}\left(\frac{37}{12}\right) - 12 \times \frac{37}{12}$

$y = 35\ln\left[\frac{37}{12} + \sqrt{\left(\frac{37}{12}\right)^2 - 1}\right] - 37$

$y = 35\ln 6 - 37$

$\therefore \left(\frac{37}{12}, 35\ln 6 - 37\right)$ //

Question 11 (**+)

$$f(x) = 3 - \cosh x, \ x \in \mathbb{R}.$$

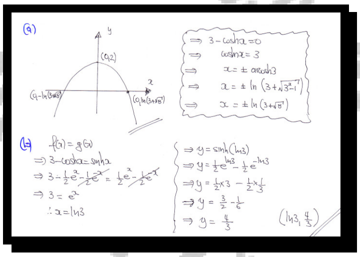
a) Sketch the graph of $f(x)$.

The graph must include the coordinates of any points where the graph meets the coordinate axes.

$$g(x) = \sinh x, \ x \in \mathbb{R}.$$

b) Find the exact coordinates of the point of intersection between the graphs of $f(x)$ and $g(x)$.

$\boxed{\left(\ln 3, \frac{4}{3}\right)}$


(a)

$\Rightarrow 3 - \cosh x = 0$

$\Rightarrow \cosh x = 3$

$\Rightarrow x = \pm \operatorname{arcosh} 3$

$\Rightarrow x = \pm \ln\left(3 + \sqrt{3^2 - 1}\right)$

$\Rightarrow x = \pm \ln\left(3 + \sqrt{8}\right)$

(b) $f(x) = g(x)$

$\Rightarrow 3 - \cosh x = \sinh x$

$\Rightarrow 3 - \frac{1}{2}e^{x} - \cancel{\frac{1}{2}e^{-x}} = \frac{1}{2}e^{x} - \cancel{\frac{1}{2}e^{-x}}$

$\Rightarrow 3 = e^{x}$

$\therefore x = \ln 3$

$\Rightarrow y = \sinh(\ln 3)$

$\Rightarrow y = \frac{1}{2}e^{\ln 3} - \frac{1}{2}e^{-\ln 3}$

$\Rightarrow y = \frac{1}{2} \times 3 - \frac{1}{2} \times \frac{1}{3}$

$\Rightarrow y = \frac{3}{2} - \frac{1}{6}$

$\Rightarrow y = \frac{4}{3}$

$\left(\ln 3, \frac{4}{3}\right)$

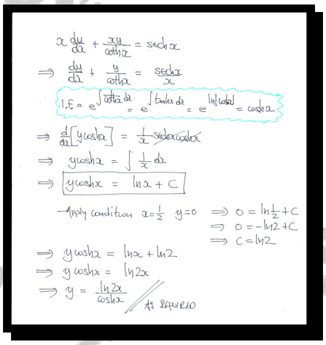
Question 12 (**+)

$$x\frac{dy}{dx}+\frac{xy}{\coth x}=\operatorname{sech} x\ ,\ x>0\,.$$

Given that $y=0$ at $x=\frac{1}{2}$, show that the solution of the above differential equation is

$$y=\frac{\ln 2x}{\cosh x}\,.$$

$\boxed{\text{proof}}$



$$x\frac{dy}{dx}+\frac{xy}{\coth x}=\operatorname{sech}x$$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{\coth x}=\frac{\operatorname{sech}x}{x}$$

$$\text{I.F.}=e^{\int \coth x\,dx}=e^{\int \tanh x\,dx}=e^{\ln|\cosh x|}=\cosh x$$

$$\Rightarrow \frac{d}{dx}\left[y\cosh x\right]=\frac{1}{x}\cancel{\operatorname{sech}x\cosh x}$$

$$\Rightarrow y\cosh x=\int\frac{1}{x}\,dx$$

$$\Rightarrow \boxed{y\cosh x=\ln x+C}$$

Apply condition $x=\frac{1}{2}$ $y=0$ $\Rightarrow 0=\ln\frac{1}{2}+C$
$\Rightarrow 0=-\ln 2+C$
$\Rightarrow C=\ln 2$

$$\Rightarrow y\cosh x=\ln x+\ln 2$$

$$\Rightarrow y\cosh x=\ln 2x$$

$$\Rightarrow y=\frac{\ln 2x}{\cosh x}$$

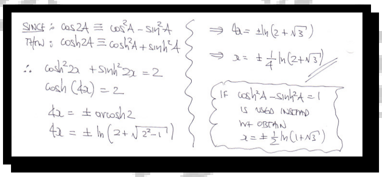
AS REQUIRED

Question 13 (**+)

Find in exact logarithmic form the solutions of the following equation.

$$\cosh^2 2x+\sinh^2 2x=2\,.$$

$$\boxed{x=\pm\frac{1}{4}\ln\left(2+\sqrt{3}\right)=\pm\frac{1}{2}\ln\left(1+\sqrt{3}\right)}$$



SINCE: $\cos 2A\equiv\cos^2A-\sin^2A$
THEN: $\cosh 2A\equiv\cosh^2A+\sinh^2A$

$\therefore\ \cosh^2 2x+\sinh^2 2x=2$
$\cosh(4x)=2$
$4x=\pm\operatorname{arcosh}2$
$4x=\pm\ln\left(2+\sqrt{2^2-1}\right)$

$\Rightarrow 4x=\pm\ln\left(2+\sqrt{3}\right)$

$\Rightarrow x=\pm\frac{1}{4}\ln\left(2+\sqrt{3}\right)$

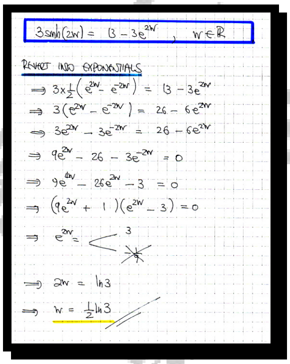
IF $\cosh^2A-\sinh^2A=1$
IS USED INSTEAD
WE OBTAIN
$x=\pm\frac{1}{2}\ln\left(1+\sqrt{3}\right)$

Question 14 **(**+)**

Find, in exact logarithmic form, the solution of the following equation.

$$3\sinh(2w) = 13 - 3e^{2w}, \ w \in \mathbb{R}.$$

$\boxed{}$, $\boxed{w = \frac{1}{2}\ln 3}$



$\boxed{3\sinh(2w) = 13 - 3e^{2w} \ , \quad w \in \mathbb{R}}$

REVERT INTO EXPONENTIALS

$\Rightarrow 3 \times \frac{1}{2}\left(e^{2w} - e^{-2w}\right) = 13 - 3e^{2w}$

$\Rightarrow 3\left(e^{2w} - e^{-2w}\right) = 26 - 6e^{2w}$

$\Rightarrow 3e^{2w} - 3e^{-2w} = 26 - 6e^{2w}$

$\Rightarrow 9e^{2w} - 26 - 3e^{-2w} = 0$

$\Rightarrow 9e^{4w} - 26e^{2w} - 3 = 0$

$\Rightarrow \left(9e^{2w} + 1\right)\left(e^{2w} - 3\right) = 0$

$\Rightarrow e^{2w} = \begin{cases} 3 \\ \times \end{cases}$

$\Rightarrow 2w = \ln 3$

$\Rightarrow w = \frac{1}{2}\ln 3$ //

Question 15 (***)

It is given that

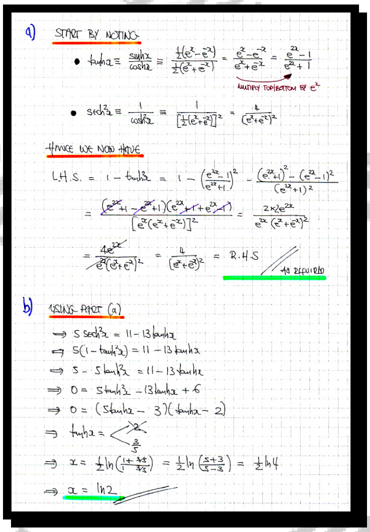
$$1-\tanh^2 x \equiv \operatorname{sech}^2 x .$$

a) Use the definitions of hyperbolic functions, in terms of exponentials, to prove the validity of the above identity.

b) Hence find in exact logarithmic form the solution of the following equation.

$$5\operatorname{sech}^2 x = 11-13\tanh x, \quad x \in \mathbb{R}.$$

$\boxed{\quad}$, $\boxed{x=\ln 2}$



a) START BY NOTING

- $\tanh x \equiv \frac{\sinh x}{\cosh x} \equiv \frac{\frac{1}{2}(e^x-e^{-x})}{\frac{1}{2}(e^x+e^{-x})} = \frac{e^x-e^{-x}}{e^x+e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$

MULTIPLY TOP/BOTTOM BY e^x

- $\operatorname{sech}^2 x \equiv \frac{1}{\cosh^2 x} \equiv \frac{1}{\left[\frac{1}{2}(e^x+e^{-x})\right]^2} = \frac{4}{(e^x+e^{-x})^2}$

HENCE WE NOW HAVE

$$\text{L.H.S.} = 1-\tanh^2 x = 1-\left(\frac{e^{2x}-1}{e^{2x}+1}\right)^2 = \frac{(e^{2x}+1)^2-(e^{2x}-1)^2}{(e^{2x}+1)^2}$$

$$= \frac{(\cancel{e^{2x}}+1-\cancel{e^{2x}}+1)(e^{2x}\cancel{+1}+e^{2x}\cancel{-1})}{\left[e^x(e^x+e^{-x})\right]^2} = \frac{2\times 2e^{2x}}{e^{2x}(e^x+e^{-x})^2}$$

$$= \frac{4\cancel{e^{2x}}}{\cancel{e^{2x}}(e^x+e^{-x})^2} = \frac{4}{(e^x+e^{-x})^2} = \text{R.H.S}$$

AS REQUIRED

b) USING PART (a)

$$\Rightarrow 5\operatorname{sech}^2 x = 11-13\tanh x$$
$$\Rightarrow 5(1-\tanh^2 x) = 11-13\tanh x$$
$$\Rightarrow 5-5\tanh^2 x = 11-13\tanh x$$
$$\Rightarrow 0 = 5\tanh^2 x-13\tanh x+6$$
$$\Rightarrow 0 = (5\tanh x-3)(\tanh x-2)$$
$$\Rightarrow \tanh x = \cancel{2},\ \frac{3}{5}$$
$$\Rightarrow x = \frac{1}{2}\ln\left(\frac{1+3/5}{1-3/5}\right) = \frac{1}{2}\ln\left(\frac{5+3}{5-3}\right) = \frac{1}{2}\ln 4$$
$$\Rightarrow x = \ln 2$$

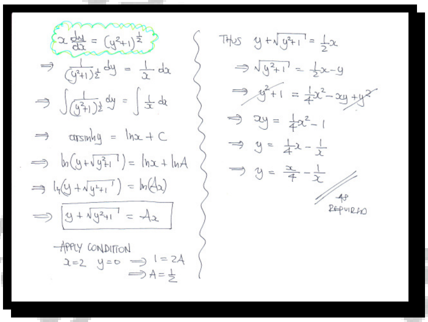
Question 16 (***)

$$x\frac{dy}{dx}=\sqrt{y^2+1}\ ,\ x>0 .$$

Given that $y=0$ at $x=2$, show that the solution of the above differential equation is

$$y=\frac{x}{4}-\frac{1}{x}.$$

$\boxed{\text{proof}}$



$$x\frac{dy}{dx}=(y^2+1)^{\frac{1}{2}}$$

$$\Rightarrow \frac{1}{(y^2+1)^{\frac{1}{2}}}\,dy=\frac{1}{x}\,dx$$

$$\Rightarrow \int\frac{1}{(y^2+1)^{\frac{1}{2}}}\,dy=\int\frac{1}{x}\,dx$$

$$\Rightarrow \text{arsinh}\,y=\ln x+C$$

$$\Rightarrow \ln\left(y+\sqrt{y^2+1}\right)=\ln x+\ln A$$

$$\Rightarrow \ln\left(y+\sqrt{y^2+1}\right)=\ln(Ax)$$

$$\Rightarrow \boxed{y+\sqrt{y^2+1}=Ax}$$

–APPLY CONDITION

$x=2\quad y=0 \Rightarrow 1=2A$

$\Rightarrow A=\frac{1}{2}$

THUS $y+\sqrt{y^2+1}=\frac{1}{2}x$

$$\Rightarrow \sqrt{y^2+1}=\frac{1}{2}x-y$$

$$\Rightarrow \cancel{y^2}+1=\frac{1}{4}x^2-xy+\cancel{y^2}$$

$$\Rightarrow xy=\frac{1}{4}x^2-1$$

$$\Rightarrow y=\frac{1}{4}x-\frac{1}{x}$$

$$\Rightarrow y=\frac{x}{4}-\frac{1}{x}$$

// AS REQUIRED

Question 17 (***)

The curves C_1 and C_2 have respective equation

$$y = \sinh x \quad \text{and} \quad y = \frac{1}{2}\operatorname{sech} x .$$

a) Sketch in the same diagram the graphs of C_1 and C_2.

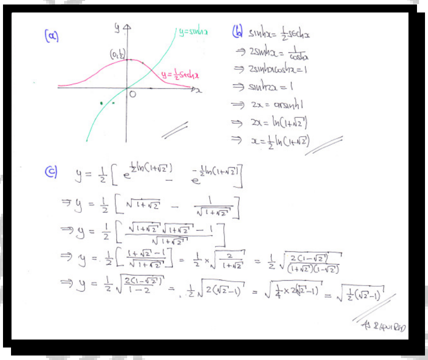
The two graphs intersect at the point P.

b) Find the x coordinates of P.

c) Hence show that the y coordinates of P is.

$$\sqrt{\frac{1}{2}\left(\sqrt{2}-1\right)} .$$

$$\boxed{x = \frac{1}{2}\ln\left(1+\sqrt{2}\right)}$$



(a)

(b) $\sinh x = \frac{1}{2}\operatorname{sech} x$

$\Rightarrow 2\sinh x = \frac{1}{\cosh x}$

$\Rightarrow 2\sinh x\cosh x = 1$

$\Rightarrow \sinh 2x = 1$

$\Rightarrow 2x = \operatorname{arsinh} 1$

$\Rightarrow 2x = \ln(1+\sqrt{2})$

$\Rightarrow x = \frac{1}{2}\ln(1+\sqrt{2})$

(c) $y = \frac{1}{2}\left[e^{\frac{1}{2}\ln(1+\sqrt{2})} - e^{-\frac{1}{2}\ln(1+\sqrt{2})}\right]$

$\Rightarrow y = \frac{1}{2}\left[\sqrt{1+\sqrt{2}} - \frac{1}{\sqrt{1+\sqrt{2}}}\right]$

$\Rightarrow y = \frac{1}{2}\left[\frac{\sqrt{1+\sqrt{2}}\sqrt{1+\sqrt{2}} - 1}{\sqrt{1+\sqrt{2}}}\right]$

$\Rightarrow y = \frac{1}{2}\left[\frac{1+\sqrt{2}-1}{\sqrt{1+\sqrt{2}}}\right] = \frac{1}{2}\times\sqrt{\frac{2}{1+\sqrt{2}}} = \frac{1}{2}\sqrt{\frac{2(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})}}$

$\Rightarrow y = \frac{1}{2}\sqrt{\frac{2(1-\sqrt{2})}{1-2}} = \frac{1}{2}\sqrt{2(\sqrt{2}-1)} = \sqrt{\frac{1}{4}\times 2(\sqrt{2}-1)} = \sqrt{\frac{1}{2}(\sqrt{2}-1)}$

Question 18 (***)

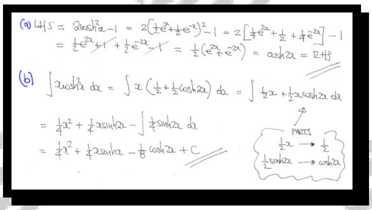
$$2\cosh^2 x - 1 \equiv \cosh 2x .$$

a) Prove the validity of the above hyperbolic identity by using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials.

b) Hence find

$$\int x\cosh^2 x \, dx .$$

$$\boxed{\frac{1}{4}x^2 + \frac{1}{4}x\sinh 2x - \frac{1}{8}\cosh 2x + C}$$



(a) LHS $= 2\cosh^2 x - 1 = 2\left(\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}\right)^2 - 1 = 2\left[\frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}\right] - 1$

$= \frac{1}{2}e^{2x} \cancel{+1} + \frac{1}{2}e^{-2x} \cancel{-1} = \frac{1}{2}\left(e^{2x} + e^{-2x}\right) = \cosh 2x =$ RHS

(b) $\int x\cosh^2 x \, dx = \int x\left(\frac{1}{2} + \frac{1}{2}\cosh 2x\right) dx = \int \frac{1}{2}x + \frac{1}{2}x\cosh 2x \, dx$

$= \frac{1}{4}x^2 + \frac{1}{4}x\sinh 2x - \int \frac{1}{4}\sinh 2x \, dx$

$= \frac{1}{4}x^2 + \frac{1}{4}x\sinh 2x - \frac{1}{8}\cosh 2x + C$

PARTS

$\frac{1}{2}x \rightarrow \frac{1}{2}$

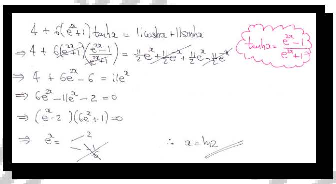
$\frac{1}{2}\sinh 2x \rightarrow \cosh 2x$

Question 19 (***)

Solve the hyperbolic equation

$$4 + 6\left(e^{2x} + 1\right)\tanh x = 11\cosh x + 11\sinh x .$$

$$\boxed{x = \ln 2}$$



$4 + 6(e^{2x}+1)\tanh x = 11\cosh x + 11\sinh x$

$\Rightarrow 4 + 6\cancel{(e^{2x}+1)}\left(\frac{e^{2x}-1}{\cancel{e^{2x}+1}}\right) = \frac{11}{2}e^{x} \cancel{+\frac{11}{2}e^{-x}} + \frac{11}{2}e^{x} \cancel{-\frac{11}{2}e^{-x}}$

$\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$

$\Rightarrow 4 + 6e^{2x} - 6 = 11e^{x}$

$\Rightarrow 6e^{2x} - 11e^{x} - 2 = 0$

$\Rightarrow (e^{x} - 2)(6e^{x} + 1) = 0$

$\Rightarrow e^{x} = 2$, $\cancel{-\frac{1}{6}}$

$\therefore x = \ln 2$

Question 20 (***)

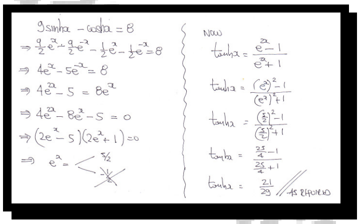
Given that

$$9\sinh x - \cosh x = 8$$

show clearly that

$$\tanh x = \frac{21}{29}.$$

$\boxed{\text{proof}}$



$$9\sinh x - \cosh x = 8$$

$$\Rightarrow \tfrac{9}{2}e^{x} - \tfrac{9}{2}e^{-x} - \tfrac{1}{2}e^{x} - \tfrac{1}{2}e^{-x} = 8$$

$$\Rightarrow 4e^{x} - 5e^{-x} = 8$$

$$\Rightarrow 4e^{2x} - 5 = 8e^{x}$$

$$\Rightarrow 4e^{2x} - 8e^{x} - 5 = 0$$

$$\Rightarrow (2e^{x} - 5)(2e^{x} + 1) = 0$$

$$\Rightarrow e^{x} = \frac{5}{2} \quad \cancel{-\tfrac{1}{2}}$$

Now

$$\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$$

$$\tanh x = \frac{(e^{x})^{2}-1}{(e^{x})^{2}+1}$$

$$\tanh x = \frac{\left(\frac{5}{2}\right)^{2}-1}{\left(\frac{5}{2}\right)^{2}+1}$$

$$\tanh x = \frac{\frac{25}{4}-1}{\frac{25}{4}+1}$$

$$\tanh x = \frac{21}{29}$$ // AS REQUIRED

Question 21 (***)

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

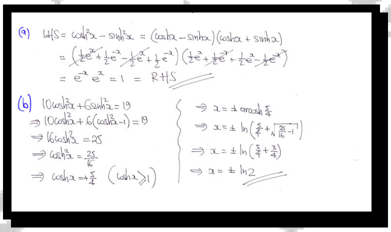
a) Prove the validity of the above hyperbolic identity by using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials.

b) Hence solve the equation

$$10\cosh^2 x + 6\sinh^2 x = 19$$

giving the answers as exact natural logarithms.

$\boxed{x = \pm\ln 2}$



(a) $\text{LHS} = \cosh^2 x - \sinh^2 x = (\cosh x - \sinh x)(\cosh x + \sinh x)$

$$= \left(\cancel{\tfrac{1}{2}e^{x}} + \tfrac{1}{2}e^{-x} - \cancel{\tfrac{1}{2}e^{x}} + \tfrac{1}{2}e^{-x}\right)\left(\tfrac{1}{2}e^{x} + \cancel{\tfrac{1}{2}e^{-x}} + \tfrac{1}{2}e^{x} - \cancel{\tfrac{1}{2}e^{-x}}\right)$$

$$= e^{-x}e^{x} = 1 = \text{RHS}$$

(b) $10\cosh^2 x + 6\sinh^2 x = 19$

$\Rightarrow 10\cosh^2 x + 6(\cosh^2 x - 1) = 19$

$\Rightarrow 16\cosh^2 x = 25$

$\Rightarrow \cosh^2 x = \frac{25}{16}$

$\Rightarrow \cosh x = +\frac{5}{4} \quad (\cosh x \geqslant 1)$

$\Rightarrow x = \pm \operatorname{arcosh}\frac{5}{4}$

$\Rightarrow x = \pm\ln\left(\frac{5}{4} + \sqrt{\frac{25}{16} - 1}\right)$

$\Rightarrow x = \pm\ln\left(\frac{5}{4} + \frac{3}{4}\right)$

$\Rightarrow x = \pm\ln 2$

Question 22 (***)

$$2\cosh 3x \cosh x \equiv \cosh 4x + \cosh 2x\ .$$

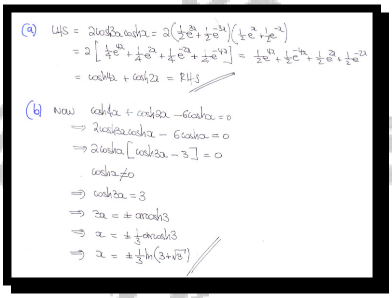
a) Prove the validity of the above hyperbolic identity by using the definitions of $\cosh x$ in terms of exponentials.

a) Hence solve the equation

$$\cosh 4x + \cosh 2x - 6\cosh x = 0$$

giving the answer as an expression involving exact natural logarithms.

$$\boxed{x = \pm\frac{1}{3}\ln\left(3+\sqrt{8}\right)}$$



(a) $\text{LHS} = 2\cosh 3x \cosh x = 2\left(\frac{1}{2}e^{3x}+\frac{1}{2}e^{-3x}\right)\left(\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}\right)$

$= 2\left[\frac{1}{4}e^{4x}+\frac{1}{4}e^{2x}+\frac{1}{4}e^{-2x}+\frac{1}{4}e^{-4x}\right] = \frac{1}{2}e^{4x}+\frac{1}{2}e^{-4x}+\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}$

$= \cosh 4x + \cosh 2x = \text{RHS}$

(b) Now $\cosh 4x + \cosh 2x - 6\cosh x = 0$

$\Rightarrow 2\cosh 3x \cosh x - 6\cosh x = 0$

$\Rightarrow 2\cosh x\left[\cosh 3x - 3\right] = 0$

$\cosh x \neq 0$

$\Rightarrow \cosh 3x = 3$

$\Rightarrow 3x = \pm \text{arcosh} 3$

$\Rightarrow x = \pm\frac{1}{3}\text{arcosh} 3$

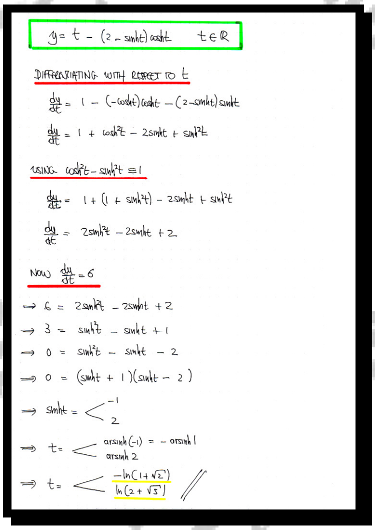
$\Rightarrow x = \pm\frac{1}{3}\ln\left(3+\sqrt{8}\right)$

Question 23 (***)

$$y = t - (2 - \sinh t)\cosh t \, , \ t \in \mathbb{R} \, .$$

Determine the values of t for which $\frac{dy}{dt} = 6$, giving the answers as exact simplified natural logarithms.

$\boxed{}$, $\boxed{t = -\ln\left(1+\sqrt{2}\right) \ \cup \ t = \ln\left(2+\sqrt{5}\right)}$



$\boxed{y = t - (2 - \sinh t)\cosh t \qquad t \in \mathbb{R}}$

DIFFERENTIATING WITH RESPECT TO t

$\frac{dy}{dt} = 1 - (-\cosh t)\cosh t - (2 - \sinh t)\sinh t$

$\frac{dy}{dt} = 1 + \cosh^2 t - 2\sinh t + \sinh^2 t$

USING $\cosh^2 t - \sinh^2 t \equiv 1$

$\frac{dy}{dt} = 1 + (1 + \sinh^2 t) - 2\sinh t + \sinh^2 t$

$\frac{dy}{dt} = 2\sinh^2 t - 2\sinh t + 2$

NOW $\frac{dy}{dt} = 6$

$\Rightarrow 6 = 2\sinh^2 t - 2\sinh t + 2$

$\Rightarrow 3 = \sinh^2 t - \sinh t + 1$

$\Rightarrow 0 = \sinh^2 t - \sinh t - 2$

$\Rightarrow 0 = (\sinh t + 1)(\sinh t - 2)$

$\Rightarrow \sinh t = \begin{cases} -1 \\ 2 \end{cases}$

$\Rightarrow t = \begin{cases} \text{arsinh}(-1) = -\text{arsinh}\,1 \\ \text{arsinh}\,2 \end{cases}$

$\Rightarrow t = \begin{cases} -\ln(1+\sqrt{2}) \\ \ln(2+\sqrt{5}) \end{cases}$ //

Question 24 (***)

$$\cosh(A-B) \equiv \cosh A \cosh B - \sinh A \sinh B \, .$$

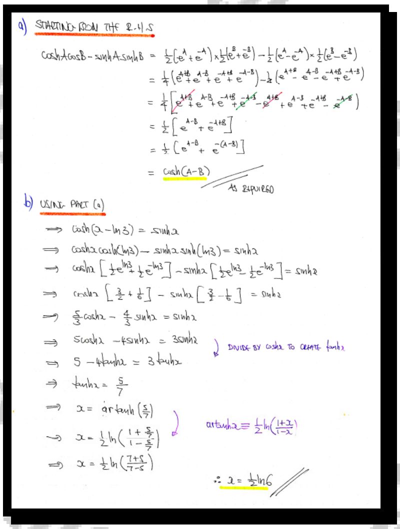
a) Prove the validity of the above hyperbolic identity by using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials.

b) Hence solve the equation

$$\cosh(x-\ln 3) = \sinh x$$

giving the answer as an exact natural logarithm.

$\boxed{\text{FP2}}$, $\boxed{x = \frac{1}{2}\ln 6}$



a) STARTING FROM THE R.H.S

$$\cosh A \cos B - \sinh A \sinh B = \frac{1}{2}(e^{A}+e^{-A}) \times \frac{1}{2}(e^{B}+e^{-B}) - \frac{1}{2}(e^{A}-e^{-A}) \times \frac{1}{2}(e^{B}-e^{-B})$$

$$= \frac{1}{4}(e^{A+B}+e^{A-B}+e^{-A+B}+e^{-A-B}) - \frac{1}{4}(e^{A+B}-e^{A-B}-e^{-A+B}+e^{-A-B})$$

$$= \frac{1}{4}\left[\cancel{e^{A+B}}+e^{A-B}+e^{-A+B}+\cancel{e^{-A-B}}-\cancel{e^{A+B}}+e^{A-B}+e^{-A+B}-\cancel{e^{-A-B}}\right)$$

$$= \frac{1}{2}\left[e^{A-B}+e^{-A+B}\right]$$

$$= \frac{1}{2}\left[e^{A-B}+e^{-(A-B)}\right]$$

$$= \cosh(A-B)$$

AS REQUIRED

b) USING PART (a)

$$\Rightarrow \cosh(x-\ln 3) = \sinh x$$

$$\Rightarrow \cosh x \cosh(\ln 3) - \sinh x \sinh(\ln 3) = \sinh x$$

$$\Rightarrow \cosh x \left[\frac{1}{2}e^{\ln 3}+\frac{1}{2}e^{-\ln 3}\right] - \sinh x\left[\frac{1}{2}e^{\ln 3}-\frac{1}{2}e^{-\ln 3}\right] = \sinh x$$

$$\Rightarrow \cosh x\left[\frac{3}{2}+\frac{1}{6}\right] - \sinh x\left[\frac{3}{2}-\frac{1}{6}\right] = \sinh x$$

$$\Rightarrow \frac{5}{3}\cosh x - \frac{4}{3}\sinh x = \sinh x$$

$$\Rightarrow 5\cosh x - 4\sinh x = 3\sinh x$$

DIVIDE BY $\cosh x$ TO CREATE $\tanh x$

$$\Rightarrow 5 - 4\tanh x = 3\tanh x$$

$$\Rightarrow \tanh x = \frac{5}{7}$$

$$\Rightarrow x = \text{artanh}\left(\frac{5}{7}\right)$$

$$\text{artanh}\, x \equiv \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow x = \frac{1}{2}\ln\left(\frac{1+\frac{5}{7}}{1-\frac{5}{7}}\right)$$

$$\Rightarrow x = \frac{1}{2}\ln\left(\frac{7+5}{7-5}\right)$$

$$\therefore x = \frac{1}{2}\ln 6$$

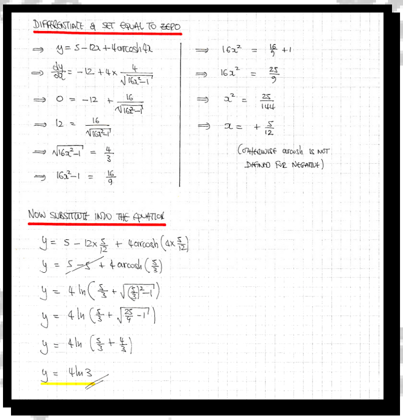
Question 25 (***)

Find, in exact simplified logarithmic form, the y coordinate of the stationary point of the curve with equation

$$y = 5 - 12x + 4\operatorname{arcosh}(4x).$$

Detailed workings must be shown.

$\boxed{}$, $\boxed{4\ln 3}$



DIFFERENTIATE & SET EQUAL TO ZERO

$\Rightarrow y = 5 - 12x + 4\operatorname{arcosh} 4x$

$\Rightarrow \frac{dy}{dx} = -12 + 4 \times \frac{4}{\sqrt{16x^2-1}}$

$\Rightarrow 0 = -12 + \frac{16}{\sqrt{16x^2-1}}$

$\Rightarrow 12 = \frac{16}{\sqrt{16x^2-1}}$

$\Rightarrow \sqrt{16x^2-1} = \frac{4}{3}$

$\Rightarrow 16x^2 - 1 = \frac{16}{9}$

$\Rightarrow 16x^2 = \frac{16}{9} + 1$

$\Rightarrow 16x^2 = \frac{25}{9}$

$\Rightarrow x^2 = \frac{25}{144}$

$\Rightarrow x = +\frac{5}{12}$

(OTHERWISE arcosh IS NOT DEFINED FOR NEGATIVE)

NOW SUBSTITUTE INTO THE EQUATION

$y = 5 - 12 \times \frac{5}{12} + 4\operatorname{arcosh}\left(4 \times \frac{5}{12}\right)$

$y = \cancel{5-5} + 4\operatorname{arcosh}\left(\frac{5}{3}\right)$

$y = 4\ln\left(\frac{5}{3} + \sqrt{\left(\frac{5}{3}\right)^2 - 1}\right)$

$y = 4\ln\left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1}\right)$

$y = 4\ln\left(\frac{5}{3} + \frac{4}{3}\right)$

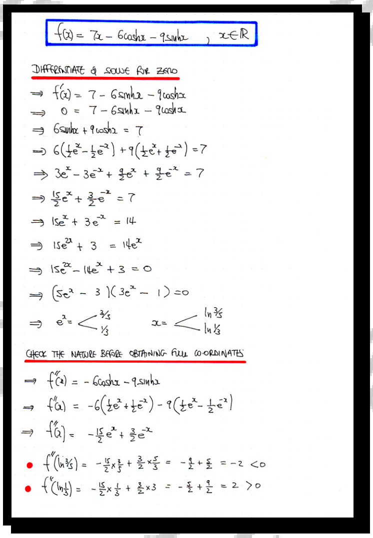
$y = 4\ln 3$

Question 26 (***)

$$f(x) \equiv 7x - 6\cosh x - 9\sinh x, \ \ x \in \mathbb{R}.$$

Find the exact coordinates of the stationary points of $f(x)$, and determine their nature. Give the coordinates in terms of simplified natural logarithms.

$$\boxed{}, \ \boxed{\left[\ln\left(\tfrac{3}{5}\right), -2 + 7\ln\left(\tfrac{3}{5}\right)\right] \cup \left[\ln\left(\tfrac{1}{3}\right), 2 - 7\ln 3\right]}$$



$$\boxed{f(x) = 7x - 6\cosh x - 9\sinh x \ , \ x \in \mathbb{R}}$$

DIFFERENTIATE & SOLVE FOR ZERO

$\Rightarrow f'(x) = 7 - 6\sinh x - 9\cosh x$

$\Rightarrow 0 = 7 - 6\sinh x - 9\cosh x$

$\Rightarrow 6\sinh x + 9\cosh x = 7$

$\Rightarrow 6\left(\frac{1}{2}e^{x} - \frac{1}{2}e^{-x}\right) + 9\left(\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}\right) = 7$

$\Rightarrow 3e^{x} - 3e^{-x} + \frac{9}{2}e^{x} + \frac{9}{2}e^{-x} = 7$

$\Rightarrow \frac{15}{2}e^{x} + \frac{3}{2}e^{-x} = 7$

$\Rightarrow 15e^{x} + 3e^{-x} = 14$

$\Rightarrow 15e^{2x} + 3 = 14e^{x}$

$\Rightarrow 15e^{2x} - 14e^{x} + 3 = 0$

$\Rightarrow (5e^{x} - 3)(3e^{x} - 1) = 0$

$\Rightarrow e^{x} = \genfrac{<}{}{0pt}{}{3/5}{1/3} \qquad x = \genfrac{<}{}{0pt}{}{\ln 3/5}{\ln 1/3}$

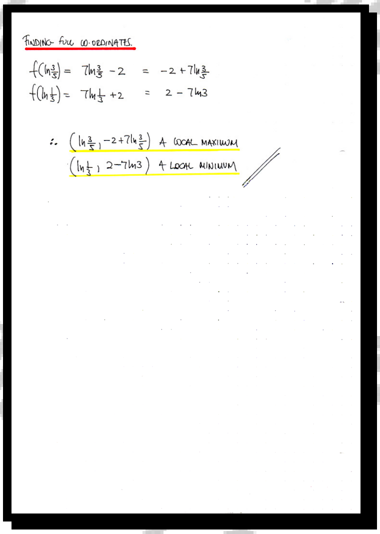
CHECK THE NATURE BEFORE OBTAINING FULL CO-ORDINATES

$\Rightarrow f''(x) = -6\cosh x - 9\sinh x$

$\Rightarrow f''(x) = -6\left(\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}\right) - 9\left(\frac{1}{2}e^{x} - \frac{1}{2}e^{-x}\right)$

$\Rightarrow f''(x) = -\frac{15}{2}e^{x} + \frac{3}{2}e^{-x}$

- $f''(\ln 3/5) = -\frac{15}{2} \times \frac{3}{5} + \frac{3}{2} \times \frac{5}{3} = -\frac{9}{2} + \frac{5}{2} = -2 < 0$
- $f''(\ln \frac{1}{3}) = -\frac{15}{2} \times \frac{1}{3} + \frac{3}{2} \times 3 = -\frac{5}{2} + \frac{9}{2} = 2 > 0$



FINDING FULL CO-ORDINATES.

$f(\ln \frac{3}{5}) = 7\ln\frac{3}{5} - 2 = -2 + 7\ln\frac{3}{5}$

$f(\ln \frac{1}{3}) = 7\ln\frac{1}{3} + 2 = 2 - 7\ln 3$

$\therefore \left(\ln\frac{3}{5}, -2 + 7\ln\frac{3}{5}\right)$ A LOCAL MAXIMUM

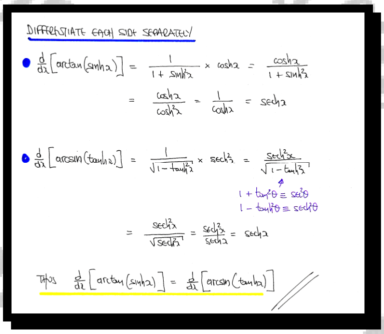
$\left(\ln\frac{1}{3}, 2 - 7\ln 3\right)$ A LOCAL MINIMUM

Question 27 (***)

Show with detailed workings that

$$\frac{d}{dx}\Big[\arctan\left(\sinh x\right)\Big] = \frac{d}{dx}\Big[\arcsin\left(\tanh x\right)\Big]$$

$$\boxed{}, \ \boxed{\text{proof}}$$



DIFFERENTIATE EACH SIDE SEPARATELY

- $\frac{d}{dx}\left[\arctan(\sinh x)\right] = \frac{1}{1 + \sinh^2 x} \times \cosh x = \frac{\cosh x}{1 + \sinh^2 x}$

$= \frac{\cosh x}{\cosh^2 x} = \frac{1}{\cosh x} = \text{sech}\, x$

- $\frac{d}{dx}\left[\arcsin(\tanh x)\right] = \frac{1}{\sqrt{1 - \tanh^2 x}} \times \text{sech}^2 x = \frac{\text{sech}^2 x}{\sqrt{1 - \tanh^2 x}}$

$1 + \tan^2\theta \equiv \sec^2\theta$

$1 - \tanh^2\theta \equiv \text{sech}^2\theta$

$= \frac{\text{sech}^2 x}{\sqrt{\text{sech}^2 x}} = \frac{\text{sech}^2 x}{\text{sech}\, x} = \text{sech}\, x$

Thus $\frac{d}{dx}\left[\arctan(\sinh x)\right] = \frac{d}{dx}\left[\arcsin(\tanh x)\right]$

Question 28 (***)

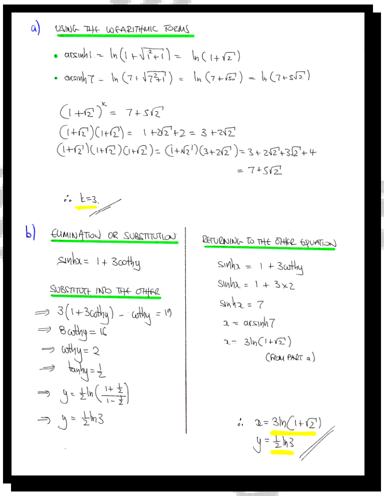
a) Given that $\text{arsinh}\,7 = k\,\text{arsinh}\,1$ determine the value of k.

b) Solve the following simultaneous equations.

$$\sinh x - 3\coth y = 1$$
$$3\sinh x - \coth y = 19$$

Give the answers in simplified logarithmic form.

$\boxed{}$, $\boxed{k = 3}$, $\boxed{[x, y] = \left[3\ln\left(1+\sqrt{2}\right),\ \tfrac{1}{2}\ln 3\right]}$



a) USING THE LOGARITHMIC FORMS

- $\text{arsinh}1 = \ln\left(1+\sqrt{1^2+1}\right) = \ln\left(1+\sqrt{2}\right)$
- $\text{arsinh}7 = \ln\left(7+\sqrt{7^2+1}\right) = \ln\left(7+\sqrt{50}\right) = \ln\left(7+5\sqrt{2}\right)$

$(1+\sqrt{2})^k = 7+5\sqrt{2}$

$(1+\sqrt{2})(1+\sqrt{2}) = 1+2\sqrt{2}+2 = 3+2\sqrt{2}$

$(1+\sqrt{2})(1+\sqrt{2})(1+\sqrt{2}) = (1+\sqrt{2})(3+2\sqrt{2}) = 3+2\sqrt{2}+3\sqrt{2}+4$
$= 7+5\sqrt{2}$

$\therefore\ k=3.$

b) ELIMINATION OR SUBSTITUTION

$\sinh x = 1 + 3\coth y$

SUBSTITUTE INTO THE OTHER

$\Rightarrow 3(1+3\coth y) - \coth y = 19$

$\Rightarrow 8\coth y = 16$

$\Rightarrow \coth y = 2$

$\Rightarrow \tanh y = \frac{1}{2}$

$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)$

$\Rightarrow y = \frac{1}{2}\ln 3$

RETURNING TO THE OTHER EQUATION

$\sinh x = 1 + 3\coth y$

$\sinh x = 1 + 3\times 2$

$\sinh x = 7$

$x = \text{arsinh}7$

$x = 3\ln(1+\sqrt{2})$

(FROM PART a)

$\therefore\ x = 3\ln(1+\sqrt{2})$

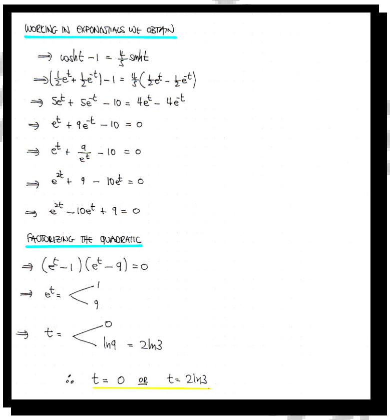
$y = \frac{1}{2}\ln 3$

Question 29 **(***)**

Solve the following equation, giving the answers as exact logarithms where appropriate.

$$\cosh t - 1 = \frac{4}{5}\sinh t .$$

$\boxed{}$, $\boxed{t = 0, \quad t = 2\ln 3}$



WORKING IN EXPONENTIALS WE OBTAIN

$\Rightarrow \cosh t - 1 = \frac{4}{5}\sinh t$

$\Rightarrow \left(\frac{1}{2}e^{t} + \frac{1}{2}e^{-t}\right) - 1 = \frac{4}{5}\left(\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\right)$

$\Rightarrow 5e^{t} + 5e^{-t} - 10 = 4e^{t} - 4e^{-t}$

$\Rightarrow e^{t} + 9e^{-t} - 10 = 0$

$\Rightarrow e^{t} + \frac{9}{e^{t}} - 10 = 0$

$\Rightarrow e^{2t} + 9 - 10e^{t} = 0$

$\Rightarrow e^{2t} - 10e^{t} + 9 = 0$

FACTORIZING THE QUADRATIC

$\Rightarrow (e^{t} - 1)(e^{t} - 9) = 0$

$\Rightarrow e^{t} = \left< \begin{matrix} 1 \\ 9 \end{matrix} \right.$

$\Rightarrow t = \left< \begin{matrix} 0 \\ \ln 9 = 2\ln 3 \end{matrix} \right.$

$\therefore$ $t = 0$ OR $t = 2\ln 3$

Question 30 **(***)**

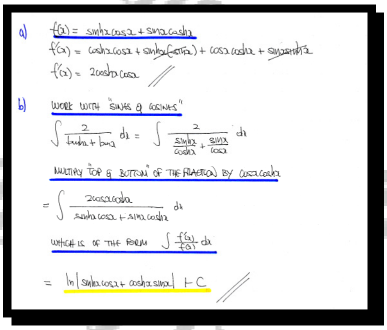
$$f(x) = \sinh x \cos x + \sin x \cosh x\,,\ \ x \in \mathbb{R}\,.$$

a) Find a simplified expression for $f'(x)$.

b) Use the answer to part (a) to find

$$\int \frac{2}{\tanh x + \tan x}\,dx\,.$$

$\boxed{\quad}$, $\boxed{f'(x) = 2\cosh x \cos x}$, $\boxed{\ln\left|\sinh x \cos x + \sin x \cosh x\right| + C}$



a) $f(x) = \sinh x \cos x + \sin x \cosh x$

$f'(x) = \cosh x \cos x + \cancel{\sinh x(-\sin x)} + \cos x \cosh x + \cancel{\sin x \sinh x}$

$f'(x) = 2\cosh x \cos x$ //

b) WORK WITH "SINES & COSINES"

$$\int \frac{2}{\tanh x + \tan x}\,dx = \int \frac{2}{\frac{\sinh x}{\cosh x} + \frac{\sin x}{\cos x}}\,dx$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY $\cos x \cosh x$

$$= \int \frac{2\cos x \cosh x}{\sinh x \cos x + \sin x \cosh x}\,dx$$

WHICH IS OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx$

$= \ln\left|\sinh x \cos x + \cosh x \sin x\right| + C$ //

Question 31 (***)

It is given that for all real x

$$\cosh 2x \equiv 1 + 2\sinh^2 x \,.$$

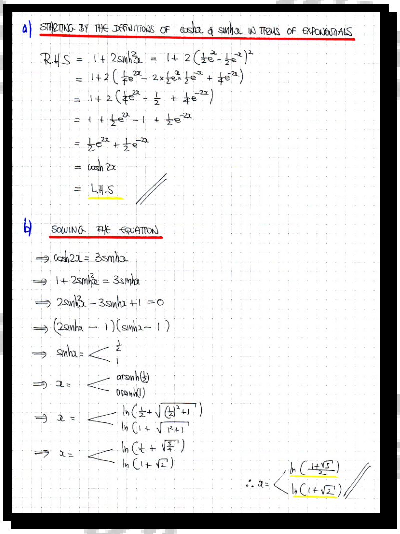
a) Prove the validity of the above hyperbolic identity, by using the definitions of the hyperbolic functions in terms of exponentials.

b) Hence solve the equation

$$\cosh 2x = 3\sinh x \,,$$

giving the final answers as exact simplified natural logarithms.

$\boxed{\text{FP2}}$, $\boxed{x = \ln\left(1+\sqrt{2}\right) \quad \cup \quad x = \ln\left(\frac{1+\sqrt{5}}{2}\right)}$



a) STARTING BY THE DEFINITIONS OF $\cosh x$ & $\sinh x$ IN TERMS OF EXPONENTIALS

$$\begin{aligned} R.H.S &= 1 + 2\sinh^2 x = 1 + 2\left(\tfrac{1}{2}e^{x} - \tfrac{1}{2}e^{-x}\right)^2 \\ &= 1 + 2\left(\tfrac{1}{4}e^{2x} - 2\times\tfrac{1}{2}e^{x}\tfrac{1}{2}e^{-x} + \tfrac{1}{4}e^{-2x}\right) \\ &= 1 + 2\left(\tfrac{1}{4}e^{2x} - \tfrac{1}{2} + \tfrac{1}{4}e^{-2x}\right) \\ &= 1 + \tfrac{1}{2}e^{2x} - 1 + \tfrac{1}{2}e^{-2x} \\ &= \tfrac{1}{2}e^{2x} + \tfrac{1}{2}e^{-2x} \\ &= \cosh 2x \\ &= L.H.S \quad // \end{aligned}$$

b) SOLVING THE EQUATION

$$\begin{aligned} &\Rightarrow \cosh 2x = 3\sinh x \\ &\Rightarrow 1 + 2\sinh^2 x = 3\sinh x \\ &\Rightarrow 2\sinh^2 x - 3\sinh x + 1 = 0 \\ &\Rightarrow (2\sinh x - 1)(\sinh x - 1) \\ &\Rightarrow \sinh x = \left\langle \begin{matrix} \frac{1}{2} \\ 1 \end{matrix} \right. \\ &\Rightarrow x = \left\langle \begin{matrix} \operatorname{arsinh}\left(\frac{1}{2}\right) \\ \operatorname{arsinh}(1) \end{matrix} \right. \\ &\Rightarrow x = \left\langle \begin{matrix} \ln\left(\frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + 1}\right) \\ \ln\left(1 + \sqrt{1^2+1}\right) \end{matrix} \right. \\ &\Rightarrow x = \left\langle \begin{matrix} \ln\left(\frac{1}{2} + \sqrt{\frac{5}{4}}\right) \\ \ln\left(1 + \sqrt{2}\right) \end{matrix} \right. \end{aligned}$$

$$\therefore x = \left\langle \begin{matrix} \ln\left(\frac{1+\sqrt{5}}{2}\right) \\ \ln\left(1+\sqrt{2}\right) \end{matrix} \right. //$$

Question 32 (***)

It is given that for all real x

$$\cosh 2x \equiv 2\cosh^2 x - 1 .$$

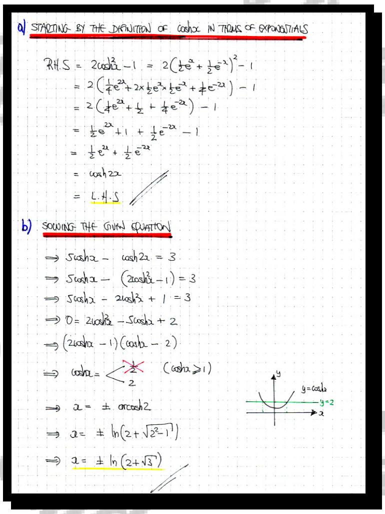
a) Prove the validity of the above hyperbolic identity, by using the definitions of the hyperbolic functions in terms of exponentials.

b) Hence solve the equation

$$5\cosh x - \cosh 2x = 3 ,$$

giving the final answers as exact simplified natural logarithms.

$\boxed{}$, $\boxed{x = \pm \ln\left(2+\sqrt{3}\right)}$



a) STARTING BY THE DEFINITION OF $\cosh x$ IN TERMS OF EXPONENTIALS

$$\text{R.H.S} = 2\cosh^2 x - 1 = 2\left(\tfrac{1}{2}e^{x} + \tfrac{1}{2}e^{-x}\right)^2 - 1$$
$$= 2\left(\tfrac{1}{4}e^{2x} + 2\times\tfrac{1}{2}e^{x}\times\tfrac{1}{2}e^{-x} + \tfrac{1}{4}e^{-2x}\right) - 1$$
$$= 2\left(\tfrac{1}{4}e^{2x} + \tfrac{1}{2} + \tfrac{1}{4}e^{-2x}\right) - 1$$
$$= \tfrac{1}{2}e^{2x} + 1 + \tfrac{1}{2}e^{-2x} - 1$$
$$= \tfrac{1}{2}e^{2x} + \tfrac{1}{2}e^{-2x}$$
$$= \cosh 2x$$
$$= \text{L.H.S}$$

b) SOLVING THE GIVEN EQUATION

$$\Rightarrow 5\cosh x - \cosh 2x = 3$$
$$\Rightarrow 5\cosh x - (2\cosh^2 x - 1) = 3$$
$$\Rightarrow 5\cosh x - 2\cosh^2 x + 1 = 3$$
$$\Rightarrow 0 = 2\cosh^2 x - 5\cosh x + 2$$
$$\Rightarrow (2\cosh x - 1)(\cosh x - 2)$$
$$\Rightarrow \cosh x = \begin{cases} \cancel{\tfrac{1}{2}} \\ 2 \end{cases} \quad (\cosh x \geqslant 1)$$
$$\Rightarrow x = \pm \operatorname{arcosh} 2$$
$$\Rightarrow x = \pm \ln\left(2 + \sqrt{2^2 - 1}\right)$$
$$\Rightarrow x = \pm \ln\left(2 + \sqrt{3}\right)$$

Question 33 (***+)

It is given that for all real x

$$\cosh 3x \equiv 4\cosh^3 x - 3\cosh x .$$

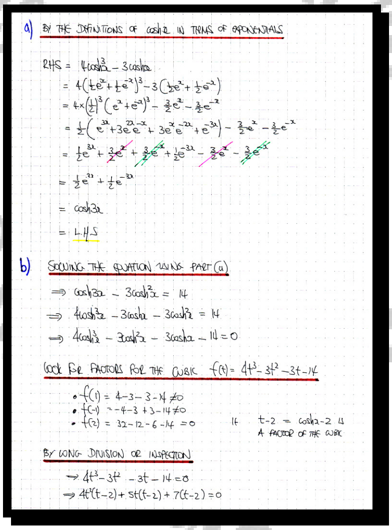
a) Prove the validity of the above hyperbolic identity, by using the definitions of the hyperbolic functions in terms of exponentials.

b) Hence solve the equation

$$\cosh 3x - 3\cosh^2 x = 14 ,$$

giving the final answers as exact simplified natural logarithms.

$\boxed{}$, $\boxed{x = \pm\ln\left(2+\sqrt{3}\right)}$



a) BY THE DEFINITIONS OF $\cosh x$ IN TERMS OF EXPONENTIALS

$RHS = 4\cosh^3 x - 3\cosh x$

$= 4\left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)^3 - 3\left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)$

$= 4\times\left(\frac{1}{2}\right)^3\left(e^x + e^{-x}\right)^3 - \frac{3}{2}e^x - \frac{3}{2}e^{-x}$

$= \frac{1}{2}\left(e^{3x} + 3e^{2x}e^{-x} + 3e^{x}e^{-2x} + e^{-3x}\right) - \frac{3}{2}e^x - \frac{3}{2}e^{-x}$

$= \frac{1}{2}e^{3x} + \cancel{\frac{3}{2}e^{x}} + \cancel{\frac{3}{2}e^{-x}} + \frac{1}{2}e^{-3x} - \cancel{\frac{3}{2}e^{x}} - \cancel{\frac{3}{2}e^{-x}}$

$= \frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}$

$= \cosh 3x$

$= L.H.S$

b) SOLVING THE EQUATION USING PART (a)

$\Rightarrow \cosh 3x - 3\cosh^2 x = 14$

$\Rightarrow 4\cosh^3 x - 3\cosh x - 3\cosh^2 x = 14$

$\Rightarrow 4\cosh^3 x - 3\cosh^2 x - 3\cosh x - 14 = 0$

LOOK FOR FACTORS FOR THE CUBIC $f(t) = 4t^3 - 3t^2 - 3t - 14$

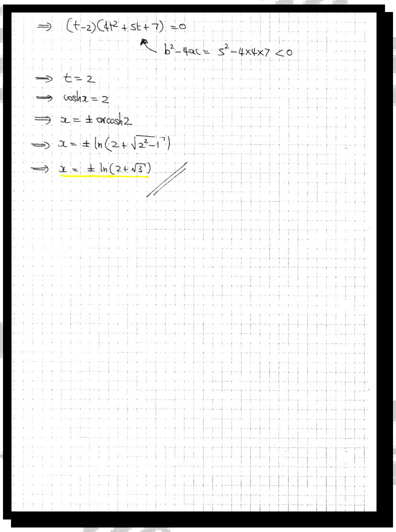
- $f(1) = 4 - 3 - 3 - 14 \neq 0$
- $f(-1) = -4 - 3 + 3 - 14 \neq 0$
- $f(2) = 32 - 12 - 6 - 14 = 0$

∴ $t - 2 = \cosh x - 2$ IS A FACTOR OF THE CUBIC

BY LONG DIVISION OR INSPECTION

$\Rightarrow 4t^3 - 3t^2 - 3t - 14 = 0$

$\Rightarrow 4t^2(t-2) + 5t(t-2) + 7(t-2) = 0$



$\Rightarrow (t-2)(4t^2 + 5t + 7) = 0$

$b^2 - 4ac = 5^2 - 4\times4\times7 < 0$

$\Rightarrow t = 2$

$\Rightarrow \cosh x = 2$

$\Rightarrow x = \pm \operatorname{arcosh} 2$

$\Rightarrow x = \pm\ln\left(2 + \sqrt{2^2 - 1}\right)$

$\Rightarrow x = \pm\ln\left(2 + \sqrt{3}\right)$ //

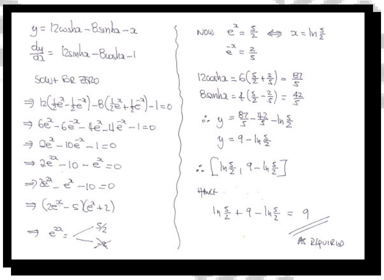
Question 34 **(***+)**

A curve C has equation

$$y = 12\cosh x - 8\sinh x - x, \ x \in \mathbb{R}.$$

Show that the sum of the coordinates of the turning point of C is 9.

proof



$y = 12\cosh x - 8\sinh x - x$

$\frac{dy}{dx} = 12\sinh x - 8\cosh x - 1$

SOLVE FOR ZERO

$\Rightarrow 12\left(\frac{1}{2}e^{x} - \frac{1}{2}e^{-x}\right) - 8\left(\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}\right) - 1 = 0$

$\Rightarrow 6e^{x} - 6e^{-x} - 4e^{x} - 4e^{-x} - 1 = 0$

$\Rightarrow 2e^{x} - 10e^{-x} - 1 = 0$

$\Rightarrow 2e^{2x} - 10 - e^{x} = 0$

$\Rightarrow 2e^{2x} - e^{x} - 10 = 0$

$\Rightarrow (2e^{x} - 5)(e^{x} + 2)$

$\Rightarrow e^{2x} = \frac{5}{2}, \ \cancel{-2}$

NOW $e^{x} = \frac{5}{2} \Leftrightarrow x = \ln\frac{5}{2}$

$e^{-x} = \frac{2}{5}$

$12\cosh x = 6\left(\frac{5}{2} + \frac{2}{5}\right) = \frac{87}{5}$

$8\sinh x = 4\left(\frac{5}{2} - \frac{2}{5}\right) = \frac{42}{5}$

$\therefore y = \frac{87}{5} - \frac{42}{5} - \ln\frac{5}{2}$

$y = 9 - \ln\frac{5}{2}$

$\therefore \left[\ln\frac{5}{2}, 9 - \ln\frac{5}{2}\right]$

HENCE

$\ln\frac{5}{2} + 9 - \ln\frac{5}{2} = 9$

AS REQUIRED

Question 35 (***+)

$$y = \operatorname{artanh} x, \ -1 < x < 1$$

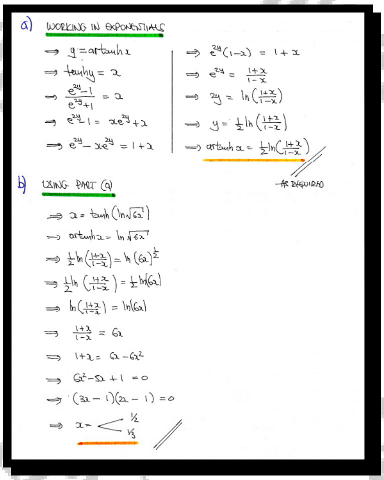
a) By using the definitions of hyperbolic functions in terms of exponentials prove that

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

b) Hence solve the equation

$$x = \tanh\left(\ln\sqrt{6x}\right).$$

$\boxed{\text{EP2-J}}$, $\boxed{x = \frac{1}{2}, \frac{1}{3}}$



a) WORKING IN EXPONENTIALS

$\Rightarrow y = \operatorname{artanh} x$

$\Rightarrow \tanh y = x$

$\Rightarrow \dfrac{e^{2y}-1}{e^{2y}+1} = x$

$\Rightarrow e^{2y} - 1 = xe^{2y} + x$

$\Rightarrow e^{2y} - xe^{2y} = 1 + x$

$\Rightarrow e^{2y}(1-x) = 1 + x$

$\Rightarrow e^{2y} = \dfrac{1+x}{1-x}$

$\Rightarrow 2y = \ln\left(\dfrac{1+x}{1-x}\right)$

$\Rightarrow y = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$

$\Rightarrow \operatorname{artanh} x = \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$ //

— AS REQUIRED

b) USING PART (a)

$\Rightarrow x = \tanh\left(\ln\sqrt{6x}\right)$

$\Rightarrow \operatorname{artanh} x = \ln\sqrt{6x}$

$\Rightarrow \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) = \ln(6x)^{\frac{1}{2}}$

$\Rightarrow \frac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) = \frac{1}{2}\ln(6x)$

$\Rightarrow \ln\left(\dfrac{1+x}{1-x}\right) = \ln(6x)$

$\Rightarrow \dfrac{1+x}{1-x} = 6x$

$\Rightarrow 1 + x = 6x - 6x^2$

$\Rightarrow 6x^2 - 5x + 1 = 0$

$\Rightarrow (3x-1)(2x-1) = 0$

$\Rightarrow x = \begin{cases} \frac{1}{2} \\ \frac{1}{3} \end{cases}$ //

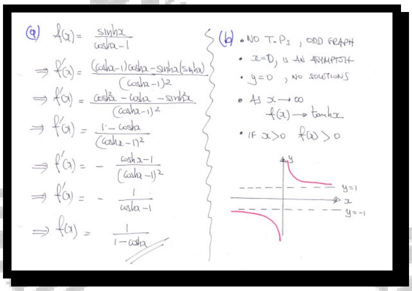
Question 36 (*+)**

$$f(x) = \frac{\sinh x}{\cosh x - 1}, \quad x \in \mathbb{R}, \; x \neq 0.$$

a) Find a simplified expression for $f'(x)$.

b) Sketch the graph of $f(x)$.

$$\boxed{f'(x) = \frac{1}{1 - \cosh x}}$$



(a) $f(x) = \frac{\sinh x}{\cosh x - 1}$

$\Rightarrow f'(x) = \frac{(\cosh x - 1)\cosh x - \sinh x(\sinh x)}{(\cosh x - 1)^2}$

$\Rightarrow f'(x) = \frac{\cosh^2 x - \cosh x - \sinh^2 x}{(\cosh x - 1)^2}$

$\Rightarrow f'(x) = \frac{1 - \cosh x}{(\cosh x - 1)^2}$

$\Rightarrow f'(x) = -\frac{\cosh x - 1}{(\cosh x - 1)^2}$

$\Rightarrow f'(x) = -\frac{1}{\cosh x - 1}$

$\Rightarrow f(x) = \frac{1}{1 - \cosh x}$

(b)
- NO T.Ps, ODD GRAPH
- $x = 0$, IS AN ASYMPTOTE
- $y = 0$, NO SOLUTIONS
- AS $x \to \infty$ $f(x) \to \tanh x$
- IF $x > 0$ $f(x) > 0$

Question 37 (*+)**

$$y = \operatorname{arsinh} x\,,\ \ x \in \mathbb{R}\,.$$

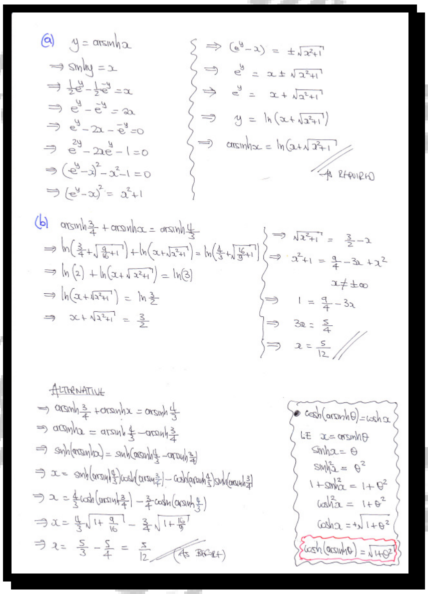
a) Show that

$$\operatorname{arsinh} x = \ln\left[x + \sqrt{x^2+1}\right].$$

b) Solve the equation

$$\operatorname{arsinh}\frac{3}{4} + \operatorname{arsinh} x = \operatorname{arsinh}\frac{4}{3}\,.$$

$$\boxed{x = \frac{5}{12}}$$



(a) $y = \operatorname{arsinh} x$

$\Rightarrow \sinh y = x$

$\Rightarrow \frac{1}{2}e^{y} - \frac{1}{2}e^{-y} = x$

$\Rightarrow e^{y} - e^{-y} = 2x$

$\Rightarrow e^{y} - 2x - e^{-y} = 0$

$\Rightarrow e^{2y} - 2xe^{y} - 1 = 0$

$\Rightarrow (e^{y} - x)^2 - x^2 - 1 = 0$

$\Rightarrow (e^{y} - x)^2 = x^2 + 1$

$\Rightarrow (e^{y} - x) = \pm\sqrt{x^2+1}$

$\Rightarrow e^{y} = x \pm \sqrt{x^2+1}$

$\Rightarrow e^{y} = x + \sqrt{x^2+1}$

$\Rightarrow y = \ln\left(x + \sqrt{x^2+1}\right)$

$\Rightarrow \operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$ // AS REQUIRED

(b) $\operatorname{arsinh}\frac{3}{4} + \operatorname{arsinh} x = \operatorname{arsinh}\frac{4}{3}$

$\Rightarrow \ln\left(\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right) + \ln\left(x + \sqrt{x^2+1}\right) = \ln\left(\frac{4}{3} + \sqrt{\frac{16}{9}+1}\right)$

$\Rightarrow \ln(2) + \ln\left(x + \sqrt{x^2+1}\right) = \ln(3)$

$\Rightarrow \ln\left(x + \sqrt{x^2+1}\right) = \ln\frac{3}{2}$

$\Rightarrow x + \sqrt{x^2+1} = \frac{3}{2}$

$\Rightarrow \sqrt{x^2+1} = \frac{3}{2} - x$

$\Rightarrow x^2 + 1 = \frac{9}{4} - 3x + x^2$

$x \neq \pm\infty$

$\Rightarrow 1 = \frac{9}{4} - 3x$

$\Rightarrow 3x = \frac{5}{4}$

$\Rightarrow x = \frac{5}{12}$ //

ALTERNATIVE

$\Rightarrow \operatorname{arsinh}\frac{3}{4} + \operatorname{arsinh} x = \operatorname{arsinh}\frac{4}{3}$

$\Rightarrow \operatorname{arsinh} x = \operatorname{arsinh}\frac{4}{3} - \operatorname{arsinh}\frac{3}{4}$

$\Rightarrow \sinh(\operatorname{arsinh} x) = \sinh\left(\operatorname{arsinh}\frac{4}{3} - \operatorname{arsinh}\frac{3}{4}\right)$

$\Rightarrow x = \sinh\left(\operatorname{arsinh}\frac{4}{3}\right)\cosh\left(\operatorname{arsinh}\frac{3}{4}\right) - \cosh\left(\operatorname{arsinh}\frac{4}{3}\right)\sinh\left(\operatorname{arsinh}\frac{3}{4}\right)$

$\Rightarrow x = \frac{4}{3}\cosh\left(\operatorname{arsinh}\frac{3}{4}\right) - \frac{3}{4}\cosh\left(\operatorname{arsinh}\frac{4}{3}\right)$

$\Rightarrow x = \frac{4}{3}\sqrt{1+\frac{9}{16}} - \frac{3}{4}\sqrt{1+\frac{16}{9}}$

$\Rightarrow x = \frac{5}{3} - \frac{5}{4} = \frac{5}{12}$ // (AS BEFORE)

• $\cosh(\operatorname{arsinh}\theta) = \cosh x$

I.E $x = \operatorname{arsinh}\theta$

$\sinh x = \theta$

$\sinh^2 x = \theta^2$

$1 + \sinh^2 x = 1 + \theta^2$

$\cosh^2 x = 1 + \theta^2$

$\cosh x = +\sqrt{1+\theta^2}$

$\boxed{\cosh(\operatorname{arsinh}\theta) = \sqrt{1+\theta^2}}$

Question 38 (***+)

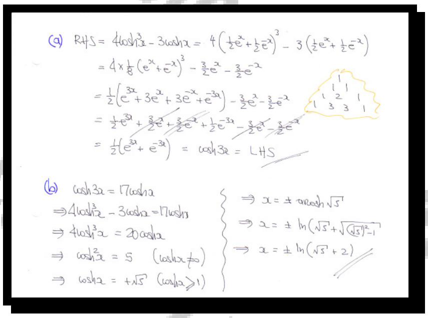
$$\cosh 3x \equiv 4\cosh^3 x - 3\cosh x .$$

a) Prove the validity of the above hyperbolic identity by using the definition of $\cosh x$ in terms of exponential functions.

b) Hence find in exact logarithmic form the solutions of the equation

$$\cosh 3x = 17\cosh x .$$

$$\boxed{x = \pm\ln\left(2+\sqrt{5}\right) = \mp\ln\left(-2+\sqrt{5}\right)}$$



(a) $\text{RHS} = 4\cosh^3 x - 3\cosh x = 4\left(\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}\right)^3 - 3\left(\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}\right)$

$= 4\times\frac{1}{8}\left(e^{x}+e^{-x}\right)^3 - \frac{3}{2}e^{x} - \frac{3}{2}e^{-x}$

$= \frac{1}{2}\left(e^{3x}+3e^{x}+3e^{-x}+e^{-3x}\right) - \frac{3}{2}e^{x} - \frac{3}{2}e^{-x}$

$= \frac{1}{2}e^{3x} + \cancel{\frac{3}{2}e^{x}} + \cancel{\frac{3}{2}e^{-x}} + \frac{1}{2}e^{-3x} - \cancel{\frac{3}{2}e^{x}} - \cancel{\frac{3}{2}e^{-x}}$

$= \frac{1}{2}\left(e^{3x}+e^{-3x}\right) = \cosh 3x = \text{LHS}$

(b) $\cosh 3x = 17\cosh x$

$\Rightarrow 4\cosh^3 x - 3\cosh x = 17\cosh x$

$\Rightarrow 4\cosh^3 x = 20\cosh x$

$\Rightarrow \cosh^2 x = 5 \quad (\cosh x \neq 0)$

$\Rightarrow \cosh x = +\sqrt{5} \quad (\cosh x \geqslant 1)$

$\Rightarrow x = \pm\,\text{arcosh}\sqrt{5}$

$\Rightarrow x = \pm\ln\left(\sqrt{5}+\sqrt{(\sqrt{5})^2-1}\right)$

$\Rightarrow x = \pm\ln\left(\sqrt{5}+2\right)$

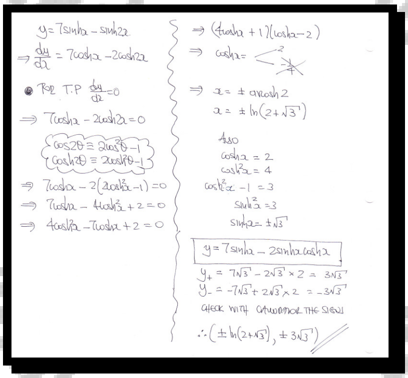
Question 39 **(***+)**

The curve C has equation

$$y = 7\sinh x - \sinh 2x, \ x \in \mathbb{R}.$$

Find in terms of natural logarithms and/or surds the exact coordinates of the stationary points of C.

$$\boxed{\pm\left(\ln\left(2+\sqrt{3}\right), 3\sqrt{3}\right)}$$



$y = 7\sinh x - \sinh 2x$

$\Rightarrow \frac{dy}{dx} = 7\cosh x - 2\cosh 2x$

● For T.P $\frac{dy}{dx} = 0$

$\Rightarrow 7\cosh x - 2\cosh 2x = 0$

$\cos 2\theta \equiv 2\cos^2\theta - 1$

$\cosh 2\theta \equiv 2\cosh^2\theta - 1$

$\Rightarrow 7\cosh x - 2(2\cosh^2 x - 1) = 0$

$\Rightarrow 7\cosh x - 4\cosh^2 x + 2 = 0$

$\Rightarrow 4\cosh^2 x - 7\cosh x + 2 = 0$

$\Rightarrow (4\cosh x + 1)(\cosh x - 2)$

$\Rightarrow \cosh x = \begin{cases} 2 \\ \cancel{-\frac{1}{4}} \end{cases}$

$\Rightarrow x = \pm \text{arcosh}\, 2$

$x = \pm \ln(2+\sqrt{3})$

Also

$\cosh x = 2$

$\cosh^2 x = 4$

$\cosh^2 x - 1 = 3$

$\sinh^2 x = 3$

$\sinh x = \pm\sqrt{3}$

$\boxed{y = 7\sinh x - 2\sinh x\cosh x}$

$y_+ = 7\sqrt{3} - 2\sqrt{3} \times 2 = 3\sqrt{3}$

$y_- = -7\sqrt{3} + 2\sqrt{3} \times 2 = -3\sqrt{3}$

CHECK WITH CALCULATOR THE SIGN

$\therefore \left(\pm\ln(2+\sqrt{3}), \pm 3\sqrt{3}\right)$

Question 40 (***+)

The curves C_1 and C_2 have respective equations

$$y = 18\cosh x,\ x \in \mathbb{R} \qquad \text{and} \qquad y = 12 + 14\sinh x,\ x \in \mathbb{R}.$$

a) Find the exact coordinates of the points of intersection between C_1 and C_2.

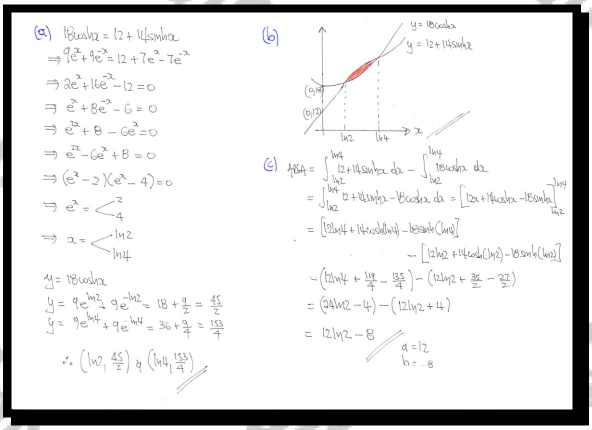
b) Sketch in the same diagram the graph of C_1 and the graph of C_2.

c) Show that the finite region bounded by the graphs of C_1 and C_2 has an area of

$$a\ln 2 + b,$$

where a and b are integers to be found.

$\boxed{\left(\ln 2, \frac{45}{2}\right) \ \& \ \left(\ln 4, \frac{153}{4}\right)}$, $\boxed{12\ln 2 - 8}$



(a) $18\cosh x = 12 + 14\sinh x$

$\Rightarrow 9e^{x} + 9e^{-x} = 12 + 7e^{x} - 7e^{-x}$

$\Rightarrow 2e^{x} + 16e^{-x} - 12 = 0$

$\Rightarrow e^{x} + 8e^{-x} - 6 = 0$

$\Rightarrow e^{2x} + 8 - 6e^{x} = 0$

$\Rightarrow e^{2x} - 6e^{x} + 8 = 0$

$\Rightarrow (e^{x} - 2)(e^{x} - 4) = 0$

$\Rightarrow e^{x} = \begin{cases} 2 \\ 4 \end{cases}$

$\Rightarrow x = \begin{cases} \ln 2 \\ \ln 4 \end{cases}$

$y = 18\cosh x$

$y = 9e^{\ln 2} + 9e^{-\ln 2} = 18 + \frac{9}{2} = \frac{45}{2}$

$y = 9e^{\ln 4} + 9e^{-\ln 4} = 36 + \frac{9}{4} = \frac{153}{4}$

$\therefore \left(\ln 2, \frac{45}{2}\right)$ & $\left(\ln 4, \frac{153}{4}\right)$ //

(b)

(c) $\text{AREA} = \int_{\ln 2}^{\ln 4} 12 + 14\sinh x \, dx - \int_{\ln 2}^{\ln 4} 18\cosh x \, dx$

$= \int_{\ln 2}^{\ln 4} 12 + 14\sinh x - 18\cosh x \, dx = \left[12x + 14\cosh x - 18\sinh x\right]_{\ln 2}^{\ln 4}$

$= \left[12\ln 4 + 14\cosh(\ln 4) - 18\sinh(\ln 4)\right] - \left[12\ln 2 + 14\cosh(\ln 2) - 18\sinh(\ln 2)\right]$

$= \left(12\ln 4 + \frac{119}{4} - \frac{135}{4}\right) - \left(12\ln 2 + \frac{35}{2} - \frac{27}{2}\right)$

$= (24\ln 2 - 4) - (12\ln 2 + 4)$

$= 12\ln 2 - 8$ //

$a = 12$
$b = -8$

Question 41 (***+)

It is given that

$$\cosh(A+B) \equiv \cosh A \cosh B + \sinh A \sinh B .$$

a) Prove the validity of the above hyperbolic identity by using the definitions of the hyperbolic functions in terms of exponential functions.

It is now given that

$$5\cosh x + 4\sinh x \equiv R\cosh(x+\alpha),$$

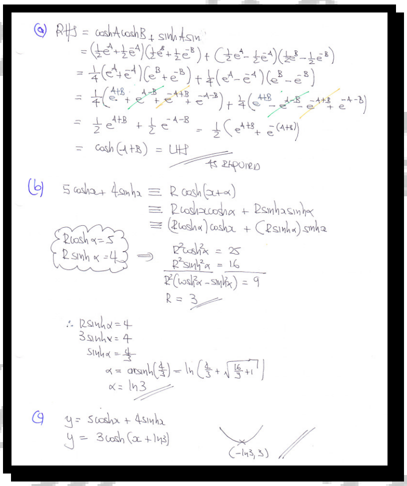
where R and α are positive constants.

b) Determine, in terms of natural logarithms where appropriate, the exact values of R and α.

c) Hence state the coordinates of the minimum point on the graph of

$$y = 5\cosh x + 4\sinh x .$$

$\boxed{R=3}$, $\boxed{\alpha=\ln 3}$, $\boxed{(-\ln 3, 3)}$



(a) RHS $= \cosh A\cosh B + \sinh A \sinh$

$= \left(\frac{1}{2}e^{A}+\frac{1}{2}e^{-A}\right)\left(\frac{1}{2}e^{B}+\frac{1}{2}e^{-B}\right)+\left(\frac{1}{2}e^{A}-\frac{1}{2}e^{-A}\right)\left(\frac{1}{2}e^{B}-\frac{1}{2}e^{-B}\right)$

$= \frac{1}{4}\left(e^{A}+e^{-A}\right)\left(e^{B}+e^{-B}\right)+\frac{1}{4}\left(e^{A}-e^{-A}\right)\left(e^{B}-e^{-B}\right)$

$= \frac{1}{4}\left(e^{A+B}+\cancel{e^{A-B}}+\cancel{e^{-A+B}}+e^{-A-B}\right)+\frac{1}{4}\left(e^{A+B}-\cancel{e^{A-B}}-\cancel{e^{-A+B}}+e^{-A-B}\right)$

$= \frac{1}{2}e^{A+B}+\frac{1}{2}e^{-A-B} = \frac{1}{2}\left(e^{A+B}+e^{-(A+B)}\right)$

$= \cosh(A+B) = \text{LHS}$

AS REQUIRED

(b) $5\cosh x + 4\sinh x \equiv R\cosh(x+\alpha)$

$\equiv R\cosh x\cosh\alpha + R\sinh x\sinh\alpha$

$\equiv (R\cosh\alpha)\cosh x + (R\sinh\alpha)\sinh x$

$R\cosh\alpha = 5$
$R\sinh\alpha = 4$ $\Rightarrow$

$R^2\cosh^2\alpha = 25$

$R^2\sinh^2\alpha = 16$

$R^2(\cosh^2\alpha - \sinh^2\alpha) = 9$

$R = 3$

$\therefore R\sinh\alpha = 4$

$3\sinh\alpha = 4$

$\sinh\alpha = \frac{4}{3}$

$\alpha = \text{arsinh}\left(\frac{4}{3}\right) = \ln\left(\frac{4}{3}+\sqrt{\frac{16}{9}+1}\right)$

$\alpha = \ln 3$

(c) $y = 5\cosh x + 4\sinh x$

$y = 3\cosh(x+\ln 3)$

$(-\ln 3, 3)$

Question 42 **(***+)**

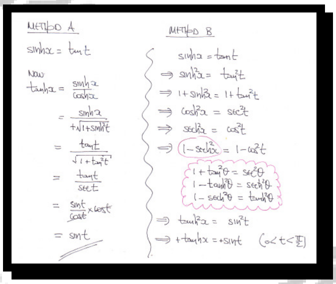
Given that

$$\sinh x = \tan t \,, \ 0 < t < \frac{\pi}{2} ,$$

show clearly that

$$\tanh x = \sin t \,.$$

$\boxed{\text{proof}}$



METHOD A

$\sinh x = \tan t$

Now

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$= \frac{\sinh x}{+\sqrt{1+\sinh^2 t}}$$

$$= \frac{\tan t}{\sqrt{1+\tan^2 t}}$$

$$= \frac{\tan t}{\sec t}$$

$$= \frac{\sin t}{\cos t} \times \cos t$$

$$= \sin t$$

METHOD B

$$\sinh x = \tan t$$

$$\Rightarrow \sinh^2 x = \tan^2 t$$

$$\Rightarrow 1 + \sinh^2 x = 1 + \tan^2 t$$

$$\Rightarrow \cosh^2 x = \sec^2 t$$

$$\Rightarrow \operatorname{sech}^2 x = \cos^2 t$$

$$\Rightarrow 1 - \operatorname{sech}^2 x = 1 - \cos^2 t$$

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 - \tanh^2\theta = \operatorname{sech}^2\theta$$
$$1 - \operatorname{sech}^2\theta = \tanh^2\theta$$

$$\Rightarrow \tanh^2 x = \sin^2 t$$

$$\Rightarrow +\tanh x = +\sin t \quad \left(0 < t < \tfrac{\pi}{2}\right)$$

Question 43 (***+)

$$f(x) \equiv \operatorname{artanh} x \ , \ x \in \mathbb{R}, \ |x| < 1$$

a) Use the definition of the hyperbolic tangent to prove that

$$f(x) \equiv \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right].$$

b) Use a method involving complex numbers and the trigonometric identity

$$1+\tan^2 x \equiv \sec^2 x ,$$

to obtain the hyperbolic equivalent

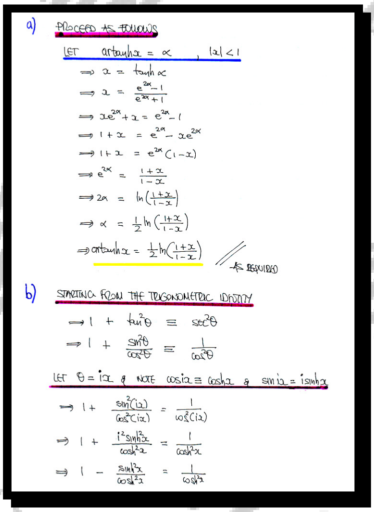
$$1-\tanh^2 x \equiv \operatorname{sech}^2 x .$$

c) Hence solve the equation

$$6\operatorname{sech}^2 x - \tanh x = 4 ,$$

giving the two solutions in the form $\pm\frac{1}{2}\ln k$, where k are two distinct integers.

$\boxed{\text{FP3}}$, $\boxed{x = \frac{1}{2}\ln 3}$, $\boxed{x = -\frac{1}{2}\ln 5}$



a) PROCEED AS FOLLOWS

LET $\operatorname{artanh} x = \alpha$, $|x| < 1$

$\Rightarrow x = \tanh \alpha$

$\Rightarrow x = \frac{e^{2\alpha}-1}{e^{2\alpha}+1}$

$\Rightarrow xe^{2\alpha} + x = e^{2\alpha} - 1$

$\Rightarrow 1 + x = e^{2\alpha} - xe^{2\alpha}$

$\Rightarrow 1 + x = e^{2\alpha}(1-x)$

$\Rightarrow e^{2\alpha} = \frac{1+x}{1-x}$

$\Rightarrow 2\alpha = \ln\left(\frac{1+x}{1-x}\right)$

$\Rightarrow \alpha = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$\Rightarrow \operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ // AS REQUIRED

b) STARTING FROM THE TRIGONOMETRIC IDENTITY

$\Rightarrow 1 + \tan^2\theta \equiv \sec^2\theta$

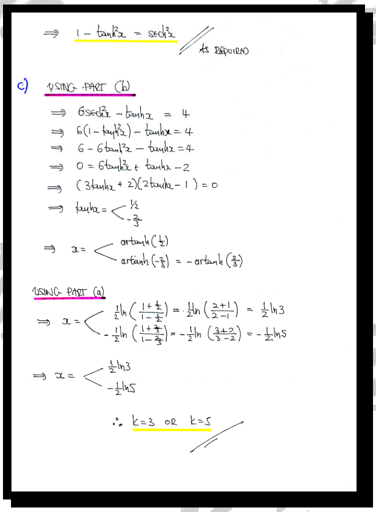
$\Rightarrow 1 + \frac{\sin^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$

LET $\theta = ix$ & NOTE $\cos ix \equiv \cosh x$ & $\sin ix = i\sinh x$

$\Rightarrow 1 + \frac{\sin^2(ix)}{\cos^2(ix)} = \frac{1}{\cos^2(ix)}$

$\Rightarrow 1 + \frac{i^2\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$

$\Rightarrow 1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$



$\Rightarrow 1 - \tanh^2 x = \operatorname{sech}^2 x$ // AS REQUIRED

c) USING PART (b)

$\Rightarrow 6\operatorname{sech}^2 x - \tanh x = 4$

$\Rightarrow 6(1-\tanh^2 x) - \tanh x = 4$

$\Rightarrow 6 - 6\tanh^2 x - \tanh x = 4$

$\Rightarrow 0 = 6\tanh^2 x + \tanh x - 2$

$\Rightarrow (3\tanh x + 2)(2\tanh x - 1) = 0$

$\Rightarrow \tanh x = \begin{cases} \frac{1}{2} \\ -\frac{2}{3} \end{cases}$

$\Rightarrow x = \begin{cases} \operatorname{artanh}\left(\frac{1}{2}\right) \\ \operatorname{artanh}\left(-\frac{2}{3}\right) = -\operatorname{artanh}\left(\frac{2}{3}\right) \end{cases}$

USING PART (a)

$\Rightarrow x = \begin{cases} \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \frac{1}{2}\ln\left(\frac{2+1}{2-1}\right) = \frac{1}{2}\ln 3 \\ -\frac{1}{2}\ln\left(\frac{1+\frac{2}{3}}{1-\frac{2}{3}}\right) = -\frac{1}{2}\ln\left(\frac{3+2}{3-2}\right) = -\frac{1}{2}\ln 5 \end{cases}$

$\Rightarrow x = \begin{cases} \frac{1}{2}\ln 3 \\ -\frac{1}{2}\ln 5 \end{cases}$

$\therefore k = 3$ OR $k = 5$ //

Question 44 (***+)

a) Sketch a detailed graph of the curve with equation

$$y = \operatorname{artanh} x\,,$$

defined in the largest real domain.

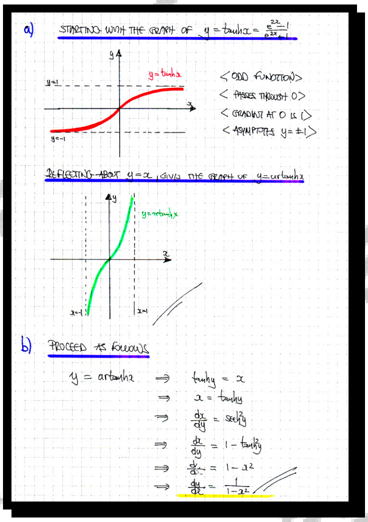
b) Obtain a simplified expression for $\frac{dy}{dx}$, in terms of x only.

c) Use integration and the answer of part **(b)** to show that

$$\operatorname{artanh} x = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right].$$

No credit will be given for any alternative methods used in part **(*c*)**.

$\boxed{}$, $\boxed{\frac{dy}{dx} = \frac{1}{1-x^2}}$



a) STARTING WITH THE GRAPH OF $y = \tanh x = \frac{e^{2x}-1}{e^{2x}+1}$

< ODD FUNCTION >

< PASSES THROUGH O >

< GRADIENT AT O IS 1 >

< ASYMPTOTES $y = \pm 1$ >

REFLECTING ABOUT $y = x$, GIVES THE GRAPH OF $y = \operatorname{artanh} x$

b) PROCEED AS FOLLOWS

$y = \operatorname{artanh} x \Rightarrow \tanh y = x$

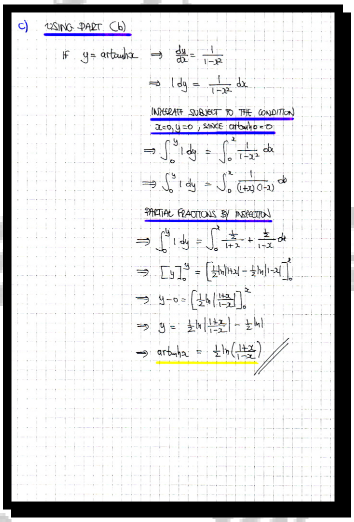
$\Rightarrow x = \tanh y$

$\Rightarrow \frac{dx}{dy} = \operatorname{sech}^2 y$

$\Rightarrow \frac{dx}{dy} = 1 - \tanh^2 y$

$\Rightarrow \frac{dx}{dy} = 1 - x^2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1-x^2}$



c) USING PART (b)

IF $y = \operatorname{artanh} x \Rightarrow \frac{dy}{dx} = \frac{1}{1-x^2}$

$\Rightarrow 1\,dy = \frac{1}{1-x^2}\,dx$

INTEGRATE SUBJECT TO THE CONDITION

$x=0, y=0$, SINCE $\operatorname{artanh} 0 = 0$

$\Rightarrow \int_0^y 1\,dy = \int_0^x \frac{1}{1-x^2}\,dx$

$\Rightarrow \int_0^y 1\,dy = \int_0^x \frac{1}{(1+x)(1-x)}\,dx$

PARTIAL FRACTIONS BY INSPECTION

$\Rightarrow \int_0^y 1\,dy = \int_0^x \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}\,dx$

$\Rightarrow \left[y\right]_0^y = \left[\frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x|\right]_0^x$

$\Rightarrow y - 0 = \left[\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right]_0^x$

$\Rightarrow y = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\ln|1|$

$\Rightarrow \operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Question 45 (***+)

a) Starting from the definitions of $\cosh x$ and $\sinh x$, in terms of exponentials, show that

$$\cos(\mathrm{i}\varphi) \equiv \cosh(\varphi) \quad \text{and} \quad \sin(\mathrm{i}\varphi) \equiv \mathrm{i}\sinh(\varphi).$$

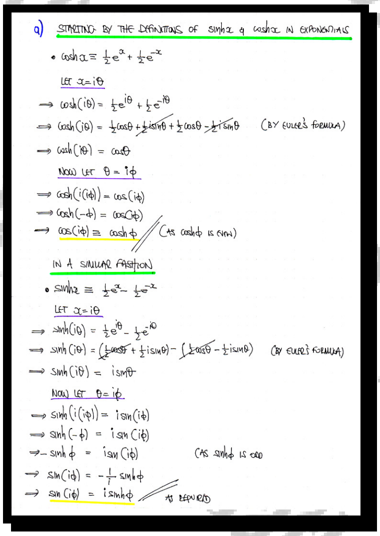
b) Use the results of part **(a)** to deduce

$$\operatorname{sech}^2\varphi + \tanh^2\varphi \equiv 1.$$

c) Hence find, in exact logarithmic form, the solutions of the following equation.

$$10\operatorname{sech} y = 5 + 3\tanh^2 y.$$

$\boxed{}$, $\boxed{y = \pm\ln\left(\frac{3+\sqrt{5}}{2}\right)}$



a) STARTING BY THE DEFINITIONS OF $\sinh x$ & $\cosh x$ IN EXPONENTIALS

- $\cosh x \equiv \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

LET $x = i\theta$

$\Longrightarrow \cosh(i\theta) = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}$

$\Longrightarrow \cosh(i\theta) = \frac{1}{2}\cos\theta + \cancel{\frac{1}{2}i\sin\theta} + \frac{1}{2}\cos\theta - \cancel{\frac{1}{2}i\sin\theta}$ (BY EULER'S FORMULA)

$\Longrightarrow \cosh(i\theta) = \cos\theta$

NOW LET $\theta = i\phi$

$\Longrightarrow \cosh(i(i\phi)) = \cos(i\phi)$

$\Longrightarrow \cosh(-\phi) = \cos(i\phi)$

$\Longrightarrow \cos(i\phi) \equiv \cosh\phi$ // (AS $\cosh\phi$ IS EVEN)

IN A SIMILAR FASHION

- $\sinh x \equiv \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$

LET $x = i\theta$

$\Longrightarrow \sinh(i\theta) = \frac{1}{2}e^{i\theta} - \frac{1}{2}e^{-i\theta}$

$\Longrightarrow \sinh(i\theta) = (\cancel{\frac{1}{2}\cos\theta} + \frac{1}{2}i\sin\theta) - (\cancel{\frac{1}{2}\cos\theta} - \frac{1}{2}i\sin\theta)$ (BY EULER'S FORMULA)

$\Longrightarrow \sinh(i\theta) = i\sin\theta$

NOW LET $\theta = i\phi$

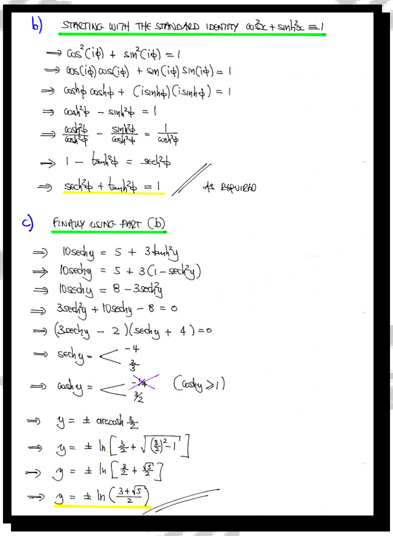
$\Longrightarrow \sinh(i(i\phi)) = i\sin(i\phi)$

$\Longrightarrow \sinh(-\phi) = i\sin(i\phi)$

$\Longrightarrow -\sinh\phi = i\sin(i\phi)$ (AS $\sinh\phi$ IS ODD

$\Longrightarrow \sin(i\phi) = -\frac{1}{i}\sinh\phi$

$\Longrightarrow \sin(i\phi) = i\sinh\phi$ // AS REQUIRED



b) STARTING WITH THE STANDARD IDENTITY $\cos^2 x + \sin^2 x \equiv 1$

$\Longrightarrow \cos^2(i\phi) + \sin^2(i\phi) = 1$

$\Longrightarrow \cos(i\phi)\cos(i\phi) + \sin(i\phi)\sin(i\phi) = 1$

$\Longrightarrow \cosh\phi\cosh\phi + (i\sinh\phi)(i\sinh\phi) = 1$

$\Longrightarrow \cosh^2\phi - \sinh^2\phi = 1$

$\Longrightarrow \frac{\cosh^2\phi}{\cosh^2\phi} - \frac{\sinh^2\phi}{\cosh^2\phi} = \frac{1}{\cosh^2\phi}$

$\Longrightarrow 1 - \tanh^2\phi = \operatorname{sech}^2\phi$

$\Longrightarrow \operatorname{sech}^2\phi + \tanh^2\phi = 1$ // AS REQUIRED

c) FINALLY USING PART (b)

$\Longrightarrow 10\operatorname{sech} y = 5 + 3\tanh^2 y$

$\Longrightarrow 10\operatorname{sech} y = 5 + 3(1 - \operatorname{sech}^2 y)$

$\Longrightarrow 10\operatorname{sech} y = 8 - 3\operatorname{sech}^2 y$

$\Longrightarrow 3\operatorname{sech}^2 y + 10\operatorname{sech} y - 8 = 0$

$\Longrightarrow (3\operatorname{sech} y - 2)(\operatorname{sech} y + 4) = 0$

$\Longrightarrow \operatorname{sech} y = \left\langle \begin{matrix} -4 \\ \frac{2}{3} \end{matrix} \right.$

$\Longrightarrow \cosh y = \left\langle \begin{matrix} \cancel{-\frac{1}{4}} \\ \frac{3}{2} \end{matrix} \right.$ $(\cosh y \geqslant 1)$

$\Longrightarrow y = \pm \operatorname{arccosh}\frac{3}{2}$

$\Longrightarrow y = \pm \ln\left[\frac{3}{2} + \sqrt{\left(\frac{3}{2}\right)^2 - 1}\right]$

$\Longrightarrow y = \pm \ln\left[\frac{3}{2} + \frac{\sqrt{5}}{2}\right]$

$\Longrightarrow y = \pm \ln\left(\frac{3+\sqrt{5}}{2}\right)$ //

Question 46 (***+)

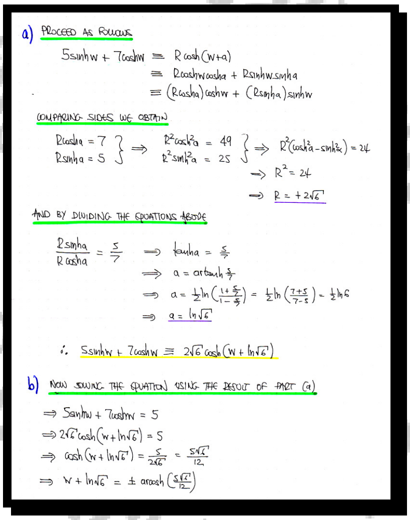
$$f(w) \equiv 5\sinh w + 7\cosh w, \ w \in \mathbb{R}$$

a) Express $f(w)$ in the form $R\cosh(w+a)$, where R and a are exact constants with $R > 0$.

b) Use the result of part **(a)** to find, in exact logarithmic form, the solutions of the following equation.

$$5\sinh w + 7\cosh w = 5.$$

$\boxed{}$, $\boxed{R=\sqrt{24}=2\sqrt{6}}$, $\boxed{a=\frac{1}{2}\ln 6=\ln\sqrt{6}}$, $\boxed{w=-\ln 2 \cup w=-\ln 3}$



a) PROCEED AS FOLLOWS

$5\sinh w + 7\cosh w \equiv R\cosh(w+a)$

$\equiv R\cosh w\cosh a + R\sinh w\sinh a$

$\equiv (R\cosh a)\cosh w + (R\sinh a)\sinh w$

COMPARING SIDES WE OBTAIN

$\left.\begin{array}{l} R\cosh a = 7 \\ R\sinh a = 5 \end{array}\right\} \Rightarrow \left.\begin{array}{l} R^2\cosh^2 a = 49 \\ R^2\sinh^2 a = 25 \end{array}\right\} \Rightarrow R^2(\cosh^2 a - \sinh^2 \alpha) = 24$

$\Rightarrow R^2 = 24$

$\Rightarrow R = +2\sqrt{6}$

AND BY DIVIDING THE EQUATIONS ABOVE

$\frac{R\sinh a}{R\cosh a} = \frac{5}{7} \Rightarrow \tanh a = \frac{5}{7}$

$\Rightarrow a = \text{artanh}\frac{5}{7}$

$\Rightarrow a = \frac{1}{2}\ln\left(\frac{1+\frac{5}{7}}{1-\frac{5}{7}}\right) = \frac{1}{2}\ln\left(\frac{7+5}{7-5}\right) = \frac{1}{2}\ln 6$

$\Rightarrow a = \ln\sqrt{6}$

$\therefore 5\sinh w + 7\cosh w \equiv 2\sqrt{6}\cosh(w + \ln\sqrt{6})$

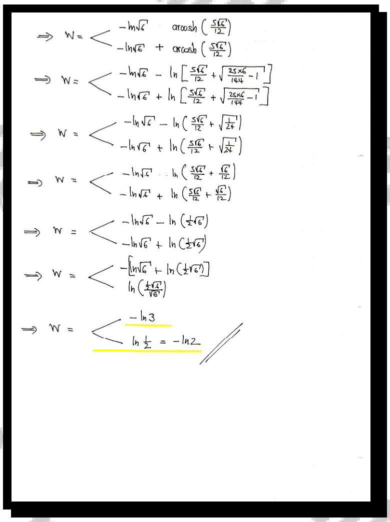
b) NOW SOLVING THE EQUATION USING THE RESULT OF PART (a)

$\Rightarrow 5\sinh w + 7\cosh w = 5$

$\Rightarrow 2\sqrt{6}\cosh(w + \ln\sqrt{6}) = 5$

$\Rightarrow \cosh(w + \ln\sqrt{6}) = \frac{5}{2\sqrt{6}} = \frac{5\sqrt{6}}{12}$

$\Rightarrow w + \ln\sqrt{6} = \pm\,\text{arcosh}\left(\frac{5\sqrt{6}}{12}\right)$



$\Rightarrow w = \left\langle \begin{array}{l} -\ln\sqrt{6} \quad \text{arcosh}\left(\frac{5\sqrt{6}}{12}\right) \\ -\ln\sqrt{6} + \text{arcosh}\left(\frac{5\sqrt{6}}{12}\right) \end{array}\right.$

$\Rightarrow w = \left\langle \begin{array}{l} -\ln\sqrt{6} - \ln\left[\frac{5\sqrt{6}}{12} + \sqrt{\frac{25\times 6}{144} - 1}\right] \\ -\ln\sqrt{6} + \ln\left[\frac{5\sqrt{6}}{12} + \sqrt{\frac{25\times 6}{144} - 1}\right] \end{array}\right.$

$\Rightarrow w = \left\langle \begin{array}{l} -\ln\sqrt{6} - \ln\left(\frac{5\sqrt{6}}{12} + \sqrt{\frac{1}{24}}\right) \\ -\ln\sqrt{6} + \ln\left(\frac{5\sqrt{6}}{12} + \sqrt{\frac{1}{24}}\right) \end{array}\right.$

$\Rightarrow w = \left\langle \begin{array}{l} -\ln\sqrt{6} \ldots \ln\left(\frac{5\sqrt{6}}{12} + \frac{\sqrt{6}}{12}\right) \\ -\ln\sqrt{6} + \ln\left(\frac{5\sqrt{6}}{12} + \frac{\sqrt{6}}{12}\right) \end{array}\right.$

$\Rightarrow w = \left\langle \begin{array}{l} -\ln\sqrt{6} - \ln\left(\frac{1}{2}\sqrt{6}\right) \\ -\ln\sqrt{6} + \ln\left(\frac{1}{2}\sqrt{6}\right) \end{array}\right.$

$\Rightarrow w = \left\langle \begin{array}{l} -\left[\ln\sqrt{6} + \ln\left(\frac{1}{2}\sqrt{6}\right)\right] \\ \ln\left(\frac{\frac{1}{2}\sqrt{6}}{\sqrt{6}}\right) \end{array}\right.$

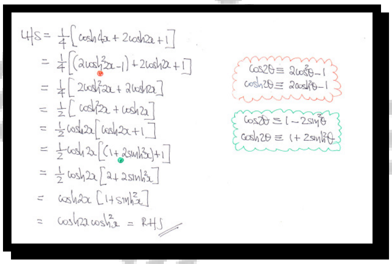
$\Rightarrow w = \left\langle \begin{array}{l} -\ln 3 \\ \ln\frac{1}{2} = -\ln 2 \end{array}\right.$ //

Question 47 **(***+)**

By using suitable hyperbolic identities, or otherwise, show that

$$\tfrac{1}{4}\left[\cosh 4x + 2\cosh 2x + 1\right] \equiv \cosh 2x \cosh^2 x .$$

$\boxed{\text{proof}}$



$$\begin{aligned} LHS &= \tfrac{1}{4}\left[\cosh 4x + 2\cosh 2x + 1\right] \\ &= \tfrac{1}{4}\left[(2\cosh^2 2x - 1) + 2\cosh 2x + 1\right] \\ &= \tfrac{1}{4}\left[2\cosh^2 2x + 2\cosh 2x\right] \\ &= \tfrac{1}{2}\left[\cosh^2 2x + \cosh 2x\right] \\ &= \tfrac{1}{2}\cosh 2x\left[\cosh 2x + 1\right] \\ &= \tfrac{1}{2}\cosh 2x\left[(1 + 2\sinh^2 x) + 1\right] \\ &= \tfrac{1}{2}\cosh 2x\left[2 + 2\sinh^2 x\right] \\ &= \cosh 2x\left[1 + \sinh^2 x\right] \\ &= \cosh 2x \cosh^2 x = RHS \end{aligned}$$

$\cos 2\theta \equiv 2\cos^2\theta - 1$

$\cosh 2\theta \equiv 2\cosh^2\theta - 1$

$\cos 2\theta \equiv 1 - 2\sin^2\theta$

$\cosh 2\theta \equiv 1 + 2\sinh^2\theta$

Question 48 **(****)**

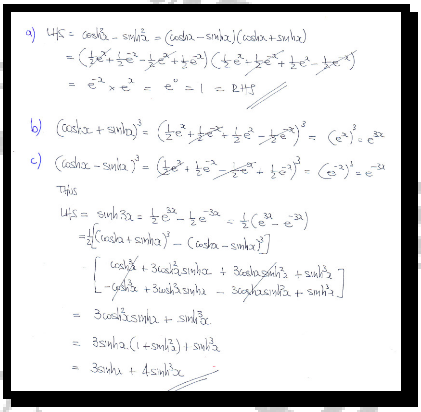
a) By expressing $\cosh x$ and $\sinh x$ in terms of exponentials, show that

$$\cosh^2 x - \sinh^2 x \equiv 1 .$$

b) Simplify $(\cosh x + \sinh x)^3$, writing the final answer as a single exponential.

c) Hence express $\sinh 3x$ in terms of $\sinh x$

$\boxed{(\cosh x + \sinh x)^3 = e^{3x}}$, $\boxed{\sinh 3x = 3\sinh x + 4\sinh^4 x}$



a) $LHS = \cosh^2 x - \sinh^2 x = (\cosh x - \sinh x)(\cosh x + \sinh x)$

$= \left(\cancel{\tfrac{1}{2}e^{x}} + \tfrac{1}{2}e^{-x} - \cancel{\tfrac{1}{2}e^{x}} + \tfrac{1}{2}e^{-x}\right)\left(\tfrac{1}{2}e^{x} + \cancel{\tfrac{1}{2}e^{-x}} + \tfrac{1}{2}e^{x} - \cancel{\tfrac{1}{2}e^{-x}}\right)$

$= e^{-x} \times e^{x} = e^{0} = 1 = RHS$

b) $(\cosh x + \sinh x)^3 = \left(\tfrac{1}{2}e^{x} + \cancel{\tfrac{1}{2}e^{-x}} + \tfrac{1}{2}e^{x} - \cancel{\tfrac{1}{2}e^{-x}}\right)^3 = (e^{x})^3 = e^{3x}$

c) $(\cosh x - \sinh x)^3 = \left(\cancel{\tfrac{1}{2}e^{x}} + \tfrac{1}{2}e^{-x} - \cancel{\tfrac{1}{2}e^{x}} + \tfrac{1}{2}e^{-x}\right)^3 = (e^{-x})^3 = e^{-3x}$

Thus

$LHS = \sinh 3x = \tfrac{1}{2}e^{3x} - \tfrac{1}{2}e^{-3x} = \tfrac{1}{2}(e^{3x} - e^{-3x})$

$= \tfrac{1}{2}\left[(\cosh x + \sinh x)^3 - (\cosh x - \sinh x)^3\right]$

$$\left[\begin{array}{l} \cancel{\cosh^3 x} + 3\cosh^2 x\sinh x + \cancel{3\cosh x\sinh^2 x} + \sinh^3 x \\ -\cancel{\cosh^3 x} + 3\cosh^2 x\sinh x - \cancel{3\cosh x\sinh^2 x} + \sinh^3 x \end{array}\right]$$

$= 3\cosh^2 x\sinh x + \sinh^3 x$

$= 3\sinh x(1 + \sinh^2 x) + \sinh^3 x$

$= 3\sinh x + 4\sinh^3 x$

Question 49 (****)

The curve C has equation

$$y = \cosh\left(2\operatorname{arsinh} x\right),\ x \in \mathbb{R}.$$

a) Find an expression for $\dfrac{dy}{dx}$.

b) Show clearly that

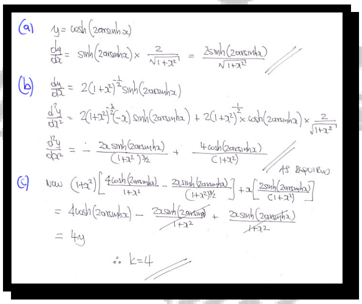
$$\frac{d^2y}{dx^2} = \frac{4}{1+x^2}\cosh\left(2\operatorname{arsinh} x\right) - \frac{2x}{\left(1+x^2\right)^{\frac{3}{2}}}\sinh\left(2\operatorname{arsinh} x\right)$$

c) Hence show further that

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - ky = 0,$$

for some value of the constant k.

$$\boxed{\frac{dy}{dx} = \frac{2\sinh\left(2\operatorname{arsinh} x\right)}{\sqrt{1+x^2}}},\ \boxed{k=4}$$



(a) $y = \cosh(2\operatorname{arsinh} x)$

$\frac{dy}{dx} = \sinh(2\operatorname{arsinh} x) \times \frac{2}{\sqrt{1+x^2}} = \frac{2\sinh(2\operatorname{arsinh} x)}{\sqrt{1+x^2}}$ //

(b) $\frac{dy}{dx} = 2(1+x^2)^{-\frac{1}{2}}\sinh(2\operatorname{arsinh} x)$

$\frac{d^2y}{dx^2} = 2(1+x^2)^{-\frac{3}{2}}(-x)\sinh(2\operatorname{arsinh} x) + 2(1+x^2)^{-\frac{1}{2}} \times \cosh(2\operatorname{arsinh} x) \times \frac{2}{\sqrt{1+x^2}}$

$\frac{d^2y}{dx^2} = -\frac{2x\sinh(2\operatorname{arsinh} x)}{(1+x^2)^{3/2}} + \frac{4\cosh(2\operatorname{arsinh} x)}{(1+x^2)}$ // AS REQUIRED

(c) Now $(1+x^2)\left[\frac{4\cosh(2\operatorname{arsinh} x)}{1+x^2} - \frac{2x\sinh(2\operatorname{arsinh} x)}{(1+x^2)^{3/2}}\right] + x\left[\frac{2\sinh(2\operatorname{arsinh} x)}{(1+x^2)}\right]$

$= 4\cosh(2\operatorname{arsinh} x) - \frac{2x\sinh(2\operatorname{arsinh} x)}{\cancel{1+x^2}} + \frac{2x\sinh(2\operatorname{arsinh} x)}{\cancel{1+x^2}}$

$= 4y$

$\therefore k=4$ //

Question 50 (****)

A function is defined in terms of exponentials by

$$f(x) = \frac{2}{e^{x} + e^{-x}}, \quad x \in \mathbb{R}.$$

a) Sketch the graph of $f(x)$.

b) Show clearly that

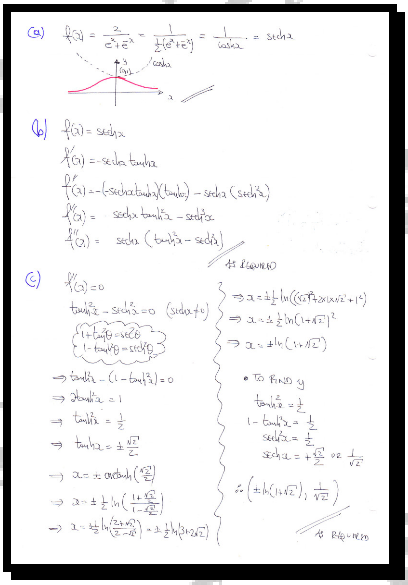
$$f''(x) = \operatorname{sech} x \left(\tanh^2 x - \operatorname{sech}^2 x\right).$$

It is given that the graph of $f(x)$ has two points of inflection.

c) Show further that the coordinates of these points are

$$\left(\pm \ln\left(1+\sqrt{2}\right), \frac{1}{\sqrt{2}}\right).$$

$\boxed{\text{proof}}$



(a) $f(x) = \frac{2}{e^{x}+e^{-x}} = \frac{1}{\frac{1}{2}(e^{x}+e^{-x})} = \frac{1}{\cosh x} = \operatorname{sech} x$

(b) $f(x) = \operatorname{sech} x$

$f'(x) = -\operatorname{sech} x \tanh x$

$f''(x) = -(-\operatorname{sech} x \tanh x)(\tanh x) - \operatorname{sech} x (\operatorname{sech}^2 x)$

$f''(x) = \operatorname{sech} x \tanh^2 x - \operatorname{sech}^3 x$

$f''(x) = \operatorname{sech} x (\tanh^2 x - \operatorname{sech}^2 x)$

AS REQUIRED

(c) $f''(x) = 0$

$\tanh^2 x - \operatorname{sech}^2 x = 0 \quad (\operatorname{sech} x \neq 0)$

$1 + \tan^2\theta = \sec^2\theta$
$1 - \tanh^2\theta = \operatorname{sech}^2\theta$

$\Rightarrow \tanh^2 x - (1 - \tanh^2 x) = 0$

$\Rightarrow 2\tanh^2 x = 1$

$\Rightarrow \tanh^2 x = \frac{1}{2}$

$\Rightarrow \tanh x = \pm \frac{\sqrt{2}}{2}$

$\Rightarrow x = \pm \operatorname{artanh}\left(\frac{\sqrt{2}}{2}\right)$

$\Rightarrow x = \pm \frac{1}{2} \ln\left(\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right)$

$\Rightarrow x = \pm\frac{1}{2} \ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) = \pm \frac{1}{2}\ln(3+2\sqrt{2})$

$\Rightarrow x = \pm\frac{1}{2}\ln\left((\sqrt{2})^2 + 2\times1\times\sqrt{2} + 1^2\right)$

$\Rightarrow x = \pm\frac{1}{2}\ln(1+\sqrt{2})^2$

$\Rightarrow x = \pm\ln(1+\sqrt{2})$

• TO FIND y

$\tanh^2 x = \frac{1}{2}$

$1 - \tanh^2 x = \frac{1}{2}$

$\operatorname{sech}^2 x = \frac{1}{2}$

$\operatorname{sech} x = +\frac{\sqrt{2}}{2}$ OR $\frac{1}{\sqrt{2}}$

$\therefore \left(\pm\ln(1+\sqrt{2}), \frac{1}{\sqrt{2}}\right)$

AS REQUIRED

Question 51 (****)

It is given that

$$\cosh(A+B) \equiv \cosh A \cosh B + \sinh A \sinh B .$$

a) Prove the validity of the above hyperbolic identity by using the definitions of the hyperbolic functions in terms of exponential functions.

It is now given that

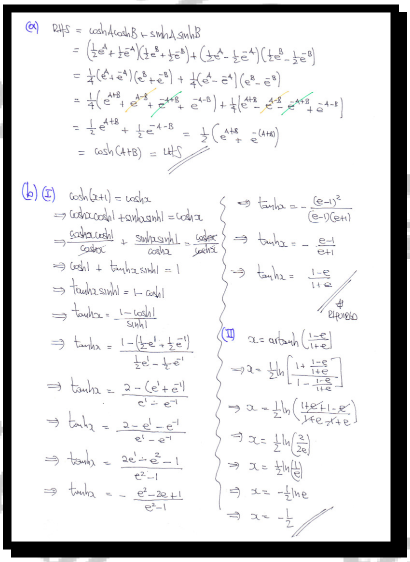
$$\cosh(x+1) = \cosh x ,$$

b) Show clearly that …

i. $\ldots \tanh x = \dfrac{1-\mathrm{e}}{1+\mathrm{e}}$.

ii. $\ldots x = -\dfrac{1}{2}$.

$\boxed{\text{proof}}$



(a) RHS $= \cosh A \cosh B + \sinh A \sinh B$

$$= \left(\tfrac{1}{2}e^{A}+\tfrac{1}{2}e^{-A}\right)\left(\tfrac{1}{2}e^{B}+\tfrac{1}{2}e^{-B}\right)+\left(\tfrac{1}{2}e^{A}-\tfrac{1}{2}e^{-A}\right)\left(\tfrac{1}{2}e^{B}-\tfrac{1}{2}e^{-B}\right)$$

$$= \tfrac{1}{4}\left(e^{A}+e^{-A}\right)\left(e^{B}+e^{-B}\right)+\tfrac{1}{4}\left(e^{A}-e^{-A}\right)\left(e^{B}-e^{-B}\right)$$

$$= \tfrac{1}{4}\left(e^{A+B}+\cancel{e^{A-B}}+\cancel{e^{-A+B}}+e^{-A-B}\right)+\tfrac{1}{4}\left(e^{A+B}-\cancel{e^{A-B}}-\cancel{e^{-A+B}}+e^{-A-B}\right)$$

$$= \tfrac{1}{2}e^{A+B}+\tfrac{1}{2}e^{-A-B} = \tfrac{1}{2}\left(e^{A+B}+e^{-(A+B)}\right)$$

$$= \cosh(A+B) = \text{LHS}$$

(b) (I) $\cosh(x+1) = \cosh x$

$\Rightarrow \cosh x \cosh 1 + \sinh x \sinh 1 = \cosh x$

$\Rightarrow \dfrac{\cosh x \cosh 1}{\cancel{\cosh x}} + \dfrac{\sinh x \sinh 1}{\cosh x} = \dfrac{\cancel{\cosh x}}{\cancel{\cosh x}}$

$\Rightarrow \cosh 1 + \tanh x \sinh 1 = 1$

$\Rightarrow \tanh x \sinh 1 = 1 - \cosh 1$

$\Rightarrow \tanh x = \dfrac{1-\cosh 1}{\sinh 1}$

$\Rightarrow \tanh x = \dfrac{1-\left(\frac{1}{2}e^{1}+\frac{1}{2}e^{-1}\right)}{\frac{1}{2}e^{1}-\frac{1}{2}e^{-1}}$

$\Rightarrow \tanh x = \dfrac{2-\left(e^{1}+e^{-1}\right)}{e^{1}-e^{-1}}$

$\Rightarrow \tanh x = \dfrac{2-e^{1}-e^{-1}}{e^{1}-e^{-1}}$

$\Rightarrow \tanh x = \dfrac{2e^{1}-e^{2}-1}{e^{2}-1}$

$\Rightarrow \tanh x = -\dfrac{e^{2}-2e+1}{e^{2}-1}$

$\Rightarrow \tanh x = -\dfrac{(e-1)^{2}}{(e-1)(e+1)}$

$\Rightarrow \tanh x = -\dfrac{e-1}{e+1}$

$\Rightarrow \tanh x = \dfrac{1-e}{1+e}$

AS REQUIRED

(II) $x = \operatorname{artanh}\left(\dfrac{1-e}{1+e}\right)$

$\Rightarrow x = \dfrac{1}{2}\ln\left[\dfrac{1+\frac{1-e}{1+e}}{1-\frac{1-e}{1+e}}\right]$

$\Rightarrow x = \dfrac{1}{2}\ln\left(\dfrac{1+\cancel{e}+1-\cancel{e}}{\cancel{1}+e-\cancel{1}+e}\right)$

$\Rightarrow x = \dfrac{1}{2}\ln\left(\dfrac{2}{2e}\right)$

$\Rightarrow x = \dfrac{1}{2}\ln\left(\dfrac{1}{e}\right)$

$\Rightarrow x = -\dfrac{1}{2}\ln e$

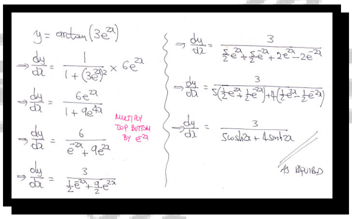
$\Rightarrow x = -\dfrac{1}{2}$

Question 52 **(****)**

Given that $y = \arctan\left(3e^{2x}\right)$, show clearly that

$$\frac{dy}{dx} = \frac{3}{5\cosh 2x + 4\sinh 2x}$$

$\boxed{\text{proof}}$



$y = \arctan\left(3e^{2x}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+\left(3e^{2x}\right)^2} \times 6e^{2x}$

$\Rightarrow \frac{dy}{dx} = \frac{6e^{2x}}{1+9e^{4x}}$

MULTIPLY TOP BOTTOM BY e^{-2x}

$\Rightarrow \frac{dy}{dx} = \frac{6}{e^{-2x}+9e^{2x}}$

$\Rightarrow \frac{dy}{dx} = \frac{3}{\frac{1}{2}e^{-2x}+\frac{9}{2}e^{2x}}$

$\Rightarrow \frac{dy}{dx} = \frac{3}{\frac{5}{2}e^{2x}+\frac{5}{2}e^{-2x}+2e^{2x}-2e^{-2x}}$

$\Rightarrow \frac{dy}{dx} = \frac{3}{5\left(\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}\right)+4\left(\frac{1}{2}e^{2x}-\frac{1}{2}e^{-2x}\right)}$

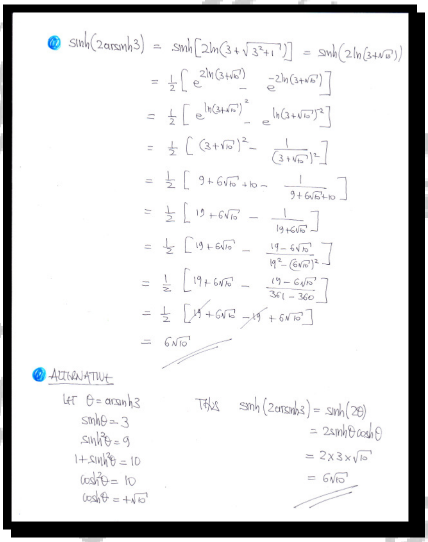
$\Rightarrow \frac{dy}{dx} = \frac{3}{5\cosh 2x + 4\sinh 2x}$

AS REQUIRED

Question 53 **(****)**

Find in exact simplified form the value of $\sinh\left(2\,\text{arsinh}\,3\right)$.

$\boxed{6\sqrt{10}}$



$\sinh(2\,\text{arsinh}\,3) = \sinh\left[2\ln\left(3+\sqrt{3^2+1}\right)\right] = \sinh\left(2\ln\left(3+\sqrt{10}\right)\right)$

$= \frac{1}{2}\left[e^{2\ln(3+\sqrt{10})} - e^{-2\ln(3+\sqrt{10})}\right]$

$= \frac{1}{2}\left[e^{\ln(3+\sqrt{10})^2} - e^{\ln(3+\sqrt{10})^{-2}}\right]$

$= \frac{1}{2}\left[(3+\sqrt{10})^2 - \frac{1}{(3+\sqrt{10})^2}\right]$

$= \frac{1}{2}\left[9+6\sqrt{10}+10 - \frac{1}{9+6\sqrt{10}+10}\right]$

$= \frac{1}{2}\left[19+6\sqrt{10} - \frac{1}{19+6\sqrt{10}}\right]$

$= \frac{1}{2}\left[19+6\sqrt{10} - \frac{19-6\sqrt{10}}{19^2-(6\sqrt{10})^2}\right]$

$= \frac{1}{2}\left[19+6\sqrt{10} - \frac{19-6\sqrt{10}}{361-360}\right]$

$= \frac{1}{2}\left[\cancel{19}+6\sqrt{10} \cancel{-19}+6\sqrt{10}\right]$

$= 6\sqrt{10}$

ALTERNATIVE

LET $\theta = \text{arsinh}\,3$

$\sinh\theta = 3$

$\sinh^2\theta = 9$

$1+\sinh^2\theta = 10$

$\cosh^2\theta = 10$

$\cosh\theta = +\sqrt{10}$

THUS $\sinh(2\,\text{arsinh}\,3) = \sinh(2\theta)$

$= 2\sinh\theta\cosh\theta$

$= 2\times 3\times\sqrt{10}$

$= 6\sqrt{10}$

Question 54 (****)

$$\cosh 2x \equiv 2\cosh^2 x - 1$$

a) Prove the validity of the above identity by using the definitions of $\cosh x$ and $\sinh x$, in terms of exponentials.

The curve C has equation

$$y = \cosh x - 1,\ x \in \mathbb{R}.$$

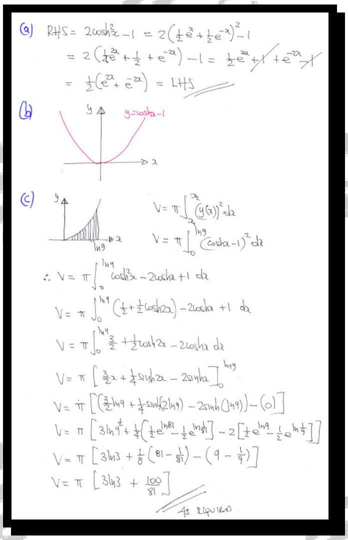
b) Sketch the graph of C.

The region bounded by C, the x axis and the line with equation $x = \ln 9$ is rotated through 2π radians about the x axis to form a volume of revolution S.

c) Show that the volume S is

$$\pi\left(3\ln 3 + \frac{100}{81}\right).$$

proof



(a) $\text{RHS} = 2\cosh^2 x - 1 = 2\left(\tfrac{1}{2}e^{x} + \tfrac{1}{2}e^{-x}\right)^2 - 1$

$= 2\left(\tfrac{1}{4}e^{2x} + \tfrac{1}{2} + e^{-2x}\right) - 1 = \tfrac{1}{2}e^{2x} + \cancel{1} + e^{-2x} \cancel{-1}$

$= \tfrac{1}{2}\left(e^{2x} + e^{-2x}\right) = \text{LHS}$

(b)

(c)

$V = \pi\int_{x_1}^{x_2} (y(x))^2\,dx$

$V = \pi\int_0^{\ln 9} (\cosh x - 1)^2\,dx$

$\therefore V = \pi\int_0^{\ln 9} \cosh^2 x - 2\cosh x + 1\,dx$

$V = \pi\int_0^{\ln 9} \left(\tfrac{1}{2} + \tfrac{1}{2}\cosh 2x\right) - 2\cosh x + 1\,dx$

$V = \pi\int_0^{\ln 9} \tfrac{3}{2} + \tfrac{1}{2}\cosh 2x - 2\cosh x\,dx$

$V = \pi\left[\tfrac{3}{2}x + \tfrac{1}{4}\sinh 2x - 2\sinh x\right]_0^{\ln 9}$

$V = \pi\left[\left(\tfrac{3}{2}\ln 9 + \tfrac{1}{4}\sinh(2\ln 9) - 2\sinh(\ln 9)\right) - (0)\right]$

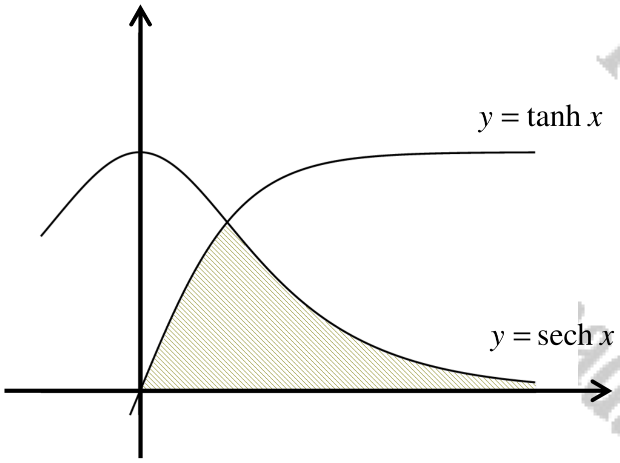
$V = \pi\left[3\ln 9^{\frac{1}{2}} + \tfrac{1}{4}\left(\tfrac{1}{2}e^{\ln 81} - \tfrac{1}{2}e^{\ln\frac{1}{81}}\right] - 2\left[\tfrac{1}{2}e^{\ln 9} - \tfrac{1}{2}e^{\ln\frac{1}{9}}\right]\right]$

$V = \pi\left[3\ln 3 + \tfrac{1}{8}\left(81 - \tfrac{1}{81}\right) - \left(9 - \tfrac{1}{9}\right)\right]$

$V = \pi\left[3\ln 3 + \tfrac{100}{81}\right]$

As required

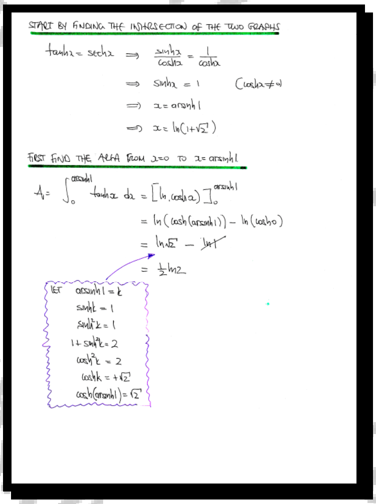
Question 55 (****)



The figure above shows the graphs of $y = \tanh x$ and $y = \text{sech}\, x$, in the first quadrant.

Show that the area shown shaded in the figure for which $x \geq 0$ is exactly $\frac{1}{4}[\pi + \ln 4]$.

☐, proof



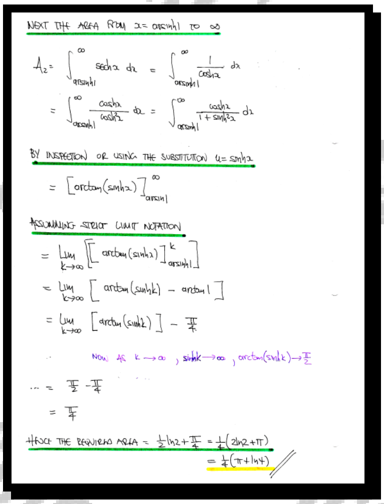
START BY FINDING THE INTERSECTION OF THE TWO GRAPHS

$$\tanh x = \text{sech}\, x \implies \frac{\sinh x}{\cosh x} = \frac{1}{\cosh x}$$
$$\implies \sinh x = 1 \qquad (\cosh x \neq 0)$$
$$\implies x = \text{arsinh}\, 1$$
$$\implies x = \ln(1+\sqrt{2})$$

FIRST FIND THE AREA FROM $x=0$ TO $x = \text{arsinh}\, 1$

$$A_1 = \int_0^{\text{arsinh}1} \tanh x \, dx = \Big[\ln(\cosh x)\Big]_0^{\text{arsinh}1}$$
$$= \ln(\cosh(\text{arsinh}1)) - \ln(\cosh 0)$$
$$= \ln\sqrt{2} - \cancel{\ln 1}$$
$$= \tfrac{1}{2}\ln 2$$

LET $\text{arsinh}\, 1 = k$
$\sinh k = 1$
$\sinh^2 k = 1$
$1 + \sinh^2 k = 2$
$\cosh^2 k = 2$
$\cosh k = +\sqrt{2}$
$\cosh(\text{arsinh}1) = \sqrt{2}$



NEXT THE AREA FROM $x = \text{arsinh}1$ TO ∞

$$A_2 = \int_{\text{arsinh}1}^{\infty} \text{sech}\, x \, dx = \int_{\text{arsinh}1}^{\infty} \frac{1}{\cosh x} \, dx$$
$$= \int_{\text{arsinh}1}^{\infty} \frac{\cosh x}{\cosh^2 x} \, dx = \int_{\text{arsinh}1}^{\infty} \frac{\cosh x}{1+\sinh^2 x} \, dx$$

BY INSPECTION OR USING THE SUBSTITUTION $u = \sinh x$

$$= \Big[\arctan(\sinh x)\Big]_{\text{arsin}1}^{\infty}$$

ASSUMING STRICT LIMIT NOTATION

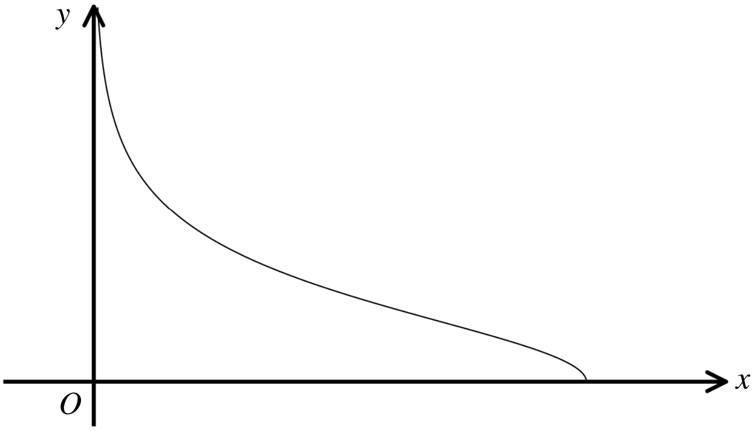
$$= \lim_{k\to\infty}\left[\Big[\arctan(\sinh x)\Big]_{\text{arsinh}1}^{k}\right]$$
$$= \lim_{k\to\infty}\Big[\arctan(\sinh k) - \arctan 1\Big]$$
$$= \lim_{k\to\infty}\Big[\arctan(\sinh k)\Big] - \frac{\pi}{4}$$

NOW AS $k \to \infty$, $\sinh k \to \infty$, $\arctan(\sinh k) \to \frac{\pi}{2}$

$$\dots = \frac{\pi}{2} - \frac{\pi}{4}$$
$$= \frac{\pi}{4}$$

HENCE THE REQUIRED AREA $= \frac{1}{2}\ln 2 + \frac{\pi}{4} = \frac{1}{4}(2\ln 2 + \pi)$
$= \frac{1}{4}(\pi + \ln 4)$ //

Question 56 (****)



The figure above shows the graph of $y = \text{arsech}\, x$, $0 < x \leq 1$.

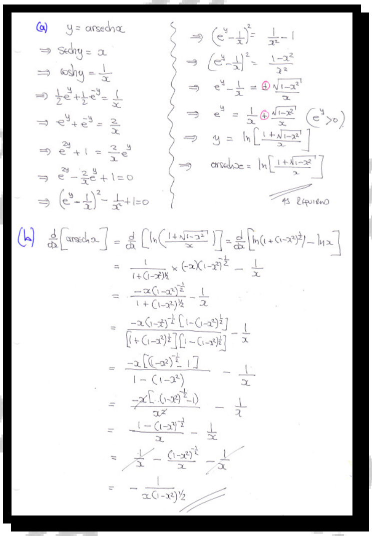
a) Show clearly that

$$\text{arsech}\, x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right).$$

b) Show further that

$$\frac{d}{dx}(\text{arsech}\, x) = -\frac{1}{x\sqrt{1-x^2}}.$$

proof



(a) $y = \text{arsech}\, x$

$\Rightarrow \text{sech}\, y = x$

$\Rightarrow \cosh y = \frac{1}{x}$

$\Rightarrow \frac{1}{2}e^{y} + \frac{1}{2}e^{-y} = \frac{1}{x}$

$\Rightarrow e^{y} + e^{-y} = \frac{2}{x}$

$\Rightarrow e^{2y} + 1 = \frac{2}{x}e^{y}$

$\Rightarrow e^{2y} - \frac{2}{x}e^{y} + 1 = 0$

$\Rightarrow \left(e^{y} - \frac{1}{x}\right)^2 - \frac{1}{x^2} + 1 = 0$

$\Rightarrow \left(e^{y} - \frac{1}{x}\right)^2 = \frac{1}{x^2} - 1$

$\Rightarrow \left(e^{y} - \frac{1}{x}\right)^2 = \frac{1-x^2}{x^2}$

$\Rightarrow e^{y} - \frac{1}{x} = \textcircled{+} \frac{\sqrt{1-x^2}}{x}$

$\Rightarrow e^{y} = \frac{1}{x} \textcircled{+} \frac{\sqrt{1-x^2}}{x} \quad (e^{y} > 0)$

$\Rightarrow y = \ln\left[\frac{1+\sqrt{1-x^2}}{x}\right]$

$\Rightarrow \text{arsech}\, x = \ln\left[\frac{1+\sqrt{1-x^2}}{x}\right]$

AS REQUIRED

(b) $\frac{d}{dx}\left[\text{arsech}\, x\right] = \frac{d}{dx}\left[\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)\right] = \frac{d}{dx}\left[\ln(1+(1-x^2)^{\frac{1}{2}}) - \ln x\right]$

$= \frac{1}{1+(1-x^2)^{1/2}} \times (-x)(1-x^2)^{-\frac{1}{2}} - \frac{1}{x}$

$= \frac{-x(1-x^2)^{-\frac{1}{2}}}{1+(1-x^2)^{1/2}} - \frac{1}{x}$

$= \frac{-x(1-x^2)^{-\frac{1}{2}}\left[1-(1-x^2)^{\frac{1}{2}}\right]}{\left[1+(1-x^2)^{\frac{1}{2}}\right]\left[1-(1-x^2)^{\frac{1}{2}}\right]} - \frac{1}{x}$

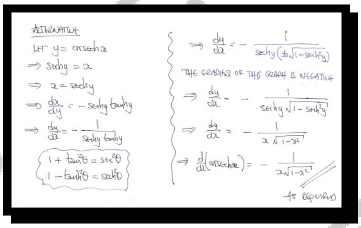
$= \frac{-x\left[(1-x^2)^{-\frac{1}{2}} - 1\right]}{1-(1-x^2)} - \frac{1}{x}$

$= \frac{-\cancel{x}\left[(1-x^2)^{-\frac{1}{2}} - 1\right)}{x^{\cancel{2}}} - \frac{1}{x}$

$= \frac{1-(1-x^2)^{-\frac{1}{2}}}{x} - \frac{1}{x}$

$= \cancel{\frac{1}{x}} - \frac{(1-x^2)^{-\frac{1}{2}}}{x} - \cancel{\frac{1}{x}}$

$= -\frac{1}{x(1-x^2)^{1/2}}$



ALTERNATIVE

LET $y = \text{arsech}\, x$

$\Rightarrow \text{sech}\, y = x$

$\Rightarrow x = \text{sech}\, y$

$\Rightarrow \frac{dx}{dy} = -\text{sech}\, y \tanh y$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\text{sech}\, y \tanh y}$

$1 + \tan^2\theta = \sec^2\theta$
$1 - \tanh^2\theta = \text{sech}^2\theta$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\text{sech}\, y\left(\pm\sqrt{1-\text{sech}^2 y}\right)}$

THE GRADIENT OF THIS GRAPH IS NEGATIVE

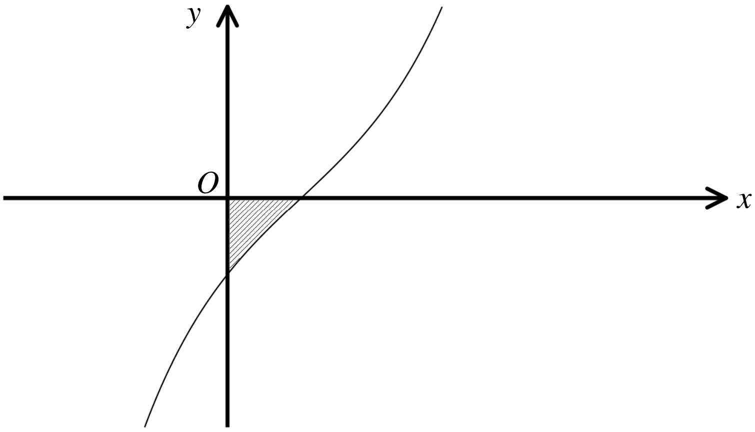
$\Rightarrow \frac{dy}{dx} = -\frac{1}{\text{sech}\, y\sqrt{1-\text{sech}^2 y}}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}}$

$\Rightarrow \frac{d}{dx}(\text{arsech}\, x) = -\frac{1}{x\sqrt{1-x^2}}$

AS REQUIRED

Question 57 (****)



The figure above shows the graph of the curve with equation

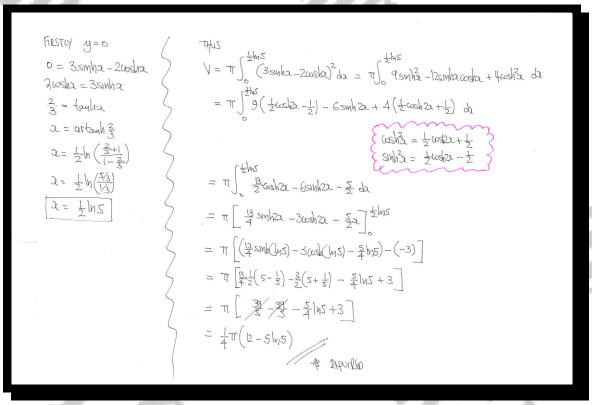
$$y = 3\sinh x - 2\cosh x\,,\;\; x \in \mathbb{R}\,.$$

The finite region bounded by the curve and the coordinate axes, shown shaded in the figure above, is revolved by 2π about the x axis to form a solid S.

Show that the volume of S is

$$\frac{1}{4}\pi\left(12 - 5\ln 5\right).$$

proof



FIRSTLY $y=0$

$0 = 3\sinh x - 2\cosh x$

$2\cosh x = 3\sinh x$

$\frac{2}{3} = \tanh x$

$x = \text{artanh}\,\frac{2}{3}$

$x = \frac{1}{2}\ln\left(\frac{\frac{2}{3}+1}{1-\frac{2}{3}}\right)$

$x = \frac{1}{2}\ln\left(\frac{5/3}{1/3}\right)$

$\boxed{x = \frac{1}{2}\ln 5}$

THUS

$V = \pi\int_0^{\frac{1}{2}\ln 5}(3\sinh x - 2\cosh x)^2\,dx = \pi\int_0^{\frac{1}{2}\ln 5} 9\sinh^2 x - 12\sinh x\cosh x + 4\cosh^2 x\;dx$

$= \pi\int_0^{\frac{1}{2}\ln 5} 9\left(\frac{1}{2}\cosh 2x - \frac{1}{2}\right) - 6\sinh 2x + 4\left(\frac{1}{2}\cosh 2x + \frac{1}{2}\right)\,dx$

$\cosh^2 x = \frac{1}{2}\cosh 2x + \frac{1}{2}$

$\sinh^2 x = \frac{1}{2}\cosh 2x - \frac{1}{2}$

$= \pi\int_0^{\frac{1}{2}\ln 5} \frac{13}{2}\cosh 2x - 6\sinh 2x - \frac{5}{2}\,dx$

$= \pi\left[\frac{13}{4}\sinh 2x - 3\cosh 2x - \frac{5}{2}x\right]_0^{\frac{1}{2}\ln 5}$

$= \pi\left[\left(\frac{13}{4}\sinh(\ln 5) - 3\cosh(\ln 5) - \frac{5}{4}\ln 5\right) - (-3)\right]$

$= \pi\left[\frac{13}{4}\times\frac{1}{2}\left(5 - \frac{1}{5}\right) - \frac{3}{2}\left(5 + \frac{1}{5}\right) - \frac{5}{4}\ln 5 + 3\right]$

$= \pi\left[\cancel{\frac{39}{5}} - \cancel{\frac{39}{5}} - \frac{5}{4}\ln 5 + 3\right]$

$= \frac{1}{4}\pi\left(12 - 5\ln 5\right)$

// AS REQUIRED

Question 58 (****)

a) Sketch the graph of $y = \text{arsech}\, x$, defined for $0 < x \leq 1$.

b) Show clearly that

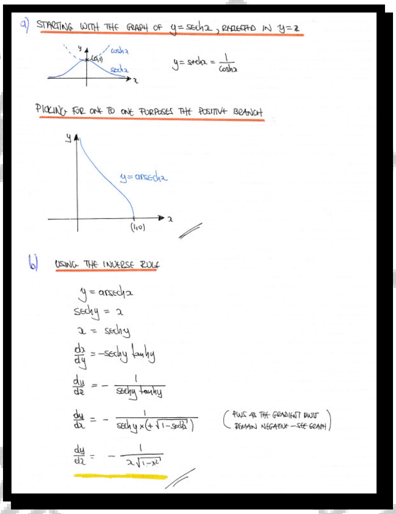
$$\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}}.$$

c) Hence evaluate

$$\int_{\frac{1}{2}}^{1} \text{arsech}\, x \;\; dx.$$

Give the answer in the form $\lambda\left[2\pi - 3\ln\left(2+\sqrt{3}\right)\right]$, where λ is a rational number to be found.

$\boxed{}$, $\boxed{\lambda = \frac{1}{6}}$



a) STARTING WITH THE GRAPH OF $y = \text{sech}\,x$, REFLECTED IN $y = x$

$y = \text{sech}\,x = \frac{1}{\cosh x}$

PICKING FOR ONE TO ONE PURPOSES THE POSITIVE BRANCH

b) USING THE INVERSE RULE

$y = \text{arsech}\,x$

$\text{sech}\,y = x$

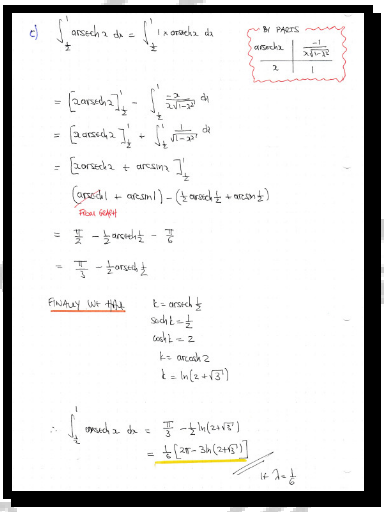
$x = \text{sech}\,y$

$\frac{dx}{dy} = -\text{sech}\,y \tanh y$

$\frac{dy}{dx} = -\frac{1}{\text{sech}\,y \tanh y}$

$\frac{dy}{dx} = -\frac{1}{\text{sech}\,y \times \left(+\sqrt{1-\text{sech}^2 y}\right)}$ (PLUS AS THE GRADIENT MUST REMAIN NEGATIVE – SEE GRAPH)

$\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}}$



c) $\int_{\frac{1}{2}}^{1} \text{arsech}\,x \; dx = \int_{\frac{1}{2}}^{1} 1 \times \text{arsech}\,x \; dx$

BY PARTS

$\text{arsech}\,x$	$\frac{-1}{x\sqrt{1-x^2}}$
x	1

$= \left[x\,\text{arsech}\,x\right]_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} \frac{-x}{x\sqrt{1-x^2}}\,dx$

$= \left[x\,\text{arsech}\,x\right]_{\frac{1}{2}}^{1} + \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$

$= \left[x\,\text{arsech}\,x + \arcsin x\right]_{\frac{1}{2}}^{1}$

$\left(\cancel{\text{arsech}\,1} + \arcsin 1\right) - \left(\frac{1}{2}\text{arsech}\,\frac{1}{2} + \arcsin\frac{1}{2}\right)$

FROM GRAPH

$= \frac{\pi}{2} - \frac{1}{2}\text{arsech}\,\frac{1}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3} - \frac{1}{2}\text{arsech}\,\frac{1}{2}$

FINALLY WE HAVE

$k = \text{arsech}\,\frac{1}{2}$

$\text{sech}\,k = \frac{1}{2}$

$\cosh k = 2$

$k = \text{arcosh}\,2$

$k = \ln\left(2+\sqrt{3}\right)$

$\therefore \int_{\frac{1}{2}}^{1} \text{arsech}\,x \; dx = \frac{\pi}{3} - \frac{1}{2}\ln\left(2+\sqrt{3}\right)$

$= \frac{1}{6}\left[2\pi - 3\ln\left(2+\sqrt{3}\right)\right]$

ie $\lambda = \frac{1}{6}$

Question 59 (****)

It is given that for all real x

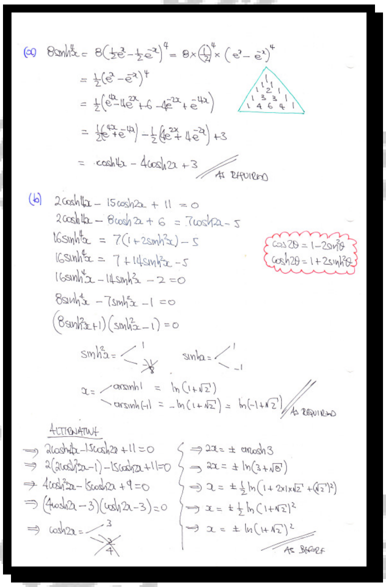
$$8\sinh^2 x \equiv \cosh 4x - 4\cosh 2x + 3 .$$

a) Prove the above hyperbolic identity, by using the definitions of the hyperbolic functions in terms of exponentials.

b) Hence, or otherwise, show that $x = \pm\ln\left(1+\sqrt{2}\right)$ are the solutions of the equation

$$2\cosh 4x - 15\cosh 2x + 11 = 0 .$$

$\boxed{\text{proof}}$



(a) $8\sinh^4 x = 8\left(\frac{1}{2}e^x - \frac{1}{2}e^{-x}\right)^4 = 8\times\left(\frac{1}{2}\right)^4\times\left(e^x - e^{-x}\right)^4$

$= \frac{1}{2}\left(e^x - e^{-x}\right)^4$

$= \frac{1}{2}\left(e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x}\right)$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$= \frac{1}{2}\left(e^{4x} + e^{-4x}\right) - \frac{1}{2}\left(4e^{2x} + 4e^{-2x}\right) + 3$

$= \cosh 4x - 4\cosh 2x + 3$ // AS REQUIRED

(b) $2\cosh 4x - 15\cosh 2x + 11 = 0$

$2\cosh 4x - 8\cosh 2x + 6 = 7\cosh 2x - 5$

$16\sinh^4 x = 7(1 + 2\sinh^2 x) - 5$

$16\sinh^4 x = 7 + 14\sinh^2 x - 5$

$16\sinh^4 x - 14\sinh^2 x - 2 = 0$

$8\sinh^4 x - 7\sinh^2 x - 1 = 0$

$(8\sinh^2 x + 1)(\sinh^2 x - 1) = 0$

$\cos 2\theta = 1 - 2\sin^2\theta$
$\cosh 2\theta = 1 + 2\sinh^2\theta$

$\sinh^2 x = \begin{cases} 1 \\ \cancel{-\frac{1}{8}} \end{cases}$ $\quad \sinh x = \begin{cases} 1 \\ -1 \end{cases}$

$x = \begin{cases} \text{arsinh}\,1 = \ln(1+\sqrt{2}) \\ \text{arsinh}(-1) = -\ln(1+\sqrt{2}) = \ln(-1+\sqrt{2}) \end{cases}$ // AS REQUIRED

ALTERNATIVE

$\Rightarrow 2\cosh 4x - 15\cosh 2x + 11 = 0$

$\Rightarrow 2(2\cosh^2 2x - 1) - 15\cosh 2x + 11 = 0$

$\Rightarrow 4\cosh^2 2x - 15\cosh 2x + 9 = 0$

$\Rightarrow (4\cosh 2x - 3)(\cosh 2x - 3) = 0$

$\Rightarrow \cosh 2x = \begin{cases} 3 \\ \cancel{\frac{3}{4}} \end{cases}$

$\Rightarrow 2x = \pm\,\text{arcosh}\,3$

$\Rightarrow 2x = \pm\ln(3 + \sqrt{8})$

$\Rightarrow x = \pm\frac{1}{2}\ln\left(1 + 2\times1\times\sqrt{2} + (\sqrt{2})^2\right)$

$\Rightarrow x = \pm\frac{1}{2}\ln(1+\sqrt{2})^2$

$\Rightarrow x = \pm\ln(1+\sqrt{2})^2$ // AS BEFORE

Question 60 **(****)**

A curve C has equation

$$y = \cosh 2x + \sinh x\,,\ x \in \mathbb{R}\,.$$

a) Show that the x coordinate of the turning point of C is

$$-\ln\left(\frac{1+\sqrt{17}}{4}\right).$$

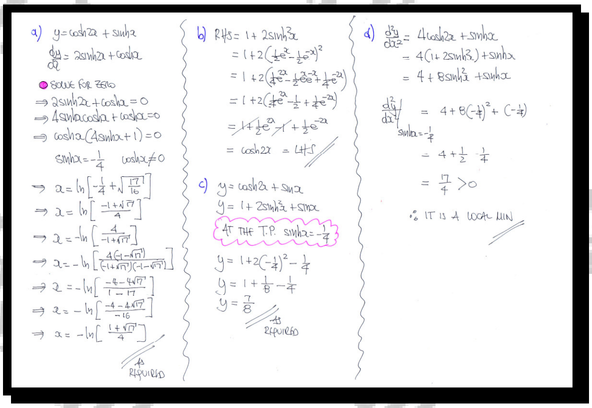
b) Using the definitions of $\cosh x$ and $\sinh x$, in terms of exponentials, prove that

$$\cosh 2x \equiv 1 + 2\sinh^2 x\,.$$

c) Hence show that the y coordinate of the turning point of C is $\frac{7}{8}$.

d) Determine the nature of the turning point.

$\boxed{\text{min}}$



a) $y = \cosh 2x + \sinh x$

$\frac{dy}{dx} = 2\sinh 2x + \cosh x$

● SOLVE FOR ZERO

$\Rightarrow 2\sinh 2x + \cosh x = 0$

$\Rightarrow 4\sinh x\cosh x + \cosh x = 0$

$\Rightarrow \cosh x(4\sinh x + 1) = 0$

$\sinh x = -\frac{1}{4} \quad \cosh x \neq 0$

$\Rightarrow x = \ln\left[-\frac{1}{4} + \sqrt{\frac{17}{16}}\right]$

$\Rightarrow x = \ln\left[\frac{-1+\sqrt{17}}{4}\right]$

$\Rightarrow x = -\ln\left[\frac{4}{-1+\sqrt{17}}\right]$

$\Rightarrow x = -\ln\left[\frac{4(-1-\sqrt{17})}{(-1+\sqrt{17})(-1-\sqrt{17})}\right]$

$\Rightarrow x = -\ln\left[\frac{-4-4\sqrt{17}}{1-17}\right]$

$\Rightarrow x = -\ln\left[\frac{-4-4\sqrt{17}}{-16}\right]$

$\Rightarrow x = -\ln\left[\frac{1+\sqrt{17}}{4}\right]$

AS REQUIRED

b) $\text{RHS} = 1 + 2\sinh^2 x$

$= 1 + 2\left(\frac{1}{2}e^{x} - \frac{1}{2}e^{-x}\right)^2$

$= 1 + 2\left(\frac{1}{4}e^{2x} - \frac{1}{2}e^{x}e^{-x} + \frac{1}{4}e^{-2x}\right)$

$= 1 + 2\left(\frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x}\right)$

$= \cancel{1} + \frac{1}{2}e^{2x} \cancel{-1} + \frac{1}{2}e^{-2x}$

$= \cosh 2x = \text{LHS}$

c) $y = \cosh 2x + \sin x$

$y = 1 + 2\sinh^2 x + \sin x$

AT THE T.P. $\sinh x = -\frac{1}{4}$

$y = 1 + 2\left(-\frac{1}{4}\right)^2 - \frac{1}{4}$

$y = 1 + \frac{1}{8} - \frac{1}{4}$

$y = \frac{7}{8}$

AS REQUIRED

d) $\frac{d^2y}{dx^2} = 4\cosh 2x + \sinh x$

$= 4(1 + 2\sinh^2 x) + \sinh x$

$= 4 + 8\sinh^2 x + \sinh x$

$\left.\frac{d^2y}{dx^2}\right|_{\sinh x = -\frac{1}{4}} = 4 + 8\left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)$

$= 4 + \frac{1}{2} - \frac{1}{4}$

$= \frac{17}{4} > 0$

$\therefore$ IT IS A LOCAL MIN

Question 61 (****)

It is given that

$$A\cosh x + B\sinh x \equiv R\cosh\left(x+\alpha\right),$$

where the A, B, R and α are constants with $A > B > 0$, $R > 0$.

a) Show clearly that …

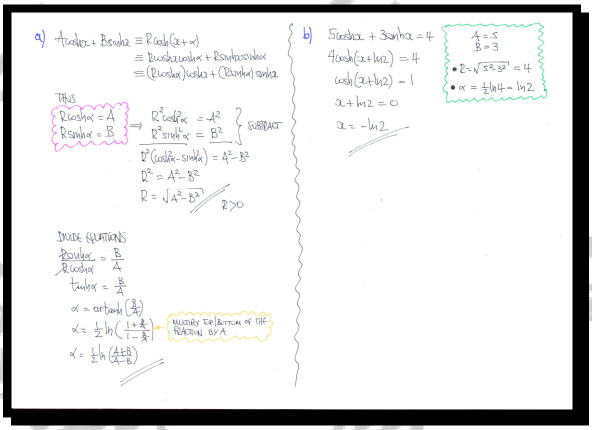
i. … $\alpha = \frac{1}{2}\ln\left(\frac{A+B}{A-B}\right)$.

ii. … $R = \sqrt{A^2 - B^2}$.

b) Use the above result to determine the exact solution of the equation

$$5\cosh x + 3\sinh x = 4.$$

$\boxed{x = -\ln 2}$



a) $A\cosh x + B\sinh x \equiv R\cosh(x+\alpha)$

$\equiv R\cosh x\cosh\alpha + R\sinh x\sinh\alpha$

$\equiv (R\cosh\alpha)\cosh x + (R\sinh\alpha)\sinh x$

THUS

$R\cosh\alpha = A$
$R\sinh\alpha = B$

$\Rightarrow$ $R^2\cosh^2\alpha = A^2$
$R^2\sinh^2\alpha = B^2$ } SUBTRACT

$R^2(\cosh^2\alpha - \sinh^2\alpha) = A^2 - B^2$

$R^2 = A^2 - B^2$

$R = \sqrt{A^2 - B^2}$ $R>0$

DIVIDE EQUATIONS

$\frac{\cancel{R}\sinh\alpha}{\cancel{R}\cosh\alpha} = \frac{B}{A}$

$\tanh\alpha = \frac{B}{A}$

$\alpha = \text{artanh}\left(\frac{B}{A}\right)$

$\alpha = \frac{1}{2}\ln\left(\frac{1+\frac{B}{A}}{1-\frac{B}{A}}\right)$ ← MULTIPLY TOP/BOTTOM OF THE FRACTION BY A

$\alpha = \frac{1}{2}\ln\left(\frac{A+B}{A-B}\right)$

b) $5\cosh x + 3\sinh x = 4$

$A=5$
$B=3$

- $R = \sqrt{5^2-3^2} = 4$
- $\alpha = \frac{1}{2}\ln 4 = \ln 2$

$4\cosh(x+\ln 2) = 4$

$\cosh(x+\ln 2) = 1$

$x + \ln 2 = 0$

$x = -\ln 2$

Question 62 (****)

$$f(x) \equiv \cosh 2x - 8\cosh x \, , \ \ x \in \mathbb{R} \, .$$

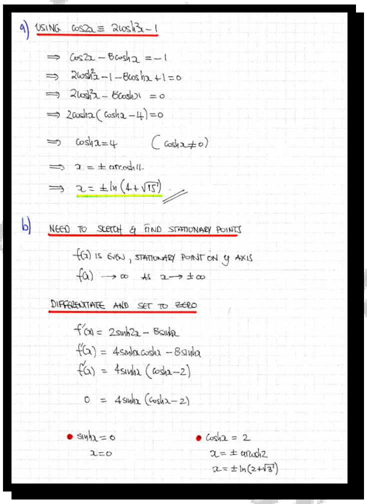
a) Determine, in exact logarithmic form, the solutions of the equation

$$f(x) = -1 \, .$$

b) If k is a real constant, determine the value, values or range of values of k, so that the equation $f(x) = k$ has…

- **i.** … one repeated real root.
- **ii.** … more than one repeated real root.
- **iii.** … two distinct real roots.
- **iv.** … four distinct real roots.
- **v.** … no real roots.

$\boxed{}$, $\boxed{x = \pm \ln\left(4 + \sqrt{15}\right)}$



a) USING $\cos 2x \equiv 2\cosh^2 x - 1$

$\Rightarrow \cos 2x - 8\cosh x = -1$

$\Rightarrow 2\cosh^2 x - 1 - 8\cosh x + 1 = 0$

$\Rightarrow 2\cosh^2 x - 8\cosh x = 0$

$\Rightarrow 2\cosh x(\cosh x - 4) = 0$

$\Rightarrow \cosh x = 4 \qquad (\cosh x \neq 0)$

$\Rightarrow x = \pm \operatorname{arcosh} 4$

$\Rightarrow x = \pm \ln(4 + \sqrt{15})$ //

b) NEED TO SKETCH & FIND STATIONARY POINTS

$f(x)$ IS EVEN, STATIONARY POINT ON y AXIS

$f(x) \to \infty$ AS $x \to \pm\infty$

DIFFERENTIATE AND SET TO ZERO

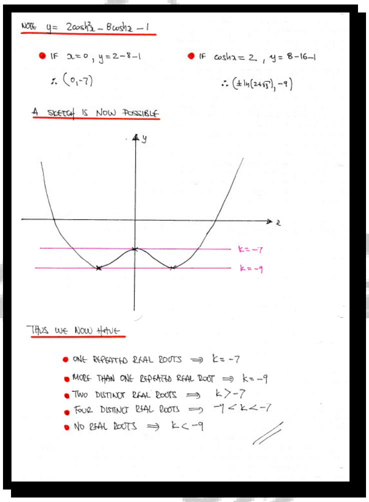
$f'(x) = 2\sinh 2x - 8\sinh x$

$f'(x) = 4\sinh x \cosh x - 8\sinh x$

$f'(x) = 4\sinh x(\cosh x - 2)$

$0 = 4\sinh x(\cosh x - 2)$

- $\sinh x = 0$
  $x = 0$
- $\cosh x = 2$
  $x = \pm \operatorname{arcosh} 2$
  $x = \pm \ln(2 + \sqrt{3})$



NOTE $y = 2\cosh^2 x - 8\cosh x - 1$

- IF $x = 0$, $y = 2 - 8 - 1$
  $\therefore (0, -7)$
- IF $\cosh x = 2$, $y = 8 - 16 - 1$
  $\therefore (\pm\ln(2+\sqrt{3}), -9)$

A SKETCH IS NOW POSSIBLE

THUS WE NOW HAVE

- ONE REPEATED REAL ROOTS $\Rightarrow k = -7$
- MORE THAN ONE REPEATED REAL ROOT $\Rightarrow k = -9$
- TWO DISTINCT REAL ROOTS $\Rightarrow k > -7$
- FOUR DISTINCT REAL ROOTS $\Rightarrow -9 < k < -7$
- NO REAL ROOTS $\Rightarrow k < -9$ //

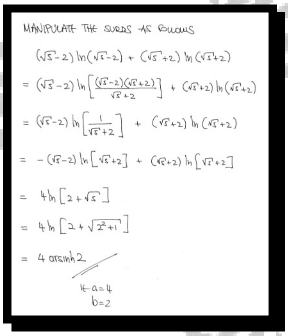
Question 63 **(****)**

Show that

$$\left(\sqrt{5}-2\right)\ln\left(\sqrt{5}-2\right)+\left(\sqrt{5}+2\right)\ln\left(\sqrt{5}+2\right),$$

can be written in the form $a \operatorname{arsinh} b$, where a and b are positive integers to be found.

$\boxed{}$, $\boxed{4\operatorname{arsinh}(2)}$



MANIPULATE THE SURDS AS FOLLOWS

$$\left(\sqrt{5}-2\right)\ln\left(\sqrt{5}-2\right)+\left(\sqrt{5}+2\right)\ln\left(\sqrt{5}+2\right)$$

$$=\left(\sqrt{5}-2\right)\ln\left[\frac{(\sqrt{5}-2)(\sqrt{5}+2)}{\sqrt{5}+2}\right]+\left(\sqrt{5}+2\right)\ln\left(\sqrt{5}+2\right)$$

$$=\left(\sqrt{5}-2\right)\ln\left[\frac{1}{\sqrt{5}+2}\right]+\left(\sqrt{5}+2\right)\ln\left(\sqrt{5}+2\right)$$

$$=-\left(\sqrt{5}-2\right)\ln\left[\sqrt{5}+2\right]+\left(\sqrt{5}+2\right)\ln\left[\sqrt{5}+2\right]$$

$$=4\ln\left[2+\sqrt{5}\right]$$

$$=4\ln\left[2+\sqrt{2^2+1}\right]$$

$$=4\operatorname{arsinh}2$$

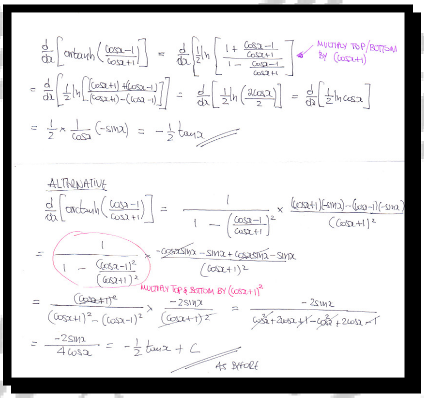
$\therefore a=4$
$b=2$

Question 64 **(****+)**

Show clearly that

$$\frac{d}{dx}\left[\operatorname{artanh}\left(\frac{\cos x+1}{\cos x-1}\right)\right]=-\frac{1}{2}\tan x.$$

$\boxed{\text{proof}}$



$$\frac{d}{dx}\left[\operatorname{artanh}\left(\frac{\cos x-1}{\cos x+1}\right)\right]=\frac{d}{dx}\left[\frac{1}{2}\ln\left[\frac{1+\frac{\cos x-1}{\cos x+1}}{1-\frac{\cos x-1}{\cos x+1}}\right]\right]$$

MULTIPLY TOP/BOTTOM BY $(\cos x+1)$

$$=\frac{d}{dx}\left[\frac{1}{2}\ln\left[\frac{(\cos x+1)+(\cos x-1)}{(\cos x+1)-(\cos x-1)}\right]\right]=\frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{2\cos x}{2}\right)\right]=\frac{d}{dx}\left[\frac{1}{2}\ln\cos x\right]$$

$$=\frac{1}{2}\times\frac{1}{\cos x}(-\sin x)=-\frac{1}{2}\tan x$$

ALTERNATIVE

$$\frac{d}{dx}\left[\operatorname{arctanh}\left(\frac{\cos x-1}{\cos x+1}\right)\right]=\frac{1}{1-\left(\frac{\cos x-1}{\cos x+1}\right)^2}\times\frac{(\cos x+1)(-\sin x)-(\cos x-1)(-\sin x)}{(\cos x+1)^2}$$

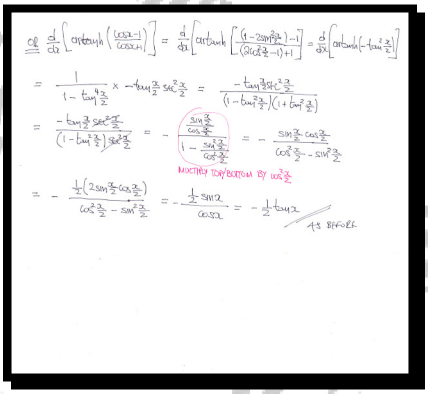
$$=\frac{1}{1-\frac{(\cos x-1)^2}{(\cos x+1)^2}}\times\frac{-\cancel{\cos x\sin x}-\sin x+\cancel{\cos x\sin x}-\sin x}{(\cos x+1)^2}$$

MULTIPLY TOP & BOTTOM BY $(\cos x+1)^2$

$$=\frac{\cancel{(\cos x+1)^2}}{(\cos x+1)^2-(\cos x-1)^2}\times\frac{-2\sin x}{\cancel{(\cos x+1)^2}}=\frac{-2\sin x}{\cancel{\cos^2 x}+2\cos x+\cancel{1}-\cancel{\cos^2 x}+2\cos x-\cancel{1}}$$

$$=\frac{-2\sin x}{4\cos x}=-\frac{1}{2}\tan x+C$$

AS BEFORE



$$\underline{\underline{\text{or}}}\ \frac{d}{dx}\left[\operatorname{artanh}\left(\frac{\cos x-1}{\cos x+1}\right)\right]=\frac{d}{dx}\left[\operatorname{artanh}\left[\frac{(1-2\sin^2\frac{x}{2})-1}{(2\cos^2\frac{x}{2}-1)+1}\right]\right]=\frac{d}{dx}\left[\operatorname{artanh}\left(-\tan^2\frac{x}{2}\right)\right]$$

$$=\frac{1}{1-\tan^4\frac{x}{2}}\times-\tan\frac{x}{2}\sec^2\frac{x}{2}=\frac{-\tan\frac{x}{2}\sec^2\frac{x}{2}}{(1-\tan^2\frac{x}{2})(1+\tan^2\frac{x}{2})}$$

$$=\frac{-\tan\frac{x}{2}\cancel{\sec^2\frac{x}{2}}}{(1-\tan^2\frac{x}{2})\cancel{\sec^2\frac{x}{2}}}=-\frac{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}=-\frac{\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}$$

MULTIPLY TOP/BOTTOM BY $\cos^2\frac{x}{2}$

$$=-\frac{\frac{1}{2}\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}=-\frac{\frac{1}{2}\sin x}{\cos x}=-\frac{1}{2}\tan x$$

AS BEFORE

Question 65 (****)

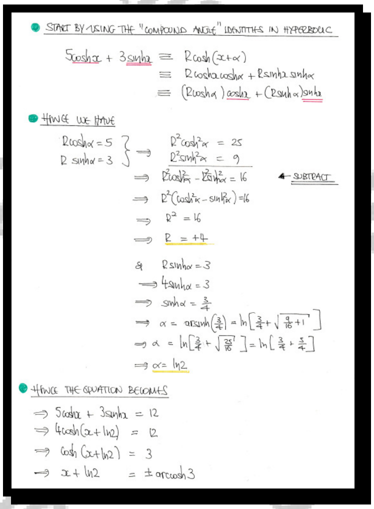
$$5\cosh x + 3\sinh x = 12$$

Express the left side of the above equation in the form $R\cosh(x+\alpha)$, where R and α are positive constants, and use it to show that

$$x = \ln\left(A \pm \sqrt{B}\right),$$

where A and B are constants to be found.

$\boxed{\text{FP2-N}}$, $\boxed{x = \ln\left(\frac{3}{2} \pm \sqrt{2}\right)}$



- START BY USING THE "COMPOUND ANGLE" IDENTITIES IN HYPERBOLIC

$$5\cosh x + 3\sinh x \equiv R\cosh(x+\alpha)$$
$$\equiv R\cosh x\cosh\alpha + R\sinh x\sinh\alpha$$
$$\equiv (R\cosh\alpha)\cosh x + (R\sinh\alpha)\sinh x$$

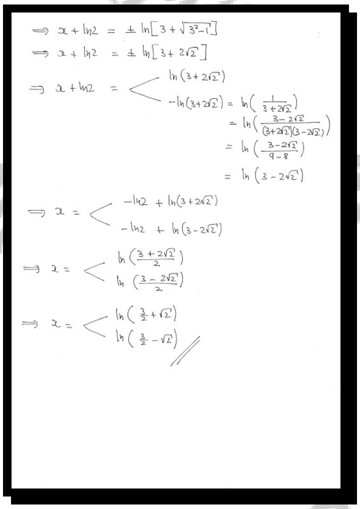
- HENCE WE HAVE

$$\left.\begin{array}{l} R\cosh\alpha = 5 \\ R\sinh\alpha = 3 \end{array}\right\} \Rightarrow \begin{array}{l} R^2\cosh^2\alpha = 25 \\ R^2\sinh^2\alpha = 9 \end{array}$$
$$\Rightarrow R^2\cosh^2\alpha - R^2\sinh^2\alpha = 16 \quad \leftarrow \text{SUBTRACT}$$
$$\Rightarrow R^2(\cosh^2\alpha - \sinh^2\alpha) = 16$$
$$\Rightarrow R^2 = 16$$
$$\Rightarrow R = +4$$

$$\& \quad R\sinh\alpha = 3$$
$$\Rightarrow 4\sinh\alpha = 3$$
$$\Rightarrow \sinh\alpha = \frac{3}{4}$$
$$\Rightarrow \alpha = \operatorname{arsinh}\left(\frac{3}{4}\right) = \ln\left[\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right]$$
$$\Rightarrow \alpha = \ln\left[\frac{3}{4} + \sqrt{\frac{25}{16}}\right] = \ln\left[\frac{3}{4} + \frac{5}{4}\right]$$
$$\Rightarrow \alpha = \ln 2$$

- HENCE THE EQUATION BECOMES

$$\Rightarrow 5\cosh x + 3\sinh x = 12$$
$$\Rightarrow 4\cosh(x+\ln 2) = 12$$
$$\Rightarrow \cosh(x+\ln 2) = 3$$
$$\Rightarrow x + \ln 2 = \pm\operatorname{arcosh} 3$$



$$\Rightarrow x + \ln 2 = \pm\ln\left[3 + \sqrt{3^2-1}\right]$$
$$\Rightarrow x + \ln 2 = \pm\ln\left[3 + 2\sqrt{2}\right]$$
$$\Rightarrow x + \ln 2 = \left\langle \begin{array}{l} \ln(3+2\sqrt{2}) \\ -\ln(3+2\sqrt{2}) = \ln\left(\frac{1}{3+2\sqrt{2}}\right) \end{array}\right.$$
$$= \ln\left(\frac{3-2\sqrt{2}}{(3+2\sqrt{2})(3-2\sqrt{2})}\right)$$
$$= \ln\left(\frac{3-2\sqrt{2}}{9-8}\right)$$
$$= \ln\left(3-2\sqrt{2}\right)$$
$$\Rightarrow x = \left\langle \begin{array}{l} -\ln 2 + \ln(3+2\sqrt{2}) \\ -\ln 2 + \ln(3-2\sqrt{2}) \end{array}\right.$$
$$\Rightarrow x = \left\langle \begin{array}{l} \ln\left(\frac{3+2\sqrt{2}}{2}\right) \\ \ln\left(\frac{3-2\sqrt{2}}{2}\right) \end{array}\right.$$
$$\Rightarrow x = \left\langle \begin{array}{l} \ln\left(\frac{3}{2}+\sqrt{2}\right) \\ \ln\left(\frac{3}{2}-\sqrt{2}\right) \end{array}\right. //$$

Question 66 (****+)

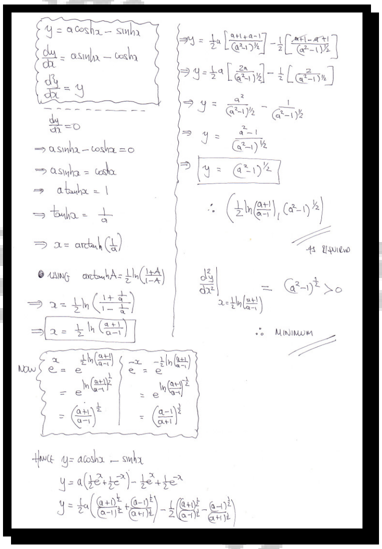
The curve C has equation

$$y = a\cosh x - \sinh x \text{, where } a > 1 .$$

Show that C has a minimum turning point with coordinates

$$\left(\frac{1}{2}\ln\left(\frac{a+1}{a-1}\right), \sqrt{a^2+1}\right).$$

$\boxed{\text{proof}}$



$$\boxed{\begin{aligned} y &= a\cosh x - \sinh x \\ \frac{dy}{dx} &= a\sinh x - \cosh x \\ \frac{d^2y}{dx^2} &= y \end{aligned}}$$

$\frac{dy}{dx} = 0$

$\Rightarrow a\sinh x - \cosh x = 0$

$\Rightarrow a\sinh x = \cosh x$

$\Rightarrow a\tanh x = 1$

$\Rightarrow \tanh x = \frac{1}{a}$

$\Rightarrow x = \text{arctanh}\left(\frac{1}{a}\right)$

● USING $\text{arctanh}A = \frac{1}{2}\ln\left(\frac{1+A}{1-A}\right)$

$\Rightarrow x = \frac{1}{2}\ln\left(\frac{1+\frac{1}{a}}{1-\frac{1}{a}}\right)$

$\Rightarrow \boxed{x = \frac{1}{2}\ln\left(\frac{a+1}{a-1}\right)}$

NOW $e^{x} = e^{\frac{1}{2}\ln\left(\frac{a+1}{a-1}\right)} = e^{\ln\left(\frac{a+1}{a-1}\right)^{\frac{1}{2}}} = \left(\frac{a+1}{a-1}\right)^{\frac{1}{2}}$

$e^{-x} = e^{-\frac{1}{2}\ln\left(\frac{a+1}{a-1}\right)} = e^{\ln\left(\frac{a+1}{a-1}\right)^{-\frac{1}{2}}} = \left(\frac{a-1}{a+1}\right)^{\frac{1}{2}}$

HENCE $y = a\cosh x - \sinh x$

$y = a\left(\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}\right) - \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

$y = \frac{1}{2}a\left(\frac{(a+1)^{\frac{1}{2}}}{(a-1)^{\frac{1}{2}}} + \frac{(a-1)^{\frac{1}{2}}}{(a+1)^{\frac{1}{2}}}\right) - \frac{1}{2}\left(\frac{(a+1)^{\frac{1}{2}}}{(a-1)^{\frac{1}{2}}} - \frac{(a-1)^{\frac{1}{2}}}{(a+1)^{\frac{1}{2}}}\right)$

$\Rightarrow y = \frac{1}{2}a\left[\frac{a+1+a-1}{(a^2-1)^{1/2}}\right] - \frac{1}{2}\left[\frac{\cancel{a}+1-\cancel{a}+1}{(a^2-1)^{1/2}}\right]$

$\Rightarrow y = \frac{1}{2}a\left[\frac{2a}{(a^2-1)^{1/2}}\right] - \frac{1}{2}\left[\frac{2}{(a^2-1)^{1/2}}\right]$

$\Rightarrow y = \frac{a^2}{(a^2-1)^{1/2}} - \frac{1}{(a^2-1)^{1/2}}$

$\Rightarrow y = \frac{a^2-1}{(a^2-1)^{1/2}}$

$\Rightarrow \boxed{y = (a^2-1)^{1/2}}$

$\therefore \left(\frac{1}{2}\ln\left(\frac{a+1}{a-1}\right), (a^2-1)^{1/2}\right)$

AS REQUIRED

$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}\ln\left(\frac{a+1}{a-1}\right)} = (a^2-1)^{\frac{1}{2}} > 0$

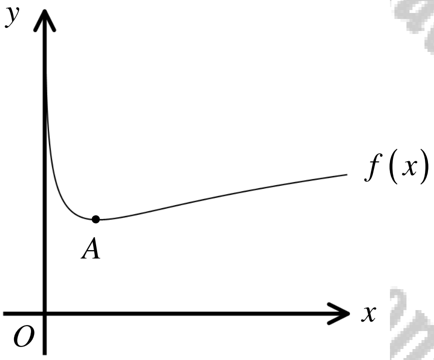
$\therefore$ MINIMUM

Question 67 **(****+)**

$$f(x) = \operatorname{arsinh} x + \operatorname{arsinh}\left(\frac{1}{x}\right), \ x \in \mathbb{R}, \ x \neq 0.$$

a) Show clearly that $f'(x) = \dfrac{x^2 - |x|}{x^2\sqrt{x^2+1}}$.

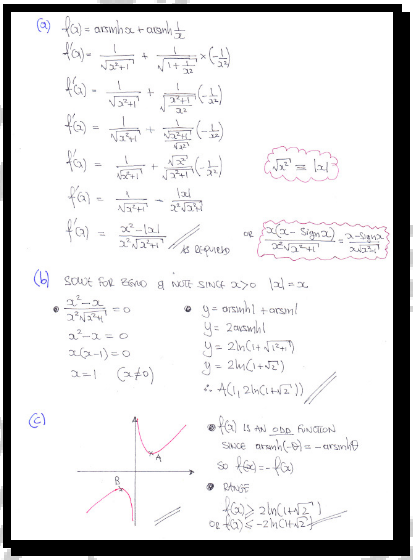
The graph of $f(x)$, for $x > 0$ is shown in the figure below.



b) Determine, in terms of natural logarithms where appropriate, the coordinates of the stationary point of $f(x)$, labelled as point A in the figure.

c) Sketch the graph of $f(x)$, fully justifying its shape for $x < 0$, and state its range.

$$\boxed{A\left[1, 2\ln\left(1+\sqrt{2}\right)\right]}, \ \boxed{f(x) \geq 2\ln\left(1+\sqrt{2}\right) \cup f(x) \leq -2\ln\left(1+\sqrt{2}\right)}$$


(a) $f(x) = \operatorname{arsinh} x + \operatorname{arsinh}\frac{1}{x}$

$f'(x) = \frac{1}{\sqrt{x^2+1}} + \frac{1}{\sqrt{1+\frac{1}{x^2}}} \times \left(-\frac{1}{x^2}\right)$

$f'(x) = \frac{1}{\sqrt{x^2+1}} + \frac{1}{\sqrt{\frac{x^2+1}{x^2}}}\left(-\frac{1}{x^2}\right)$

$f'(x) = \frac{1}{\sqrt{x^2+1}} + \frac{1}{\frac{\sqrt{x^2+1}}{\sqrt{x^2}}}\left(-\frac{1}{x^2}\right)$

$f'(x) = \frac{1}{\sqrt{x^2+1}} + \frac{\sqrt{x^2}}{\sqrt{x^2+1}}\left(-\frac{1}{x^2}\right)$ $\quad \sqrt{x^2} \equiv |x|$

$f'(x) = \frac{1}{\sqrt{x^2+1}} - \frac{|x|}{x^2\sqrt{x^2+1}}$

$f'(x) = \frac{x^2-|x|}{x^2\sqrt{x^2+1}}$ // AS REQUIRED $\quad$ OR $\frac{x(x-\operatorname{Sign}x)}{x^2\sqrt{x^2+1}} = \frac{x-\operatorname{Sign}x}{x\sqrt{x^2-1}}$

(b) SOLVE FOR ZERO & NOTE SINCE $x>0$ $|x| = x$

- $\frac{x^2-x}{x^2\sqrt{x^2+1}} = 0$
  $x^2 - x = 0$
  $x(x-1) = 0$
  $x = 1 \quad (x \neq 0)$
- $y = \operatorname{arsinh}1 + \operatorname{arsinh}1$
  $y = 2\operatorname{arsinh}1$
  $y = 2\ln(1+\sqrt{1^2+1})$
  $y = 2\ln(1+\sqrt{2})$
  $\therefore A(1, 2\ln(1+\sqrt{2}))$ //

(c)
- $f(x)$ IS AN ODD FUNCTION
  SINCE $\operatorname{arsinh}(-\theta) = -\operatorname{arsinh}\theta$
  SO $f(-x) = -f(x)$
- RANGE
  $f(x) \geq 2\ln(1+\sqrt{2})$
  OR $f(x) \leq -2\ln(1+\sqrt{2})$ //

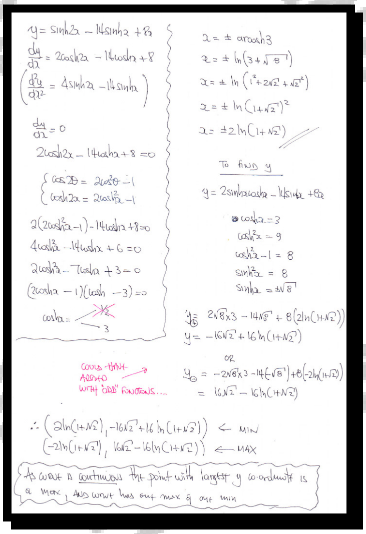
Question 68 (****+)

The curve C has equation

$$y = \sinh 2x - 14\sinh x + 8x .$$

Find the exact coordinates of the turning points of C and determine their nature.

$$\boxed{\left[2\ln\left(1+\sqrt{2}\right), -16\sqrt{2}+16\ln\left(1+\sqrt{2}\right)\right]}, \boxed{\left[-2\ln\left(1+\sqrt{2}\right), 16\sqrt{2}-16\ln\left(1+\sqrt{2}\right)\right]}$$



$y = \sinh 2x - 14\sinh x + 8x$

$\frac{dy}{dx} = 2\cosh 2x - 14\cosh x + 8$

$\left(\frac{d^2y}{dx^2} = 4\sinh 2x - 14\sinh x\right)$

$\frac{dy}{dx} = 0$

$2\cosh 2x - 14\cosh x + 8 = 0$

$\begin{cases} \cos 2\theta = 2\cos^2\theta - 1 \\ \cosh 2x = 2\cosh^2 x - 1 \end{cases}$

$2(2\cosh^2 x - 1) - 14\cosh x + 8 = 0$

$4\cosh^2 x - 14\cosh x + 6 = 0$

$2\cosh^2 x - 7\cosh x + 3 = 0$

$(2\cosh x - 1)(\cosh - 3) = 0$

$\cosh x = \cancel{\tfrac{1}{2}}$, 3

$x = \pm \text{arcosh} 3$

$x = \pm \ln\left(3 + \sqrt{8}\right)$

$x = \pm \ln\left(1^2 + 2\sqrt{2} + \sqrt{2}^2\right)$

$x = \pm \ln\left(1 + \sqrt{2}\right)^2$

$x = \pm 2\ln\left(1 + \sqrt{2}\right)$

To find y

$y = 2\sinh x\cosh x - 14\sinh x + 8x$

$\cosh x = 3$

$\cosh^2 x = 9$

$\cosh^2 x - 1 = 8$

$\sinh^2 x = 8$

$\sinh x = \pm\sqrt{8}$

$y_{+} = 2\sqrt{8}\times 3 - 14\sqrt{8} + 8\left(2\ln(1+\sqrt{2})\right)$

$y = -16\sqrt{2} + 16\ln(1+\sqrt{2})$

OR

$y_{-} = -2\sqrt{8}\times 3 - 14(-\sqrt{8}) + 8\left(-2\ln(1+\sqrt{2})\right)$

$= 16\sqrt{2} - 16\ln(1+\sqrt{2})$

COULD HAVE ARGUED WITH "ODD" FUNCTIONS...

$\therefore \left(2\ln(1+\sqrt{2}), -16\sqrt{2} + 16\ln(1+\sqrt{2})\right) \leftarrow$ MIN

$\left(-2\ln(1+\sqrt{2}), 16\sqrt{2} - 16\ln(1+\sqrt{2})\right) \leftarrow$ MAX

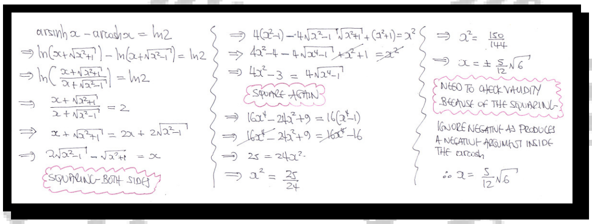
As curve is continuous the point with largest y co-ordinate is a max, and curve has one max & one min

Question 69 (****+)

Find, in exact surd form the solution of the equation

$$\text{arsinh}\, x - \text{arcosh}\, x = \ln 2 .$$

$$\boxed{x = \frac{5}{12}\sqrt{6}}$$



$\text{arsinh}\, x - \text{arcosh}\, x = \ln 2$

$\Rightarrow \ln\left(x + \sqrt{x^2+1}\right) - \ln\left(x + \sqrt{x^2-1}\right) = \ln 2$

$\Rightarrow \ln\left(\frac{x + \sqrt{x^2+1}}{x + \sqrt{x^2-1}}\right) = \ln 2$

$\Rightarrow \frac{x + \sqrt{x^2+1}}{x + \sqrt{x^2-1}} = 2$

$\Rightarrow x + \sqrt{x^2+1} = 2x + 2\sqrt{x^2-1}$

$\Rightarrow 2\sqrt{x^2-1} - \sqrt{x^2+1} = x$

SQUARING BOTH SIDES

$\Rightarrow 4(x^2-1) - 4\sqrt{x^2-1}\sqrt{x^2+1} + (x^2+1) = x^2$

$\Rightarrow 4x^2 - 4 - 4\sqrt{x^4-1} \cancel{+x^2} + 1 = \cancel{x^2}$

$\Rightarrow 4x^2 - 3 = 4\sqrt{x^4-1}$

SQUARE AGAIN

$\Rightarrow 16x^4 - 24x^2 + 9 = 16(x^4-1)$

$\Rightarrow \cancel{16x^4} - 24x^2 + 9 = \cancel{16x^4} - 16$

$\Rightarrow 25 = 24x^2$

$\Rightarrow x^2 = \frac{25}{24}$

$\Rightarrow x^2 = \frac{150}{144}$

$\Rightarrow x = \pm\frac{5}{12}\sqrt{6}$

NEED TO CHECK VALIDITY BECAUSE OF THE SQUARING

IGNORE NEGATIVE AS PRODUCES A NEGATIVE ARGUMENT INSIDE THE arcosh

$\therefore x = \frac{5}{12}\sqrt{6}$

Question 70 **(****+)**

$$\cosh x \equiv \frac{1}{2}\left(e^{x}+e^{-x}\right) \quad \text{and} \quad \sinh x \equiv \frac{1}{2}\left(e^{x}-e^{-x}\right).$$

a) Use the above definitions to show that …

i. … $\cosh^2 x - \sinh^2 x \equiv 1$.

ii. … $4\cosh^3 x - 3\cosh x \equiv \cosh 3x$.

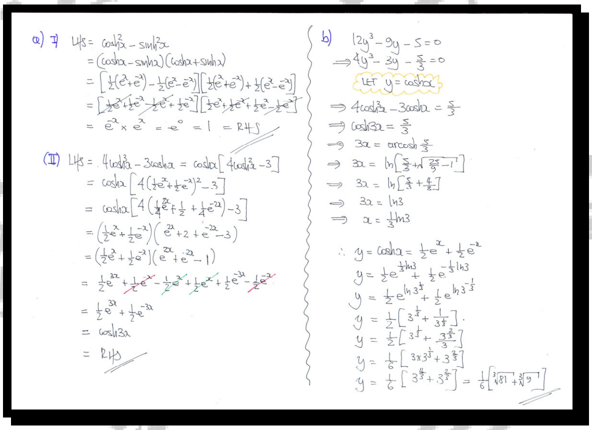
b) Hence show that the real root of the equation

$$12y^3 - 9y - 5 = 0,$$

can be written as

$$\frac{1}{6}\left(\sqrt[3]{81}+\sqrt[3]{9}\right).$$

$\boxed{\text{proof}}$



a) I LHS = $\cosh^2 x - \sinh^2 x$

$= (\cosh x - \sinh x)(\cosh x + \sinh x)$

$= \left[\frac{1}{2}(e^{x}+e^{-x}) - \frac{1}{2}(e^{x}-e^{-x})\right]\left[\frac{1}{2}(e^{x}+e^{-x}) + \frac{1}{2}(e^{x}-e^{-x})\right]$

$= \left[\cancel{\frac{1}{2}e^{x}} + \frac{1}{2}e^{-x} - \cancel{\frac{1}{2}e^{x}} + \frac{1}{2}e^{-x}\right]\left[\frac{1}{2}e^{x} + \cancel{\frac{1}{2}e^{-x}} + \frac{1}{2}e^{x} - \cancel{\frac{1}{2}e^{-x}}\right]$

$= e^{-x} \times e^{x} = e^{0} = 1 = \text{RHS}$

(II) LHS = $4\cosh^3 x - 3\cosh x = \cosh x\left[4\cosh^2 x - 3\right]$

$= \cosh x\left[4\left(\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}\right)^2 - 3\right]$

$= \cosh x\left[4\left(\frac{1}{4}e^{2x}+\frac{1}{2}+\frac{1}{4}e^{-2x}\right) - 3\right]$

$= \left(\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}\right)\left(e^{2x}+2+e^{-2x}-3\right)$

$= \left(\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}\right)\left(e^{2x}+e^{-2x}-1\right)$

$= \frac{1}{2}e^{3x} + \cancel{\frac{1}{2}e^{-x}} - \cancel{\frac{1}{2}e^{x}} + \cancel{\frac{1}{2}e^{x}} + \frac{1}{2}e^{-3x} - \cancel{\frac{1}{2}e^{-x}}$

$= \frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}$

$= \cosh 3x$

$= \text{RHS}$

b) $12y^3 - 9y - 5 = 0$

$\Rightarrow 4y^3 - 3y - \frac{5}{3} = 0$

LET $y = \cosh x$

$\Rightarrow 4\cosh^3 x - 3\cosh x = \frac{5}{3}$

$\Rightarrow \cosh 3x = \frac{5}{3}$

$\Rightarrow 3x = \text{arcosh}\,\frac{5}{3}$

$\Rightarrow 3x = \ln\left[\frac{5}{3}+\sqrt{\frac{25}{9}-1}\right]$

$\Rightarrow 3x = \ln\left[\frac{5}{3}+\frac{4}{3}\right]$

$\Rightarrow 3x = \ln 3$

$\Rightarrow x = \frac{1}{3}\ln 3$

$\therefore y = \cosh x = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

$y = \frac{1}{2}e^{\frac{1}{3}\ln 3} + \frac{1}{2}e^{-\frac{1}{3}\ln 3}$

$y = \frac{1}{2}e^{\ln 3^{\frac{1}{3}}} + \frac{1}{2}e^{\ln 3^{-\frac{1}{3}}}$

$y = \frac{1}{2}\left[3^{\frac{1}{3}} + \frac{1}{3^{\frac{1}{3}}}\right]$

$y = \frac{1}{2}\left[3^{\frac{1}{3}} + \frac{3^{\frac{2}{3}}}{3}\right]$

$y = \frac{1}{6}\left[3 \times 3^{\frac{1}{3}} + 3^{\frac{2}{3}}\right]$

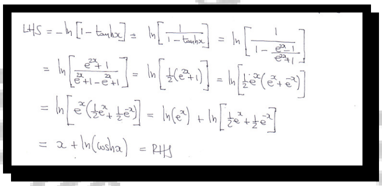
$y = \frac{1}{6}\left[3^{\frac{4}{3}} + 3^{\frac{2}{3}}\right] = \frac{1}{6}\left[\sqrt[3]{81} + \sqrt[3]{9}\right]$

Question 71 (****+)

Show clearly that

$$-\ln(1-\tanh x) \equiv x+\ln(\cosh x).$$

proof



$$\text{LHS} = -\ln\left[1-\tanh x\right] = \ln\left[\frac{1}{1-\tanh x}\right] = \ln\left[\frac{1}{1-\frac{e^{2x}-1}{e^{2x}+1}}\right]$$

$$= \ln\left[\frac{e^{2x}+1}{e^{2x}+1-e^{2x}+1}\right] = \ln\left[\frac{1}{2}(e^{2x}+1)\right] = \ln\left[\frac{1}{2}e^{x}(e^{x}+e^{-x})\right]$$

$$= \ln\left[e^{x}\left(\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}\right)\right] = \ln(e^{x}) + \ln\left[\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}\right]$$

$$= x+\ln(\cosh x) = \text{RHS}$$

Question 72 (****+)

A curve C has equation

$$y = 3\sinh x - 2\cosh x,\ x \in \mathbb{R}.$$

Sketch the graph of C.

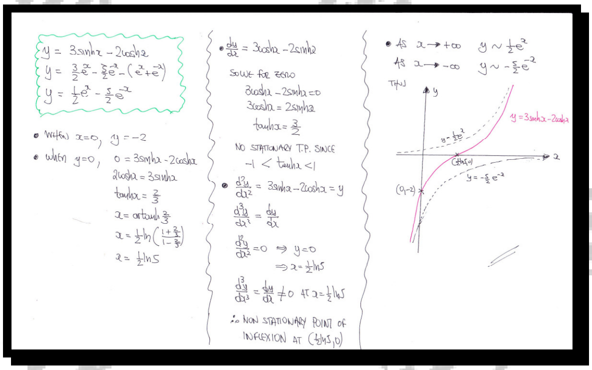
The sketch must include …

… the coordinates of any points where the graph of C meets the coordinates axes.

… the coordinates of any stationary or non stationary turning points.

… the behaviour of the curve for large positive and large negative values of x

graph



$$y = 3\sinh x - 2\cosh x$$

$$y = \frac{3}{2}e^{x} - \frac{3}{2}e^{-x} - (e^{x}+e^{-x})$$

$$y = \frac{1}{2}e^{x} - \frac{5}{2}e^{-x}$$

- WHEN $x=0$, $y=-2$
- when $y=0$, $0 = 3\sinh x - 2\cosh x$

$$2\cosh x = 3\sinh x$$

$$\tanh x = \frac{2}{3}$$

$$x = \text{artanh}\,\frac{2}{3}$$

$$x = \frac{1}{2}\ln\left(\frac{1+\frac{2}{3}}{1-\frac{2}{3}}\right)$$

$$x = \frac{1}{2}\ln 5$$

- $\frac{dy}{dx} = 3\cosh x - 2\sinh x$

SOLVE FOR ZERO

$$3\cosh x - 2\sinh x = 0$$

$$3\cosh x = 2\sinh x$$

$$\tanh x = \frac{3}{2}$$

NO STATIONARY T.P. SINCE $-1 < \tanh x < 1$

- $\frac{d^2y}{dx^2} = 3\sinh x - 2\cosh x = y$

$$\frac{d^3y}{dx^3} = \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow y=0 \Rightarrow x = \frac{1}{2}\ln 5$$

$$\frac{d^3y}{dx^3} = \frac{dy}{dx} \neq 0 \text{ AT } x = \frac{1}{2}\ln 5$$

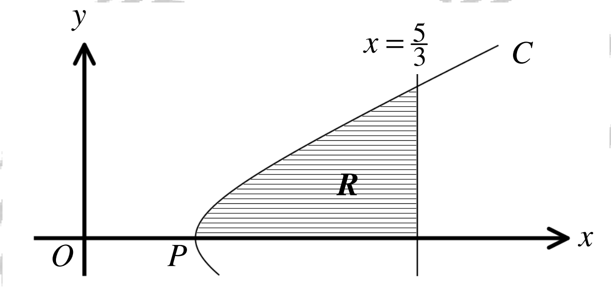
$\therefore$ NON STATIONARY POINT OF INFLEXION AT $(\frac{1}{2}\ln 5, 0)$

- AS $x \to +\infty$ $y \sim \frac{1}{2}e^{2}$

AS $x \to -\infty$ $y \sim -\frac{5}{2}e^{-2}$

THUS

Question 73 (****+)



The figure above shows part of the curve C with parametric equations

$$x = t + \frac{1}{4t}, \quad y = t - \frac{1}{4t} \quad , \quad t > 0 .$$

The curve crosses the x axis at P.

a) Determine the coordinates of P.

b) By considering $x + y$ and $x - y$ find a Cartesian equation for C.

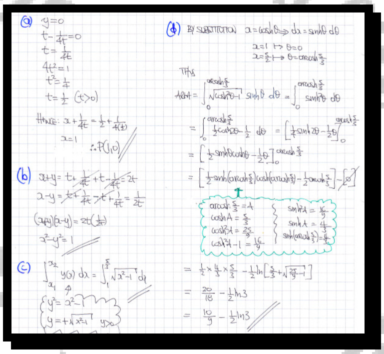
The region R bounded by C, the straight line with equation $x = \frac{5}{3}$ and the x axis is shown shaded in the figure.

c) Show that the area of R is given by

$$\int_1^{\frac{5}{3}} \sqrt{x^2 - 1} \ dx .$$

d) Hence calculate an exact value for the area of R.

$\boxed{P(1,0)}$, $\boxed{x^2 - y^2 = 1}$, $\boxed{\text{Area} = \frac{10}{9} - \frac{1}{2}\ln 3}$



(a) $y=0$

$t - \frac{1}{4t} = 0$

$t = \frac{1}{4t}$

$4t^2 = 1$

$t^2 = \frac{1}{4}$

$t = \frac{1}{2} \ (t>0)$

HENCE: $x + \frac{1}{4t} = \frac{1}{2} + \frac{1}{4(\frac{1}{2})}$

$x = 1$

$\therefore P(1,0)$

(b) $x+y = t + \cancel{\frac{1}{4t}} + t - \cancel{\frac{1}{4t}} = 2t$

$x-y = \cancel{t} + \frac{1}{4t} - \cancel{t} + \frac{1}{4t} = \frac{1}{2t}$

$(x+y)(x-y) = 2t\left(\frac{1}{2t}\right)$

$x^2 - y^2 = 1$

(c) $\int_{x_1}^{x_2} y(x)\,dx = \int_1^{\frac{5}{3}} \sqrt{x^2-1}\,dx$

$y^2 = x^2 - 1$

$y = +\sqrt{x^2-1} \quad y>0$

(d) BY SUBSTITUTION $x = \cosh\theta \Rightarrow dx = \sinh\theta\,d\theta$

$x = 1 \mapsto \theta = 0$

$x = \frac{5}{3} \mapsto \theta = \text{arcosh}\frac{5}{3}$

THUS

$\text{AREA} = \int_0^{\text{arcosh}\frac{5}{3}} \sqrt{\cosh^2\theta - 1}\,\sinh\theta\,d\theta = \int_0^{\text{arcosh}\frac{5}{3}} \sinh^2\theta\,d\theta$

$= \int_0^{\text{arcosh}\frac{5}{3}} \frac{1}{2}\cosh 2\theta - \frac{1}{2}\,d\theta = \left[\frac{1}{4}\sinh 2\theta - \frac{1}{2}\theta\right]_0^{\text{arcosh}\frac{5}{3}}$

$= \left[\frac{1}{2}\sinh\theta\cosh\theta - \frac{1}{2}\theta\right]_0^{\text{arcosh}\frac{5}{3}}$

$= \left[\frac{1}{2}\sinh(\text{arcosh}\frac{5}{3})\cosh(\text{arcosh}\frac{5}{3}) - \frac{1}{2}\text{arcosh}\frac{5}{3}\right] - [0]$

$\text{arcosh}\frac{5}{3} = A$

$\cosh A = \frac{5}{3}$

$\cosh^2 A = \frac{25}{9}$

$\cosh^2 A - 1 = \frac{16}{9}$

$\sinh^2 A = \frac{16}{9}$

$\sinh A = \frac{4}{3}$

$\sinh(\text{arcosh}\frac{5}{3}) = \frac{4}{3}$

$= \frac{1}{2} \times \frac{4}{3} \times \frac{5}{3} - \frac{1}{2}\ln\left[\frac{5}{3} + \sqrt{\frac{25}{9} - 1}\right]$

$= \frac{20}{18} - \frac{1}{2}\ln 3$

$= \frac{10}{9} - \frac{1}{2}\ln 3$

Question 74 (****+)

The function f is defined

$$f(t) \equiv \ln(1+\sin t), \quad \sin t \neq \pm 1$$

a) Show clearly that …

i. … $f(t) - f(-t) = 2\ln(\sec t + \tan t)$.

ii. … $2\ln(\sec t + \tan t) = -2\ln(\sec t - \tan t)$

A curve C is given parametrically by

$$x = f(t) + f(-t), \quad y = f(t) - f(-t).$$

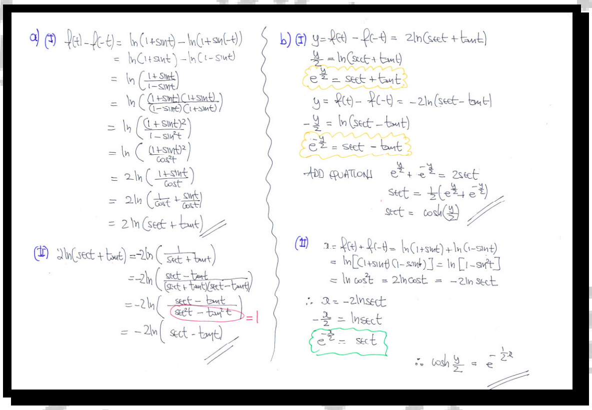
b) Show further that …

i. … $\sec t = \cosh \frac{y}{2}$

ii. … a Cartesian equation of C can be written as

$$\cosh \frac{y}{2} = e^{-\frac{1}{2}x}$$

proof



a) (I) $f(t) - f(-t) = \ln(1+\sin t) - \ln(1+\sin(-t))$

$= \ln(1+\sin t) - \ln(1-\sin t)$

$= \ln\left(\frac{1+\sin t}{1-\sin t}\right)$

$= \ln\left(\frac{(1+\sin t)(1+\sin t)}{(1-\sin t)(1+\sin t)}\right)$

$= \ln\left(\frac{(1+\sin t)^2}{1-\sin^2 t}\right)$

$= \ln\left(\frac{(1+\sin t)^2}{\cos^2 t}\right)$

$= 2\ln\left(\frac{1+\sin t}{\cos t}\right)$

$= 2\ln\left(\frac{1}{\cos t} + \frac{\sin t}{\cos t}\right)$

$= 2\ln(\sec t + \tan t)$ //

(II) $2\ln(\sec t + \tan t) = -2\ln\left(\frac{1}{\sec t + \tan t}\right)$

$= -2\ln\left(\frac{\sec t - \tan t}{(\sec t + \tan t)(\sec t - \tan t)}\right)$

$= -2\ln\left(\frac{\sec t - \tan t}{\boxed{\sec^2 t - \tan^2 t}}\right) = 1$

$= -2\ln(\sec t - \tan t)$ //

b) (I) $y = f(t) - f(-t) = 2\ln(\sec t + \tan t)$

$\frac{y}{2} = \ln(\sec t + \tan t)$

$\boxed{e^{\frac{y}{2}} = \sec t + \tan t}$

$y = f(t) - f(-t) = -2\ln(\sec t - \tan t)$

$-\frac{y}{2} = \ln(\sec t - \tan t)$

$\boxed{e^{-\frac{y}{2}} = \sec t - \tan t}$

ADD EQUATIONS $e^{\frac{y}{2}} + e^{-\frac{y}{2}} = 2\sec t$

$\sec t = \frac{1}{2}\left(e^{\frac{y}{2}} + e^{-\frac{y}{2}}\right)$

$\sec t = \cosh\left(\frac{y}{2}\right)$ //

(II) $x = f(t) + f(-t) = \ln(1+\sin t) + \ln(1-\sin t)$

$= \ln[(1+\sin t)(1-\sin t)] = \ln[1-\sin^2 t]$

$= \ln\cos^2 t = 2\ln\cos t = -2\ln\sec t$

$\therefore x = -2\ln\sec t$

$-\frac{x}{2} = \ln\sec t$

$\boxed{e^{-\frac{x}{2}} = \sec t}$

$\therefore \cosh\frac{y}{2} = e^{-\frac{1}{2}x}$ //

Question 75 **(****+)**

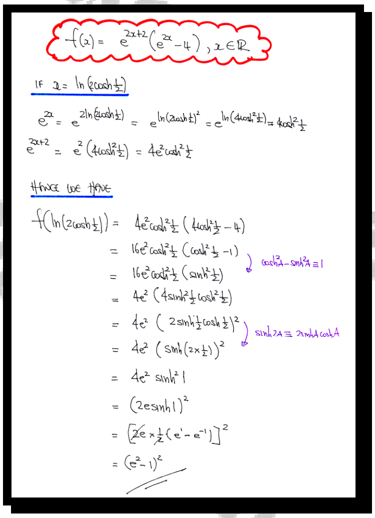
The function f is given by

$$f(x) \equiv e^{2x+2}\left(e^{2x}-4\right), \quad x \in \mathbb{R}.$$

Show that

$$f\left[\ln\left(2\cosh\tfrac{1}{2}\right)\right] = \left(e^{2}-1\right)^{2}.$$

$\boxed{}$, $\boxed{\text{proof}}$



$$f(x) = e^{2x+2}\left(e^{2x}-4\right), x \in \mathbb{R}$$

IF $x = \ln\left(2\cosh\frac{1}{2}\right)$

$$e^{2x} = e^{2\ln\left(2\cosh\frac{1}{2}\right)} = e^{\ln\left(2\cosh\frac{1}{2}\right)^2} = e^{\ln\left(4\cosh^2\frac{1}{2}\right)} = 4\cosh^2\tfrac{1}{2}$$

$$e^{2x+2} = e^{2}\left(4\cosh^2\tfrac{1}{2}\right) = 4e^2\cosh^2\tfrac{1}{2}$$

HENCE WE HAVE

$$\begin{aligned}
f\left(\ln\left(2\cosh\tfrac{1}{2}\right)\right) &= 4e^2\cosh^2\tfrac{1}{2}\left(4\cosh^2\tfrac{1}{2}-4\right)\\
&= 16e^2\cosh^2\tfrac{1}{2}\left(\cosh^2\tfrac{1}{2}-1\right) \quad \cosh^2A-\sinh^2A\equiv 1\\
&= 16e^2\cosh^2\tfrac{1}{2}\left(\sinh^2\tfrac{1}{2}\right)\\
&= 4e^2\left(4\sinh^2\tfrac{1}{2}\cosh^2\tfrac{1}{2}\right)\\
&= 4e^2\left(2\sinh\tfrac{1}{2}\cosh\tfrac{1}{2}\right)^2 \quad \sinh 2A \equiv 2\sinh A\cosh A\\
&= 4e^2\left(\sinh\left(2\times\tfrac{1}{2}\right)\right)^2\\
&= 4e^2\sinh^2 1\\
&= \left(2e\sinh 1\right)^2\\
&= \left[\cancel{2}e\times\tfrac{1}{\cancel{2}}\left(e^1-e^{-1}\right)\right]^2\\
&= \left(e^2-1\right)^2
\end{aligned}$$

Question 76 (****+)

It is given that for suitable values of x

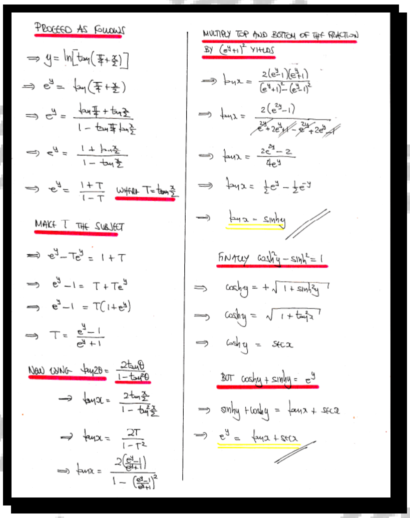
$$y = \ln\left[\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}x\right)\right] \quad .$$

Show, with detailed workings, that

$$\sinh y = \tan x ,$$

and hence deduce a simplified expression for e^y in terms of x.

$\boxed{\text{-U}}$, $\boxed{e^y = \tan x + \sec x}$



PROCEED AS FOLLOWS

$\Rightarrow y = \ln\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$

$\Rightarrow e^y = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

$\Rightarrow e^y = \frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}}$

$\Rightarrow e^y = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$

$\Rightarrow e^y = \frac{1+T}{1-T}$ WHERE $T = \tan\frac{x}{2}$

MAKE T THE SUBJECT

$\Rightarrow e^y - Te^y = 1 + T$

$\Rightarrow e^y - 1 = T + Te^y$

$\Rightarrow e^y - 1 = T(1+e^y)$

$\Rightarrow T = \frac{e^y-1}{e^y+1}$

NOW USING $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\rightarrow \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$

$\rightarrow \tan x = \frac{2T}{1-T^2}$

$\Rightarrow \tan x = \frac{2\left(\frac{e^y-1}{e^y+1}\right)}{1-\left(\frac{e^y-1}{e^y+1}\right)^2}$

MULTIPLY TOP AND BOTTOM OF THE FRACTION BY $(e^y+1)^2$ YIELDS

$\Rightarrow \tan x = \frac{2(e^y-1)(e^y+1)}{(e^y+1)^2-(e^y-1)^2}$

$\Rightarrow \tan x = \frac{2(e^{2y}-1)}{\cancel{e^{2y}}+2e^y\cancel{+1}-\cancel{e^{2y}}+2e^y\cancel{-1}}$

$\Rightarrow \tan x = \frac{2e^{2y}-2}{4e^y}$

$\Rightarrow \tan x = \frac{1}{2}e^y - \frac{1}{2}e^{-y}$

$\rightarrow \tan x = \sinh y$ //

FINALLY $\cosh^2 y - \sinh^2 = 1$

$\Rightarrow \cosh y = +\sqrt{1+\sinh^2 y}$

$\Rightarrow \cosh y = \sqrt{1+\tan^2 x}$

$\Rightarrow \cosh y = \sec x$

BUT $\cosh y + \sinh y = e^y$

$\Rightarrow \sinh y + \cosh y = \tan x + \sec x$

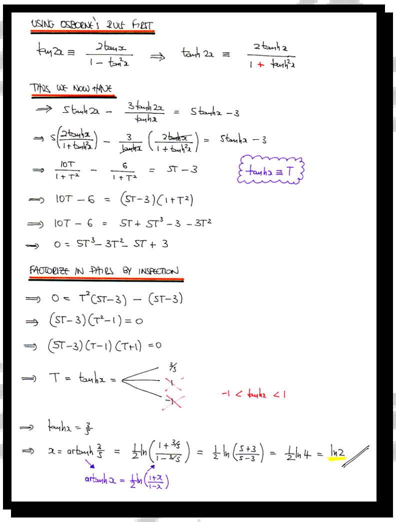
$\Rightarrow e^y = \tan x + \sec x$ //

Question 77 **(****+)**

$$5\tanh 2x - \frac{3\tan 2x}{\tanh x} = 5\tanh x - 3 .$$

Find, as an exact natural logarithm, the real solution of the above equation.

$\boxed{\quad}$, $\boxed{x = \ln 2}$



USING OSBORNE'S RULE FIRST

$\tan 2x \equiv \frac{2\tan x}{1-\tan^2 x} \Rightarrow \tanh 2x \equiv \frac{2\tanh x}{1+\tanh^2 x}$

THUS WE NOW HAVE

$\Rightarrow 5\tanh 2x - \frac{3\tanh 2x}{\tanh x} = 5\tanh x - 3$

$\Rightarrow 5\left(\frac{2\tanh x}{1+\tanh^2 x}\right) - \frac{3}{\cancel{\tanh x}}\left(\frac{2\cancel{\tanh x}}{1+\tanh^2 x}\right) = 5\tanh x - 3$

$\Rightarrow \frac{10T}{1+T^2} - \frac{6}{1+T^2} = 5T - 3$ $\tanh x \equiv T$

$\Rightarrow 10T - 6 = (5T-3)(1+T^2)$

$\Rightarrow 10T - 6 = 5T + 5T^3 - 3 - 3T^2$

$\Rightarrow 0 = 5T^3 - 3T^2 - 5T + 3$

FACTORIZE IN PAIRS BY INSPECTION

$\Rightarrow 0 = T^2(5T-3) - (5T-3)$

$\Rightarrow (5T-3)(T^2-1) = 0$

$\Rightarrow (5T-3)(T-1)(T+1) = 0$

$\Rightarrow T = \tanh x = \begin{cases} 3/5 \\ \cancel{1} \\ \cancel{-1} \end{cases}$ $-1 < \tanh x < 1$

$\Rightarrow \tanh x = \frac{3}{5}$

$\Rightarrow x = \text{artanh}\,\frac{3}{5} = \frac{1}{2}\ln\left(\frac{1+3/5}{1-3/5}\right) = \frac{1}{2}\ln\left(\frac{5+3}{5-3}\right) = \frac{1}{2}\ln 4 = \ln 2$ //

$\text{artanh}\,x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

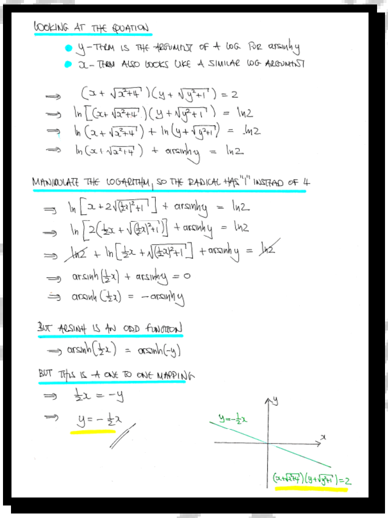
Question 78 (*****)

Sketch the graph of

$$\left[x+\sqrt{x^2+4}\right]\left[y+\sqrt{y^2+1}\right]=2\,,\quad x\in(-\infty,\infty)\,,\ y\in(-\infty,\infty)$$

You must show a detailed method in this question

$\boxed{}$, $\boxed{\text{proof}}$



LOOKING AT THE EQUATION

- y – TERM IS THE ARGUMENT OF A LOG FOR arsinh y
- x – TERM ALSO LOOKS LIKE A SIMILAR LOG ARGUMENT

$$\Rightarrow \left(x+\sqrt{x^2+4}\right)\left(y+\sqrt{y^2+1}\right)=2$$
$$\Rightarrow \ln\left[\left(x+\sqrt{x^2+4}\right)\left(y+\sqrt{y^2+1}\right)\right]=\ln 2$$
$$\Rightarrow \ln\left(x+\sqrt{x^2+4}\right)+\ln\left(y+\sqrt{y^2+1}\right)=\ln 2$$
$$\Rightarrow \ln\left(x+\sqrt{x^2+4}\right)+\text{arsinh}\,y=\ln 2$$

MANIPULATE THE LOGARITHM, SO THE RADICAL HAS "1" INSTEAD OF 4

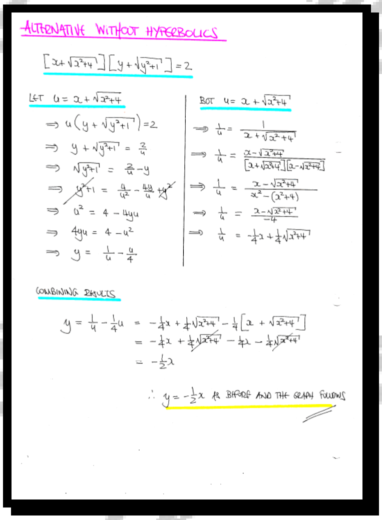
$$\Rightarrow \ln\left[x+2\sqrt{\left(\tfrac{1}{2}x\right)^2+1}\right]+\text{arsinh}\,y=\ln 2$$
$$\Rightarrow \ln\left[2\left(\tfrac{1}{2}x+\sqrt{\left(\tfrac{1}{2}x\right)^2+1}\right)\right]+\text{arsinh}\,y=\ln 2$$
$$\Rightarrow \cancel{\ln 2}+\ln\left[\tfrac{1}{2}x+\sqrt{\left(\tfrac{1}{2}x\right)^2+1}\right]+\text{arsinh}\,y=\cancel{\ln 2}$$
$$\Rightarrow \text{arsinh}\left(\tfrac{1}{2}x\right)+\text{arsinh}\,y=0$$
$$\Rightarrow \text{arsinh}\left(\tfrac{1}{2}x\right)=-\text{arsinh}\,y$$

BUT ARSINH IS AN ODD FUNCTION

$$\Rightarrow \text{arsinh}\left(\tfrac{1}{2}x\right)=\text{arsinh}(-y)$$

BUT THIS IS A ONE TO ONE MAPPING

$$\Rightarrow \tfrac{1}{2}x=-y$$
$$\Rightarrow y=-\tfrac{1}{2}x$$



ALTERNATIVE WITHOUT HYPERBOLICS

$$\left[x+\sqrt{x^2+4}\right]\left[y+\sqrt{y^2+1}\right]=2$$

LET $u=x+\sqrt{x^2+4}$

$$\Rightarrow u\left(y+\sqrt{y^2+1}\right)=2$$
$$\Rightarrow y+\sqrt{y^2+1}=\frac{2}{u}$$
$$\Rightarrow \sqrt{y^2+1}=\frac{2}{u}-y$$
$$\Rightarrow \cancel{y^2}+1=\frac{4}{u^2}-\frac{4y}{u}+\cancel{y^2}$$
$$\Rightarrow u^2=4-4yu$$
$$\Rightarrow 4yu=4-u^2$$
$$\Rightarrow y=\frac{1}{u}-\frac{u}{4}$$

BUT $u=x+\sqrt{x^2+4}$

$$\Rightarrow \frac{1}{u}=\frac{1}{x+\sqrt{x^2+4}}$$
$$\Rightarrow \frac{1}{u}=\frac{x-\sqrt{x^2+4}}{\left[x+\sqrt{x^2+4}\right]\left[x-\sqrt{x^2+4}\right]}$$
$$\Rightarrow \frac{1}{u}=\frac{x-\sqrt{x^2+4}}{x^2-(x^2+4)}$$
$$\Rightarrow \frac{1}{u}=\frac{x-\sqrt{x^2+4}}{-4}$$
$$\Rightarrow \frac{1}{u}=-\frac{1}{4}x+\frac{1}{4}\sqrt{x^2+4}$$

COMBINING RESULTS

$$\begin{aligned} y=\frac{1}{u}-\frac{1}{4}u &= -\tfrac{1}{4}x+\tfrac{1}{4}\sqrt{x^2+4}-\tfrac{1}{4}\left[x+\sqrt{x^2+4}\right] \\ &= -\tfrac{1}{4}x+\cancel{\tfrac{1}{4}\sqrt{x^2+4}}-\tfrac{1}{4}x-\cancel{\tfrac{1}{4}\sqrt{x^2+4}} \\ &= -\tfrac{1}{2}x \end{aligned}$$

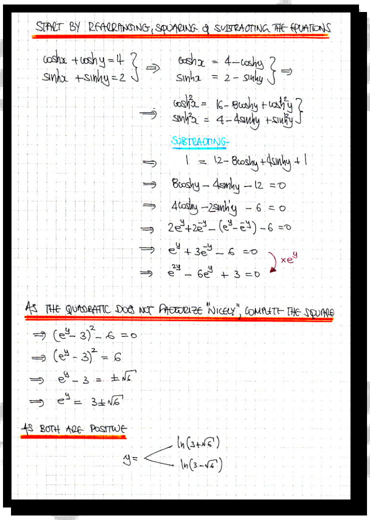
$\therefore\ y=-\frac{1}{2}x$ AS BEFORE AND THE GRAPH FOLLOWS

Question 79 (*****)

Determine, as exact simplified natural logarithms, the solutions of the following simultaneous equations

$$\cosh x + \cosh y = 4 \qquad \text{and} \qquad \sinh x + \sinh y = 2 .$$

$$[x, y] = \left[\ln\left(3-\sqrt{6}\right), \ln\left(3+\sqrt{6}\right)\right] = \left[\ln\left(3+\sqrt{6}\right), \ln\left(3-\sqrt{6}\right)\right]$$



START BY REARRANGING, SQUARING & SUBTRACTING THE EQUATIONS

$$\left.\begin{aligned}\cosh x + \cosh y &= 4\\ \sinh x + \sinh y &= 2\end{aligned}\right\} \Rightarrow \left.\begin{aligned}\cosh x &= 4 - \cosh y\\ \sinh x &= 2 - \sinh y\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned}\cosh^2 x &= 16 - 8\cosh y + \cosh^2 y\\ \sinh^2 x &= 4 - 4\sinh y + \sinh^2 y\end{aligned}\right\}$$

SUBTRACTING

$$\Rightarrow 1 = 12 - 8\cosh y + 4\sinh y + 1$$

$$\Rightarrow 8\cosh y - 4\sinh y - 12 = 0$$

$$\Rightarrow 4\cosh y - 2\sinh y - 6 = 0$$

$$\Rightarrow 2e^{y} + 2e^{-y} - (e^{y} - e^{-y}) - 6 = 0$$

$$\Rightarrow e^{y} + 3e^{-y} - 6 = 0 \quad \times e^{y}$$

$$\Rightarrow e^{2y} - 6e^{y} + 3 = 0$$

AS THE QUADRATIC DOES NOT FACTORIZE "NICELY", COMPLETE THE SQUARE

$$\Rightarrow (e^{y} - 3)^2 - 6 = 0$$

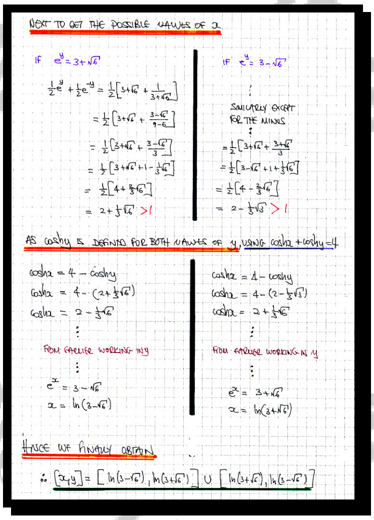
$$\Rightarrow (e^{y} - 3)^2 = 6$$

$$\Rightarrow e^{y} - 3 = \pm\sqrt{6}$$

$$\Rightarrow e^{y} = 3 \pm \sqrt{6}$$

AS BOTH ARE POSITIVE

$$y = \begin{cases}\ln(3+\sqrt{6})\\ \ln(3-\sqrt{6})\end{cases}$$



NEXT TO GET THE POSSIBLE VALUES OF x

IF $e^{y} = 3 + \sqrt{6}$

$$\frac{1}{2}e^{y} + \frac{1}{2}e^{-y} = \frac{1}{2}\left[3+\sqrt{6} + \frac{1}{3+\sqrt{6}}\right]$$

$$= \frac{1}{2}\left[3+\sqrt{6} + \frac{3-\sqrt{6}}{9-6}\right]$$

$$= \frac{1}{2}\left[3+\sqrt{6} + \frac{3-\sqrt{6}}{3}\right]$$

$$= \frac{1}{2}\left[3+\sqrt{6} + 1 - \frac{1}{3}\sqrt{6}\right]$$

$$= \frac{1}{2}\left[4 + \frac{2}{3}\sqrt{6}\right]$$

$$= 2 + \frac{1}{3}\sqrt{6} > 1$$

IF $e^{y} = 3 - \sqrt{6}$

$\vdots$

SIMILARLY EXCEPT FOR THE MINUS

$\vdots$

$$= \frac{1}{2}\left[3+\sqrt{6} + \frac{3+\sqrt{6}}{3}\right]$$

$$= \frac{1}{2}\left[3-\sqrt{6} + 1 + \frac{1}{3}\sqrt{6}\right]$$

$$= \frac{1}{2}\left[4 - \frac{2}{3}\sqrt{6}\right]$$

$$= 2 - \frac{1}{3}\sqrt{3} > 1$$

AS $\cosh y$ IS DEFINED FOR BOTH VALUES OF y, USING $\cosh x + \cosh y = 4$

$$\cosh x = 4 - \cosh y$$

$$\cosh x = 4 - \left(2 + \frac{1}{3}\sqrt{6}\right)$$

$$\cosh x = 2 - \frac{1}{3}\sqrt{6}$$

$\vdots$

FROM EARLIER WORKING IN y

$\vdots$

$$e^{x} = 3 - \sqrt{6}$$

$$x = \ln(3-\sqrt{6})$$

$$\cosh x = 4 - \cosh y$$

$$\cosh x = 4 - \left(2 - \frac{1}{3}\sqrt{3}\right)$$

$$\cosh x = 2 + \frac{1}{3}\sqrt{6}$$

$\vdots$

FROM EARLIER WORKING IN y

$\vdots$

$$e^{x} = 3 + \sqrt{6}$$

$$x = \ln(3+\sqrt{6})$$

HENCE WE FINALLY OBTAIN

$$\therefore [x, y] = \left[\ln(3-\sqrt{6}), \ln(3+\sqrt{6})\right] \cup \left[\ln(3+\sqrt{6}), \ln(3-\sqrt{6})\right]$$

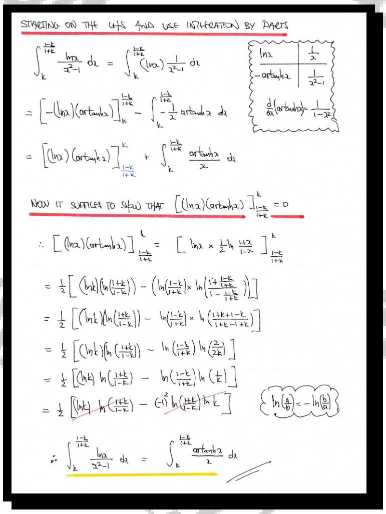
Question 80 (*****)

If $0 < k < \sqrt{2} - 1$ prove that

$$\int_{k}^{\frac{1-k}{1+k}} \frac{\ln x}{x^2 - 1} \, dx = \int_{k}^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x} \, dx .$$

You need not evaluate these integrals.

$\boxed{}$, $\boxed{\text{proof}}$



STARTING ON THE LHS AND USE INTEGRATION BY PARTS

$$\int_{k}^{\frac{1-k}{1+k}} \frac{\ln x}{x^2-1}\,dx = \int_{k}^{\frac{1-k}{1+k}} (\ln x)\frac{1}{x^2-1}\,dx$$

$$= \Big[-(\ln x)(\operatorname{artanh}x)\Big]_{k}^{\frac{1-k}{1+k}} - \int_{k}^{\frac{1-k}{1+k}} -\frac{1}{x}\operatorname{artanh}x\,dx$$

$$= \Big[(\ln x)(\operatorname{artanh}x)\Big]_{\frac{1-k}{1+k}}^{k} + \int_{k}^{\frac{1-k}{1+k}} \frac{\operatorname{artanh}x}{x}\,dx$$

$\ln x$	$\frac{1}{x}$
$-\operatorname{artanh}x$	$\frac{1}{x^2-1}$

$\frac{d}{dx}(\operatorname{artanh}x) = \frac{1}{1-x^2}$

NOW IT SUFFICES TO SHOW THAT $\Big[(\ln x)(\operatorname{artanh}x)\Big]_{\frac{1-k}{1+k}}^{k} = 0$

$$\therefore \Big[(\ln x)(\operatorname{artanh}x)\Big]_{\frac{1-k}{1+k}}^{k} = \Big[\ln x \times \tfrac{1}{2}\ln\tfrac{1+x}{1-x}\Big]_{\frac{1-k}{1+k}}^{k}$$

$$= \frac{1}{2}\left[(\ln k)\left(\ln\left(\frac{1+k}{1-k}\right)\right) - \left(\ln\left(\frac{1-k}{1+k}\right)\times\ln\left(\frac{1+\frac{1-k}{1+k}}{1-\frac{1-k}{1+k}}\right)\right)\right]$$

$$= \frac{1}{2}\left[(\ln k)\left(\ln\left(\frac{1+k}{1-k}\right)\right) - \ln\left(\frac{1-k}{1+k}\right)\times\ln\left(\frac{1+k+1-k}{1+k-1+k}\right)\right]$$

$$= \frac{1}{2}\left[(\ln k)\left(\ln\left(\frac{1+k}{1-k}\right)\right) - \ln\left(\frac{1-k}{1+k}\right)\ln\left(\frac{2}{2k}\right)\right]$$

$$= \frac{1}{2}\left[(\ln k)\ln\left(\frac{1+k}{1-k}\right) - \ln\left(\frac{1-k}{1+k}\right)\ln\left(\frac{1}{k}\right)\right]$$

$$= \frac{1}{2}\left[\cancel{(\ln k)\ln\left(\frac{1+k}{1-k}\right)} - \cancel{(-1)^2\ln\left(\frac{1+k}{1-k}\right)\ln k}\right]$$

$\ln\left(\frac{a}{b}\right) = -\ln\left(\frac{b}{a}\right)$

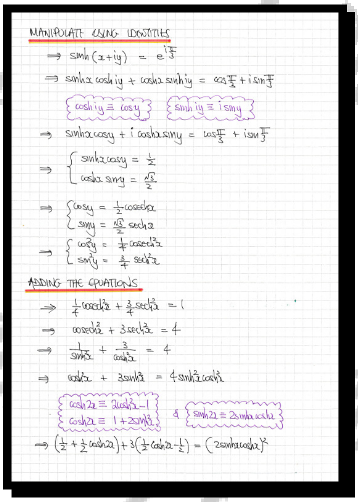
$$\therefore \int_{k}^{\frac{1-k}{1+k}} \frac{\ln x}{x^2-1}\,dx = \int_{k}^{\frac{1-k}{1+k}} \frac{\operatorname{artanh}x}{x}\,dx$$

Question 81 (*****)

Determine the general solution of the following equation.

$$\sinh(x+\mathrm{i}y) = \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}.$$

$$\boxed{}, \quad \boxed{(x,y) = \left[\ln\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right), \frac{\pi}{4}+2k\pi\right], \; k \in \mathbb{Z}}$$



MANIPULATE USING IDENTITIES

$\Rightarrow \sinh(x+iy) = e^{i\frac{\pi}{3}}$

$\Rightarrow \sinh x \cosh iy + \cosh x \sinh iy = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$\cosh iy \equiv \cos y$ $\quad$ $\sinh iy \equiv i\sin y$

$\Rightarrow \sinh x \cos y + i\cosh x \sin y = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$\Rightarrow \begin{cases} \sinh x \cos y = \frac{1}{2} \\ \cosh x \sin y = \frac{\sqrt{3}}{2} \end{cases}$

$\Rightarrow \begin{cases} \cos y = \frac{1}{2}\operatorname{cosech} x \\ \sin y = \frac{\sqrt{3}}{2}\operatorname{sech} x \end{cases}$

$\Rightarrow \begin{cases} \cos^2 y = \frac{1}{4}\operatorname{cosech}^2 x \\ \sin^2 y = \frac{3}{4}\operatorname{sech}^2 x \end{cases}$

ADDING THE EQUATIONS

$\Rightarrow \frac{1}{4}\operatorname{cosech}^2 x + \frac{3}{4}\operatorname{sech}^2 x = 1$

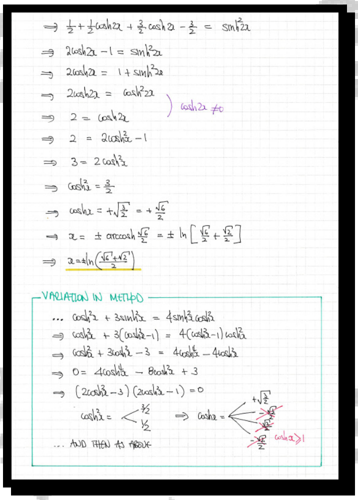
$\Rightarrow \operatorname{cosech}^2 x + 3\operatorname{sech}^2 x = 4$

$\Rightarrow \frac{1}{\sinh^2 x} + \frac{3}{\cosh^2 x} = 4$

$\Rightarrow \cosh^2 x + 3\sinh^2 x = 4\sinh^2 x\cosh^2 x$

$\cosh 2x \equiv 2\cosh^2 x - 1$
$\cosh 2x \equiv 1 + 2\sinh^2 x$ & $\sinh 2x \equiv 2\sinh x\cosh x$

$\Rightarrow \left(\frac{1}{2}+\frac{1}{2}\cosh 2x\right) + 3\left(\frac{1}{2}\cosh 2x - \frac{1}{2}\right) = (2\sinh x\cosh x)^2$



$\Rightarrow \frac{1}{2}+\frac{1}{2}\cosh 2x + \frac{3}{2}\cosh 2x - \frac{3}{2} = \sinh^2 2x$

$\Rightarrow 2\cosh 2x - 1 = \sinh^2 2x$

$\Rightarrow 2\cosh 2x = 1 + \sinh^2 2x$

$\Rightarrow 2\cosh 2x = \cosh^2 2x$ $\quad$) $\cosh 2x \neq 0$

$\Rightarrow 2 = \cosh 2x$

$\Rightarrow 2 = 2\cosh^2 x - 1$

$\Rightarrow 3 = 2\cosh^2 x$

$\Rightarrow \cosh^2 x = \frac{3}{2}$

$\Rightarrow \cosh x = +\sqrt{\frac{3}{2}} = +\frac{\sqrt{6}}{2}$

$\Rightarrow x = \pm\operatorname{arccosh}\frac{\sqrt{6}}{2} = \pm\ln\left[\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}\right]$

$\Rightarrow x = \pm\ln\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)$

VARIATION IN METHOD

$\ldots \cosh^2 x + 3\sinh^2 x = 4\sinh^2 x\cosh^2 x$

$\Rightarrow \cosh^2 x + 3(\cosh^2 x - 1) = 4(\cosh^2 x - 1)\cosh^2 x$

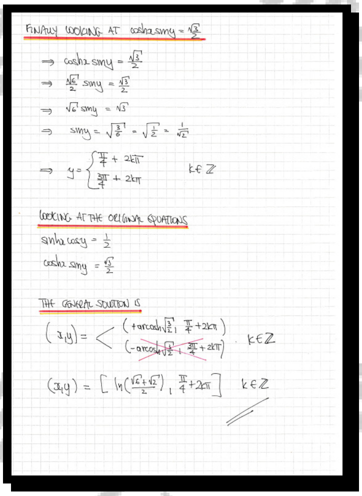
$\Rightarrow \cosh^2 x + 3\cosh^2 x - 3 = 4\cosh^4 x - 4\cosh^2 x$

$\Rightarrow 0 = 4\cosh^4 x - 8\cosh^2 x + 3$

$\Rightarrow (2\cosh^2 x - 3)(2\cosh^2 x - 1) = 0$

$\cosh^2 x = \begin{cases} 3/2 \\ 1/2 \end{cases} \Rightarrow \cosh x = \begin{cases} +\sqrt{\frac{3}{2}} \\ \cancel{-\frac{\sqrt{3}}{2}} \\ \cancel{\frac{\sqrt{2}}{2}} \\ \cancel{-\frac{\sqrt{2}}{2}} \end{cases}$ $\cosh x \geqslant 1$

$\ldots$ AND THEN AS ABOVE



FINALLY LOOKING AT $\cosh x \sin y = \frac{\sqrt{3}}{2}$

$\Rightarrow \cosh x \sin y = \frac{\sqrt{3}}{2}$

$\Rightarrow \frac{\sqrt{6}}{2}\sin y = \frac{\sqrt{3}}{2}$

$\Rightarrow \sqrt{6}\sin y = \sqrt{3}$

$\Rightarrow \sin y = \sqrt{\frac{3}{6}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$\Rightarrow y = \begin{cases} \frac{\pi}{4} + 2k\pi \\ \frac{3\pi}{4} + 2k\pi \end{cases} \quad k \in \mathbb{Z}$

LOOKING AT THE ORIGINAL EQUATIONS

$\sinh x \cos y = \frac{1}{2}$

$\cosh x \sin y = \frac{\sqrt{5}}{2}$

THE GENERAL SOLUTION IS

$(x,y) = \begin{cases} \left(+\operatorname{arcosh}\sqrt{\frac{3}{2}}, \frac{\pi}{4}+2k\pi\right) \\ \cancel{\left(-\operatorname{arcosh}\sqrt{\frac{3}{2}}, \frac{3\pi}{4}+2k\pi\right)} \end{cases} \quad k \in \mathbb{Z}$

$(x,y) = \left[\ln\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right), \frac{\pi}{4}+2k\pi\right] \quad k \in \mathbb{Z}$

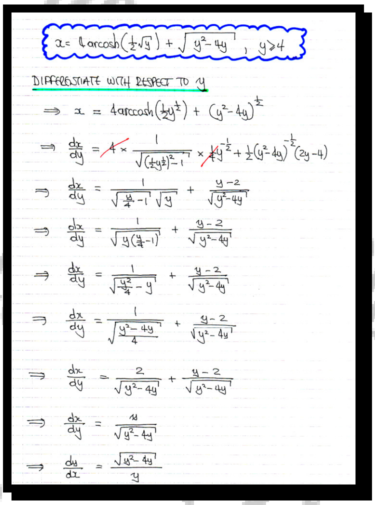
Question 82 (*****)

$$x = 4\operatorname{arcosh}\left(\tfrac{1}{2}\sqrt{y}\right) + \sqrt{y^2 - 4y}\,, \quad y \ge 4\,.$$

Use differentiation to show that

$$\frac{d^2y}{dx^2} = \frac{2}{y^2}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$



$$x = 4\operatorname{arcosh}\left(\tfrac{1}{2}\sqrt{y}\right) + \sqrt{y^2-4y}\,, \quad y \geqslant 4$$

DIFFERENTIATE WITH RESPECT TO y

$$\Rightarrow x = 4\operatorname{arccosh}\left(\tfrac{1}{2}y^{\frac{1}{2}}\right) + \left(y^2-4y\right)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dx}{dy} = \cancel{4} \times \frac{1}{\sqrt{\left(\frac{1}{2}y^{\frac{1}{2}}\right)^2 - 1}} \times \cancel{\tfrac{1}{4}}y^{-\frac{1}{2}} + \tfrac{1}{2}\left(y^2-4y\right)^{-\frac{1}{2}}(2y-4)$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{\frac{y}{4}-1}\,\sqrt{y}} + \frac{y-2}{\sqrt{y^2-4y}}$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{y\left(\frac{y}{4}-1\right)}} + \frac{y-2}{\sqrt{y^2-4y}}$$

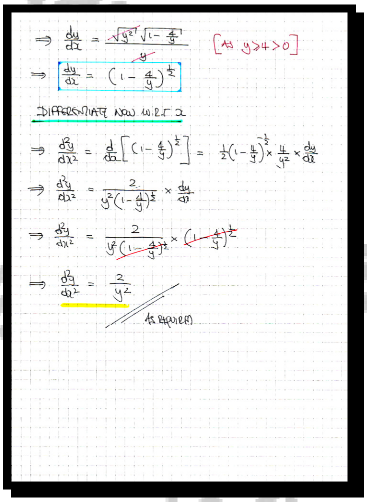
$$\Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{\frac{y^2}{4}-y}} + \frac{y-2}{\sqrt{y^2-4y}}$$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{\frac{y^2-4y}{4}}} + \frac{y-2}{\sqrt{y^2-4y}}$$

$$\Rightarrow \frac{dx}{dy} = \frac{2}{\sqrt{y^2-4y}} + \frac{y-2}{\sqrt{y^2-4y}}$$

$$\Rightarrow \frac{dx}{dy} = \frac{y}{\sqrt{y^2-4y}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{y^2-4y}}{y}$$



$$\Rightarrow \frac{dy}{dx} = \frac{\cancel{\sqrt{y^2}}\sqrt{1-\frac{4}{y}}}{\cancel{y}} \qquad \left[\text{AS } y \geqslant 4 > 0\right]$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \left(1-\frac{4}{y}\right)^{\frac{1}{2}}}$$

DIFFERENTIATE NOW W.R.T x

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\left(1-\frac{4}{y}\right)^{\frac{1}{2}}\right] = \frac{1}{2}\left(1-\frac{4}{y}\right)^{-\frac{1}{2}} \times \frac{4}{y^2} \times \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2}{y^2\left(1-\frac{4}{y}\right)^{\frac{1}{2}}} \times \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2}{y^2\cancel{\left(1-\frac{4}{y}\right)^{\frac{1}{2}}}} \times \cancel{\left(1-\frac{4}{y}\right)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2}{y^2}$$

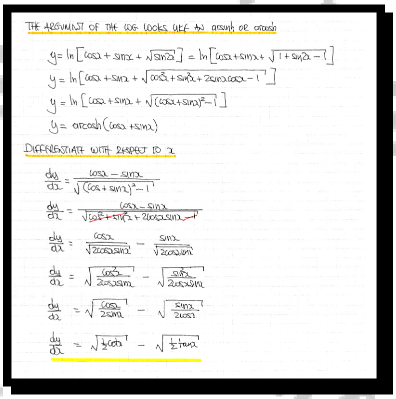
AS REQUIRED

Question 83 (*****)

Use inverse hyperbolic functions to show that

$$\frac{d}{dx}\left[\ln\left(\cos x+\sin x+\sqrt{\sin 2x}\right)\right] = \sqrt{\frac{1}{2}\cot x} - \sqrt{\frac{1}{2}\tan x}.$$

$\boxed{}$, $\boxed{\text{proof}}$



THE ARGUMENT OF THE LOG LOOKS LIKE AN arsinh OR arcosh

$y=\ln\left[\cos x+\sin x+\sqrt{\sin 2x}\right]=\ln\left[\cos x+\sin x+\sqrt{1+\sin 2x-1}\right]$

$y=\ln\left[\cos x+\sin x+\sqrt{\cos^2 x+\sin^2 x+2\sin x\cos x-1}\right]$

$y=\ln\left[\cos x+\sin x+\sqrt{(\cos x+\sin x)^2-1}\right]$

$y=\text{arcosh}(\cos x+\sin x)$

DIFFERENTIATE WITH RESPECT TO x

$\frac{dy}{dx}=\frac{\cos x-\sin x}{\sqrt{(\cos+\sin x)^2-1}}$

$\frac{dy}{dx}=\frac{\cos x-\sin x}{\sqrt{\cancel{\cos^2 x+\sin^2 x}+2\cos x\sin x\cancel{-1}}}$

$\frac{dy}{dx}=\frac{\cos x}{\sqrt{2\cos x\sin x}}-\frac{\sin x}{\sqrt{2\cos x\sin x}}$

$\frac{dy}{dx}=\sqrt{\frac{\cos^2 x}{2\cos x\sin x}}-\sqrt{\frac{\sin^2 x}{2\cos x\sin x}}$

$\frac{dy}{dx}=\sqrt{\frac{\cos x}{2\sin x}}-\sqrt{\frac{\sin x}{2\cos x}}$

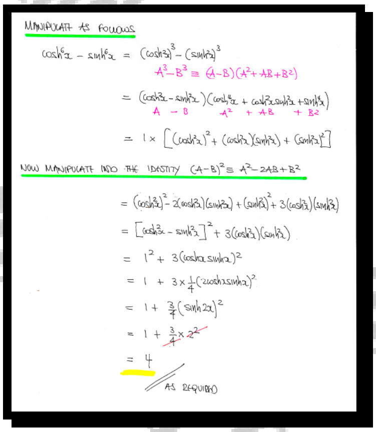
$\frac{dy}{dx}=\sqrt{\frac{1}{2}\cot x}-\sqrt{\frac{1}{2}\tan x}$

Question 84 (*****)

Show, with detailed workings, that

$$\sinh 2x = 2 \quad \Rightarrow \quad \cosh^6 x - \sinh^6 x = 4$$

$\boxed{}$, $\boxed{\text{proof}}$



MANIPULATE AS FOLLOWS

$\cosh^6 x-\sinh^6 x=(\cosh^2 x)^3-(\sinh^2 x)^3$

$A^3-B^3\equiv(A-B)(A^2+AB+B^2)$

$=\underbrace{(\cosh^2 x-\sinh^2 x)}_{A\ -\ B}\underbrace{(\cosh^4 x+\cosh^2 x\sinh^2 x+\sinh^4 x)}_{A^2\ +\ AB\ +\ B^2}$

$=1\times\left[(\cosh^2 x)^2+(\cosh^2 x)(\sinh^2 x)+(\sinh^2 x)^2\right]$

NOW MANIPULATE INTO THE IDENTITY $(A-B)^2\equiv A^2-2AB+B^2$

$=(\cosh^2 x)^2-2(\cosh^2 x)(\sinh^2 x)+(\sinh^2 x)^2+3(\cosh^2 x)(\sinh^2 x)$

$=\left[\cosh^2 x-\sinh^2 x\right]^2+3(\cosh^2 x)(\sinh^2 x)$

$=1^2+3(\cosh x\sinh x)^2$

$=1+3\times\frac{1}{4}(2\cosh x\sinh x)^2$

$=1+\frac{3}{4}(\sinh 2x)^2$

$=1+\frac{3}{\cancel{4}}\times\cancel{2^2}$

$=4$

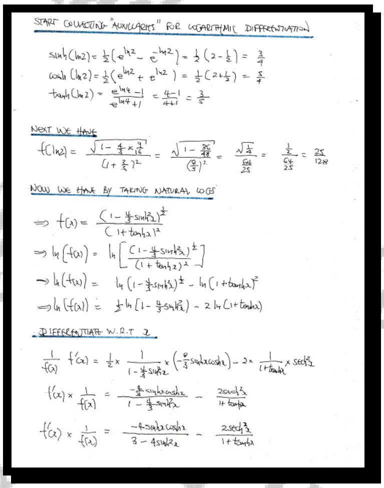
// AS REQUIRED

Question 85 (*****)

$$f(x) \equiv \frac{\sqrt{1-\frac{4}{3}\sinh^2 x}}{(1+\tanh x)^2}.$$

Determine the value of $f'(\ln 2)$.

V, ☐, $\boxed{f'(\ln 2) = -\frac{145}{256}}$



START COLLECTING "AUXILLARIES" FOR LOGARITHMIC DIFFERENTIATION

$\sinh(\ln 2) = \frac{1}{2}(e^{\ln 2} - e^{-\ln 2}) = \frac{1}{2}(2-\frac{1}{2}) = \frac{3}{4}$

$\cosh(\ln 2) = \frac{1}{2}(e^{\ln 2} + e^{\ln 2}) = \frac{1}{2}(2+\frac{1}{2}) = \frac{5}{4}$

$\tanh(\ln 2) = \frac{e^{\ln 4}-1}{e^{\ln 4}+1} = \frac{4-1}{4+1} = \frac{3}{5}$

NEXT WE HAVE

$f(\ln 2) = \frac{\sqrt{1-\frac{4}{3}\times\frac{9}{16}}}{(1+\frac{3}{5})^2} = \frac{\sqrt{1-\frac{36}{48}}}{(\frac{8}{5})^2} = \frac{\sqrt{\frac{1}{4}}}{\frac{64}{25}} = \frac{\frac{1}{2}}{\frac{64}{25}} = \frac{25}{128}$

NOW WE HAVE BY TAKING NATURAL LOGS

$\Rightarrow f(x) = \frac{(1-\frac{4}{3}\sinh^2 x)^{\frac{1}{2}}}{(1+\tanh x)^2}$

$\Rightarrow \ln(f(x)) = \ln\left[\frac{(1-\frac{4}{3}\sinh^2 x)^{\frac{1}{2}}}{(1+\tanh x)^2}\right]$

$\Rightarrow \ln(f(x)) = \ln(1-\frac{4}{3}\sinh^2 x)^{\frac{1}{2}} - \ln(1+\tanh x)^2$

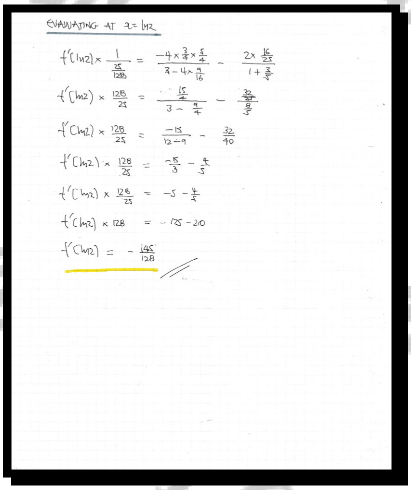
$\Rightarrow \ln(f(x)) = \frac{1}{2}\ln(1-\frac{4}{3}\sinh^2 x) - 2\ln(1+\tanh x)$

DIFFERENTIATE W.R.T x

$\frac{1}{f(x)} f'(x) = \frac{1}{2} \times \frac{1}{1-\frac{4}{3}\sinh^2 x} \times (-\frac{8}{3}\sinh x\cosh x) - 2 \times \frac{1}{1+\tanh x} \times \operatorname{sech}^2 x$

$f'(x) \times \frac{1}{f(x)} = \frac{-\frac{4}{3}\sinh x\cosh x}{1-\frac{4}{3}\sinh^2 x} - \frac{2\operatorname{sech}^2 x}{1+\tanh x}$

$f'(x) \times \frac{1}{f(x)} = \frac{-4\sinh x\cosh x}{3-4\sinh^2 x} - \frac{2\operatorname{sech}^2 x}{1+\tanh x}$



EVALUATING AT $x = \ln 2$

$f'(\ln 2) \times \frac{1}{\frac{25}{128}} = \frac{-4\times\frac{3}{4}\times\frac{5}{4}}{3-4\times\frac{9}{16}} - \frac{2\times\frac{16}{25}}{1+\frac{3}{5}}$

$f'(\ln 2) \times \frac{128}{25} = \frac{-\frac{15}{4}}{3-\frac{9}{4}} - \frac{\frac{32}{25}}{\frac{8}{5}}$

$f'(\ln 2) \times \frac{128}{25} = \frac{-15}{12-9} - \frac{32}{40}$

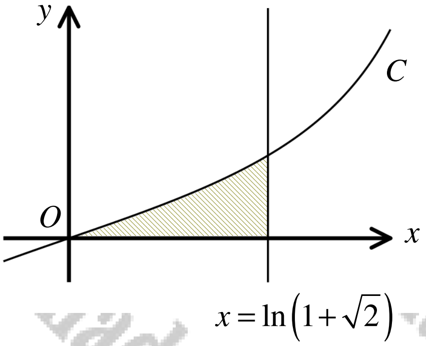
$f'(\ln 2) \times \frac{128}{25} = \frac{-15}{3} - \frac{4}{5}$

$f'(\ln 2) \times \frac{128}{25} = -5 - \frac{4}{5}$

$f'(\ln 2) \times 128 = -125 - 20$

$f'(\ln 2) = -\frac{145}{128}$ //

Question 86 (*****)

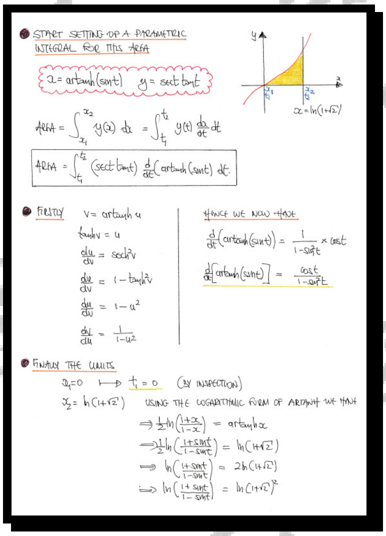


The figure above shows the curve C whose parametric equations are

$$x = \operatorname{artanh}(\sin t), \quad y = \sec t \tan t, \quad -\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi .$$

Find the area of the finite region bounded by the x axis, the curve and the straight line with equation $x = \ln\left(1+\sqrt{2}\right)$.

$\boxed{\quad}$, $\boxed{\text{area} = \frac{1}{2}}$


START SETTING UP A PARAMETRIC INTEGRAL FOR THIS AREA

$x = \operatorname{artanh}(\sin t) \qquad y = \sec t \tan t$

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt$$

$$\boxed{\text{AREA} = \int_{t_1}^{t_2} (\sec t \tan t)\frac{d}{dt}(\operatorname{artanh}(\sin t)\,dt.}$$

FIRSTLY

$v = \operatorname{artanh} u$

$\tanh v = u$

$\frac{du}{dv} = \operatorname{sech}^2 v$

$\frac{du}{dv} = 1 - \tanh^2 v$

$\frac{du}{dv} = 1 - u^2$

$\frac{dv}{du} = \frac{1}{1-u^2}$

HENCE WE NOW HAVE

$\frac{d}{dt}(\operatorname{artanh}(\sin t)) = \frac{1}{1-\sin^2 t} \times \cos t$

$\frac{d}{dt}[\operatorname{artanh}(\sin t)] = \frac{\cos t}{1-\sin^2 t}$

FINALLY THE LIMITS

$x_1 = 0 \longmapsto t_1 = 0$ (BY INSPECTION)

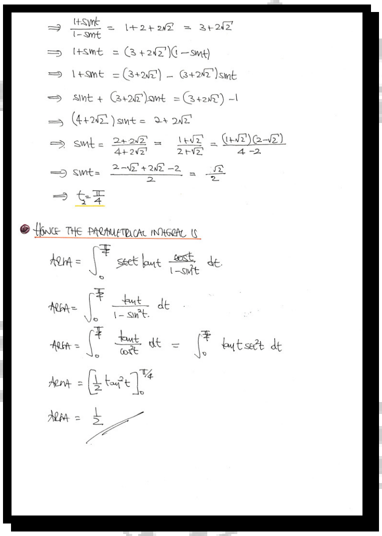
$x_2 = \ln(1+\sqrt{2})$ USING THE LOGARITHMIC FORM OF ARTANH WE HAVE

$\Rightarrow \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \operatorname{artanh} x$

$\Rightarrow \frac{1}{2}\ln\left(\frac{1+\sin t}{1-\sin t}\right) = \ln(1+\sqrt{2})$

$\Rightarrow \ln\left(\frac{1+\sin t}{1-\sin t}\right) = 2\ln(1+\sqrt{2})$

$\Rightarrow \ln\left(\frac{1+\sin t}{1-\sin t}\right) = \ln(1+\sqrt{2})^2$


$\Rightarrow \frac{1+\sin t}{1-\sin t} = 1+2+2\sqrt{2} = 3+2\sqrt{2}$

$\Rightarrow 1+\sin t = (3+2\sqrt{2})(1-\sin t)$

$\Rightarrow 1+\sin t = (3+2\sqrt{2}) - (3+2\sqrt{2})\sin t$

$\Rightarrow \sin t + (3+2\sqrt{2})\sin t = (3+2\sqrt{2}) - 1$

$\Rightarrow (4+2\sqrt{2})\sin t = 2+2\sqrt{2}$

$\Rightarrow \sin t = \frac{2+2\sqrt{2}}{4+2\sqrt{2}} = \frac{1+\sqrt{2}}{2+\sqrt{2}} = \frac{(1+\sqrt{2})(2-\sqrt{2})}{4-2}$

$\Rightarrow \sin t = \frac{2-\sqrt{2}+2\sqrt{2}-2}{2} = \frac{\sqrt{2}}{2}$

$\Rightarrow t_2 = \frac{\pi}{4}$

HENCE THE PARAMETRICAL INTEGRAL IS

$$\text{AREA} = \int_0^{\frac{\pi}{4}} \cancel{\sec t} \tan t \frac{\cancel{\cos t}}{1-\sin^2 t}\,dt.$$

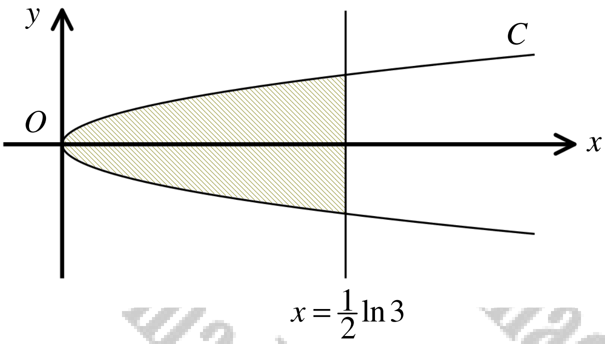
$$\text{AREA} = \int_0^{\frac{\pi}{4}} \frac{\tan t}{1-\sin^2 t}\,dt$$

$$\text{AREA} = \int_0^{\frac{\pi}{4}} \frac{\tan t}{\cos^2 t}\,dt = \int_0^{\frac{\pi}{4}} \tan t \sec^2 t\,dt$$

$$\text{AREA} = \left[\tfrac{1}{2}\tan^2 t\right]_0^{\pi/4}$$

$$\text{AREA} = \tfrac{1}{2}$$

Question 87 (*****)



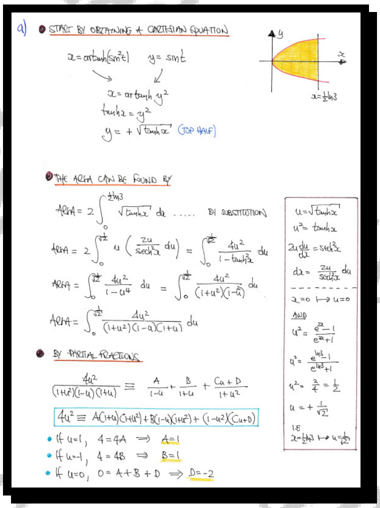
The figure above shows the curve C whose parametric equations are

$$x = \text{artanh}\left(\sin^2 t\right), \quad y = \sin t, \quad -\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi .$$

a) Use integration in Cartesian coordinates to find the exact area of the finite region bounded by the curve and the straight line with equation $x = \frac{1}{2}\ln 3$.

b) Use integration in parametric to verify the validity of the result of part **(a)**.

$\boxed{}$, $\boxed{\text{area} = 2\ln\left(1+\sqrt{2}\right) - 2\arctan\left(\frac{1}{\sqrt{2}}\right)}$



a) • START BY OBTAINING A CARTESIAN EQUATION

$x = \text{artanh}(\sin^2 t)$ $\quad$ $y = \sin t$

$x = \text{artanh}\, y^2$

$\tanh x = y^2$

$y = +\sqrt{\tanh x}$ (TOP HALF)

• THE AREA CAN BE FOUND BY

AREA $= 2\int_0^{\frac{1}{2}\ln 3} \sqrt{\tanh x}\, dx$ ..... BY SUBSTITUTION

AREA $= 2\int_0^{\frac{1}{\sqrt{2}}} u\left(\frac{2u}{\text{sech}^2 x}\, du\right) = \int_0^{\frac{1}{\sqrt{2}}} \frac{4u^2}{1-\tanh^2 x}\, du$

AREA $= \int_0^{\frac{1}{\sqrt{2}}} \frac{4u^2}{1-u^4}\, du = \int_0^{\frac{1}{\sqrt{2}}} \frac{4u^2}{(1+u^2)(1-u^2)}\, du$

AREA $= \int_0^{\frac{1}{\sqrt{2}}} \frac{4u^2}{(1+u^2)(1-u)(1+u)}\, du$

$u = \sqrt{\tanh x}$

$u^2 = \tanh x$

$2u\frac{du}{dx} = \text{sech}^2 x$

$dx = \frac{2u}{\text{sech}^2 x}\, du$

$x = 0 \longmapsto u = 0$

AND

$u^2 = \frac{e^{2x}-1}{e^{2x}+1}$

$u^2 = \frac{e^{\ln 3}-1}{e^{\ln 3}+1}$

$u^2 = \frac{2}{4} = \frac{1}{2}$

$u = +\frac{1}{\sqrt{2}}$

I.E

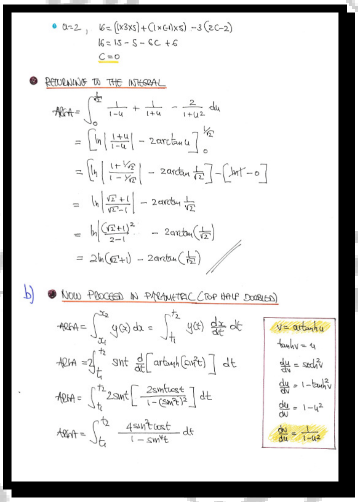
$x = \frac{1}{2}\ln 3 \longmapsto u = \frac{1}{\sqrt{2}}$

• BY PARTIAL FRACTIONS

$\frac{4u^2}{(1+u^2)(1-u)(1+u)} \equiv \frac{A}{1-u} + \frac{B}{1+u} + \frac{Cu+D}{1+u^2}$

$\boxed{4u^2 \equiv A(1+u)(1+u^2) + B(1-u)(1+u^2) + (1-u^2)(Cu+D)}$

- If $u=1$, $4 = 4A \Rightarrow A=1$
- If $u=-1$, $4 = 4B \Rightarrow B=1$
- If $u=0$, $0 = A+B+D \Rightarrow D=-2$



- $u=2$, $16 = (1\times3\times5) + (1\times(-1)\times5) - 3(2C-2)$

$16 = 15 - 5 - 6C + 6$

$C = 0$

• RETURNING TO THE INTEGRAL

AREA $= \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{1-u} + \frac{1}{1+u} - \frac{2}{1+u^2}\, du$

$= \left[\ln\left|\frac{1+u}{1-u}\right| - 2\arctan u\right]_0^{1/\sqrt{2}}$

$= \left[\ln\left|\frac{1+1/\sqrt{2}}{1-1/\sqrt{2}}\right| - 2\arctan\frac{1}{\sqrt{2}}\right] - \left[\ln 1 - 0\right]$

$= \ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| - 2\arctan\frac{1}{\sqrt{2}}$

$= \ln\left|\frac{(\sqrt{2}+1)^2}{2-1}\right. - 2\arctan\left(\frac{1}{\sqrt{2}}\right)$

$= 2\ln(\sqrt{2}+1) - 2\arctan\left(\frac{1}{\sqrt{2}}\right)$ //

b) • NOW PROCEED IN PARAMETRIC (TOP HALF DOUBLED)

AREA $= \int_{x_1}^{x_2} y(x)\, dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\, dt$

AREA $= 2\int_{t_1}^{t_2} \sin t \frac{d}{dt}\left[\text{artanh}(\sin^2 t)\right] dt$

AREA $= \int_{t_1}^{t_2} 2\sin t\left[\frac{2\sin t\cos t}{1-(\sin^2 t)^2}\right] dt$

AREA $= \int_{t_1}^{t_2} \frac{4\sin^2 t\cos t}{1-\sin^4 t}\, dt$

$v = \text{artanh}\, u$

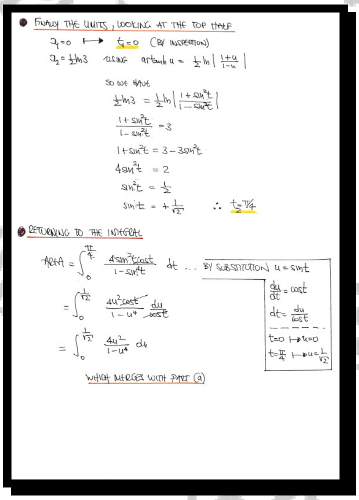
$\tanh v = u$

$\frac{du}{dv} = \text{sech}^2 v$

$\frac{du}{dv} = 1 - \tanh^2 v$

$\frac{du}{dv} = 1 - u^2$

$\frac{dv}{du} = \frac{1}{1-u^2}$



• FINALLY THE LIMITS, LOOKING AT THE TOP HALF

$x_1 = 0 \longmapsto t_1 = 0$ (BY INSPECTION)

$x_2 = \frac{1}{2}\ln 3$ USING $\text{artanh}\, u = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|$

SO WE HAVE

$\frac{1}{2}\ln 3 = \frac{1}{2}\ln\left|\frac{1+\sin^2 t}{1-\sin^2 t}\right|$

$\frac{1+\sin^2 t}{1-\sin^2 t} = 3$

$1 + \sin^2 t = 3 - 3\sin^2 t$

$4\sin^2 t = 2$

$\sin^2 t = \frac{1}{2}$

$\sin t = +\frac{1}{\sqrt{2}}$ $\quad \therefore t_2 = \pi/4$

• RETURNING TO THE INTEGRAL

AREA $= \int_0^{\frac{\pi}{4}} \frac{4\sin^2 t\cos t}{1-\sin^4 t}\, dt$ ... BY SUBSTITUTION $u = \sin t$

$= \int_0^{\frac{1}{\sqrt{2}}} \frac{4u^2 \cancel{\cos t}}{1-u^4} \frac{du}{\cancel{\cos t}}$

$= \int_0^{\frac{1}{\sqrt{2}}} \frac{4u^2}{1-u^4}\, du$

$\frac{du}{dt} = \cos t$

$dt = \frac{du}{\cos t}$

$t=0 \longmapsto u=0$

$t=\frac{\pi}{4} \longmapsto u=\frac{1}{\sqrt{2}}$

WHICH MERGES WITH PART (a)

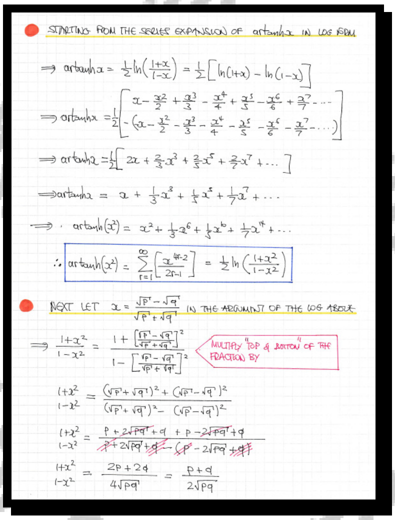
Question 88 (*****)

Given that p and q are positive, show that the natural logarithm of their arithmetic mean exceeds the arithmetic mean of their natural logarithms by

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right].$$

You may find the series expansion of $\operatorname{artanh}\left(x^2\right)$ useful in this question.

$\boxed{}$, $\boxed{\text{proof}}$



● STARTING FROM THE SERIES EXPANSION OF artanh$x$ IN LOG FORM

$$\Rightarrow \operatorname{artanh}x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \tfrac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$$

$$\Rightarrow \operatorname{artanh}x = \tfrac{1}{2}\left[\begin{matrix} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \dots \\ -\left(x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} - \dots\right)\end{matrix}\right]$$

$$\Rightarrow \operatorname{artanh}x = \tfrac{1}{2}\left[2x + \tfrac{2}{3}x^3 + \tfrac{2}{5}x^5 + \tfrac{2}{7}x^7 + \dots\right]$$

$$\Rightarrow \operatorname{artanh}x = x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \dots$$

$$\Rightarrow \operatorname{artanh}(x^2) = x^2 + \tfrac{1}{3}x^6 + \tfrac{1}{5}x^{10} + \tfrac{1}{7}x^{14} + \dots$$

$$\therefore \boxed{\operatorname{artanh}(x^2) = \sum_{r=1}^{\infty}\left[\frac{x^{4r-2}}{2r-1}\right] = \tfrac{1}{2}\ln\left(\frac{1+x^2}{1-x^2}\right)}$$

● NEXT LET $x = \frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}$ IN THE ARGUMENT OF THE LOG ABOVE

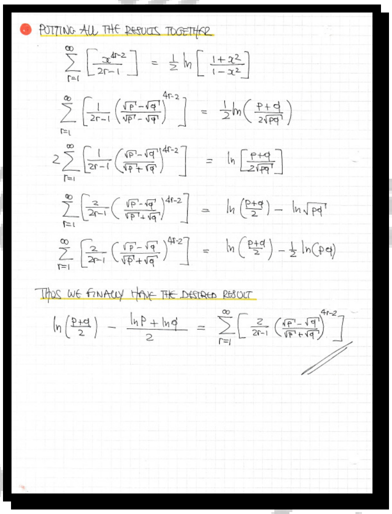
$$\Rightarrow \frac{1+x^2}{1-x^2} = \frac{1 + \left[\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right]^2}{1 - \left[\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right]^2}$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY

$$\frac{1+x^2}{1-x^2} = \frac{(\sqrt{p}+\sqrt{q})^2 + (\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2 - (\sqrt{p}-\sqrt{q})^2}$$

$$\frac{1+x^2}{1-x^2} = \frac{p + \cancel{2\sqrt{pq}} + q + p - \cancel{2\sqrt{pq}} + q}{\cancel{p} + 2\sqrt{pq} + \cancel{q} - (\cancel{p} - 2\sqrt{pq} + \cancel{q})}$$

$$\frac{1+x^2}{1-x^2} = \frac{2p+2q}{4\sqrt{pq}} = \frac{p+q}{2\sqrt{pq}}$$



● PUTTING ALL THE RESULTS TOGETHER

$$\sum_{r=1}^{\infty}\left[\frac{x^{4r-2}}{2r-1}\right] = \tfrac{1}{2}\ln\left[\frac{1+x^2}{1-x^2}\right]$$

$$\sum_{r=1}^{\infty}\left[\frac{1}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}-\sqrt{q}}\right)^{4r-2}\right] = \tfrac{1}{2}\ln\left(\frac{p+q}{2\sqrt{pq}}\right)$$

$$2\sum_{r=1}^{\infty}\left[\frac{1}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right] = \ln\left[\frac{p+q}{2\sqrt{pq}}\right]$$

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right] = \ln\left(\frac{p+q}{2}\right) - \ln\sqrt{pq}$$

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right] = \ln\left(\frac{p+q}{2}\right) - \tfrac{1}{2}\ln(pq)$$

THUS WE FINALLY HAVE THE DESIRED RESULT

$$\ln\left(\frac{p+q}{2}\right) - \frac{\ln p + \ln q}{2} = \sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right]$$