POLAR COORDINATES

CONVERTING BETWEEN CARTESIANS AND POLARS

Question 1

A curve C has Cartesian equation

$$\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2\right),\ a \neq 0 .$$

Determine a polar equation for C.

$\boxed{r^2 = a^2 \cos 2\theta}$

$$\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2\right)$$
$$\Rightarrow \left(r^2\right)^2 = a^2\left(r^2\cos^2\theta - r^2\sin^2\theta\right)$$
$$\Rightarrow r^4 = a^2 r^2\left(\cos^2\theta - \sin^2\theta\right)$$
$$\Rightarrow r^2 = a^2\cos 2\theta$$

Question 2

A curve C has Cartesian equation

$$\left(x^2+y^2\right)^3 = (x+y)^4 .$$

Show that a polar equation for C is given by

$$r = 1+\sin 2\theta,\ r \geq 0 .$$

$\boxed{\text{proof}}$

$$\left(x^2+y^2\right)^3 = (x+y)^4$$
$$\Rightarrow \left(r^2\right)^3 = \left(r\cos\theta + r\sin\theta\right)^4$$
$$\Rightarrow r^6 = r^4\left(\cos\theta+\sin\theta\right)^4$$
$$\Rightarrow r^2 = \left(\cos\theta+\sin\theta\right)^4$$
$$\Rightarrow r = \left(\cos\theta+\sin\theta\right)^2$$
$$\Rightarrow r = \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta$$
$$\Rightarrow r = 1 + 2\sin\theta\cos\theta$$
$$\Rightarrow r = 1 + \sin 2\theta$$

Question 3

A curve C has Cartesian equation

$$\left(x^2+y^2\right)^2 = 2x^2 y.$$

Show that a polar equation for C can be written as

$$r = \sin 2\theta \cos\theta.$$

$\boxed{\text{proof}}$

$$\begin{aligned}
&\left(x^2+y^2\right)^2 = 2x^2y\\
&\Rightarrow \left(r^2\right)^2 = 2(r\cos\theta)^2(r\sin\theta)\\
&\Rightarrow r^4 = 2\left(r^2\cos^2\theta\right)(r\sin\theta)\\
&\Rightarrow r^4 = 2r^3\cos^2\theta\sin\theta \qquad r\neq 0\\
&\Rightarrow r = 2\cos^2\theta\sin\theta\\
&\Rightarrow r = (2\cos\theta\sin\theta)\cos\theta\\
&\Rightarrow r = \sin 2\theta\cos\theta \quad \text{// AS REQUIRED}
\end{aligned}$$

$$\begin{aligned}
x^2+y^2 &= r^2\\
x &= r\cos\theta\\
y &= r\sin\theta
\end{aligned}$$

Question 4

A circle has Cartesian equation

$$(x-3)^2+(y-4)^2 = 25.$$

Show that a polar equation for the circle is given by

$$r = A\cos\theta + B\sin\theta,$$

where A and B are constants.

$\boxed{r = 6\cos\theta - 4\sin\theta}$

$$\begin{aligned}
&(x-3)^2+(y-4)^2 = 25\\
&\Rightarrow x^2-6x+9+y^2-8y+16 = 25\\
&\Rightarrow x^2+y^2-6x-8y = 0\\
&\Rightarrow r^2 - 6r\cos\theta - 8r\sin\theta = 0\\
&\Rightarrow r - 6\cos\theta - 8\sin\theta = 0\\
&\Rightarrow r = 6\cos\theta + 8\sin\theta \quad //
\end{aligned}$$

$$x^2+y^2 = r^2$$

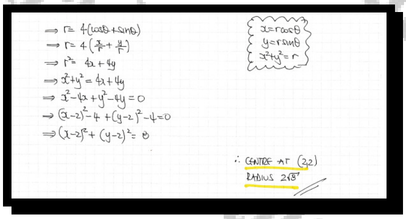
Question 5

A circle has polar equation

$$r = 4(\cos\theta + \sin\theta) \quad 0 \le \theta < 2\pi \, .$$

Determine the Cartesian coordinates of the centre of the circle and the length of its radius.

$$\boxed{(2,2), \text{ radius} = \sqrt{8}}$$



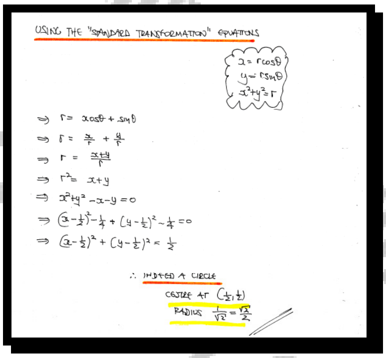
Question 6

Write the polar equation

$$r = \cos\theta + \sin\theta, \; 0 \le \theta < 2\pi$$

in Cartesian form, and hence show that it represents a circle, further determining the coordinates of its centre and the size of its radius.

$$\boxed{\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}}$$



Question 7

A curve C has polar equation

$$r = 1 + \cos 2\theta .$$

Determine a Cartesian equation for C.

$$\boxed{\left(x^2 + y^2\right)^3 = 4x^4}$$

$$r = 1 + \cos 2\theta$$
$$\Rightarrow r = 1 + \left(2\cos^2\theta - 1\right)$$
$$\Rightarrow r = 2\cos^2\theta \qquad x = r\cos\theta$$
$$\Rightarrow r = 2\left(\frac{x}{r}\right)^2$$
$$\Rightarrow r = 2\frac{x^2}{r^2}$$
$$\Rightarrow r^3 = 2x^2$$
$$\Rightarrow \left(r^3\right)^2 = \left(2x^2\right)^2$$
$$\Rightarrow \left(r^2\right)^3 = 4x^4 \qquad x^2 + y^2 = r^2$$
$$\Rightarrow \left(x^2 + y^2\right)^3 = 4x^4 \; //$$

Question 8

A curve C has polar equation

$$r = \sec\theta + \tan\theta .$$

Determine a Cartesian equation for C.

$$\boxed{x^2 + y^2 = \frac{y^2}{(x-1)^2}}$$

$$r = \sec\theta + \tan\theta$$
$$\Rightarrow r = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$
$$\Rightarrow r = \frac{1 + \sin\theta}{\cos\theta}$$
$$x = r\cos\theta \qquad y = r\sin\theta$$
$$\Rightarrow r = \frac{1 + \frac{y}{r}}{\frac{x}{r}}$$

MULTIPLY TOP/BOTTOM OF FRACTION BY r

$$\Rightarrow r = \frac{r + y}{x}$$
$$\Rightarrow rx = r + y$$
$$\Rightarrow rx - r = y$$
$$\Rightarrow r(x-1) = y$$
$$\Rightarrow r = \frac{y}{x-1}$$
$$\Rightarrow r^2 = \frac{y^2}{(x-1)^2} \qquad x^2 + y^2 = r^2$$
$$\Rightarrow x^2 + y^2 = \frac{y^2}{(x-1)^2} \; //$$

Question 9

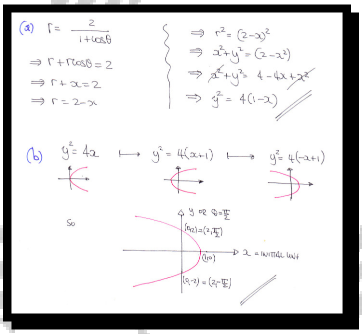
A curve C has polar equation

$$r = \frac{2}{1+\cos\theta}, \quad 0 \le \theta < 2\pi .$$

a) Find a Cartesian equation for C.

b) Sketch the graph of C.

$$\boxed{y^2 = 4(1-x)}$$



Question 10

The curve C has Cartesian equation

$$\left(x^2+y^2\right)(x-1)^2 = x^2 .$$

Find a polar equation of C in the form $r = f(\theta)$.

$$\boxed{r = 1+\sec\theta}$$

(a) $\left(x^2+y^2\right)(x-1)^2 = x^2$

$\Rightarrow r^2(x-1)^2 = x^2$

$\Rightarrow r^2 = \frac{x^2}{(x-1)^2}$

$\Rightarrow r = \frac{x}{x-1}$

$\Rightarrow r = \frac{r\cos\theta}{r\cos\theta - 1}$

$\Rightarrow 1 = \frac{\cos\theta}{r\cos\theta - 1}$

$\Rightarrow r\cos\theta - 1 = \cos\theta$

$\Rightarrow r\cos\theta = 1 + \cos\theta$

$\Rightarrow r = \frac{1}{\cos\theta} + 1$

$\Rightarrow r = \sec\theta + 1$

Question 11

A curve C_1 has polar equation

$$r = 2\sin\theta, \; 0 \le \theta < 2\pi .$$

a) Find a Cartesian equation for C_1, and describe it geometrically.

A different curve C_2 has Cartesian equation

$$y^2 = \frac{x^4}{1-x^2}, \; x \ne \pm 1 .$$

b) Find a polar equation for C_2, in the form $r = f(\theta)$.

$\boxed{x^2 + (y-1)^2 = 1}$, $\boxed{r = \tan\theta}$

(a) $r = 2\sin\theta$

$\Rightarrow r = 2\left(\frac{y}{r}\right)$

$\Rightarrow r^2 = 2y$

$\Rightarrow x^2 + y^2 - 2y = 0$

$\Rightarrow x^2 + (y-1)^2 - 1 = 0$

$\Rightarrow x^2 + (y-1)^2 = 1$

CIRCLE CENTRE $(0,1)$ RADIUS 1

(b) $y^2 = \frac{x^4}{1-x^2}$

$\Rightarrow y^2 - x^2y^2 = x^4$

$\Rightarrow y^2 = x^4 + x^2y^2$

$\Rightarrow y^2 = x^2(x^2+y^2)$

$\Rightarrow y^2 = x^2r^2$

$\Rightarrow r^2 = \frac{y^2}{x^2}$

$\Rightarrow r^2 = \frac{\cancel{r^2}\sin^2\theta}{\cancel{r^2}\cos^2\theta}$

$\Rightarrow r^2 = \tan^2\theta$

$\Rightarrow r = \tan\theta$

Question 12

Show that the polar equation of the top half of the parabola with Cartesian equation

$$y = \sqrt{2x+1}, \ x \geq -\tfrac{1}{2},$$

is given by the polar equation

$$r = \frac{1}{1-\cos\theta}, \ r \geq 0.$$

$\boxed{\text{proof}}$

$$y = \sqrt{2x+1}$$
$$\Rightarrow y^2 = 2x+1$$
$$\Rightarrow y^2 + x^2 = x^2 + 2x + 1$$
$$\Rightarrow r^2 = (x+1)^2$$
$$\Rightarrow r = x+1$$
$$\Rightarrow r - 1 = r\cos\theta$$
$$\Rightarrow r - r\cos\theta = 1$$
$$\Rightarrow r(1-\cos\theta) = 1$$
$$\Rightarrow r = \frac{1}{1-\cos\theta}$$

Question 13

The points A and B have respective coordinates $(-1,0)$ and $(1,0)$. The locus of the point $P(x,y)$ traces a curve in such a way so that $|AP||BP|=1$.

a) By forming a Cartesian equation of the locus of P, show that the polar equation of the curve is

$$r^2 = 2\cos 2\theta, \ 0 \le \theta < 2\pi .$$

b) Sketch the curve.

proof

a) DETERMINING THE CARTESIAN LOCUS

$A(-1,0) \quad B(1,0) \quad P(x,y)$

- $|AP| = \sqrt{(x+1)^2+y^2}$
- $|BP| = \sqrt{(x-1)^2+y^2}$

$$\Rightarrow |AP||BP| = 1$$
$$\Rightarrow |AP|^2|BP|^2 = 1$$
$$\Rightarrow \left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right] = 1$$
$$\Rightarrow \left\{\begin{array}{l} y^2(x+1)^2 + (x+1)^2(x-1)^2 \\ y^2(x-1)^2 + y^4 \end{array}\right\} = 1$$
$$\Rightarrow y^2\left[(x+1)^2+(x-1)^2\right] + y^4 + (x+1)^2(x-1)^2 = 1$$
$$\rightarrow y^2\left[x^2+2x+1+x^2-2x+1\right] + y^4 + (x^2-1)^2 = 1$$
$$\Rightarrow y^2\left[2x^2+2\right] + y^4 + x^4 - 2x^2 + 1 = 1$$
$$\Rightarrow 2x^2y^2 + 2y^2 + y^4 + x^4 - 2x^2 = 0$$

REGROUPING

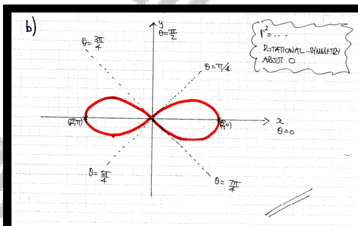
$$\Rightarrow (y^4 + 2x^2y^2 + x^4) + 2(y^2 - x^2) = 0$$
$$\Rightarrow (x^2+y^2)^2 + 2(y^2-x^2) = 0$$

TRANSFORM INTO POLARS

$$\Rightarrow (r^2)^2 + 2(r^2\sin^2\theta - r^2\cos^2\theta) = 0$$
$$\Rightarrow r^4 + 2r^2(\sin^2\theta - \cos^2\theta) = 0$$
$$\Rightarrow r^2 + 2(\sin^2\theta - \cos^2\theta) = 0$$
$$\Rightarrow r^2 = 2(\cos^2\theta - \sin^2\theta)$$
$$\Rightarrow r^2 = 2\cos 2\theta$$

AS REQUIRED

b)



Question 14

The curve C has polar equation

$$r = \tan\theta\,,\ 0 \le \theta < \frac{\pi}{2}.$$

Find a Cartesian equation of C in the form $y = f(x)$.

$$\boxed{y = \frac{x^2}{\sqrt{1-x^2}}}$$

$$r = \tan\theta$$
$$\Rightarrow r^2 = \tan^2\theta$$
$$\Rightarrow r^2 = \frac{\sin^2\theta}{\cos^2\theta}$$
$$\Rightarrow r^2\cos^2\theta = \sin^2\theta$$
$$\Rightarrow (r\cos\theta)^2 = \sin^2\theta$$
$$\Rightarrow x^2 = \sin^2\theta$$
$$\Rightarrow x^2 = 1-\cos^2\theta$$
$$\Rightarrow \cos^2\theta = 1-x^2$$
$$\Rightarrow \sec^2\theta = \frac{1}{1-x^2}$$
$$\Rightarrow 1+\tan^2\theta = \frac{1}{1-x^2}$$
$$\Rightarrow \tan^2\theta = \frac{1}{1-x^2} - 1$$
$$\Rightarrow r^2 = \frac{1-(1-x^2)}{1-x^2}$$
$$\Rightarrow x^2+y^2 = \frac{x^2}{1-x^2}$$
$$\Rightarrow y^2 = \frac{x^2}{1-x^2} - x^2$$
$$\Rightarrow y^2 = \frac{x^2-(x^2-x^4)}{1-x^2}$$
$$\Rightarrow y^2 = \frac{x^4}{1-x^2}$$
$$\Rightarrow y = +\sqrt{\frac{x^4}{1-x^2}}$$
$$\Rightarrow y = \frac{x^2}{\sqrt{1-x^2}}$$

TANGENTS TO POLAR CURVES

Question 1

A Cardioid has polar equation

$$r = 1 + 2\cos\theta, \; 0 \le \theta \le \frac{\pi}{2}.$$

The point P lies on the Cardioid so that the tangent to the Cardioid at P is parallel to the initial line.

Determine the exact length of OP, where O is the pole.

$$\boxed{\frac{1}{4}\left(3+\sqrt{33}\right)}$$

PARALLEL TO THE INITIAL LINE IMPLIES $\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy/d\theta}{dx/d\theta} = 0 \Rightarrow \frac{dy}{d\theta} = 0$

$\Rightarrow \frac{d}{d\theta}(y) = 0$

$\Rightarrow \frac{d}{d\theta}(r\sin\theta) = 0$

$\Rightarrow \frac{d}{d\theta}\left((1+2\cos\theta)\sin\theta\right) = 0$

$\Rightarrow \frac{d}{d\theta}(\sin\theta + \sin 2\theta) = 0$

$\Rightarrow \cos\theta + 2\cos 2\theta = 0$

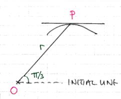
$\Rightarrow \cos\theta + 2(2\cos^2\theta - 1) = 0$

$\Rightarrow 4\cos^2\theta + \cos\theta - 2 = 0$

$\therefore \cos\theta = \frac{-1 \pm \sqrt{33}}{8}$

$\cos\theta = \frac{-1+\sqrt{33}}{8} \quad \left(0 < \theta < \frac{\pi}{2}\right)$

LOOKING AT THE DIAGRAM



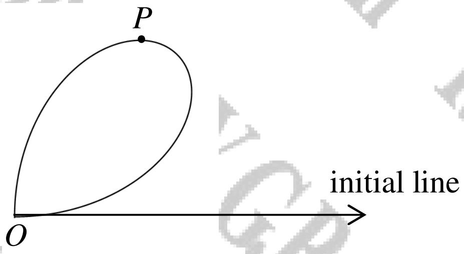
$\therefore |OP| = 1 + 2\cos\theta$

$= 1 + 2\left(\frac{-1+\sqrt{33}}{8}\right)$

$= 1 + \frac{-1+\sqrt{33}}{4}$

$= \frac{3+\sqrt{33}}{4}$

Question 2



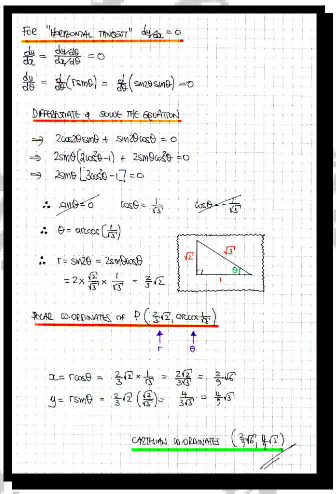
The figure above shows the polar curve with equation

$$r = \sin 2\theta, \; 0 \le \theta \le \frac{\pi}{2}.$$

The point P lies on the curve so that the tangent at P is parallel to the initial line.

Find the **Cartesian** coordinates of P.

$$\boxed{\left(\frac{2}{9}\sqrt{6}, \frac{4}{9}\sqrt{3}\right)}$$



FOR "HORIZONTAL TANGENT" $dy/dx = 0$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = 0$

$\frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = \frac{d}{d\theta}(\sin 2\theta \sin\theta) = 0$

DIFFERENTIATE & SOLVE THE EQUATION

$\Rightarrow 2\cos 2\theta \sin\theta + \sin 2\theta \cos\theta = 0$

$\Rightarrow 2\sin\theta(2\cos^2\theta - 1) + 2\sin\theta\cos^2\theta = 0$

$\Rightarrow 2\sin\theta[3\cos^2\theta - 1] = 0$

$\therefore \cancel{\sin\theta = 0} \qquad \cos\theta = \frac{1}{\sqrt{3}} \qquad \cancel{\cos\theta = -\frac{1}{\sqrt{3}}}$

$\therefore \theta = \arccos\left(\frac{1}{\sqrt{3}}\right)$

$\therefore r = \sin 2\theta = 2\sin\theta\cos\theta$

$= 2 \times \frac{\sqrt{2}}{\sqrt{3}} \times \frac{1}{\sqrt{3}} = \frac{2}{3}\sqrt{2}$

$\sqrt{2}$ $\sqrt{3}$ θ 1

POLAR CO-ORDINATES OF $P\left(\frac{2}{3}\sqrt{2}, \arccos\frac{1}{\sqrt{3}}\right)$

r θ

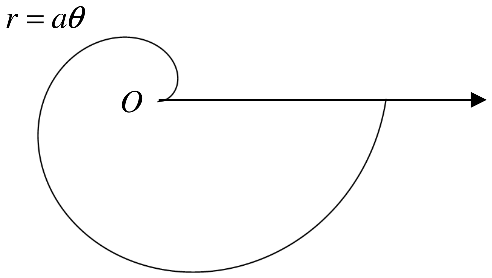
$x = r\cos\theta = \frac{2}{3}\sqrt{2} \times \frac{1}{\sqrt{3}} = \frac{2\sqrt{2}}{3\sqrt{3}} = \frac{2}{9}\sqrt{6}$

$y = r\sin\theta = \frac{2}{3}\sqrt{2}\left(\frac{\sqrt{2}}{\sqrt{3}}\right) = \frac{4}{3\sqrt{3}} = \frac{4}{9}\sqrt{3}$

CARTESIAN CO-ORDINATES $\left(\frac{2}{9}\sqrt{6}, \frac{4}{9}\sqrt{3}\right)$

POLAR CURVE AREAS

Question 1



The figure above shows a spiral curve with polar equation

$$r = a\theta \, , \; 0 \le \theta \le 2\pi \, ,$$

where a is a positive constant.

Find the area of the finite region bounded by the spiral and the initial line.

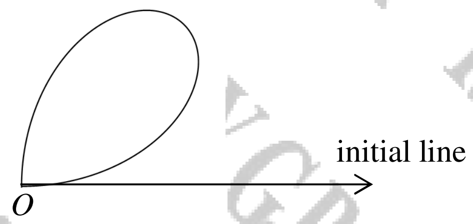
$$\boxed{\text{area} = \frac{4}{3}a^2\pi^3}$$

USING THE STANDARD FORMULA FOR POLAR AREA

$$\text{AREA} = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2 \, d\theta = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2}(a\theta)^2 \, d\theta = \frac{1}{2}a^2 \int_0^{2\pi} \theta^2 \, d\theta$$

$$= \frac{1}{2}a^2 \left[\frac{1}{3}\theta^3 \right]_0^{2\pi} = \frac{1}{6}a^2 \left[8\pi^3 - 0 \right] = \frac{4}{3}\pi^3 a^2$$

Question 2



The figure above shows the polar curve C with equation

$$r = \sin 2\theta, \ 0 \le \theta \le \frac{\pi}{2}.$$

Find the exact value of the area enclosed by the curve.

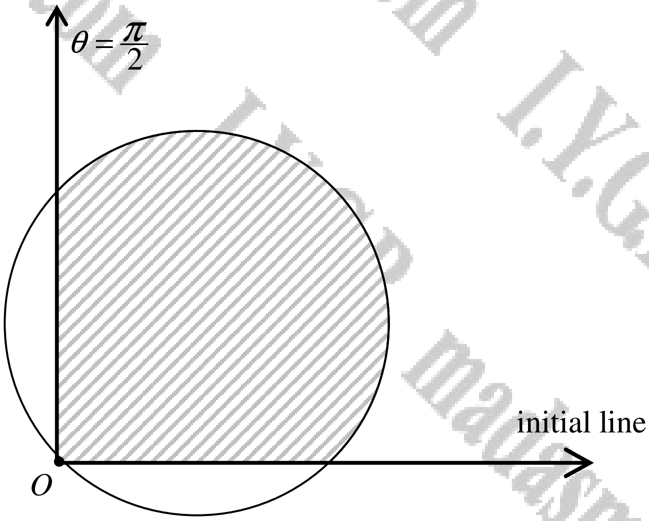
$\boxed{\text{area} = \frac{\pi}{8}}$

(a) Area $= \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}(\sin 2\theta)^2 \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin^2 2\theta \, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4} - \frac{1}{4}\cos 4\theta \, d\theta$

$= \left[\frac{1}{4}\theta - \frac{1}{16}\sin 4\theta\right]_0^{\frac{\pi}{2}} = \left(\frac{\pi}{8} - 0\right) - (0) = \frac{\pi}{8}$

Question 3

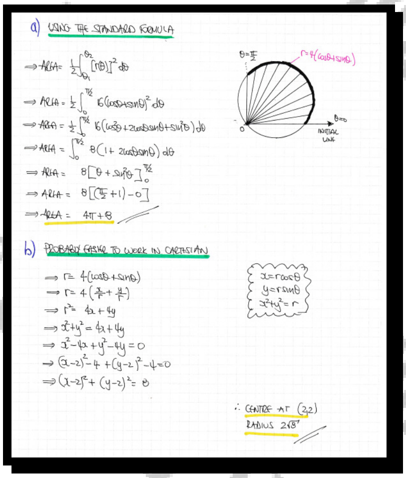


The figure above shows a circle with polar equation

$$r = 4(\cos\theta + \sin\theta) \quad 0 \le \theta < 2\pi .$$

Find the exact area of the shaded region bounded by the circle, the initial line and the half line $\theta = \frac{\pi}{2}$.

$\boxed{\text{area} = 4\pi + 8}$


a) USING THE STANDARD FORMULA

$\Rightarrow \text{AREA} = \frac{1}{2}\int_{\theta_1}^{\theta_2} [r(\theta)]^2 \, d\theta$

$\Rightarrow \text{AREA} = \frac{1}{2}\int_0^{\pi/2} 16(\cos\theta + \sin\theta)^2 \, d\theta$

$\Rightarrow \text{AREA} = \frac{1}{2}\int_0^{\pi/2} 16(\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta) \, d\theta$

$\Rightarrow \text{AREA} = \int_0^{\pi/2} 8(1 + 2\cos\theta\sin\theta) \, d\theta$

$\Rightarrow \text{AREA} = 8\left[\theta + \sin^2\theta\right]_0^{\pi/2}$

$\Rightarrow \text{AREA} = 8\left[\left(\frac{\pi}{2} + 1\right) - 0\right]$

$\Rightarrow \text{AREA} = 4\pi + 8$

$\theta = \frac{\pi}{2}$

$r = 4(\cos\theta + \sin\theta)$

$\theta = 0$

INITIAL LINE

O

b) PROBABLY EASIER TO WORK IN CARTESIAN

$\Rightarrow r = 4(\cos\theta + \sin\theta)$

$\Rightarrow r = 4\left(\frac{x}{r} + \frac{y}{r}\right)$

$\Rightarrow r^2 = 4x + 4y$

$\Rightarrow x^2 + y^2 = 4x + 4y$

$\Rightarrow x^2 - 4x + y^2 - 4y = 0$

$\Rightarrow (x-2)^2 - 4 + (y-2)^2 - 4 = 0$

$\Rightarrow (x-2)^2 + (y-2)^2 = 8$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r$

$\therefore$ CENTRE AT $(2,2)$
RADIUS $2\sqrt{2}$

Question 4

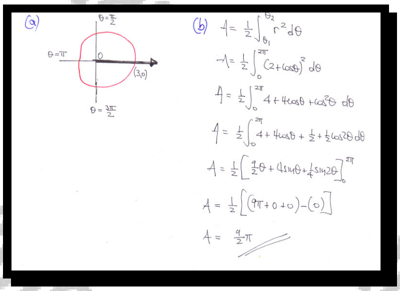
The polar curve C has equation

$$r = 2 + \cos\theta\,,\ 0 \le \theta < 2\pi\,.$$

a) Sketch the graph of C.

b) Show that the area enclosed by the curve is $\frac{9}{2}\pi$.

$\boxed{\text{proof}}$



(a)

$\theta = \frac{\pi}{2}$, $\theta = \pi$, O, $(3,0)$, $\theta = \frac{3\pi}{2}$

(b)

$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$

$A = \frac{1}{2}\int_0^{2\pi} (2+\cos\theta)^2 \, d\theta$

$A = \frac{1}{2}\int_0^{2\pi} 4 + 4\cos\theta + \cos^2\theta \, d\theta$

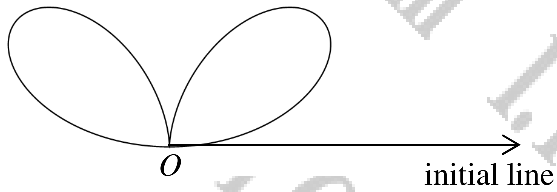
$A = \frac{1}{2}\int_0^{2\pi} 4 + 4\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta \, d\theta$

$A = \frac{1}{2}\left[\frac{9}{2}\theta + 4\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$

$A = \frac{1}{2}\left[(9\pi + 0 + 0) - (0)\right]$

$A = \frac{9}{2}\pi$

Question 5



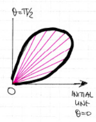
The figure above shows the polar curve C with equation

$$r = 2\sin 2\theta\sqrt{\cos\theta}\,,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

Show that the area enclosed by one of the two identical loops of the curve is $\dfrac{16}{15}$.

proof

LOOKING AT THE LOOP ON THE RIGHT



$$\text{AREA} = \frac{1}{2}\int_{\theta_1}^{\theta_2} (r(\theta))^2\, d\theta$$

$$\text{AREA} = \frac{1}{2}\int_{\theta=0}^{\theta=\pi/2} \left[2\sin 2\theta\sqrt{\cos\theta}\,\right]^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 4\sin^2 2\theta\cos\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 4(2\sin\theta\cos\theta)^2\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} 8\sin^2\theta\cos^2\theta\cos\theta\, d\theta$$

MANIPULATE AS FOLLOWS, OR USE THE SUBSTITUTION $u = \sin\theta$

$$= \int_0^{\pi/2} 8\sin^2\theta(1-\sin^2\theta)\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 8\sin^2\theta\cos\theta - 8\sin^4\theta\cos\theta\, d\theta$$

BY RECOGNITION WE HAVE

$$= \left[\frac{8}{3}\sin^3\theta - \frac{8}{5}\sin^5\theta\right]_0^{\pi/2}$$

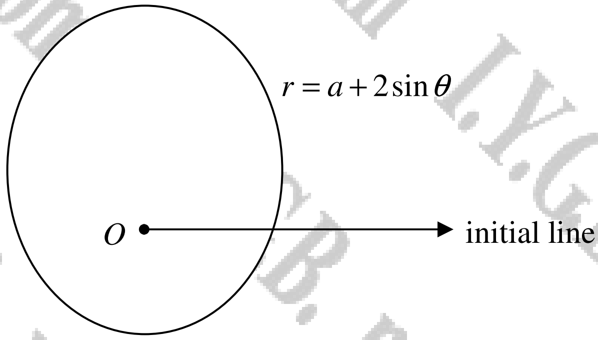
$$= \left(\frac{8}{3} - \frac{8}{5}\right) - (0 - 0)$$

$$= 8\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{16}{15}$$

AS REQUIRED

Question 6



The diagram above shows the curve with polar equation

$$r = a + 2\sin\theta,\ 0 \le \theta < 2\pi,$$

where a is a positive constant.

Determine the value of a given that the area bounded by the curve is 38π.

$\boxed{a = 6}$

$$\boxed{A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta}$$

$\Rightarrow 38\pi = \int_0^{2\pi} \frac{1}{2}(a + 2\sin\theta)^2 \, d\theta$

$\Rightarrow 38\pi = \frac{1}{2}\int_0^{2\pi} a^2 + 2a\sin\theta + 4\sin^2\theta \, d\theta$

$\Rightarrow 76\pi = \int_0^{2\pi} a^2 + 2a\sin\theta + 4\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$

$\Rightarrow 76\pi = \int_0^{2\pi} a^2 + 2a\sin\theta + 2 - 2\cos 2\theta \, d\theta$

$\Rightarrow 76\pi = \left[a^2\theta - 2a\cos\theta + 2\theta - \sin 2\theta\right]_0^{2\pi}$

$\Rightarrow 76\pi = (2\pi a^2 \cancel{- 2a} + 4\pi - 0) - (0 \cancel{- 2a} + 0 - 0)$

$\Rightarrow 76\pi = 2\pi a^2 + 4\pi$

$\Rightarrow 38 = a^2 + 2$

$\Rightarrow a^2 = 36$

$\Rightarrow a = 6$ // $a > 0$