PROOF BY INDUCTION

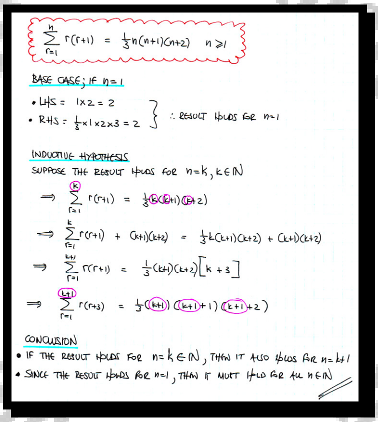
SUMMATION RESULTS

Question 1 (**)

Prove by induction that

$$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2), \quad n \geq 1, \ n \in \mathbb{N}.$$

proof

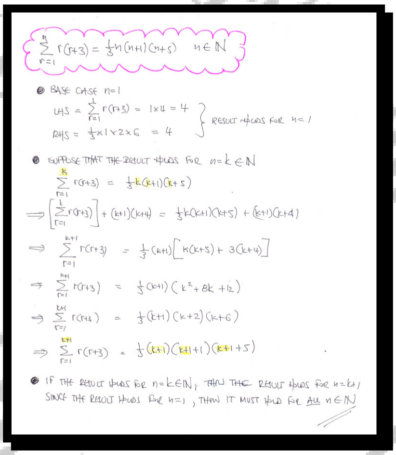


Question 2 (**)

Prove by induction that

$$\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5), \quad n \geq 1, \ n \in \mathbb{N}.$$

proof

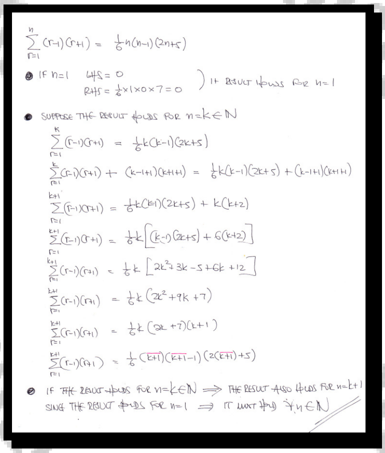


Question 3 (+)**

Prove by induction that

$$\sum_{r=1}^{n}(r-1)(r+1)=\frac{1}{6}n(n-1)(2n+5),\ \ n\geq 1,\ n\in\mathbb{N}.$$

proof



$$\sum_{r=1}^{n}(r-1)(r+1) = \frac{1}{6}n(n-1)(2n+5)$$

- IF $n=1$ LHS $=0$, RHS $=\frac{1}{6}\times1\times0\times7=0$ ∴ RESULT HOLDS FOR $n=1$
- SUPPOSE THE RESULT HOLDS FOR $n=k\in\mathbb{N}$

$$\sum_{r=1}^{k}(r-1)(r+1) = \frac{1}{6}k(k-1)(2k+5)$$

$$\sum_{r=1}^{k}(r-1)(r+1) + (k-1+1)(k+1+1) = \frac{1}{6}k(k-1)(2k+5)+(k-1+1)(k+1+1)$$

$$\sum_{r=1}^{k+1}(r-1)(r+1) = \frac{1}{6}k(k-1)(2k+5)+k(k+2)$$

$$\sum_{r=1}^{k+1}(r-1)(r+1) = \frac{1}{6}k\left[(k-1)(2k+5)+6(k+2)\right]$$

$$\sum_{r=1}^{k+1}(r-1)(r+1) = \frac{1}{6}k\left[2k^2+3k-5+6k+12\right]$$

$$\sum_{r=1}^{k+1}(r-1)(r+1) = \frac{1}{6}k(2k^2+9k+7)$$

$$\sum_{r=1}^{k+1}(r-1)(r+1) = \frac{1}{6}k(2k+7)(k+1)$$

$$\sum_{r=1}^{k+1}(r-1)(r+1) = \frac{1}{6}(k+1)(k+1-1)(2(k+1)+5)$$

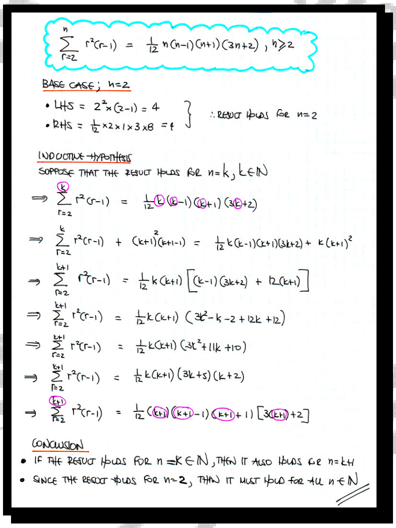
- IF THE RESULT HOLDS FOR $n=k\in\mathbb{N}$ ⟹ THE RESULT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$ ⟹ IT MUST HOLD $\forall n\in\mathbb{N}$

Question 4 **(**+)**

Prove by induction that

$$\sum_{r=2}^{n} r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2), \quad n \geq 2, \; n \in \mathbb{N}.$$

[illegible], proof



$$\sum_{r=2}^{n} r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2), \; n \geq 2$$

BASE CASE; $n=2$

- $LHS = 2^2 \times (2-1) = 4$
- $RHS = \frac{1}{12} \times 2 \times 1 \times 3 \times 8 = 4$

$\therefore$ RESULT HOLDS FOR $n=2$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$

$$\Rightarrow \sum_{r=2}^{k} r^2(r-1) = \frac{1}{12}k(k-1)(k+1)(3k+2)$$

$$\Rightarrow \sum_{r=2}^{k} r^2(r-1) + (k+1)^2(k+1-1) = \frac{1}{12}k(k-1)(k+1)(3k+2) + k(k+1)^2$$

$$\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}k(k+1)\left[(k-1)(3k+2) + 12(k+1)\right]$$

$$\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}k(k+1)(3k^2 - k - 2 + 12k + 12)$$

$$\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}k(k+1)(3k^2 + 11k + 10)$$

$$\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}k(k+1)(3k+5)(k+2)$$

$$\Rightarrow \sum_{r=2}^{k+1} r^2(r-1) = \frac{1}{12}(k+1)((k+1)-1)((k+1)+1)[3(k+1)+2]$$

CONCLUSION

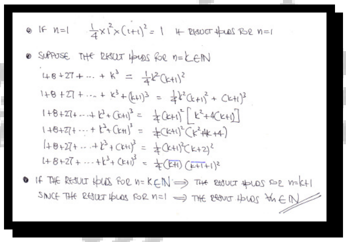
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
- SINCE THE RESULT HOLDS FOR $n=2$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 5 (**+)

Prove by induction that

$$1+8+27+64+\ldots+n^3=\frac{1}{4}n^2\left(n+1\right)^2,\quad n\geq 1,\ n\in\mathbb{N}.$$

proof

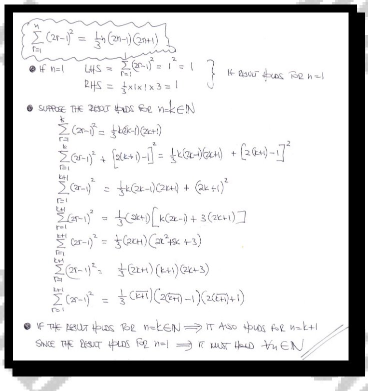


Question 6 (***)

Prove by induction that

$$\sum_{r=1}^{n}\left(2r-1\right)^2=\frac{1}{3}n\left(2n-1\right)\left(2n+1\right),\quad n\geq 1,\ n\in\mathbb{N}.$$

proof

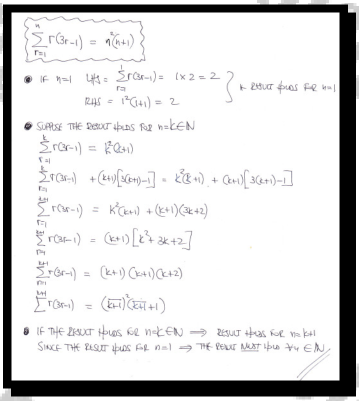


Question 7 (***)

Prove by induction that

$$\sum_{r=1}^{n} r(3r-1) = n^2(n+1), \quad n \ge 1, \ n \in \mathbb{N}.$$

proof



$$\sum_{r=1}^{n} r(3r-1) = n^2(n+1)$$

- IF $n=1$ $\quad LHS = \sum_{r=1}^{1} r(3r-1) = 1 \times 2 = 2$, $\quad RHS = 1^2(1+1) = 2$ $\}$ ∴ RESULT HOLDS FOR $n=1$
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\sum_{r=1}^{k} r(3r-1) = k^2(k+1)$$

$$\sum_{r=1}^{k} r(3r-1) + (k+1)[3(k+1)-1] = k^2(k+1) + (k+1)[3(k+1)-1]$$

$$\sum_{r=1}^{k+1} r(3r-1) = k^2(k+1) + (k+1)(3k+2)$$

$$\sum_{r=1}^{k+1} r(3r-1) = (k+1)[k^2+3k+2]$$

$$\sum_{r=1}^{k+1} r(3r-1) = (k+1)(k+1)(k+2)$$

$$\sum^{k+1} r(3r-1) = (\overline{k+1})^2(\overline{k+1}+1)$$

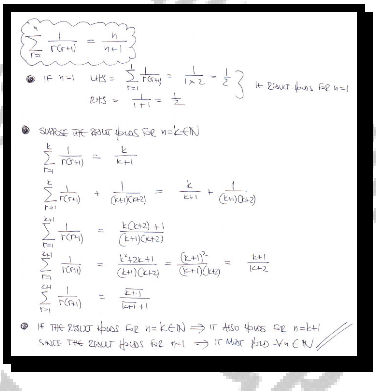
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N} \Rightarrow$ RESULT HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1 \Rightarrow$ THE RESULT MUST HOLD $\forall n \in \mathbb{N}$

Question 8 (***)

Prove by induction that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}, \quad n \ge 1, \ n \in \mathbb{N}.$$

proof



$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

- IF $n=1$ $\quad LHS = \sum_{r=1}^{1} \frac{1}{r(r+1)} = \frac{1}{1 \times 2} = \frac{1}{2}$, $\quad RHS = \frac{1}{1+1} = \frac{1}{2}$ $\}$ ∴ RESULT HOLDS FOR $n=1$
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$$

$$\sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{\overline{k+1}}{\overline{k+1}+1}$$

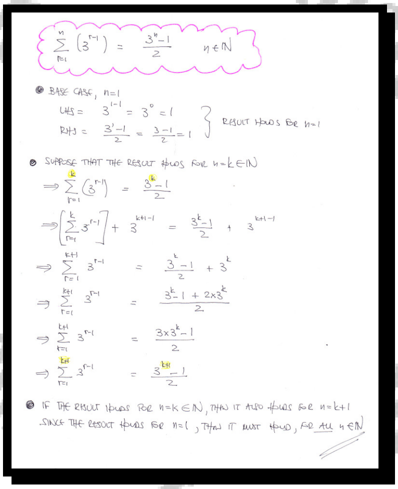
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N} \Rightarrow$ IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1 \Rightarrow$ IT MUST HOLD $\forall n \in \mathbb{N}$

Question 9 **(***)**

Prove by induction that

$$\sum_{r=1}^{n}\left(3^{r-1}\right)=\frac{3^{n}-1}{2},\ \ n\geq 1,\ n\in\mathbb{N}.$$

proof



$$\sum_{r=1}^{n}\left(3^{r-1}\right)=\frac{3^{n}-1}{2}\qquad n\in\mathbb{N}$$

- BASE CASE, $n=1$

$LHS = 3^{1-1}=3^{0}=1$

$RHS = \frac{3^{1}-1}{2}=\frac{3-1}{2}=1$

RESULT HOLDS FOR $n=1$

- SUPPOSE THAT THE RESULT HOLDS FOR $n=k\in\mathbb{N}$

$$\Rightarrow \sum_{r=1}^{k}\left(3^{r-1}\right)=\frac{3^{k}-1}{2}$$

$$\Rightarrow \left[\sum_{r=1}^{k}3^{r-1}\right]+3^{k+1-1}=\frac{3^{k}-1}{2}+3^{k+1-1}$$

$$\Rightarrow \sum_{r=1}^{k+1}3^{r-1}=\frac{3^{k}-1}{2}+3^{k}$$

$$\Rightarrow \sum_{r=1}^{k+1}3^{r-1}=\frac{3^{k}-1+2\times3^{k}}{2}$$

$$\Rightarrow \sum_{r=1}^{k+1}3^{r-1}=\frac{3\times3^{k}-1}{2}$$

$$\Rightarrow \sum_{r=1}^{k+1}3^{r-1}=\frac{3^{k+1}-1}{2}$$

- IF THE RESULT HOLDS FOR $n=k\in\mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$

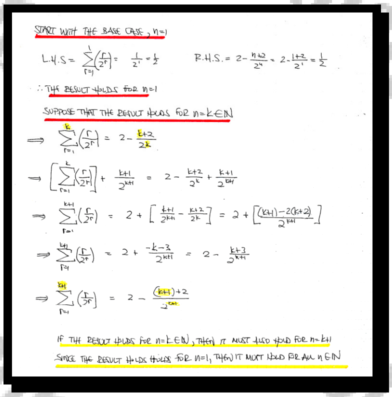
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD, FOR ALL $n\in\mathbb{N}$

Question 10 (***+)

Prove by induction that

$$\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}, \quad n \geq 1, \ n \in \mathbb{N}.$$

proof



START WITH THE BASE CASE, $n=1$

$\text{L.H.S.} = \sum_{r=1}^{1} \left(\frac{r}{2^r}\right) = \frac{1}{2^1} = \frac{1}{2}$ $\qquad$ $\text{R.H.S.} = 2 - \frac{n+2}{2^n} = 2 - \frac{1+2}{2^1} = \frac{1}{2}$

$\therefore$ THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n = k \in \mathbb{N}$

$\Rightarrow \sum_{r=1}^{k} \left(\frac{r}{2^r}\right) = 2 - \frac{k+2}{2^k}$

$\Rightarrow \left[\sum_{r=1}^{k} \left(\frac{r}{2^r}\right)\right] + \frac{k+1}{2^{k+1}} = 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$

$\Rightarrow \sum_{r=1}^{k+1} \left(\frac{r}{2^r}\right) = 2 + \left[\frac{k+1}{2^{k+1}} - \frac{k+2}{2^k}\right] = 2 + \left[\frac{(k+1) - 2(k+2)}{2^{k+1}}\right]$

$\Rightarrow \sum_{r=1}^{k+1} \left(\frac{r}{2^r}\right) = 2 + \frac{-k-3}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}}$

$\Rightarrow \sum_{r=1}^{k+1} \left(\frac{r}{2^r}\right) = 2 - \frac{(k+1)+2}{2^{k+1}}$

IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n = k+1$

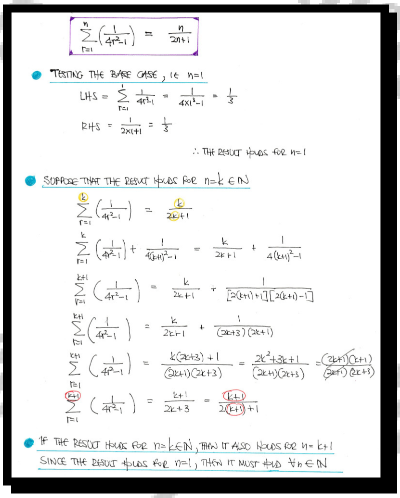
SINCE THE RESULT HOLDS HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 11 (***+)

Prove by induction that

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{n}{2n+1}, \quad n \geq 1, \; n \in \mathbb{N}.$$

proof

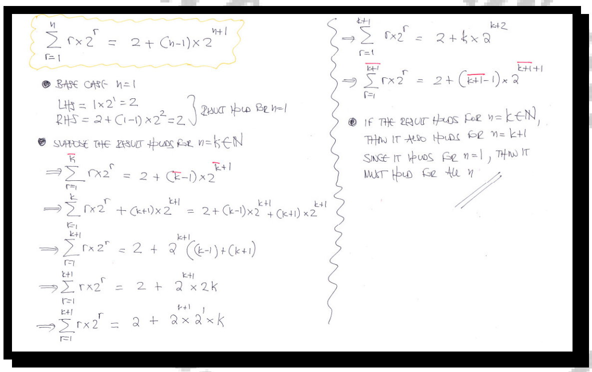


Question 12 (***+)

Prove by induction that

$$\sum_{r=1}^{n} r \times 2^r = 2 + (n-1)2^{n+1}, \quad n \geq 1, \; n \in \mathbb{N}.$$

proof

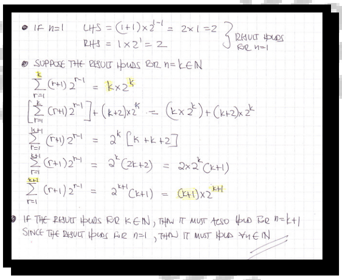


Question 13 (***+)

Prove by induction that

$$\sum_{r=1}^{n}\left[(r+1)\times 2^{r}\right]=n\times 2^{n},\quad n\geq 1,\ n\in\mathbb{N}.$$

proof

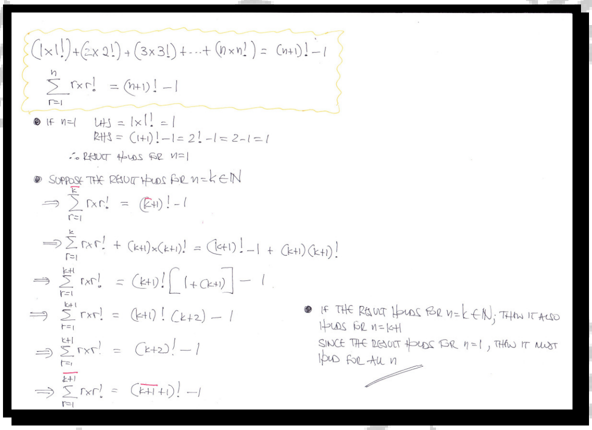


Question 14 (***+)

If $n\geq 1,\ n\in\mathbb{N}$, prove by induction that

$$1\times 1!+2\times 2!+3\times 3!+\ldots+n\times n!=(n+1)!-1.$$

proof

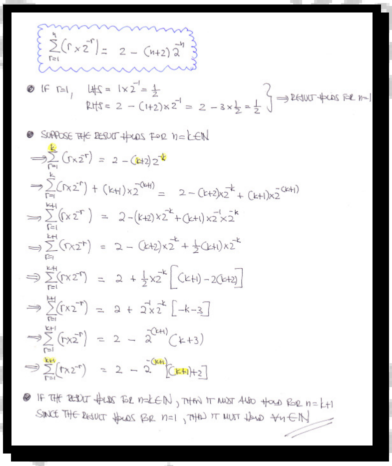


Question 15 (***+)

Prove by induction that

$$\sum_{r=1}^{n} r \times 2^{-r} = 2-(n+2)2^{-n}, \quad n \geq 1, \; n \in \mathbb{N}.$$

proof



$$\sum_{r=1}^{n}(r\times 2^{-r}) = 2-(n+2)2^{-n}$$

- IF $r=1$, $LHS = 1\times 2^{-1} = \frac{1}{2}$, $RHS = 2-(1+2)\times 2^{-1} = 2-3\times\frac{1}{2} = \frac{1}{2}$ $\Rightarrow$ RESULT HOLDS FOR $n=1$

- SUPPOSE THE RESULT HOLDS FOR $n=k\in\mathbb{N}$

$\Rightarrow \sum_{r=1}^{k}(r\times 2^{-r}) = 2-(k+2)2^{-k}$

$\Rightarrow \sum_{r=1}^{k}(r\times 2^{-r}) + (k+1)\times 2^{-(k+1)} = 2-(k+2)\times 2^{-k} + (k+1)\times 2^{-(k+1)}$

$\Rightarrow \sum_{r=1}^{k+1}(r\times 2^{-r}) = 2-(k+2)\times 2^{-k} + (k+1)\times 2^{-1}\times 2^{-k}$

$\Rightarrow \sum_{r=1}^{k+1}(r\times 2^{-r}) = 2-(k+2)\times 2^{-k} + \frac{1}{2}(k+1)\times 2^{-k}$

$\Rightarrow \sum_{r=1}^{k+1}(r\times 2^{-r}) = 2+\frac{1}{2}\times 2^{-k}\left[(k+1)-2(k+2)\right]$

$\Rightarrow \sum_{r=1}^{k+1}(r\times 2^{-r}) = 2+2^{-1}\times 2^{-k}\left[-k-3\right]$

$\Rightarrow \sum_{r=1}^{k+1}(r\times 2^{-r}) = 2-2^{-(k+1)}(k+3)$

$\Rightarrow \sum_{r=1}^{k+1}(r\times 2^{-r}) = 2-2^{-(k+1)}\left[(k+1)+2\right]$

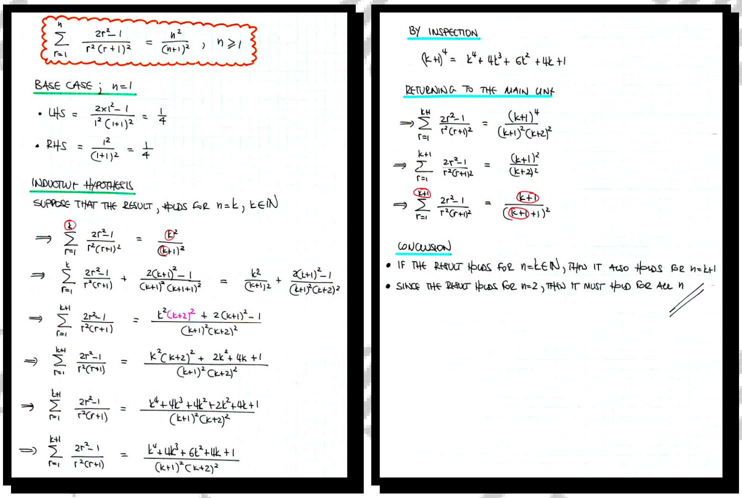
- IF THE RESULT HOLDS FOR $n=k\in\mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$. SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n\in\mathbb{N}$

Question 16 (****)

Prove by induction that

$$\sum_{r=1}^{n} \frac{2r^2-1}{r^2(r+1)^2} = \frac{n^2}{(n+1)^2}, \quad n \geq 1,\ n \in \mathbb{N}.$$

[illegible], proof



$$\sum_{r=1}^{n} \frac{2r^2-1}{r^2(r+1)^2} = \frac{n^2}{(n+1)^2}, \quad n \geq 1$$

BASE CASE ; $n=1$

- LHS $= \frac{2\times 1^2 - 1}{1^2(1+1)^2} = \frac{1}{4}$
- RHS $= \frac{1^2}{(1+1)^2} = \frac{1}{4}$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT, HOLDS FOR $n=k$, $k \in \mathbb{N}$

$$\Rightarrow \sum_{r=1}^{k} \frac{2r^2-1}{r^2(r+1)^2} = \frac{k^2}{(k+1)^2}$$

$$\Rightarrow \sum_{r=1}^{k} \frac{2r^2-1}{r^2(r+1)} + \frac{2(k+1)^2-1}{(k+1)^2(k+1+1)^2} = \frac{k^2}{(k+1)^2} + \frac{2(k+1)^2-1}{(k+1)^2(k+2)^2}$$

$$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)} = \frac{k^2(k+2)^2 + 2(k+1)^2 - 1}{(k+1)^2(k+2)^2}$$

$$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)} = \frac{k^2(k+2)^2 + 2k^2 + 4k + 1}{(k+1)^2(k+2)^2}$$

$$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)} = \frac{k^4+4k^3+4k^2+2k^2+4k+1}{(k+1)^2(k+2)^2}$$

$$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)} = \frac{k^4+4k^3+6k^2+4k+1}{(k+1)^2(k+2)^2}$$

BY INSPECTION

$$(k+1)^4 = k^4+4k^3+6k^2+4k+1$$

RETURNING TO THE MAIN LINE

$$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{(k+1)^4}{(k+1)^2(k+2)^2}$$

$$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{(k+1)^2}{(k+2)^2}$$

$$\Rightarrow \sum_{r=1}^{k+1} \frac{2r^2-1}{r^2(r+1)^2} = \frac{(k+1)^2}{((k+1)+1)^2}$$

CONCLUSION

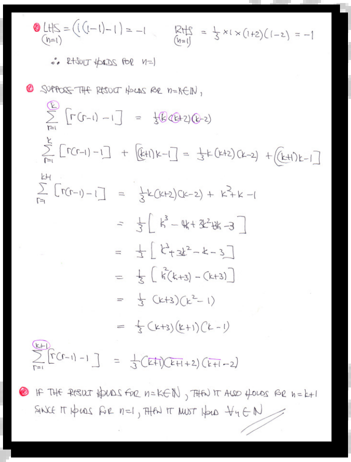
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
- SINCE THE RESULT HOLDS FOR $n=2$, THEN IT MUST HOLD FOR ALL n

Question 17 **(****)**

Prove by induction that

$$\sum_{r=1}^{n}\left[r(r-1)-1\right]=\tfrac{1}{3}n(n+2)(n-2), \quad n \ge 1, \ n \in \mathbb{N}.$$

proof



- LHS $(n=1)$ $= 1(1-1)-1 = -1$ RHS $(n=1)$ $= \frac{1}{3}\times 1\times(1+2)(1-2) = -1$

  $\therefore$ RESULT HOLDS FOR $n=1$

- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$,

$$\sum_{r=1}^{k}\left[r(r-1)-1\right] = \tfrac{1}{3}k(k+2)(k-2)$$

$$\sum_{r=1}^{k}\left[r(r-1)-1\right] + \left[(k+1)k-1\right] = \tfrac{1}{3}k(k+2)(k-2) + \left[(k+1)k-1\right]$$

$$\sum_{r=1}^{k+1}\left[r(r-1)-1\right] = \tfrac{1}{3}k(k+2)(k-2) + k^2+k-1$$

$$= \tfrac{1}{3}\left[k^3-4k+3k^2+3k-3\right]$$

$$= \tfrac{1}{3}\left[k^3+3k^2-k-3\right]$$

$$= \tfrac{1}{3}\left[k^2(k+3)-(k+3)\right]$$

$$= \tfrac{1}{3}(k+3)(k^2-1)$$

$$= \tfrac{1}{3}(k+3)(k+1)(k-1)$$

$$\sum_{r=1}^{k+1}\left[r(r-1)-1\right] = \tfrac{1}{3}(k+1)(k+1+2)(k+1-2)$$

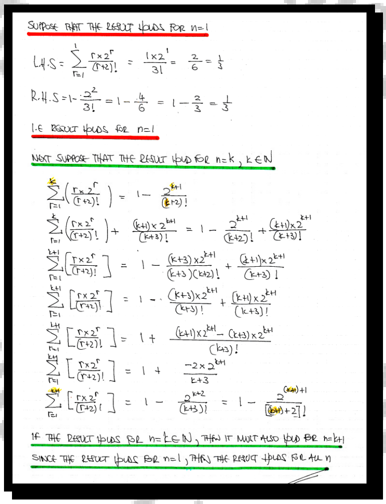
- IF THE RESULT HOLDS FOR $n=k\in\mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$. SINCE IT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 18 (****)

Prove by induction that

$$\sum_{r=1}^{n} \frac{r \times 2^r}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}, \quad n \geq 1, \ n \in \mathbb{N}.$$

proof

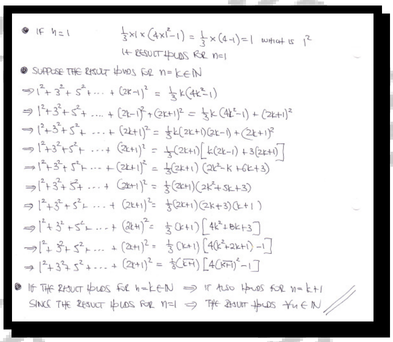


Question 19 (****)

Prove by induction that

$$1^2 + 3^2 + 5^2 + 7^2 + \ldots + (2n-1)^2 \equiv \frac{1}{3}n\left(4n^2 - 1\right), \quad n \geq 1, \ n \in \mathbb{N}.$$

proof



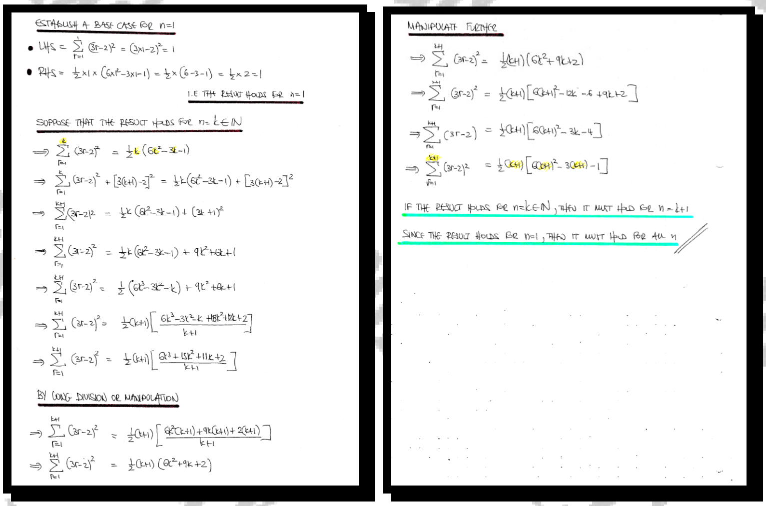
Question 20 (****)

Prove by mathematical induction that if n is a positive integer then

$$\sum_{r=1}^{n}(3r-2)^2 = \frac{1}{2}n\left(6n^2-3n-1\right).$$

You may not use other methods of proof in this question.

[ ], [proof]



ESTABLISH A BASE CASE FOR $n=1$

- $\text{LHS} = \sum_{r=1}^{1}(3r-2)^2 = (3\times1-2)^2 = 1$
- $\text{RHS} = \frac{1}{2}\times1\times(6\times1^2-3\times1-1) = \frac{1}{2}\times(6-3-1) = \frac{1}{2}\times2 = 1$

I.E THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n=k\in\mathbb{N}$

$$\Rightarrow \sum_{r=1}^{k}(3r-2)^2 = \frac{1}{2}k(6k^2-3k-1)$$

$$\Rightarrow \sum_{r=1}^{k}(3r-2)^2 + [3(k+1)-2]^2 = \frac{1}{2}k(6k^2-3k-1) + [3(k+1)-2]^2$$

$$\Rightarrow \sum_{r=1}^{k+1}(3r-2|^2 = \frac{1}{2}k(6k^2-3k-1) + (3k+1)^2$$

$$\Rightarrow \sum_{r=1}^{k+1}(3r-2)^2 = \frac{1}{2}k(6k^2-3k-1) + 9k^2+6k+1$$

$$\Rightarrow \sum_{r=1}^{k+1}(3r-2)^2 = \frac{1}{2}(6k^3-3k^2-k) + 9k^2+6k+1$$

$$\Rightarrow \sum_{r=1}^{k+1}(3r-2)^2 = \frac{1}{2}(k+1)\left[\frac{6k^3-3k^2-k+18k^2+12k+2}{k+1}\right]$$

$$\Rightarrow \sum_{r=1}^{k+1}(3r-2)^2 = \frac{1}{2}(k+1)\left[\frac{6k^3+15k^2+11k+2}{k+1}\right]$$

BY LONG DIVISION OR MANIPULATION

$$\Rightarrow \sum_{r=1}^{k+1}(3r-2)^2 = \frac{1}{2}(k+1)\left[\frac{6k^2(k+1)+9k(k+1)+2(k+1)}{k+1}\right]$$

$$\Rightarrow \sum_{r=1}^{k+1}(3r-2)^2 = \frac{1}{2}(k+1)(6k^2+9k+2)$$

MANIPULATE FURTHER

$$\Rightarrow \sum_{r=1}^{k+1}(3r-2)^2 = \frac{1}{2}(k+1)(6k^2+9k+2)$$

$$\Rightarrow \sum_{r=1}^{k+1}(3r-2)^2 = \frac{1}{2}(k+1)\left[6(k+1)^2-12k-6+9k+2\right]$$

$$\Rightarrow \sum_{r=1}^{k+1}(3r-2) = \frac{1}{2}(k+1)\left[6(k+1)^2-3k-4\right]$$

$$\Rightarrow \sum_{r=1}^{k+1}(3r-2)^2 = \frac{1}{2}(k+1)\left[6(k+1)^2-3(k+1)-1\right]$$

IF THE RESULT HOLDS FOR $n=k\in\mathbb{N}$, THEN IT MUST HOLD FOR $n=k+1$

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n

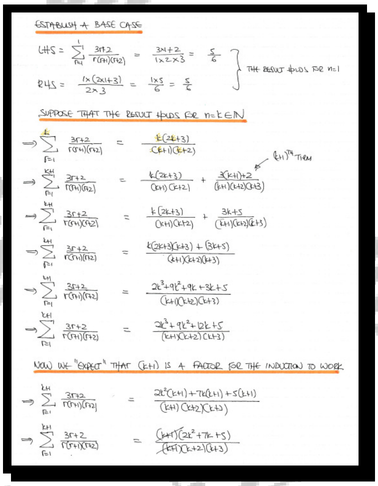
Question 21 (****)

Prove by mathematical induction that if n is a positive integer then

$$\sum_{r=1}^{n} \frac{3r+2}{r(r+1)(r+2)} = \frac{n(2n+3)}{(n+1)(n+2)}.$$

You may not use other methods of proof in this question.

[ ] , [proof]



ESTABLISH A BASE CASE

$$\text{LHS} = \sum_{r=1}^{1} \frac{3r+2}{r(r+1)(r+2)} = \frac{3\times1+2}{1\times2\times3} = \frac{5}{6}$$

$$\text{RHS} = \frac{1\times(2\times1+3)}{2\times3} = \frac{1\times5}{6} = \frac{5}{6}$$

THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n=k\in\mathbb{N}$

$$\Longrightarrow \sum_{r=1}^{k} \frac{3r+2}{r(r+1)(r+2)} = \frac{k(2k+3)}{(k+1)(k+2)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{k(2k+3)}{(k+1)(k+2)} + \frac{3(k+1)+2}{(k+1)(k+2)(k+3)}$$

$(k+1)^{\text{th}}$ TERM

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{k(2k+3)}{(k+1)(k+2)} + \frac{3k+5}{(k+1)(k+2)(k+3)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{k(2k+3)(k+3)+(3k+5)}{(k+1)(k+2)(k+3)}$$

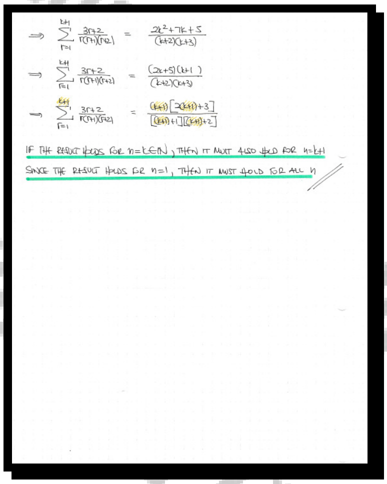
$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{2k^3+9k^2+9k+3k+5}{(k+1)(k+2)(k+3)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{2k^3+9k^2+12k+5}{(k+1)(k+2)(k+3)}$$

NOW WE "EXPECT" THAT $(k+1)$ IS A FACTOR FOR THE INDUCTION TO WORK

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{2k^2(k+1)+7k(k+1)+5(k+1)}{(k+1)(k+2)(k+3)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{(k+1)(2k^2+7k+5)}{(k+1)(k+2)(k+3)}$$



$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{2k^2+7k+5}{(k+2)(k+3)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{(2k+5)(k+1)}{(k+2)(k+3)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{(k+1)[2(k+1)+3]}{[(k+1)+1][(k+1)+2]}$$

IF THE RESULT HOLDS FOR $n=k\in\mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$

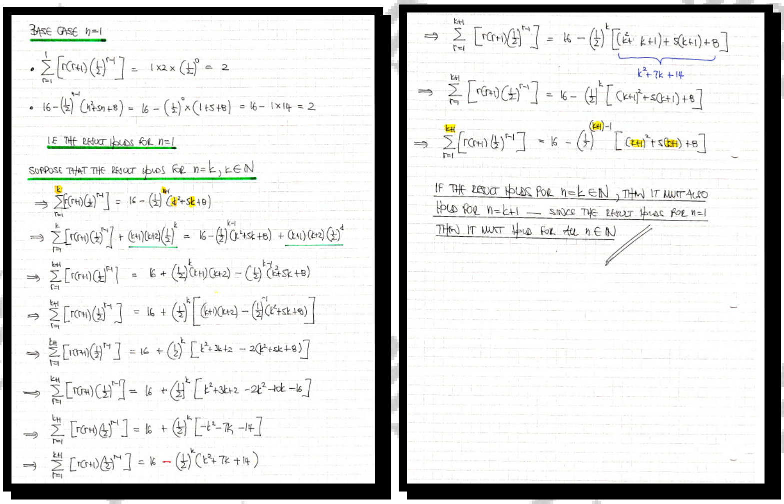
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n

Question 22 (****)

Prove by induction that

$$\sum_{r=1}^{n}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right]=16-\left(\frac{1}{2}\right)^{n-1}\left(n^2+5n+8\right),\ \ n\geq 1,\ n\in\mathbb{N}.$$

I.Y.G.B, proof


BASE CASE $n=1$

- $\sum_{r=1}^{1}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right] = 1\times2\times\left(\frac{1}{2}\right)^0 = 2$
- $16-\left(\frac{1}{2}\right)^{n-1}(n^2+5n+8) = 16-\left(\frac{1}{2}\right)^0\times(1+5+8) = 16-1\times14 = 2$

I.E. THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n=k,\ k\in\mathbb{N}$

$\Rightarrow \sum_{r=1}^{k}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right] = 16-\left(\frac{1}{2}\right)^{k-1}(k^2+5k+8)$

$\Rightarrow \sum_{r=1}^{k}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right] + (k+1)(k+2)\left(\frac{1}{2}\right)^{k} = 16-\left(\frac{1}{2}\right)^{k-1}(k^2+5k+8) + (k+1)(k+2)\left(\frac{1}{2}\right)^{k}$

$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right] = 16+\left(\frac{1}{2}\right)^{k}(k+1)(k+2)-\left(\frac{1}{2}\right)^{k-1}(k^2+5k+8)$

$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right] = 16+\left(\frac{1}{2}\right)^{k}\left[(k+1)(k+2)-\left(\frac{1}{2}\right)^{-1}(k^2+5k+8)\right]$

$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right] = 16+\left(\frac{1}{2}\right)^{k}\left[k^2+3k+2-2(k^2+5k+8)\right]$

$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right] = 16+\left(\frac{1}{2}\right)^{k}\left[k^2+3k+2-2k^2-10k-16\right]$

$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right] = 16+\left(\frac{1}{2}\right)^{k}\left[-k^2-7k-14\right]$

$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right] = 16-\left(\frac{1}{2}\right)^{k}(k^2+7k+14)$

$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right] = 16-\left(\frac{1}{2}\right)^{k}\left[(k^2+2k+1)+5(k+1)+8\right]$ ($k^2+7k+14$)

$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right] = 16-\left(\frac{1}{2}\right)^{k}\left[(k+1)^2+5(k+1)+8\right]$

$\Rightarrow \sum_{r=1}^{k+1}\left[r(r+1)\left(\frac{1}{2}\right)^{r-1}\right] = 16-\left(\frac{1}{2}\right)^{(k+1)-1}\left[(k+1)^2+5(k+1)+8\right]$

IF THE RESULT HOLDS FOR $n=k\in\mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$ — SINCE THE RESULT HOLDS FOR $n=1$ THEN IT MUST HOLD FOR ALL $n\in\mathbb{N}$

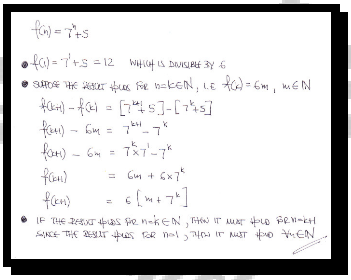
DIVISIBILITY RESULTS

Question 1 (**)

$$f(n) = 7^n + 5, \ n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 6, for all $n \in \mathbb{N}$.

proof



$f(k) = 7^k + 5$

- $f(1) = 7^1 + 5 = 12$ WHICH IS DIVISIBLE BY 6
- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, i.e. $f(k) = 6m$, $m \in \mathbb{N}$

$f(k+1) - f(k) = [7^{k+1} + 5] - [7^k + 5]$

$f(k+1) - 6m = 7^{k+1} - 7^k$

$f(k+1) - 6m = 7^k 7^1 - 7^k$

$f(k+1) = 6m + 6 \times 7^k$

$f(k+1) = 6[m + 7^k]$

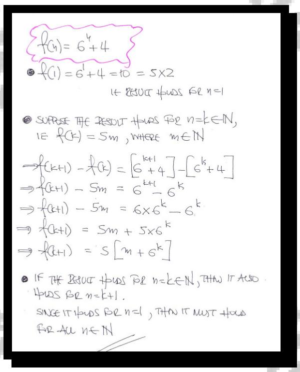
- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT MUST HOLD FOR $n = k+1$. SINCE THE RESULT HOLDS FOR $n = 1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 2 (**)

$$f(n) = 6^n + 4, \ n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 5, for all $n \in \mathbb{N}$.

proof



$f(k) = 6^k + 4$

- $f(1) = 6^1 + 4 = 10 = 5 \times 2$

  i.e. RESULT HOLDS FOR $n = 1$

- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, i.e. $f(k) = 5m$, WHERE $m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [6^{k+1} + 4] - [6^k + 4]$

$\Rightarrow f(k+1) - 5m = 6^{k+1} - 6^k$

$\Rightarrow f(k+1) - 5m = 6 \times 6^k - 6^k$

$\Rightarrow f(k+1) = 5m + 5 \times 6^k$

$\Rightarrow f(k+1) = 5[m + 6^k]$

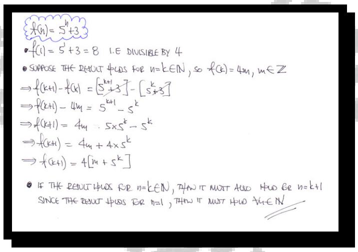
- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$. SINCE IT HOLDS FOR $n = 1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 3 **(**)**

$$f(n) = 5^n + 3, \ n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 4, for all $n \in \mathbb{N}$.

proof



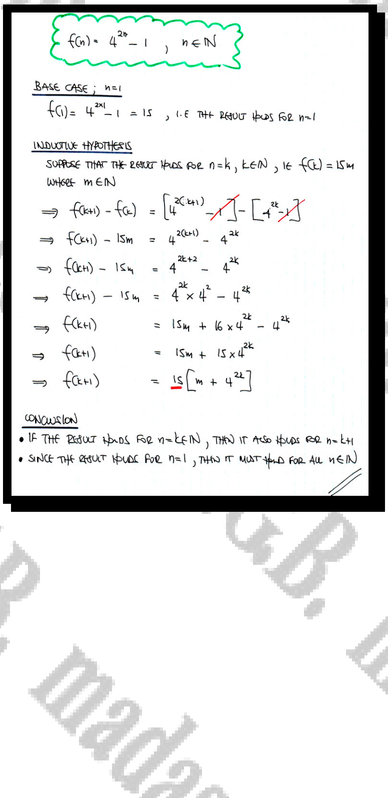
Question 4 **(**)**

Prove by induction that for all natural numbers n,

$$4^{2n} - 1$$

is divisible by 15.

, proof



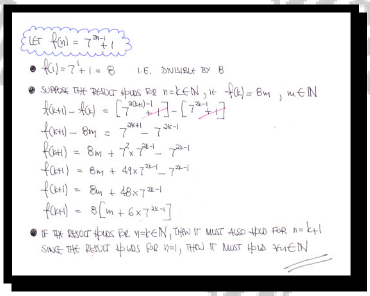
Question 5 (**)

Prove by induction that for all natural numbers n,

$$7^{2n-1}+1$$

is divisible by 8.

proof


LET $f(n) = 7^{2n-1}+1$

- $f(1) = 7^1+1 = 8$ I.E. DIVISIBLE BY 8
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, IE $f(k) = 8m$, $m \in \mathbb{N}$

$f(k+1) - f(k) = \left[7^{2(k+1)-1}+1\right] - \left[7^{2k-1}+1\right]$

$f(k+1) - 8m = 7^{2k+1} - 7^{2k-1}$

$f(k+1) = 8m + 7^2 \times 7^{2k-1} - 7^{2k-1}$

$f(k+1) = 8m + 49 \times 7^{2k-1} - 7^{2k-1}$

$f(k+1) = 8m + 48 \times 7^{2k-1}$

$f(k+1) = 8\left[m + 6 \times 7^{2k-1}\right]$

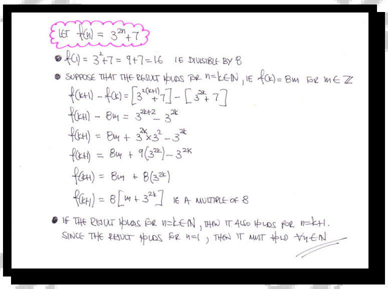
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 6 (**)

Prove by induction that for all natural numbers n,

$$3^{2n}+7 \text{ is divisible by } 8.$$

proof


LET $f(n) = 3^{2n}+7$

- $f(1) = 3^2+7 = 9+7 = 16$ IE DIVISIBLE BY 8
- SUPPOSE THAT THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, IE $f(k) = 8m$ FOR $m \in \mathbb{Z}$

$f(k+1) - f(k) = \left[3^{2(k+1)}+7\right] - \left[3^{2k}+7\right]$

$f(k+1) - 8m = 3^{2k+2} - 3^{2k}$

$f(k+1) = 8m + 3^{2k} \times 3^2 - 3^{2k}$

$f(k+1) = 8m + 9(3^{2k}) - 3^{2k}$

$f(k+1) = 8m + 8(3^{2k})$

$f(k+1) = 8\left[m + 3^{2k}\right]$ IE A MULTIPLE OF 8

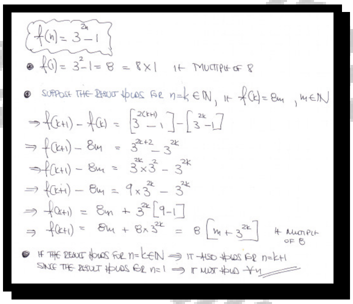
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$.
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 7 (**)

$$f(n) = 3^{2n} - 1, \ n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is a multiple of 8, for all $n \in \mathbb{N}$.

proof


$f(n) = 3^{2n} - 1$

- $f(1) = 3^2 - 1 = 8 = 8 \times 1$ IE MULTIPLE OF 8
- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, IE $f(k) = 8m$, $m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [3^{2(k+1)} - 1] - [3^{2k} - 1]$

$\Rightarrow f(k+1) - 8m = 3^{2k+2} - 3^{2k}$

$\Rightarrow f(k+1) - 8m = 3^{2k} \times 3^2 - 3^{2k}$

$\Rightarrow f(k+1) - 8m = 9 \times 3^{2k} - 3^{2k}$

$\Rightarrow f(k+1) = 8m + 3^{2k}[9-1]$

$\Rightarrow f(k+1) = 8m + 8 \times 3^{2k} = 8[m + 3^{2k}]$ IE MULTIPLE OF 8

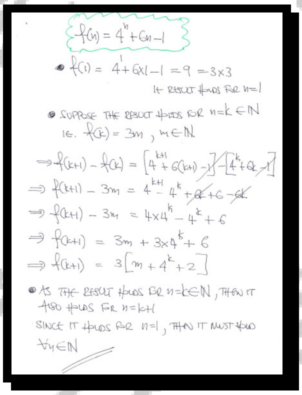
- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N} \Rightarrow$ IT ALSO HOLDS FOR $n = k+1$
SINCE THE RESULT HOLDS FOR $n = 1 \Rightarrow$ IT MUST HOLD $\forall n$

Question 8 (+)**

$$f(n) = 4^n + 6n - 1, \ n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 3, for all $n \in \mathbb{N}$.

proof


$f(n) = 4^n + 6n - 1$

- $f(1) = 4^1 + 6 \times 1 - 1 = 9 = 3 \times 3$
IE RESULT HOLDS FOR $n = 1$
- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$
IE. $f(k) = 3m$, $m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [4^{k+1} + 6(k+1) - 1] - [4^k + 6k - 1]$

$\Rightarrow f(k+1) - 3m = 4^{k+1} - 4^k + 6k + 6 - 6k$

$\Rightarrow f(k+1) - 3m = 4 \times 4^k - 4^k + 6$

$\Rightarrow f(k+1) = 3m + 3 \times 4^k + 6$

$\Rightarrow f(k+1) = 3[m + 4^k + 2]$

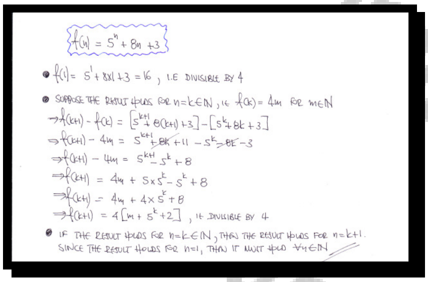
- AS THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$
SINCE IT HOLDS FOR $n = 1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 9 **(**+)**

$$f(n) = 5^n + 8n + 3 \ , \ n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 4, for all $n \in \mathbb{N}$.

proof



$f(n) = 5^n + 8n + 3$

- $f(1) = 5^1 + 8\times1 + 3 = 16$, I.E DIVISIBLE BY 4
- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, IE $f(k) = 4m$ FOR $m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [5^{k+1} + 8(k+1) + 3] - [5^k + 8k + 3]$

$\Rightarrow f(k+1) - 4m = 5^{k+1} + 8k + 11 - 5^k - 8k - 3$

$\Rightarrow f(k+1) - 4m = 5^{k+1} - 5^k + 8$

$\Rightarrow f(k+1) = 4m + 5\times5^k - 5^k + 8$

$\Rightarrow f(k+1) = 4m + 4\times5^k + 8$

$\Rightarrow f(k+1) = 4[m + 5^k + 2]$, IE DIVISIBLE BY 4

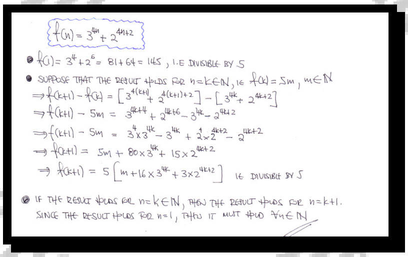
- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN THE RESULT HOLDS FOR $n = k+1$. SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 10 **(**+)**

$$f(n) = 3^{4n} + 2^{4n+2} \ , \ n \in \mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 5, for all $n \in \mathbb{N}$.

proof



$f(n) = 3^{4n} + 2^{4n+2}$

- $f(1) = 3^4 + 2^6 = 81 + 64 = 145$, I.E DIVISIBLE BY 5
- SUPPOSE THAT THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, IE $f(k) = 5m$, $m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [3^{4(k+1)} + 2^{4(k+1)+2}] - [3^{4k} + 2^{4k+2}]$

$\Rightarrow f(k+1) - 5m = 3^{4k+4} + 2^{4k+6} - 3^{4k} - 2^{4k+2}$

$\Rightarrow f(k+1) - 5m = 3^4\times3^{4k} - 3^{4k} + 2^4\times2^{4k+2} - 2^{4k+2}$

$\Rightarrow f(k+1) = 5m + 80\times3^{4k} + 15\times2^{4k+2}$

$\Rightarrow f(k+1) = 5[m + 16\times3^{4k} + 3\times2^{4k+2}]$ IE DIVISIBLE BY 5

- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN THE RESULT HOLDS FOR $n = k+1$. SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

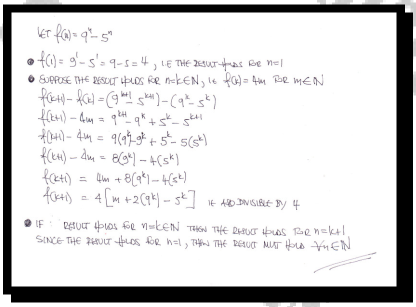
Question 11 **(**+)**

Prove by induction that for all natural numbers n,

$$9^n - 5^n$$

is divisible by 4.

proof

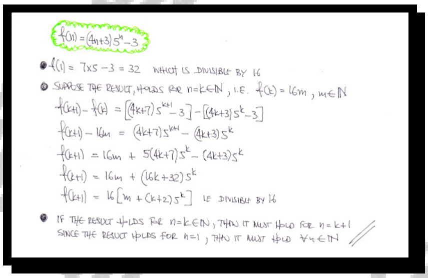


Question 12 **(**+)**

$$f(n) = (4n+3)5^n - 3, \; n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 16, for all $n \in \mathbb{N}$.

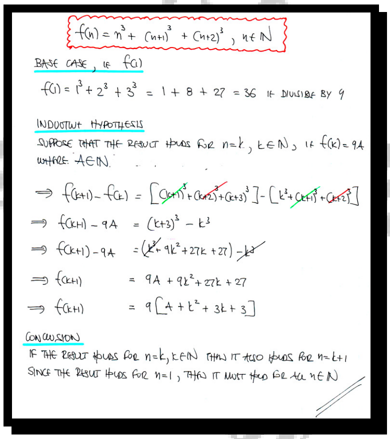
proof



Question 13 (***)

Prove by induction that the sum of the cubes of any three consecutive positive integers is always divisible by 9.

[ ], proof


$f(n) = n^3 + (n+1)^3 + (n+2)^3, \ n \in \mathbb{N}$

BASE CASE, IE $f(1)$

$f(1) = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$ IE DIVISIBLE BY 9

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n = k$, $k \in \mathbb{N}$, IE $f(k) = 9A$ WHERE $A \in \mathbb{N}$.

$\Rightarrow f(k+1) - f(k) = \left[(k+1)^3 + (k+2)^3 + (k+3)^3\right] - \left[k^3 + (k+1)^3 + (k+2)^3\right]$

$\Rightarrow f(k+1) - 9A = (k+3)^3 - k^3$

$\Rightarrow f(k+1) - 9A = (k^3 + 9k^2 + 27k + 27) - k^3$

$\Rightarrow f(k+1) = 9A + 9k^2 + 27k + 27$

$\Rightarrow f(k+1) = 9\left[A + k^2 + 3k + 3\right]$

CONCLUSION

IF THE RESULT HOLDS FOR $n = k$, $k \in \mathbb{N}$ THEN IT ALSO HOLDS FOR $n = k+1$. SINCE THE RESULT HOLDS FOR $n = 1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

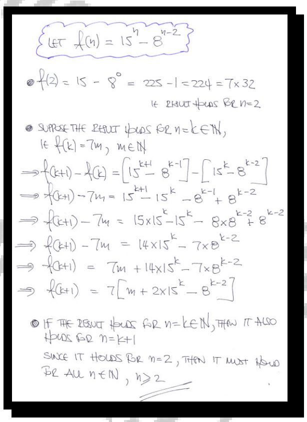
Question 14 (***)

Prove by induction that for all natural numbers n, such that $n \geq 2$,

$$15^n - 8^{n-2},$$

is divisible by 7.

proof


LET $f(n) = 15^n - 8^{n-2}$

- $f(2) = 15^2 - 8^0 = 225 - 1 = 224 = 7 \times 32$

IE RESULT HOLDS FOR $n = 2$

- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, IE $f(k) = 7m$, $m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = \left[15^{k+1} - 8^{k-1}\right] - \left[15^k - 8^{k-2}\right]$

$\Rightarrow f(k+1) - 7m = 15^{k+1} - 15^k - 8^{k-1} + 8^{k-2}$

$\Rightarrow f(k+1) - 7m = 15 \times 15^k - 15^k - 8 \times 8^{k-2} + 8^{k-2}$

$\Rightarrow f(k+1) - 7m = 14 \times 15^k - 7 \times 8^{k-2}$

$\Rightarrow f(k+1) = 7m + 14 \times 15^k - 7 \times 8^{k-2}$

$\Rightarrow f(k+1) = 7\left[m + 2 \times 15^k - 8^{k-2}\right]$

- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$

SINCE IT HOLDS FOR $n = 2$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$, $n \geq 2$

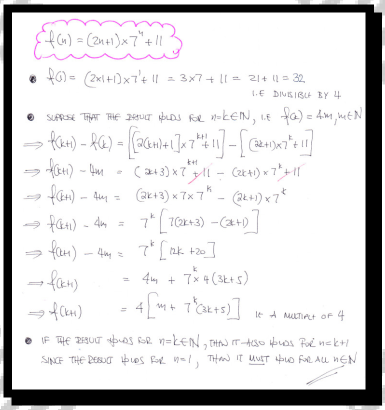
Question 15 (***)

Prove by induction that for all natural numbers n,

$$(2n+1)7^n + 11 ,$$

is divisible by 4.

proof



$f(n) = (2n+1)\times 7^n + 11$

- $f(1) = (2\times1+1)\times 7^1 + 11 = 3\times 7 + 11 = 21 + 11 = 32$ I.E. DIVISIBLE BY 4
- SUPPOSE THAT THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, I.E. $f(k) = 4m, m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = \left[[2(k+1)+1]\times 7^{k+1} + 11\right] - \left[(2k+1)\times 7^k + 11\right]$

$\Rightarrow f(k+1) - 4m = (2k+3)\times 7^{k+1} + 11 - (2k+1)\times 7^k + 11$

$\Rightarrow f(k+1) - 4m = (2k+3)\times 7 \times 7^k - (2k+1)\times 7^k$

$\Rightarrow f(k+1) - 4m = 7^k\left[7(2k+3) - (2k+1)\right]$

$\Rightarrow f(k+1) - 4m = 7^k\left[12k + 20\right]$

$\Rightarrow f(k+1) = 4m + 7^k \times 4(3k+5)$

$\Rightarrow f(k+1) = 4\left[m + 7^k(3k+5)\right]$ IE A MULTIPLE OF 4

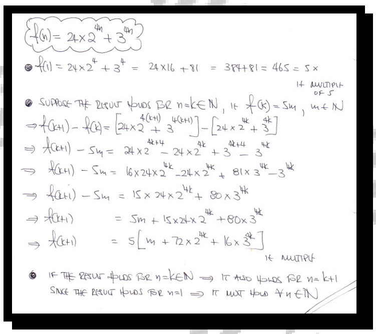
- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THAN IT ALSO HOLDS FOR $n = k+1$ SINCE THE RESULT HOLDS FOR $n=1$, THAN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 16 (***)

$$f(n) = 24\times 2^{4n} + 3^{4n}, \; n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 5, for all $n \in \mathbb{N}$.

proof



$f(n) = 24\times 2^{4n} + 3^{4n}$

- $f(1) = 24\times 2^4 + 3^4 = 24\times 16 + 81 = 384 + 81 = 465 = 5\times$ IE MULTIPLE OF 5
- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, IE $f(k) = 5m, m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = \left[24\times 2^{4(k+1)} + 3^{4(k+1)}\right] - \left[24\times 2^{4k} + 3^{4k}\right]$

$\Rightarrow f(k+1) - 5m = 24\times 2^{4k+4} - 24\times 2^{4k} + 3^{4k+4} - 3^{4k}$

$\Rightarrow f(k+1) - 5m = 16\times 24\times 2^{4k} - 24\times 2^{4k} + 81\times 3^{4k} - 3^{4k}$

$\Rightarrow f(k+1) - 5m = 15\times 24\times 2^{4k} + 80\times 3^{4k}$

$\Rightarrow f(k+1) = 5m + 15\times 24\times 2^{4k} + 80\times 3^{4k}$

$\Rightarrow f(k+1) = 5\left[m + 72\times 2^{4k} + 16\times 3^{4k}\right]$ IE MULTIPLE

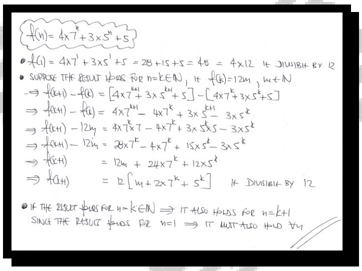
- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N} \Rightarrow$ IT ALSO HOLDS FOR $n = k+1$ SINCE THE RESULT HOLDS FOR $n=1 \Rightarrow$ IT MUST HOLD $\forall n \in \mathbb{N}$

Question 17 **(***)**

$$f(n) = 4 \times 7^n + 3 \times 5^n + 5, \; n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 12, for all $n \in \mathbb{N}$.

proof


$f(n) = 4\times7^n + 3\times5^n + 5$

- $f(1) = 4\times7^1 + 3\times5^1 + 5 = 28+15+5 = 48 = 4\times12$ ∴ DIVISIBLE BY 12
- SUPPOSE THE RESULT HOLDS FOR $n=k\in\mathbb{N}$, ∴ $f(k)=12m$, $m\in\mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [4\times7^{k+1} + 3\times5^{k+1}+5] - [4\times7^k + 3\times5^k + 5]$

$\Rightarrow f(k+1) - f(k) = 4\times7^{k+1} - 4\times7^k + 3\times5^{k+1} - 3\times5^k$

$\Rightarrow f(k+1) - 12m = 4\times7^k\times7 - 4\times7^k + 3\times5^k\times5 - 3\times5^k$

$\Rightarrow f(k+1) - 12m = 28\times7^k - 4\times7^k + 15\times5^k - 3\times5^k$

$\Rightarrow f(k+1) = 12m + 24\times7^k + 12\times5^k$

$\Rightarrow f(k+1) = 12[m + 2\times7^k + 5^k]$ ∴ DIVISIBLE BY 12

- IF THE RESULT HOLDS FOR $n=k\in\mathbb{N} \Rightarrow$ IT ALSO HOLDS FOR $n=k+1$

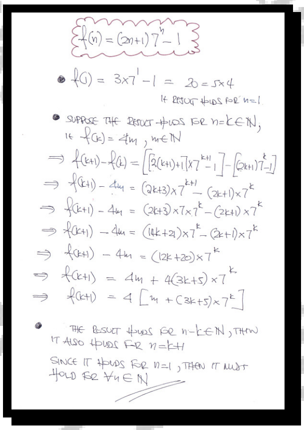
SINCE THE RESULT HOLDS FOR $n=1 \Rightarrow$ IT MUST ALSO HOLD $\forall n$

Question 18 **(***)**

$$f(n) = (2n+1)7^n - 1, \; n \in \mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 4, for all $n \in \mathbb{N}$.

proof


$f(n) = (2n+1)7^n - 1$

- $f(1) = 3\times7^1 - 1 = 20 = 5\times4$

∴ RESULT HOLDS FOR $n=1$

- SUPPOSE THE RESULT HOLDS FOR $n=k\in\mathbb{N}$,

∴ $f(k) = 4m$, $m\in\mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [[2(k+1)+1]\times7^{k+1} - 1] - [(2k+1)7^k - 1]$

$\Rightarrow f(k+1) - 4m = (2k+3)\times7^{k+1} - (2k+1)\times7^k$

$\Rightarrow f(k+1) - 4m = (2k+3)\times7\times7^k - (2k+1)\times7^k$

$\Rightarrow f(k+1) - 4m = (14k+21)\times7^k - (2k+1)\times7^k$

$\Rightarrow f(k+1) - 4m = (12k+20)\times7^k$

$\Rightarrow f(k+1) = 4m + 4(3k+5)\times7^k$

$\Rightarrow f(k+1) = 4[m + (3k+5)\times7^k]$

- THE RESULT HOLDS FOR $n=k\in\mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$

SINCE IT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR $\forall n\in\mathbb{N}$

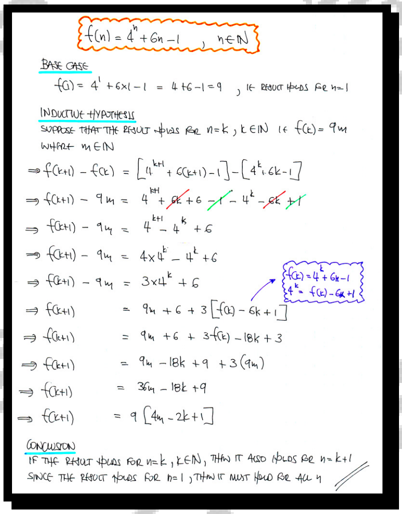
Question 19 **(***+)**

Prove by induction that for all natural numbers n,

$$4^n + 6n - 1$$

is divisible by 9.

proof



$f(n) = 4^n + 6n - 1, \quad n \in \mathbb{N}$

BASE CASE

$f(1) = 4^1 + 6 \times 1 - 1 = 4 + 6 - 1 = 9$, IE RESULT HOLDS FOR $n=1$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$ IE $f(k) = 9m$ WHERE $m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [4^{k+1} + 6(k+1) - 1] - [4^k + 6k - 1]$

$\Rightarrow f(k+1) - 9m = 4^{k+1} + 6k + 6 - 1 - 4^k - 6k + 1$

$\Rightarrow f(k+1) - 9m = 4^{k+1} - 4^k + 6$

$\Rightarrow f(k+1) - 9m = 4 \times 4^k - 4^k + 6$

$\Rightarrow f(k+1) - 9m = 3 \times 4^k + 6$

$f(k) = 4^k + 6k - 1$
$4^k = f(k) - 6k + 1$

$\Rightarrow f(k+1) = 9m + 6 + 3[f(k) - 6k + 1]$

$\Rightarrow f(k+1) = 9m + 6 + 3f(k) - 18k + 3$

$\Rightarrow f(k+1) = 9m - 18k + 9 + 3(9m)$

$\Rightarrow f(k+1) = 36m - 18k + 9$

$\Rightarrow f(k+1) = 9[4m - 2k + 1]$

CONCLUSION

IF THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n

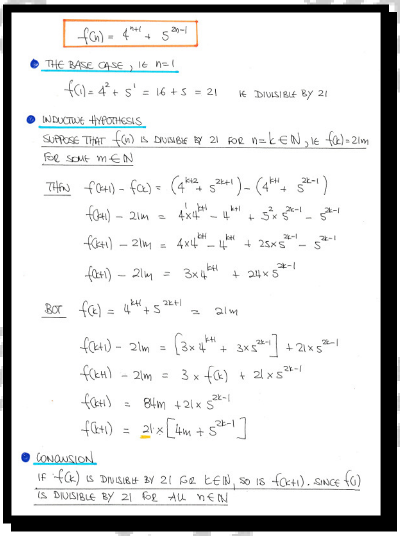
Question 20 (***+)

Prove by induction that for all natural numbers n,

$$4^{n+1} + 5^{2n-1}$$

is divisible by 21.

[ ], [proof]



$f(n) = 4^{n+1} + 5^{2n-1}$

- THE BASE CASE, IE $n=1$

$f(1) = 4^2 + 5^1 = 16 + 5 = 21$ IE DIVISIBLE BY 21

- INDUCTIVE HYPOTHESIS

SUPPOSE THAT $f(n)$ IS DIVISIBLE BY 21 FOR $n = k \in \mathbb{N}$, IE $f(k) = 21m$ FOR SOME $m \in \mathbb{N}$

THEN $f(k+1) - f(k) = (4^{k+2} + 5^{2k+1}) - (4^{k+1} + 5^{2k-1})$

$f(k+1) - 21m = 4^1 \times 4^{k+1} - 4^{k+1} + 5^2 \times 5^{2k-1} - 5^{2k-1}$

$f(k+1) - 21m = 4 \times 4^{k+1} - 4^{k+1} + 25 \times 5^{2k-1} - 5^{2k-1}$

$f(k+1) - 21m = 3 \times 4^{k+1} + 24 \times 5^{2k-1}$

BUT $f(k) = 4^{k+1} + 5^{2k+1} = 21m$

$f(k+1) - 21m = [3 \times 4^{k+1} + 3 \times 5^{2k-1}] + 21 \times 5^{2k-1}$

$f(k+1) - 21m = 3 \times f(k) + 21 \times 5^{2k-1}$

$f(k+1) = 84m + 21 \times 5^{2k-1}$

$f(k+1) = 21 \times [4m + 5^{2k-1}]$

- CONCLUSION

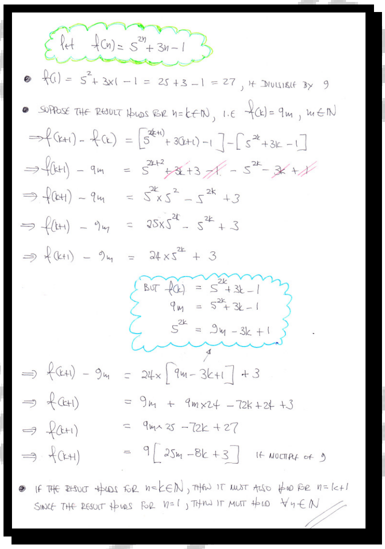
IF $f(k)$ IS DIVISIBLE BY 21 FOR $k \in \mathbb{N}$, SO IS $f(k+1)$. SINCE $f(1)$ IS DIVISIBLE BY 21 FOR ALL $n \in \mathbb{N}$

Question 21 (***+)

$$f(n) = 5^{2n} + 3n - 1, \; n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 9, for all $n \in \mathbb{N}$.

proof



Let $f(n) = 5^{2n} + 3n - 1$

- $f(1) = 5^2 + 3\times1 - 1 = 25 + 3 - 1 = 27$, IS DIVISIBLE BY 9
- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, I.E $f(k) = 9m$, $m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [5^{2(k+1)} + 3(k+1) - 1] - [5^{2k} + 3k - 1]$

$\Rightarrow f(k+1) - 9m = 5^{2k+2} + 3k + 3 - 1 - 5^{2k} - 3k + 1$

$\Rightarrow f(k+1) - 9m = 5^{2k} \times 5^2 - 5^{2k} + 3$

$\Rightarrow f(k+1) - 9m = 25 \times 5^{2k} - 5^{2k} + 3$

$\Rightarrow f(k+1) - 9m = 24 \times 5^{2k} + 3$

BUT $f(k) = 5^{2k} + 3k - 1$

$9m = 5^{2k} + 3k - 1$

$5^{2k} = 9m - 3k + 1$

$\Rightarrow f(k+1) - 9m = 24 \times [9m - 3k + 1] + 3$

$\Rightarrow f(k+1) = 9m + 9m \times 24 - 72k + 24 + 3$

$\Rightarrow f(k+1) = 9m \times 25 - 72k + 27$

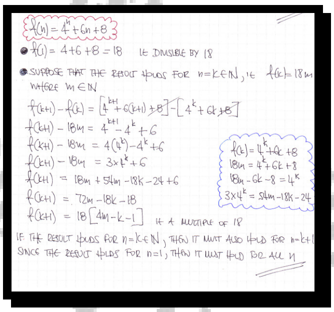
$\Rightarrow f(k+1) = 9[25m - 8k + 3]$ IE MULTIPLE OF 9

- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n = k+1$. SINCE THE RESULT HOLDS FOR $n = 1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 22 (****)

Prove by induction that 18 is a factor of $4^n + 6n + 8$, for all $n \in \mathbb{N}$.

proof



$f(n) = 4^n + 6n + 8$

- $f(1) = 4 + 6 + 8 = 18$ IE DIVISIBLE BY 18
- SUPPOSE THAT THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, IE $f(k) = 18m$ WHERE $m \in \mathbb{N}$

$f(k+1) - f(k) = [4^{k+1} + 6(k+1) + 8] - [4^k + 6k + 8]$

$f(k+1) - 18m = 4^{k+1} - 4^k + 6$

$f(k+1) - 18m = 4(4^k) - 4^k + 6$

$f(k+1) - 18m = 3 \times 4^k + 6$

$f(k) = 4^k + 6k + 8$

$18m = 4^k + 6k + 8$

$18m - 6k - 8 = 4^k$

$3 \times 4^k = 54m - 18k - 24$

$f(k+1) = 18m + 54m - 18k - 24 + 6$

$f(k+1) = 72m - 18k - 18$

$f(k+1) = 18[4m - k - 1]$ IE A MULTIPLE OF 18

IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n = k+1$. SINCE THE RESULT HOLDS FOR $n = 1$, THEN IT MUST HOLD FOR ALL n

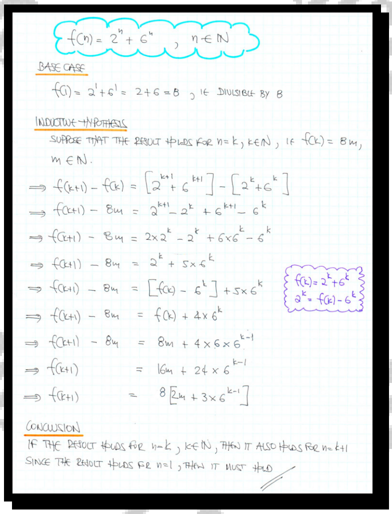
Question 23 (****)

Prove by induction that for all natural numbers n,

$$2^n + 6^n$$

is divisible by 8.

[illegible], proof



$f(n) = 2^n + 6^n \quad , \quad n \in \mathbb{N}$

BASE CASE

$f(1) = 2^1 + 6^1 = 2 + 6 = 8$, IE DIVISIBLE BY 8

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n = k$, $k \in \mathbb{N}$, IE $f(k) = 8m$, $m \in \mathbb{N}$.

$\Rightarrow f(k+1) - f(k) = \left[2^{k+1} + 6^{k+1}\right] - \left[2^k + 6^k\right]$

$\Rightarrow f(k+1) - 8m = 2^{k+1} - 2^k + 6^{k+1} - 6^k$

$\Rightarrow f(k+1) - 8m = 2 \times 2^k - 2^k + 6 \times 6^k - 6^k$

$\Rightarrow f(k+1) - 8m = 2^k + 5 \times 6^k$

$\Rightarrow f(k+1) - 8m = \left[f(k) - 6^k\right] + 5 \times 6^k$

$f(k) = 2^k + 6^k$

$2^k = f(k) - 6^k$

$\Rightarrow f(k+1) - 8m = f(k) + 4 \times 6^k$

$\Rightarrow f(k+1) - 8m = 8m + 4 \times 6 \times 6^{k-1}$

$\Rightarrow f(k+1) = 16m + 24 \times 6^{k-1}$

$\Rightarrow f(k+1) = 8\left[2m + 3 \times 6^{k-1}\right]$

CONCLUSION

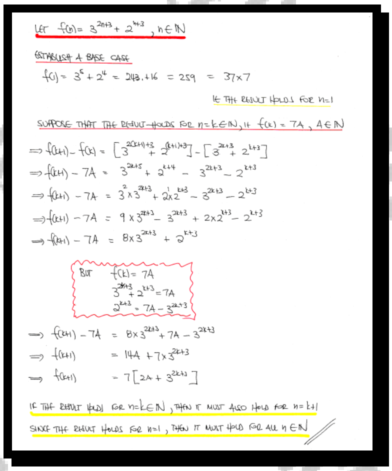
IF THE RESULT HOLDS FOR $n = k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$

SINCE THE RESULT HOLDS FOR $n = 1$, THEN IT MUST HOLD

Question 24 (***+)

Prove by mathematical induction that if n is a positive integer then $3^{2n+3} + 2^{n+3}$ is always divisible by 7.

[ ] , [proof]



LET $f(n) = 3^{2n+3} + 2^{n+3}, \ n \in \mathbb{N}$

ESTABLISH A BASE CASE

$f(1) = 3^5 + 2^4 = 243 + 16 = 259 = 37 \times 7$

IE THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, IE $f(k) = 7A, \ A \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = \left[3^{2(k+1)+3} + 2^{(k+1)+3}\right] - \left[3^{2k+3} + 2^{k+3}\right]$

$\Rightarrow f(k+1) - 7A = 3^{2k+5} + 2^{k+4} - 3^{2k+3} - 2^{k+3}$

$\Rightarrow f(k+1) - 7A = 3^2 \times 3^{2k+3} + 2^1 \times 2^{k+3} - 3^{2k+3} - 2^{k+3}$

$\Rightarrow f(k+1) - 7A = 9 \times 3^{2k+3} - 3^{2k+3} + 2 \times 2^{k+3} - 2^{k+3}$

$\Rightarrow f(k+1) - 7A = 8 \times 3^{2k+3} + 2^{k+3}$

BUT $f(k) = 7A$
$3^{2k+3} + 2^{k+3} = 7A$
$2^{k+3} = 7A - 3^{2k+3}$

$\Rightarrow f(k+1) - 7A = 8 \times 3^{2k+3} + 7A - 3^{2k+3}$

$\Rightarrow f(k+1) = 14A + 7 \times 3^{2k+3}$

$\Rightarrow f(k+1) = 7\left[2A + 3^{2k+3}\right]$

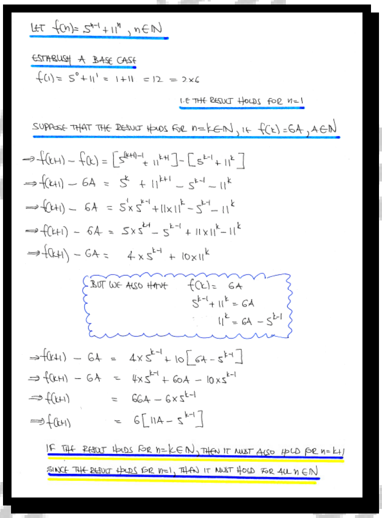
IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n = k+1$

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 25 (***+)

Prove by mathematical induction that if n is a positive integer then $5^{n-1}+11^{n}$ is always divisible by 6.

[ ], [proof]



LET $f(n) = 5^{n-1} + 11^{n}, \; n \in \mathbb{N}$

ESTABLISH A BASE CASE

$f(1) = 5^{0} + 11^{1} = 1 + 11 = 12 = 2 \times 6$

I.E THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, I.E $f(k) = 6A, \; A \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = \left[5^{(k+1)-1} + 11^{k+1}\right] - \left[5^{k-1} + 11^{k}\right]$

$\Rightarrow f(k+1) - 6A = 5^{k} + 11^{k+1} - 5^{k-1} - 11^{k}$

$\Rightarrow f(k+1) - 6A = 5^{1} \times 5^{k-1} + 11 \times 11^{k} - 5^{k-1} - 11^{k}$

$\Rightarrow f(k+1) - 6A = 5 \times 5^{k-1} - 5^{k-1} + 11 \times 11^{k} - 11^{k}$

$\Rightarrow f(k+1) - 6A = 4 \times 5^{k-1} + 10 \times 11^{k}$

BUT WE ALSO HAVE
$f(k) = 6A$
$5^{k-1} + 11^{k} = 6A$
$11^{k} = 6A - 5^{k-1}$

$\Rightarrow f(k+1) - 6A = 4 \times 5^{k-1} + 10\left[6A - 5^{k-1}\right]$

$\Rightarrow f(k+1) - 6A = 4 \times 5^{k-1} + 60A - 10 \times 5^{k-1}$

$\Rightarrow f(k+1) = 66A - 6 \times 5^{k-1}$

$\Rightarrow f(k+1) = 6\left[11A - 5^{k-1}\right]$

IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n = k+1$

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

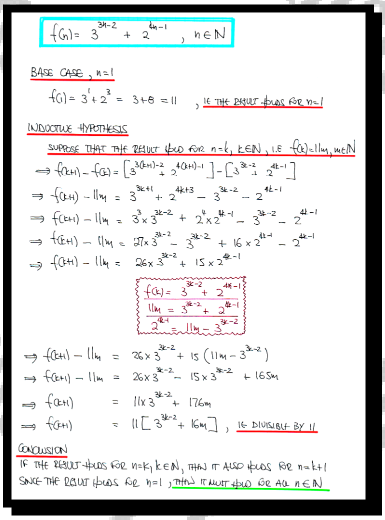
Question 26 **(***+)**

Prove by the method of induction that

$$3^{3n-2}+2^{4n-1}\ ,\ n\in\mathbb{N}\ ,$$

is divisible by 11.

I.Y.G , proof


$f(n) = 3^{3n-2} + 2^{4n-1}\ ,\ n \in \mathbb{N}$

BASE CASE, $n=1$

$f(1) = 3^1 + 2^3 = 3 + 8 = 11$, I.E. THE RESULT HOLDS FOR $n=1$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLD FOR $n=k$, $k \in \mathbb{N}$, I.E. $f(k) = 11m$, $m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = \left[3^{3(k+1)-2} + 2^{4(k+1)-1}\right] - \left[3^{3k-2} + 2^{4k-1}\right]$

$\Rightarrow f(k+1) - 11m = 3^{3k+1} + 2^{4k+3} - 3^{3k-2} - 2^{4k-1}$

$\Rightarrow f(k+1) - 11m = 3^3 \times 3^{3k-2} + 2^4 \times 2^{4k-1} - 3^{3k-2} - 2^{4k-1}$

$\Rightarrow f(k+1) - 11m = 27 \times 3^{3k-2} - 3^{3k-2} + 16 \times 2^{4k-1} - 2^{4k-1}$

$\Rightarrow f(k+1) - 11m = 26 \times 3^{3k-2} + 15 \times 2^{4k-1}$

$f(k) = 3^{3k-2} + 2^{4k-1}$

$11m = 3^{3k-2} + 2^{4k-1}$

$2^{4k-1} = 11m - 3^{3k-2}$

$\Rightarrow f(k+1) - 11m = 26 \times 3^{3k-2} + 15\left(11m - 3^{3k-2}\right)$

$\Rightarrow f(k+1) - 11m = 26 \times 3^{3k-2} - 15 \times 3^{3k-2} + 165m$

$\Rightarrow f(k+1) = 11 \times 3^{3k-2} + 176m$

$\Rightarrow f(k+1) = 11\left[3^{3k-2} + 16m\right]$, I.E. DIVISIBLE BY 11

CONCLUSION

IF THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$

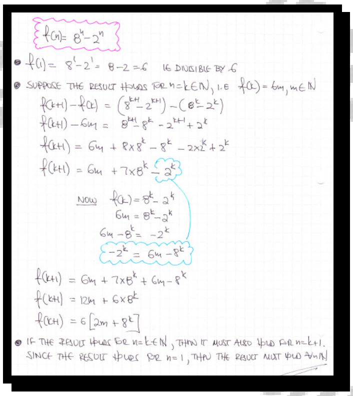
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 27 (***+)

$$f(n) = 8^n - 2^n, \ n \in \mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 6, for all $n \in \mathbb{N}$.

proof



$f(n) = 8^n - 2^n$

- $f(1) = 8^1 - 2^1 = 8 - 2 = 6$ IS DIVISIBLE BY 6
- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, I.E. $f(k) = 6m$, $m \in \mathbb{N}$

$f(k+1) - f(k) = (8^{k+1} - 2^{k+1}) - (8^k - 2^k)$

$f(k+1) - 6m = 8^{k+1} - 8^k - 2^{k+1} + 2^k$

$f(k+1) = 6m + 8 \times 8^k - 8^k - 2 \times 2^k + 2^k$

$f(k+1) = 6m + 7 \times 8^k - 2^k$

NOW $f(k) = 8^k - 2^k$

$6m = 8^k - 2^k$

$6m - 8^k = -2^k$

$-2^k = 6m - 8^k$

$f(k+1) = 6m + 7 \times 8^k + 6m - 8^k$

$f(k+1) = 12m + 6 \times 8^k$

$f(k+1) = 6[2m + 8^k]$

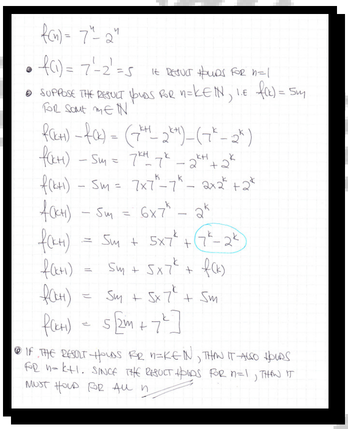
- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n = k+1$. SINCE THE RESULT HOLDS FOR $n = 1$, THEN THE RESULT MUST HOLD $\forall n \in \mathbb{N}$

Question 28 (***+)

$$f(n) = 7^n - 2^n, \ n \in \mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 5, for all $n \in \mathbb{N}$.

proof



$f(n) = 7^n - 2^n$

- $f(1) = 7^1 - 2^1 = 5$ IE RESULT HOLDS FOR $n = 1$
- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, I.E. $f(k) = 5m$ FOR SOME $m \in \mathbb{N}$

$f(k+1) - f(k) = (7^{k+1} - 2^{k+1}) - (7^k - 2^k)$

$f(k+1) - 5m = 7^{k+1} - 7^k - 2^{k+1} + 2^k$

$f(k+1) - 5m = 7 \times 7^k - 7^k - 2 \times 2^k + 2^k$

$f(k+1) - 5m = 6 \times 7^k - 2^k$

$f(k+1) = 5m + 5 \times 7^k + 7^k - 2^k$

$f(k+1) = 5m + 5 \times 7^k + f(k)$

$f(k+1) = 5m + 5 \times 7^k + 5m$

$f(k+1) = 5[2m + 7^k]$

- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$. SINCE THE RESULT HOLDS FOR $n = 1$, THEN IT MUST HOLD FOR ALL n

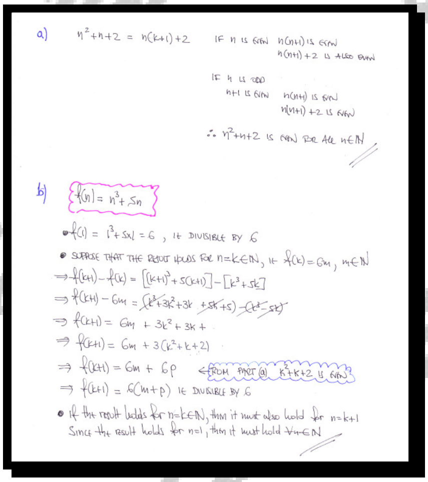
Question 29 (***+)

$$f(n) = n^3 + 5n, \ n \in \mathbb{N}.$$

a) Show that $n^2 + n + 2$ is always even for all $n \in \mathbb{N}$.

b) Hence, prove by induction that $f(n)$ is divisible by 6, for all $n \in \mathbb{N}$.

proof



a) $n^2+n+2 = n(k+1)+2$

IF n IS EVEN $n(n+1)$ IS EVEN, $n(n+1)+2$ IS ALSO EVEN

IF n IS ODD, $n+1$ IS EVEN, $n(n+1)$ IS EVEN, $n(n+1)+2$ IS EVEN

$\therefore$ n^2+n+2 IS EVEN FOR ALL $n \in \mathbb{N}$

b) $f(n) = n^3+5n$

- $f(1) = 1^3+5\times1 = 6$, IE DIVISIBLE BY 6
- SUPPOSE THAT THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, IE $f(k) = 6m$, $m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [(k+1)^3+5(k+1)] - [k^3+5k]$

$\Rightarrow f(k+1) - 6m = (k^3+3k^2+3k+5k+5) - (k^3-5k)$

$\Rightarrow f(k+1) = 6m + 3k^2+3k+$

$\Rightarrow f(k+1) = 6m + 3(k^2+k+2)$

$\Rightarrow f(k+1) = 6m + 6p$ ← FROM PART (a) k^2+k+2 IS EVEN

$\Rightarrow f(k+1) = 6(m+p)$ IE DIVISIBLE BY 6

- If the result holds for $n=k \in \mathbb{N}$, then it must also hold for $n=k+1$. Since the result holds for $n=1$, then it must hold $\forall n \in \mathbb{N}$

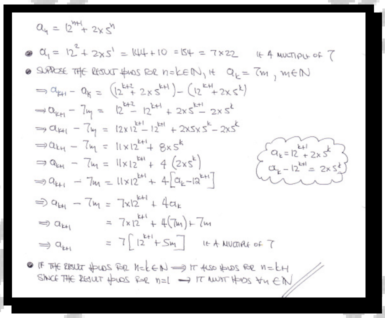
Question 30 (***+)

A sequence of positive numbers is given by

$$a_n = 12^{n+1} + 2 \times 5^n, \ n \in \mathbb{N}.$$

Prove by induction that every term of the sequence is a multiple of 7

proof



$a_n = 12^{n+1} + 2\times5^n$

- $a_1 = 12^2 + 2\times5^1 = 144+10 = 154 = 7\times22$ IE A MULTIPLE OF 7
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, IE $a_k = 7m$, $m \in \mathbb{N}$

$\Rightarrow a_{k+1} - a_k = (12^{k+2} + 2\times5^{k+1}) - (12^{k+1} + 2\times5^k)$

$\Rightarrow a_{k+1} - 7m = 12^{k+2} - 12^{k+1} + 2\times5^{k+1} - 2\times5^k$

$\Rightarrow a_{k+1} - 7m = 12\times12^{k+1} - 12^{k+1} + 2\times5\times5^k - 2\times5^k$

$\Rightarrow a_{k+1} - 7m = 11\times12^{k+1} + 8\times5^k$

$\Rightarrow a_{k+1} - 7m = 11\times12^{k+1} + 4(2\times5^k)$

$\Rightarrow a_{k+1} - 7m = 11\times12^{k+1} + 4[a_k - 12^{k+1}]$

$a_k = 12^{k+1} + 2\times5^k$

$a_k - 12^{k+1} = 2\times5^k$

$\Rightarrow a_{k+1} - 7m = 7\times12^{k+1} + 4a_k$

$\Rightarrow a_{k+1} = 7\times12^{k+1} + 4(7m) + 7m$

$\Rightarrow a_{k+1} = 7[12^{k+1} + 5m]$ IE A MULTIPLE OF 7

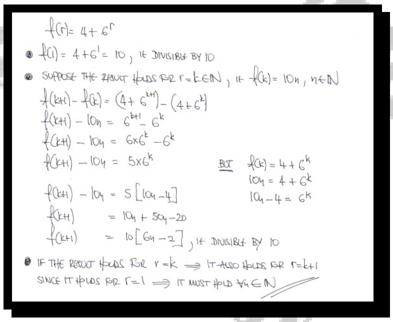
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N} \Rightarrow$ IT ALSO HOLDS FOR $n=k+1$. SINCE THE RESULT HOLDS FOR $n=1 \rightarrow$ IT MUST HOLD $\forall n \in \mathbb{N}$

Question 31 (***+)

$$f(r) = 4 + 6^r, \; r \in \mathbb{N}.$$

Prove by induction that $f(r)$ is divisible by 10

proof



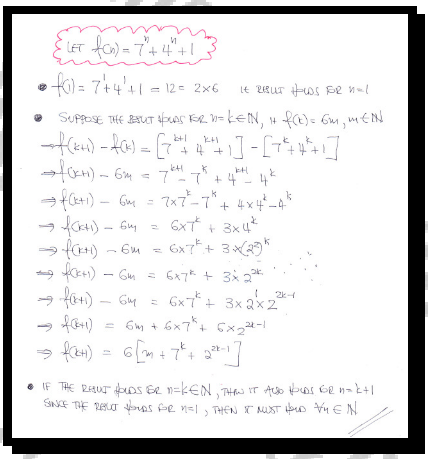
Question 32 (***+)

Prove by induction that for all natural numbers n, the following expression

$$7^n + 4^n + 1$$

is divisible by 6.

proof



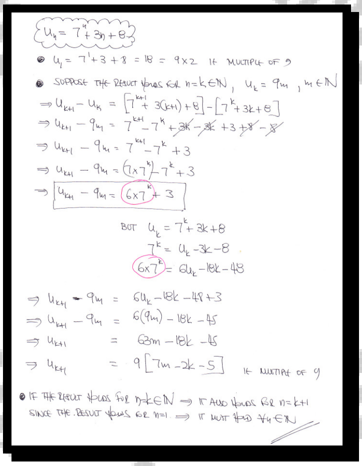
Question 33 **(***+)**

A sequence of positive numbers is given by

$$u_n = 7^n + 3n + 8, \ n \in \mathbb{N}.$$

Prove by induction that every term of the sequence is a multiple of 9

proof



$u_n = 7^n + 3n + 8$

• $u_1 = 7^1 + 3 + 8 = 18 = 9 \times 2$ IE MULTIPLE OF 9

• SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, $u_k = 9m$, $m \in \mathbb{N}$

$\Rightarrow u_{k+1} - u_k = [7^{k+1} + 3(k+1) + 8] - [7^k + 3k + 8]$

$\Rightarrow u_{k+1} - 9m = 7^{k+1} - 7^k + 3k - 3k + 3 + 8 - 8$

$\Rightarrow u_{k+1} - 9m = 7^{k+1} - 7^k + 3$

$\Rightarrow u_{k+1} - 9m = (7 \times 7^k) - 7^k + 3$

$\Rightarrow u_{k+1} - 9m = 6 \times 7^k + 3$

BUT $u_k = 7^k + 3k + 8$

$7^k = u_k - 3k - 8$

$6 \times 7^k = 6u_k - 18k - 48$

$\Rightarrow u_{k+1} - 9m = 6u_k - 18k - 48 + 3$

$\Rightarrow u_{k+1} - 9m = 6(9m) - 18k - 45$

$\Rightarrow u_{k+1} = 63m - 18k - 45$

$\Rightarrow u_{k+1} = 9[7m - 2k - 5]$ IE MULTIPLE OF 9

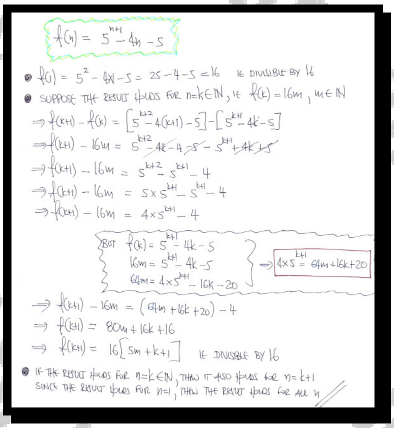
• IF THE RESULT HOLDS FOR $n = k \in \mathbb{N} \Rightarrow$ IT ALSO HOLDS FOR $n = k+1$
SINCE THE RESULT HOLDS FOR $n = 1 \Rightarrow$ IT MUST HOLD $\forall n \in \mathbb{N}$

Question 34 **(***+)**

$$f(n) = 5^{n+1} - 4n - 5, \ n \in \mathbb{N}.$$

Prove by induction that $f(n)$ is divisible by 16

proof



$f(n) = 5^{n+1} - 4n - 5$

• $f(1) = 5^2 - 4 \times 1 - 5 = 25 - 4 - 5 = 16$ IE DIVISIBLE BY 16

• SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, IE $f(k) = 16m$, $m \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = [5^{k+2} - 4(k+1) - 5] - [5^{k+1} - 4k - 5]$

$\Rightarrow f(k+1) - 16m = 5^{k+2} - 4k - 4 - 5 - 5^{k+1} + 4k + 5$

$\Rightarrow f(k+1) - 16m = 5^{k+2} - 5^{k+1} - 4$

$\Rightarrow f(k+1) - 16m = 5 \times 5^{k+1} - 5^{k+1} - 4$

$\Rightarrow f(k+1) - 16m = 4 \times 5^{k+1} - 4$

BUT $f(k) = 5^{k+1} - 4k - 5$

$16m = 5^{k+1} - 4k - 5$

$64m = 4 \times 5^{k+1} - 16k - 20$

$\Rightarrow 4 \times 5^{k+1} = 64m + 16k + 20$

$\Rightarrow f(k+1) - 16m = (64m + 16k + 20) - 4$

$\Rightarrow f(k+1) = 80m + 16k + 16$

$\Rightarrow f(k+1) = 16[5m + k + 1]$ IE DIVISIBLE BY 16

• IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$
SINCE THE RESULT HOLDS FOR $n = 1$, THEN THE RESULT HOLDS FOR ALL n

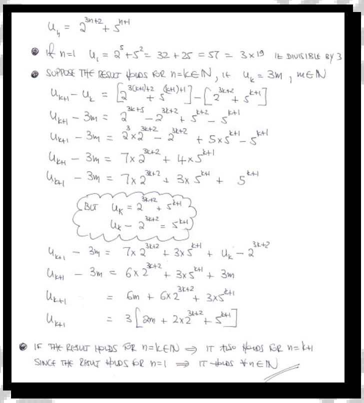
Question 35 **(***+)**

A sequence of positive numbers is given by

$$u_n = 2^{3n+2} + 5^{n+1}, \quad n \in \mathbb{N}.$$

Prove by induction that every term of the sequence is a multiple of 3.

proof



$u_n = 2^{3n+2} + 5^{n+1}$

- If $n=1$ $u_1 = 2^5 + 5^2 = 32 + 25 = 57 = 3 \times 19$ IS DIVISIBLE BY 3
- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, IE $u_k = 3m$, $m \in \mathbb{N}$

$u_{k+1} - u_k = \left[2^{3(k+1)+2} + 5^{(k+1)+1}\right] - \left[2^{3k+2} + 5^{k+1}\right]$

$u_{k+1} - 3m = 2^{3k+5} - 2^{3k+2} + 5^{k+2} - 5^{k+1}$

$u_{k+1} - 3m = 2^3 \times 2^{3k+2} - 2^{3k+2} + 5 \times 5^{k+1} - 5^{k+1}$

$u_{k+1} - 3m = 7 \times 2^{3k+2} + 4 \times 5^{k+1}$

$u_{k+1} - 3m = 7 \times 2^{3k+2} + 3 \times 5^{k+1} + 5^{k+1}$

BUT $u_k = 2^{3k+2} + 5^{k+1}$

$u_k - 2^{3k+2} = 5^{k+1}$

$u_{k+1} - 3m = 7 \times 2^{3k+2} + 3 \times 5^{k+1} + u_k - 2^{3k+2}$

$u_{k+1} - 3m = 6 \times 2^{3k+2} + 3 \times 5^{k+1} + 3m$

$u_{k+1} = 6m + 6 \times 2^{3k+2} + 3 \times 5^{k+1}$

$u_{k+1} = 3\left[2m + 2 \times 2^{3k+2} + 5^{k+1}\right]$

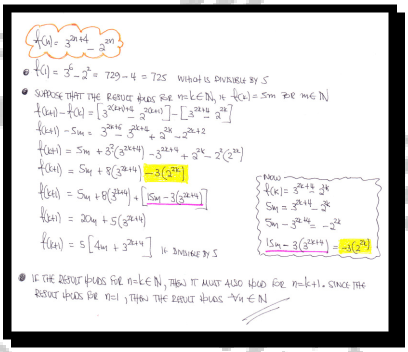
- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N} \Rightarrow$ IT ALSO HOLDS FOR $n = k+1$
SINCE THE RESULT HOLDS FOR $n=1 \Rightarrow$ IT HOLDS $\forall n \in \mathbb{N}$

Question 36 **(***+)**

$$f(n) = 3^{2n+4} - 2^{2n}, \ n \in \mathbb{N}$$

Prove by induction that $f(n)$ is divisible by 5, for all $n \in \mathbb{N}$.

proof



$f(n) = 3^{2n+4} - 2^{2n}$

- $f(1) = 3^6 - 2^2 = 729 - 4 = 725$ WHICH IS DIVISIBLE BY 5
- SUPPOSE THAT THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, IE $f(k) = 5m$ FOR $m \in \mathbb{N}$

$f(k+1) - f(k) = \left[3^{2(k+1)+4} - 2^{2(k+1)}\right] - \left[3^{2k+4} - 2^{2k}\right]$

$f(k+1) - 5m = 3^{2k+6} - 3^{2k+4} + 2^{2k} - 2^{2k+2}$

$f(k+1) = 5m + 3^2(3^{2k+4}) - 3^{2k+4} + 2^{2k} - 2^2(2^{2k})$

$f(k+1) = 5m + 8(3^{2k+4}) - 3(2^{2k})$

$f(k+1) = 5m + 8(3^{2k+4}) + \left[15m - 3(3^{2k+4})\right]$

$f(k+1) = 20m + 5(3^{2k+4})$

$f(k+1) = 5\left[4m + 3^{2k+4}\right]$ IE DIVISIBLE BY 5

NOW

$f(k) = 3^{2k+4} - 2^{2k}$

$5m = 3^{2k+4} - 2^{2k}$

$5m - 3^{2k+4} = -2^{2k}$

$15m - 3(3^{2k+4}) = -3(2^{2k})$

- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n = k+1$. SINCE THE RESULT HOLDS FOR $n=1$, THEN THE RESULT HOLDS $\forall n \in \mathbb{N}$

RECURRENCE RELATIONS

Question 1 (**)

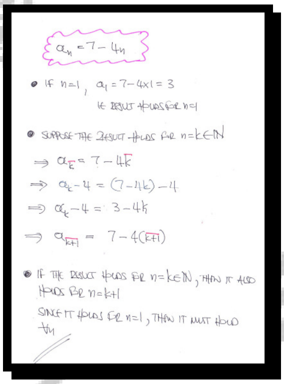
A sequence of integers is defined recursively by the relation

$$a_{n+1} = a_n - 4, \; a_1 = 3, \; n = 1, 2, 3, \ldots$$

Prove by induction that its n^{th} term is given by

$$a_n = 7 - 4n, \; n = 1, 2, 3, \ldots$$

proof



Question 2 (**)

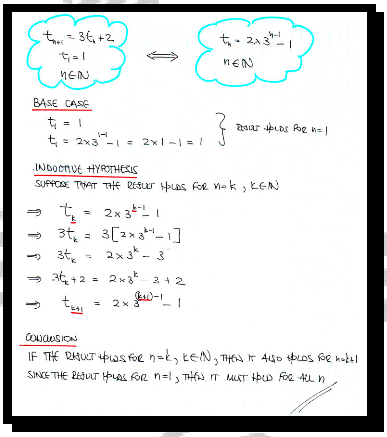
A sequence of integers t_1, t_2, t_3, … is given by the recurrence relation

$$t_{n+1} = 3t_n + 2, \quad t_1 = 1, \quad n \in \mathbb{N} .$$

Prove by induction that its n^{th} term of the sequence is given by

$$t_n = 2 \times 3^{n-1} - 1 , \; n \in \mathbb{N} .$$

[ ], [proof]



$t_{n+1} = 3t_n + 2$, $t_1 = 1$, $n \in \mathbb{N}$ $\iff$ $t_n = 2 \times 3^{n-1} - 1$, $n \in \mathbb{N}$

BASE CASE

$t_1 = 1$

$t_1 = 2 \times 3^{1-1} - 1 = 2 \times 1 - 1 = 1$ } RESULT HOLDS FOR $n = 1$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n = k$, $k \in \mathbb{N}$

$\Rightarrow t_k = 2 \times 3^{k-1} - 1$

$\Rightarrow 3t_k = 3[2 \times 3^{k-1} - 1]$

$\Rightarrow 3t_k = 2 \times 3^k - 3$

$\Rightarrow 3t_k + 2 = 2 \times 3^k - 3 + 2$

$\Rightarrow t_{k+1} = 2 \times 3^{(k+1)-1} - 1$

CONCLUSION

IF THE RESULT HOLDS FOR $n = k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$

SINCE THE RESULT HOLDS FOR $n = 1$, THEN IT MUST HOLD FOR ALL n

Question 3 (**)

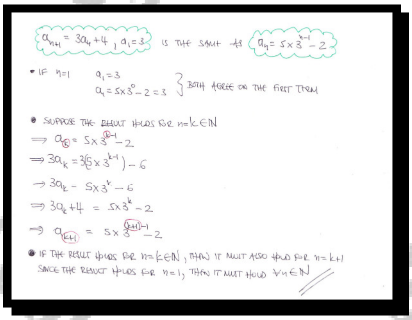
A sequence of integers is defined inductively by the relation

$$a_{n+1} = 3a_n + 4, \quad a_1 = 3, \quad n = 1,2,3,...$$

Prove by induction that its n^{th} term is given by

$$a_n = 5 \times 3^{n-1} - 2, \quad n = 1,2,3,...$$

proof


$a_{n+1} = 3a_n + 4$, $a_1 = 3$ IS THE SAME AS $a_n = 5 \times 3^{n-1} - 2$

- IF $n=1$ $a_1 = 3$, $a_1 = 5 \times 3^0 - 2 = 3$ } BOTH AGREE ON THE FIRST TERM
- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$

$\Rightarrow a_k = 5 \times 3^{k-1} - 2$

$\Rightarrow 3a_k = 3(5 \times 3^{k-1}) - 6$

$\Rightarrow 3a_k = 5 \times 3^k - 6$

$\Rightarrow 3a_k + 4 = 5 \times 3^k - 2$

$\Rightarrow a_{k+1} = 5 \times 3^{(k+1)-1} - 2$

- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n = k+1$. SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 4 (**)

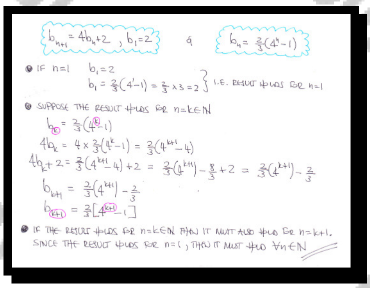
The terms of a sequence can be generated by the recurrence relation

$$b_{n+1} = 4b_n + 2, \quad b_1 = 2, \quad n = 1,2,3,...$$

Prove by induction that the n^{th} term of the sequence is given by

$$b_n = \frac{2}{3}\left(4^n - 1\right), \quad n = 1,2,3,...$$

proof


$b_{n+1} = 4b_n + 2$, $b_1 = 2$ & $b_n = \frac{2}{3}(4^n - 1)$

- IF $n=1$ $b_1 = 2$, $b_1 = \frac{2}{3}(4^1 - 1) = \frac{2}{3} \times 3 = 2$ } I.E. RESULT HOLDS FOR $n=1$
- SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$

$b_k = \frac{2}{3}(4^k - 1)$

$4b_k = 4 \times \frac{2}{3}(4^k - 1) = \frac{2}{3}(4^{k+1} - 4)$

$4b_k + 2 = \frac{2}{3}(4^{k+1} - 4) + 2 = \frac{2}{3}(4^{k+1}) - \frac{8}{3} + 2 = \frac{2}{3}(4^{k+1}) - \frac{2}{3}$

$b_{k+1} = \frac{2}{3}(4^{k+1}) - \frac{2}{3}$

$b_{k+1} = \frac{2}{3}[4^{k+1} - 1]$

- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$ THEN IT MUST ALSO HOLD FOR $n = k+1$. SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 5 (**)

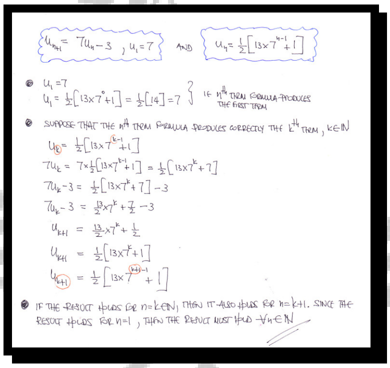
A sequence is defined by the recurrence relation

$$u_{n+1} = 7u_n - 3, \; u_1 = 7, \; n = 1, 2, 3, \ldots$$

Prove by induction that its n^{th} term is given by

$$u_n = \frac{1}{2}\left(13 \times 7^{n-1} + 1\right), \; n = 1, 2, 3, \ldots$$

proof



$u_{n+1} = 7u_n - 3, \; u_1 = 7$ AND $u_n = \frac{1}{2}\left[13 \times 7^{n-1} + 1\right]$

- $u_1 = 7$
  $u_1 = \frac{1}{2}\left[13 \times 7^0 + 1\right] = \frac{1}{2}[14] = 7$ } IF n^{th} TERM FORMULA PRODUCES THE FIRST TERM
- SUPPOSE THAT THE n^{th} TERM FORMULA PRODUCES CORRECTLY THE k^{th} TERM, $k \in \mathbb{N}$

$$u_k = \frac{1}{2}\left[13 \times 7^{k-1} + 1\right]$$
$$7u_k = 7 \times \frac{1}{2}\left[13 \times 7^{k-1} + 1\right] = \frac{1}{2}\left[13 \times 7^k + 7\right]$$
$$7u_k - 3 = \frac{1}{2}\left[13 \times 7^k + 7\right] - 3$$
$$7u_k - 3 = \frac{13}{2} \times 7^k + \frac{7}{2} - 3$$
$$u_{k+1} = \frac{13}{2} \times 7^k + \frac{1}{2}$$
$$u_{k+1} = \frac{1}{2}\left[13 \times 7^k + 1\right]$$
$$u_{k+1} = \frac{1}{2}\left[13 \times 7^{(k+1)-1} + 1\right]$$

- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$. SINCE THE RESULT HOLDS FOR $n = 1$, THEN THE RESULT MUST HOLD $\forall n \in \mathbb{N}$

Question 6 (**)

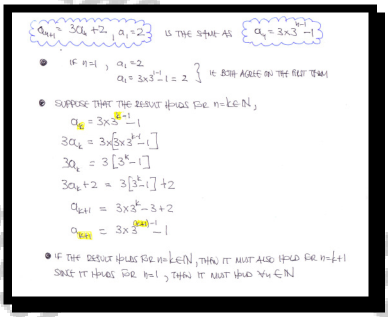
A sequence of integers a_1, a_2, a_3, a_4, … is given by

$$a_{n+1} = 3a_n + 2 \, , \; a_1 = 2 \, , \; n = 1,2,3, \; ...$$

Prove by induction that its n^{th} term is given by

$$a_n = 3 \times 3^{n-1} - 1 \, , \; n = 1,2,3, \; ...$$

proof



Question 7 **(**+)**

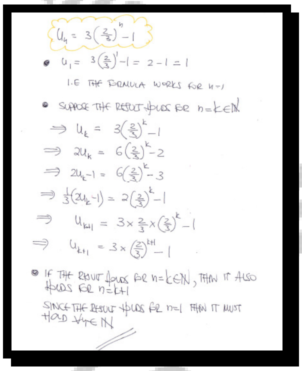
A certain sequence can be generated by the recurrence relation

$$u_{n+1} = \frac{1}{3}\left(2u_n - 1\right), \quad u_1 = 1, \quad n = 1, 2, 3, \ldots$$

Prove by induction that the n^{th} term of the sequence is given by

$$u_n = 3\left(\frac{2}{3}\right)^n - 1, \quad n = 1, 2, 3, \ldots$$

proof



Question 8 (***+)

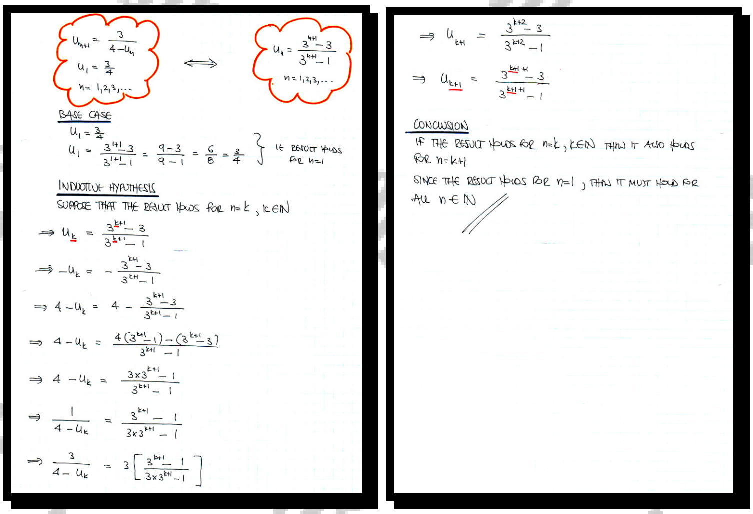
A sequence is defined recursively by

$$u_{n+1} = \frac{3}{4-u_n}, \quad u_1 = \frac{3}{4}, \quad n = 1, 2, 3, \ldots$$

Prove by induction that

$$u_n = \frac{3^{n+1}-3}{3^{n+1}-1}, \quad n = 1, 2, 3, \ldots$$

[ ] , [proof]



$u_{n+1} = \frac{3}{4-u_n}$, $u_1 = \frac{3}{4}$, $n = 1,2,3,\ldots$ $\Longleftrightarrow$ $u_n = \frac{3^{n+1}-3}{3^{n+1}-1}$, $n = 1,2,3,\ldots$

BASE CASE

$u_1 = \frac{3}{4}$

$u_1 = \frac{3^{1+1}-3}{3^{1+1}-1} = \frac{9-3}{9-1} = \frac{6}{8} = \frac{3}{4}$ } ∴ RESULT HOLDS FOR $n=1$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$

$\Rightarrow u_k = \frac{3^{k+1}-3}{3^{k+1}-1}$

$\Rightarrow -u_k = -\frac{3^{k+1}-3}{3^{k+1}-1}$

$\Rightarrow 4-u_k = 4 - \frac{3^{k+1}-3}{3^{k+1}-1}$

$\Rightarrow 4-u_k = \frac{4(3^{k+1}-1)-(3^{k+1}-3)}{3^{k+1}-1}$

$\Rightarrow 4-u_k = \frac{3\times 3^{k+1}-1}{3^{k+1}-1}$

$\Rightarrow \frac{1}{4-u_k} = \frac{3^{k+1}-1}{3\times 3^{k+1}-1}$

$\Rightarrow \frac{3}{4-u_k} = 3\left[\frac{3^{k+1}-1}{3\times 3^{k+1}-1}\right]$

$\Rightarrow u_{k+1} = \frac{3^{k+2}-3}{3^{k+2}-1}$

$\Rightarrow u_{k+1} = \frac{3^{k+1+1}-3}{3^{k+1+1}-1}$

CONCLUSION

IF THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$ THEN IT ALSO HOLDS FOR $n=k+1$

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 9 (***+)

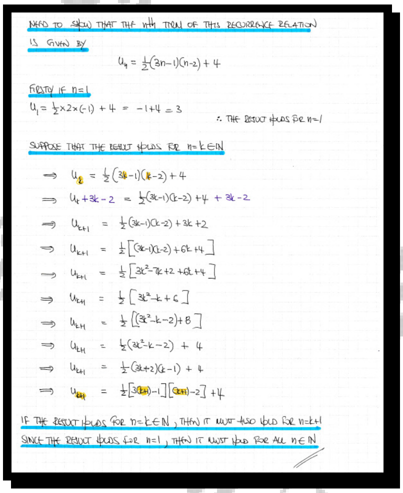
A sequence is defined recursively by

$$u_{n+1} = u_n + 3k - 2\,,\ u_1 = 3\,,\quad n = 1,2,3,\ldots$$

Prove by induction that

$$u_n = \tfrac{1}{2}(3n-1)(n-2) + 4\,,\ n = 1,2,3,\ldots$$

[ ] , [proof]


NEED TO SHOW THAT THE nth TERM OF THIS RECURRENCE RELATION IS GIVEN BY

$$u_n = \tfrac{1}{2}(3n-1)(n-2) + 4$$

FIRSTLY IF n=1

$u_1 = \tfrac{1}{2}\times 2 \times (-1) + 4 = -1+4 = 3$

∴ THE RESULT HOLDS FOR n=1

SUPPOSE THAT THE RESULT HOLDS FOR $n = k \in \mathbb{N}$

$\Rightarrow u_k = \tfrac{1}{2}(3k-1)(k-2) + 4$

$\Rightarrow u_k + 3k - 2 = \tfrac{1}{2}(3k-1)(k-2) + 4 + 3k - 2$

$\Rightarrow u_{k+1} = \tfrac{1}{2}(3k-1)(k-2) + 3k + 2$

$\Rightarrow u_{k+1} = \tfrac{1}{2}\left[(3k-1)(k-2) + 6k + 4\right]$

$\Rightarrow u_{k+1} = \tfrac{1}{2}\left[3k^2 - 7k + 2 + 6k + 4\right]$

$\Rightarrow u_{k+1} = \tfrac{1}{2}\left[3k^2 - k + 6\right]$

$\Rightarrow u_{k+1} = \tfrac{1}{2}\left[(3k^2 - k - 2) + 8\right]$

$\Rightarrow u_{k+1} = \tfrac{1}{2}(3k^2 - k - 2) + 4$

$\Rightarrow u_{k+1} = \tfrac{1}{2}(3k+2)(k-1) + 4$

$\Rightarrow u_{k+1} = \tfrac{1}{2}\left[3(k+1)-1\right]\left[(k+1)-2\right] + 4$

IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n = k+1$

SINCE THE RESULT HOLDS FOR n=1, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 10 (***+)

A sequence is defined recursively by

$$u_{n+1} = \frac{u_n}{u_n + 1}, \; u_1 = 2, \; n \in \mathbb{N} \; .$$

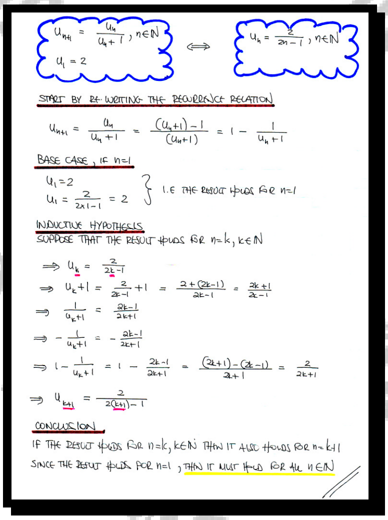
By writing the above recurrence relation in the form

$$u_{n+1} = A + \frac{B}{u_n + 1},$$

where A and B are integers, use proof by induction to show that

$$u_n = \frac{2}{2n-1}, \; n \in \mathbb{N} \; .$$

[ ], [proof]



$u_{n+1} = \frac{u_n}{u_n+1}, n \in \mathbb{N}$, $u_1 = 2$ $\Longleftrightarrow$ $u_n = \frac{2}{2n-1}, n \in \mathbb{N}$

START BY RE-WRITING THE RECURRENCE RELATION

$u_{n+1} = \frac{u_n}{u_n+1} = \frac{(u_n+1)-1}{(u_n+1)} = 1 - \frac{1}{u_n+1}$

BASE CASE, IF $n=1$

$u_1 = 2$
$u_1 = \frac{2}{2\times1-1} = 2$ } I.E THE RESULT HOLDS FOR $n=1$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n=k, k \in \mathbb{N}$

$\Rightarrow u_k = \frac{2}{2k-1}$

$\Rightarrow u_k + 1 = \frac{2}{2k-1} + 1 = \frac{2+(2k-1)}{2k-1} = \frac{2k+1}{2k-1}$

$\Rightarrow \frac{1}{u_k+1} = \frac{2k-1}{2k+1}$

$\Rightarrow -\frac{1}{u_k+1} = -\frac{2k-1}{2k+1}$

$\Rightarrow 1 - \frac{1}{u_k+1} = 1 - \frac{2k-1}{2k+1} = \frac{(2k+1)-(2k-1)}{2k+1} = \frac{2}{2k+1}$

$\Rightarrow u_{k+1} = \frac{2}{2(k+1)-1}$

CONCLUSION

IF THE RESULT HOLDS FOR $n=k, k \in \mathbb{N}$ THEN IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 11 **(***+)**

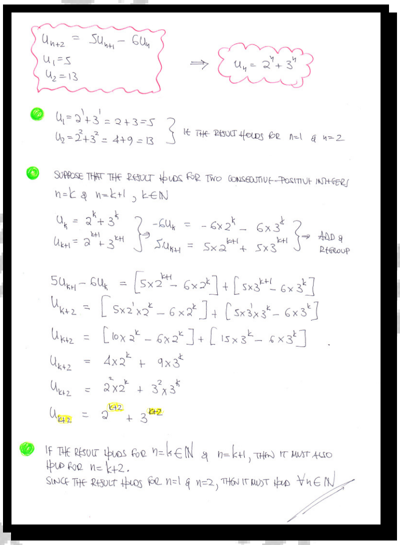
A sequence is generated by the recurrence relation

$$u_{n+2} = 5u_{n+1} - 6u_n \,, \; u_1 = 5 \,, \; u_2 = 13 \,, \quad n = 1, 2, 3, \ldots$$

Prove by induction that n^{th} term of this sequence is given by

$$u_n = 2^n + 3^n \,, \; n = 1, 2, 3, \ldots$$

proof



$u_{n+2} = 5u_{n+1} - 6u_n$, $u_1 = 5$, $u_2 = 13$ $\Rightarrow$ $u_n = 2^n + 3^n$

$u_1 = 2^1 + 3^1 = 2 + 3 = 5$, $u_2 = 2^2 + 3^2 = 4 + 9 = 13$ } ∴ THE RESULT HOLDS FOR $n=1$ & $n=2$

SUPPOSE THAT THE RESULT HOLDS FOR TWO CONSECUTIVE POSITIVE INTEGERS $n=k$ & $n=k+1$, $k \in \mathbb{N}$

$u_k = 2^k + 3^k$, $u_{k+1} = 2^{k+1} + 3^{k+1}$ } $\Rightarrow$ $-6u_k = -6 \times 2^k - 6 \times 3^k$, $5u_{k+1} = 5 \times 2^{k+1} + 5 \times 3^{k+1}$ } $\Rightarrow$ ADD & REGROUP

$$5u_{k+1} - 6u_k = \left[5 \times 2^{k+1} - 6 \times 2^k\right] + \left[5 \times 3^{k+1} - 6 \times 3^k\right]$$
$$u_{k+2} = \left[5 \times 2^1 \times 2^k - 6 \times 2^k\right] + \left[5 \times 3^1 \times 3^k - 6 \times 3^k\right]$$
$$u_{k+2} = \left[10 \times 2^k - 6 \times 2^k\right] + \left[15 \times 3^k - 6 \times 3^k\right]$$
$$u_{k+2} = 4 \times 2^k + 9 \times 3^k$$
$$u_{k+2} = 2^2 \times 2^k + 3^2 \times 3^k$$
$$u_{k+2} = 2^{k+2} + 3^{k+2}$$

IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$ & $n = k+1$, THEN IT MUST ALSO HOLD FOR $n = k+2$.
SINCE THE RESULT HOLDS FOR $n=1$ & $n=2$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 12 (****)

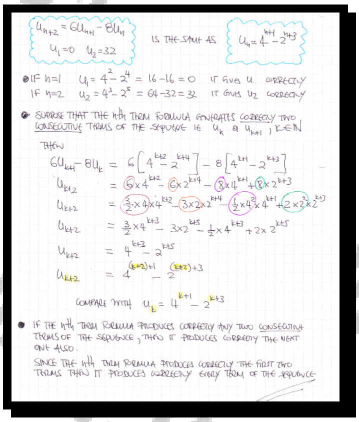
A sequence is generated by the recurrence relation

$$u_{n+2} = 6u_{n+1} - 8u_n\,,\; u_1 = 0\,,\; u_2 = 32\,,\quad n = 1,2,3,\ldots$$

Prove by induction that n^{th} term of this sequence is given by

$$u_n = 4^{n+1} - 2^{n+3}\,,\; n = 1,2,3,\ldots$$

proof



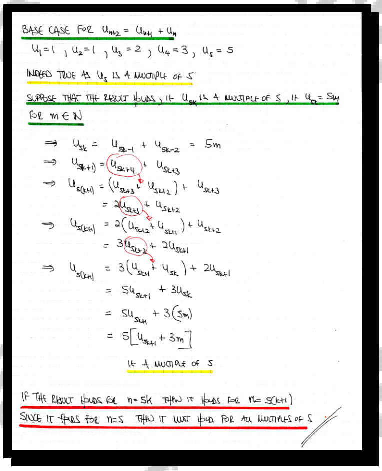
Question 13 (****)

A sequence is generated by the recurrence relation

$$u_{n+2} = u_{n+1} + u_n, \qquad u_1 = 0,\ u_2 = 1, \qquad n = 1, 2, 3, \ldots$$

Prove by induction that u_{5m} is a multiple of 5, for all $m \in \mathbb{N}$.

proof


BASE CASE FOR $u_{n+2} = u_{n+1} + u_n$

$u_1 = 1,\ u_2 = 1,\ u_3 = 2,\ u_4 = 3,\ u_5 = 5$

INDEED TRUE AS u_5 IS A MULTIPLE OF 5

SUPPOSE THAT THE RESULT HOLDS, IE u_{5k} IS A MULTIPLE OF 5, IE $u_{5k} = 5m$ FOR $m \in \mathbb{N}$

$\Rightarrow u_{5k} = u_{5k-1} + u_{5k-2} = 5m$

$\Rightarrow u_{5(k+1)} = u_{5k+4} + u_{5k+3}$

$\Rightarrow u_{5(k+1)} = (u_{5k+3} + u_{5k+2}) + u_{5k+3}$

$= 2u_{5k+3} + u_{5k+2}$

$\Rightarrow u_{5(k+1)} = 2(u_{5k+2} + u_{5k+1}) + u_{5k+2}$

$= 3u_{5k+2} + 2u_{5k+1}$

$\Rightarrow u_{5(k+1)} = 3(u_{5k+1} + u_{5k}) + 2u_{5k+1}$

$= 5u_{5k+1} + 3u_{5k}$

$= 5u_{5k+1} + 3(5m)$

$= 5[u_{5k+1} + 3m]$

IE A MULTIPLE OF 5

IF THE RESULT HOLDS FOR $n = 5k$ THEN IT HOLDS FOR $n = 5(k+1)$

SINCE IT HOLDS FOR $n = 5$ THEN IT MUST HOLD FOR ALL MULTIPLES OF 5

Question 14 (****+)

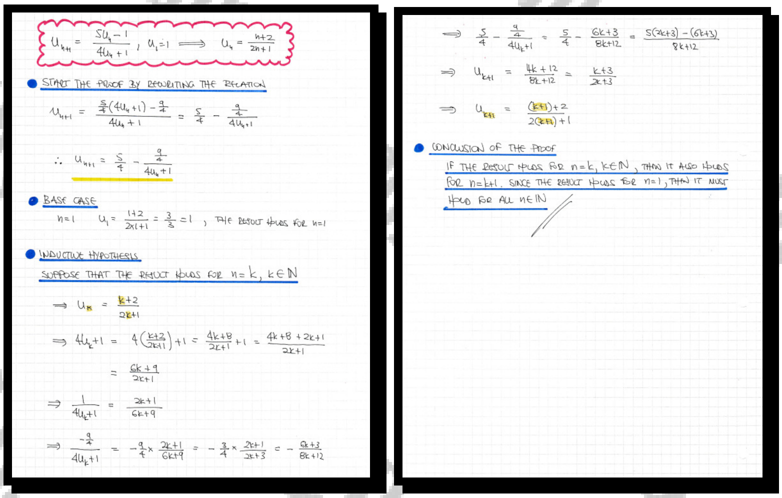
A sequence of numbers is given by the recurrence relation

$$u_{n+1} = \frac{5u_n - 1}{4u_n + 1}, \ u_1 = 1, \ n \in \mathbb{N}, \ n \geq 1.$$

Prove by induction that the n^{th} term of the sequence is given by

$$u_n = \frac{n+2}{2n+1}.$$

[ ], [proof]



Question 15 (****+)

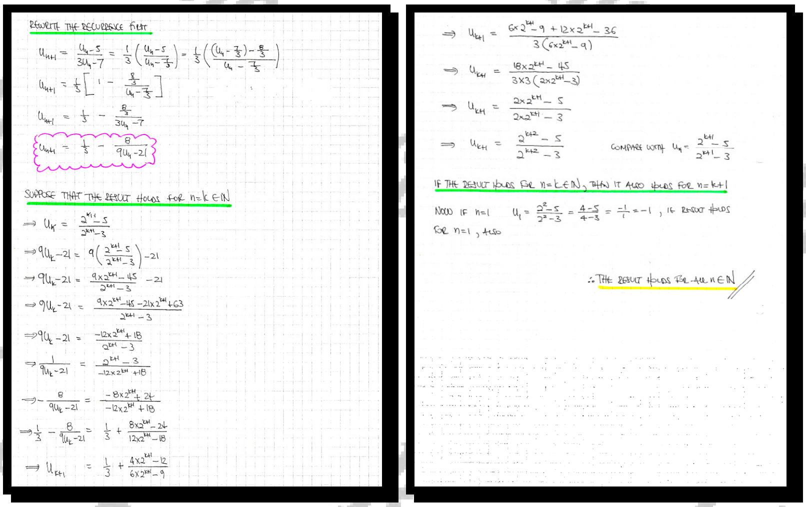
A sequence of numbers is given by the recurrence relation

$$u_{n+1} = \frac{u_n - 5}{3u_n - 7}, \ u_1 = -1, \ n \in \mathbb{N}, \ n \geq 1.$$

Prove by induction that the n^{th} term of the sequence is given by

$$u_n = \frac{2^{n+1} - 5}{2^{n+1} - 3}.$$

[ ], [proof]



REWRITE THE RECURRENCE FIRST

$$u_{n+1} = \frac{u_n - 5}{3u_n - 7} = \frac{1}{3}\left(\frac{u_n - 5}{u_n - \frac{7}{3}}\right) = \frac{1}{3}\left(\frac{\left(u_n - \frac{7}{3}\right) - \frac{8}{3}}{u_n - \frac{7}{3}}\right)$$

$$u_{n+1} = \frac{1}{3}\left[1 - \frac{\frac{8}{3}}{u_n - \frac{7}{3}}\right]$$

$$u_{n+1} = \frac{1}{3} - \frac{\frac{8}{3}}{3u_n - 7}$$

$$u_{n+1} = \frac{1}{3} - \frac{8}{9u_n - 21}$$

SUPPOSE THAT THE RESULT HOLDS FOR $n = k \in \mathbb{N}$

$$\Rightarrow u_k = \frac{2^{k+1} - 5}{2^{k+1} - 3}$$

$$\Rightarrow 9u_k - 21 = 9\left(\frac{2^{k+1} - 5}{2^{k+1} - 3}\right) - 21$$

$$\Rightarrow 9u_k - 21 = \frac{9 \times 2^{k+1} - 45}{2^{k+1} - 3} - 21$$

$$\Rightarrow 9u_k - 21 = \frac{9 \times 2^{k+1} - 45 - 21 \times 2^{k+1} + 63}{2^{k+1} - 3}$$

$$\Rightarrow 9u_k - 21 = \frac{-12 \times 2^{k+1} + 18}{2^{k+1} - 3}$$

$$\Rightarrow \frac{1}{9u_k - 21} = \frac{2^{k+1} - 3}{-12 \times 2^{k+1} + 18}$$

$$\Rightarrow -\frac{8}{9u_k - 21} = \frac{-8 \times 2^{k+1} + 24}{-12 \times 2^{k+1} + 18}$$

$$\Rightarrow \frac{1}{3} - \frac{8}{9u_k - 21} = \frac{1}{3} + \frac{8 \times 2^{k+1} - 24}{12 \times 2^{k+1} - 18}$$

$$\Rightarrow u_{k+1} = \frac{1}{3} + \frac{4 \times 2^{k+1} - 12}{6 \times 2^{k+1} - 9}$$

$$\Rightarrow u_{k+1} = \frac{6 \times 2^{k+1} - 9 + 12 \times 2^{k+1} - 36}{3\left(6 \times 2^{k+1} - 9\right)}$$

$$\Rightarrow u_{k+1} = \frac{18 \times 2^{k+1} - 45}{3 \times 3\left(2 \times 2^{k+1} - 3\right)}$$

$$\Rightarrow u_{k+1} = \frac{2 \times 2^{k+1} - 5}{2 \times 2^{k+1} - 3}$$

$$\Rightarrow u_{k+1} = \frac{2^{k+2} - 5}{2^{k+2} - 3} \qquad \text{COMPARE WITH } u_k = \frac{2^{k+1} - 5}{2^{k+1} - 3}$$

IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$

NOW IF $n = 1$ $\quad u_1 = \frac{2^2 - 5}{2^2 - 3} = \frac{4 - 5}{4 - 3} = \frac{-1}{1} = -1$, IE RESULT HOLDS FOR $n = 1$, ALSO

$\therefore$ THE RESULT HOLDS FOR ALL $n \in \mathbb{N}$

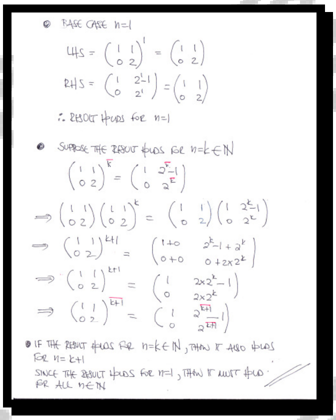
POWERS OF MATRICES

Question 1 (**)

Prove by induction that

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}, \quad n \geq 1, \ n \in \mathbb{N} .$$

proof



- BASE CASE $n=1$

$$LHS = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^1 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$RHS = \begin{pmatrix} 1 & 2^1-1 \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$\therefore$ RESULT HOLDS FOR $n=1$

- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 2^k-1 \\ 0 & 2^k \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2^k-1 \\ 0 & 2^k \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1+0 & 2^k-1+2^k \\ 0+0 & 0+2\times 2^k \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 2\times 2^k-1 \\ 0 & 2\times 2^k \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 2^{k+1}-1 \\ 0 & 2^{k+1} \end{pmatrix}$$

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 2 (**)

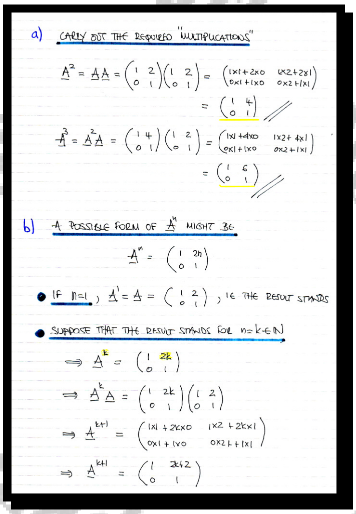
A transformation where $\mathbb{R}^2 \mapsto \mathbb{R}^2$ is defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

a) Find the elements of the matrices, $\mathbf{A}^2$ and $\mathbf{A}^3$.

b) Write down a suitable form for $\mathbf{A}^n$ and use the method of proof by induction to prove it.

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}^3 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$



a) CARRY OUT THE REQUIRED "MULTIPLICATIONS"

$$A^2 = AA = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\times1+2\times0 & 1\times2+2\times1 \\ 0\times1+1\times0 & 0\times2+1\times1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = A^2A = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\times1+4\times0 & 1\times2+4\times1 \\ 0\times1+1\times0 & 0\times2+1\times1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$$

b) A POSSIBLE FORM OF A^n MIGHT BE

$$A^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$

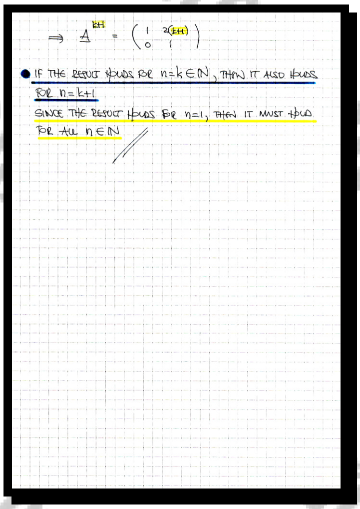
- IF $n=1$, $A^1 = A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, IE THE RESULT STANDS
- SUPPOSE THAT THE RESULT STANDS FOR $n=k \in \mathbb{N}$

$$\Rightarrow A^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow A^kA = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} 1\times1+2k\times0 & 1\times2+2k\times1 \\ 0\times1+1\times0 & 0\times2k+1\times1 \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} 1 & 2k+2 \\ 0 & 1 \end{pmatrix}$$



$$\Rightarrow A^{k+1} = \begin{pmatrix} 1 & 2(k+1) \\ 0 & 1 \end{pmatrix}$$

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$

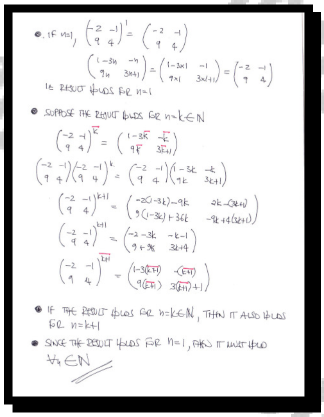
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

Question 3 (**)

Prove by induction that if $n \geq 1,\ n \in \mathbb{N}$, then

$$\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}^n = \begin{pmatrix} 1-3n & -n \\ 9n & 3n+1 \end{pmatrix}.$$

proof



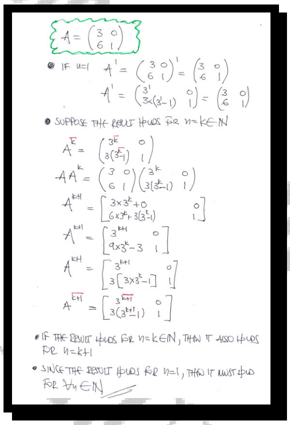
Question 4 (**)

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$$

Prove by induction that if $n \geq 1$, $n \in \mathbb{N}$, then

$$\mathbf{A}^n = \begin{pmatrix} 3^n & 0 \\ 3\left(3^n - 1\right) & 1 \end{pmatrix}.$$

proof


$A = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$

• IF $n=1$ $A^1 = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^1 = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$

$A^1 = \begin{pmatrix} 3^1 & 0 \\ 3\times(3^1-1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$

• SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$

$A^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$

$AA^k = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$

$A^{k+1} = \begin{bmatrix} 3\times 3^k + 0 & 0 \\ 6\times 3^k + 3(3^k-1) & 1 \end{bmatrix}$

$A^{k+1} = \begin{bmatrix} 3^{k+1} & 0 \\ 9\times 3^k - 3 & 1 \end{bmatrix}$

$A^{k+1} = \begin{bmatrix} 3^{k+1} & 0 \\ 3[3\times 3^k - 1] & 1 \end{bmatrix}$

$A^{k+1} = \begin{bmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}-1) & 1 \end{bmatrix}$

• IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$

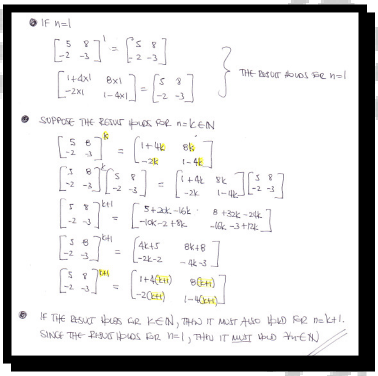
• SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR $\forall n \in \mathbb{N}$

Question 5 **(**)**

Prove by induction that

$$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}^n = \begin{pmatrix} 1+4n & 8n \\ -2n & 1-4n \end{pmatrix}, \;\; n \geq 1, \; n \in \mathbb{N} \; .$$

proof



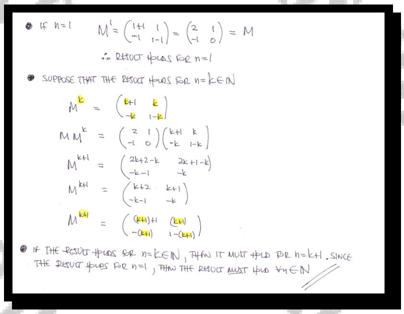
Question 6 **(**)**

$$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Prove by induction that

$$\mathbf{M}^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}, \;\; n \geq 1, \; n \in \mathbb{N} \; .$$

proof



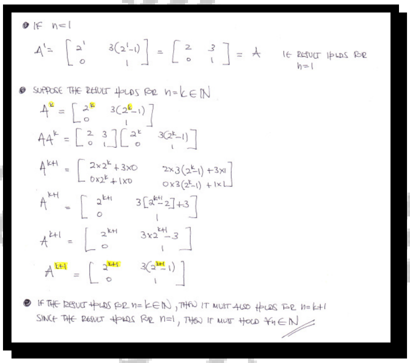
Question 7 (**+)

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

Prove by induction that

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 3\left(2^n - 1\right) \\ 0 & 1 \end{pmatrix}, \; n \geq 1, \; n \in \mathbb{N} \; .$$

proof



• IF $n=1$

$A^1 = \begin{bmatrix} 2^1 & 3(2^1-1) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = A$ ∴ RESULT HOLDS FOR $n=1$

• SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$

$A^k = \begin{bmatrix} 2^k & 3(2^k-1) \\ 0 & 1 \end{bmatrix}$

$AA^k = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2^k & 3(2^k-1) \\ 0 & 1 \end{bmatrix}$

$A^{k+1} = \begin{bmatrix} 2\times2^k + 3\times0 & 2\times3(2^k-1) + 3\times1 \\ 0\times2^k + 1\times0 & 0\times3(2^k-1) + 1\times1 \end{bmatrix}$

$A^{k+1} = \begin{bmatrix} 2^{k+1} & 3[2^{k+1}-2]+3 \\ 0 & 1 \end{bmatrix}$

$A^{k+1} = \begin{bmatrix} 2^{k+1} & 3\times2^{k+1}-3 \\ 0 & 1 \end{bmatrix}$

$A^{k+1} = \begin{bmatrix} 2^{k+1} & 3(2^{k+1}-1) \\ 0 & 1 \end{bmatrix}$

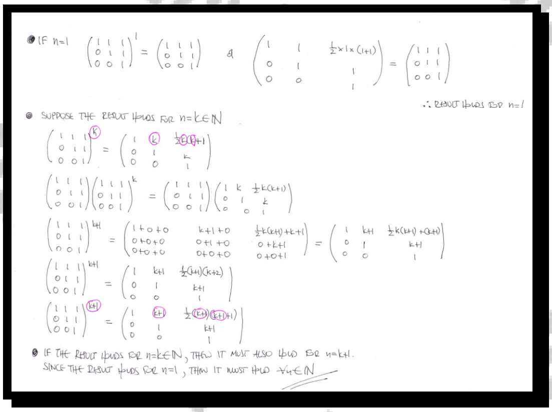
• IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT MUST ALSO HOLDS FOR $n = k+1$
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 8 (**+)

Prove by induction that

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}n(n+1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, \ n \geq 1, \ n \in \mathbb{N} .$$

proof

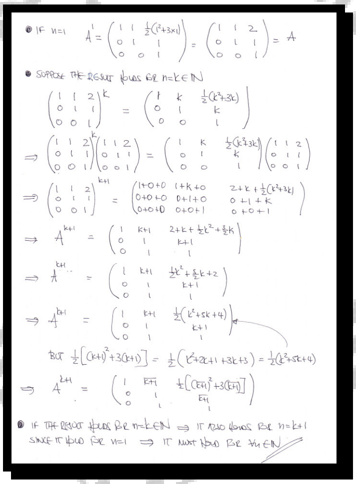


Question 9 (+)**

Prove by induction that

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{1}{2}\left(n^2+3n\right) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, \quad n \geq 1,\ n \in \mathbb{N}\ .$$

proof



• IF $n=1$ $\quad A^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1^2+3\times1) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = A$

• SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1+0+0 & 1+k+0 & 2+k+\frac{1}{2}(k^2+3k) \\ 0+0+0 & 0+1+0 & 0+1+k \\ 0+0+0 & 0+0+1 & 0+0+1 \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} 1 & k+1 & 2+k+\frac{1}{2}k^2+\frac{3}{2}k \\ 0 & 1 & k+1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}k^2+\frac{5}{2}k+2 \\ 0 & 1 & k+1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+5k+4) \\ 0 & 1 & k+1 \\ 0 & 1 & 1 \end{pmatrix}$$

BUT $\frac{1}{2}\left[(k+1)^2+3(k+1)\right] = \frac{1}{2}\left(k^2+2k+1+3k+3\right) = \frac{1}{2}(k^2+5k+4)$

$$\Rightarrow A^{k+1} = \begin{pmatrix} 1 & k+1 & \frac{1}{2}\left[(k+1)^2+3(k+1)\right] \\ 0 & 1 & k+1 \\ 0 & 1 & 1 \end{pmatrix}$$

• IF THE RESULT HOLDS FOR $n=k \in \mathbb{N} \Rightarrow$ IT ALSO HOLDS FOR $n=k+1$
SINCE IT HOLD FOR $n=1 \Rightarrow$ IT MUST HOLD FOR $\forall n \in \mathbb{N}$

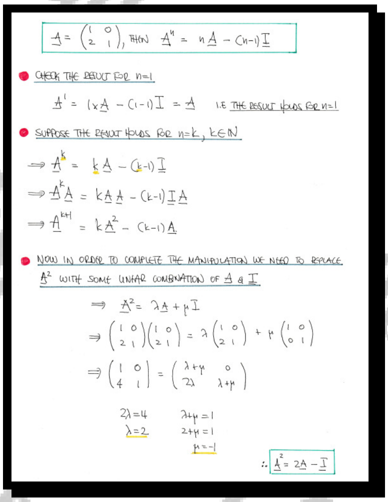
Question 10 (*****)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Prove by induction that

$$\mathbf{A}^n = n\mathbf{A} - (n-1)\mathbf{I}, \ n \geq 1, \ n \in \mathbb{N} \ .$$

[ ] , [proof]


$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, THEN $A^n = nA - (n-1)I$

- CHECK THE RESULT FOR $n=1$

$A^1 = 1 \times A - (1-1)I = A$ I.E THE RESULT HOLDS FOR $n=1$

- SUPPOSE THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$

$\Rightarrow A^k = kA - (k-1)I$

$\Rightarrow A^k A = kAA - (k-1)IA$

$\Rightarrow A^{k+1} = kA^2 - (k-1)A$

- NOW IN ORDER TO COMPLETE THE MANIPULATION WE NEED TO REPLACE A^2 WITH SOME LINEAR COMBINATION OF A & I

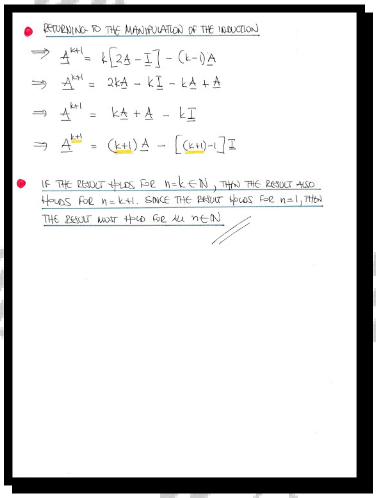
$\Rightarrow A^2 = \lambda A + \mu I$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \lambda\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \mu\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} \lambda+\mu & 0 \\ 2\lambda & \lambda+\mu \end{pmatrix}$

$2\lambda = 4$, $\lambda = 2$; $\lambda + \mu = 1$, $2 + \mu = 1$, $\mu = -1$

$\therefore A^2 = 2A - I$


- RETURNING TO THE MANIPULATION OF THE INDUCTION

$\Rightarrow A^{k+1} = k[2A - I] - (k-1)A$

$\Rightarrow A^{k+1} = 2kA - kI - kA + A$

$\Rightarrow A^{k+1} = kA + A - kI$

$\Rightarrow A^{k+1} = (k+1)A - [(k+1)-1]I$

- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN THE RESULT ALSO HOLDS FOR $n=k+1$. SINCE THE RESULT HOLDS FOR $n=1$, THEN THE RESULT MUST HOLD FOR ALL $n \in \mathbb{N}$

MISCELLANEOUS RESULTS

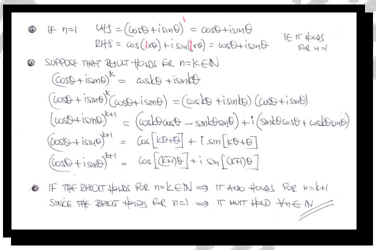
Question 1 **(**+)**

De Moivre's theorem states

$$(\cos\theta + \mathrm{i}\sin\theta)^n \equiv \cos n\theta + \mathrm{i}\sin n\theta, \; n \in \mathbb{N}.$$

Prove this theorem by induction.

proof



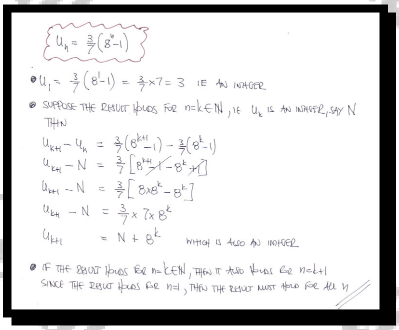
- IF $n=1$ LHS $= (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$
  RHS $= \cos(1\times\theta) + i\sin(1\times\theta) = \cos\theta + i\sin\theta$ IE IT HOLDS FOR $n=1$
- SUPPOSE THAT RESULT HOLDS FOR $n=k\in\mathbb{N}$
  $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$
  $(\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$
  $(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$
  $(\cos\theta + i\sin\theta)^{k+1} = \cos[k\theta+\theta] + i\sin[k\theta+\theta]$
  $(\cos\theta + i\sin\theta)^{k+1} = \cos[(k+1)\theta] + i\sin[(k+1)\theta]$
- IF THE RESULT HOLDS FOR $n=k\in\mathbb{N} \Rightarrow$ IT ALSO HOLDS FOR $n=k+1$
  SINCE THE RESULT HOLDS FOR $n=1 \Rightarrow$ IT MUST HOLD $\forall n\in\mathbb{N}$

Question 2 **(**+)**

$$u_n = \frac{3}{7}\left(8^n - 1\right), \; n \in \mathbb{N}.$$

Prove by induction that every term of this sequence is an integer.

proof



$u_n = \frac{3}{7}(8^n - 1)$

- $u_1 = \frac{3}{7}(8^1 - 1) = \frac{3}{7}\times 7 = 3$ IE AN INTEGER
- SUPPOSE THE RESULT HOLDS FOR $n=k\in\mathbb{N}$, IE u_k IS AN INTEGER, SAY N
  THEN
  $u_{k+1} - u_k = \frac{3}{7}(8^{k+1} - 1) - \frac{3}{7}(8^k - 1)$
  $u_{k+1} - N = \frac{3}{7}[8^{k+1} - 1 - 8^k + 1]$
  $u_{k+1} - N = \frac{3}{7}[8\times 8^k - 8^k]$
  $u_{k+1} - N = \frac{3}{7}\times 7\times 8^k$
  $u_{k+1} = N + 8^k$ WHICH IS ALSO AN INTEGER
- IF THE RESULT HOLDS FOR $n=k\in\mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
  SINCE THE RESULT HOLDS FOR $n=1$, THEN THE RESULT MUST HOLD FOR ALL n

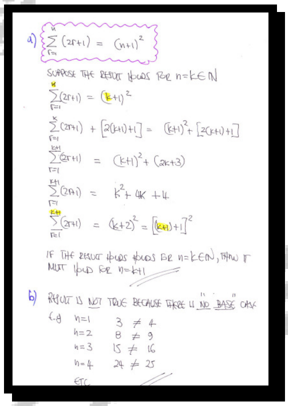
Question 3 (**+)

$$\sum_{r=1}^{n}(2r+1)=(n+1)^2,\ n\in\mathbb{N}$$

a) Show that if the above result holds for $n=k$, then it also holds for $n=k+1$.

b) Explain why the result is **not** true.

proof



a) $\sum_{r=1}^{n}(2r+1)=(n+1)^2$

SUPPOSE THE RESULT HOLDS FOR $n=k\in\mathbb{N}$

$\sum_{r=1}^{k}(2r+1)=(k+1)^2$

$\sum_{r=1}^{k}(2r+1)+[2(k+1)+1]=(k+1)^2+[2(k+1)+1]$

$\sum_{r=1}^{k+1}(2r+1)=(k+1)^2+(2k+3)$

$\sum_{r=1}^{k+1}(2r+1)=k^2+4k+4$

$\sum_{r=1}^{k+1}(2r+1)=(k+2)^2=[(k+1)+1]^2$

IF THE RESULT HOLDS FOR $n=k\in\mathbb{N}$, THEN IT MUST HOLD FOR $n=k+1$

b) RESULT IS NOT TRUE BECAUSE THERE IS "NO BASE" CASE

e.g. $n=1$ $3\neq4$

$n=2$ $8\neq9$

$n=3$ $15\neq16$

$n=4$ $24\neq25$

ETC

Question 4 (**+)

The distinct square matrices $\mathbf{A}$ and $\mathbf{B}$ have the properties

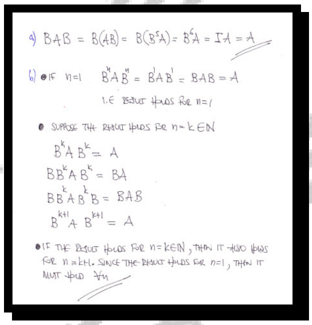
- $\mathbf{AB} = \mathbf{B}^5\mathbf{A}$
- $\mathbf{B}^6 = \mathbf{I}$

where $\mathbf{I}$ is the identity matrix.

a) Show that $\mathbf{BAB} = \mathbf{A}$.

b) Hence prove by induction that $\mathbf{B}^n\mathbf{AB}^n = \mathbf{A}$, for all $n \in \mathbb{N}$.

proof



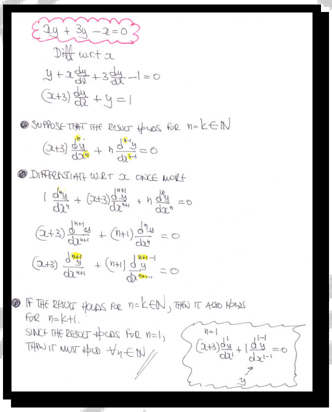
Question 5 (***)

$$xy + 3y = x.$$

Prove by induction

$$(x+3)\frac{d^n}{dx^n}(y) + (n+1)\frac{d^{n-1}}{dx^{n-1}}(y) = 0.$$

proof



$xy + 3y - x = 0$

Diff w.r.t x

$y + x\frac{dy}{dx} + 3\frac{dy}{dx} - 1 = 0$

$(x+3)\frac{dy}{dx} + y = 1$

- SUPPOSE THAT THE RESULT HOLDS FOR $n = k \in \mathbb{N}$

$(x+3)\frac{d^n y}{dx^n} + n\frac{d^{n-1}y}{dx^{n-1}} = 0$

- DIFFERENTIATE W.R.T x ONCE MORE

$1\frac{d^n y}{dx^n} + (x+3)\frac{d^{n+1}y}{dx^{n+1}} + n\frac{d^n y}{dx^n} = 0$

$(x+3)\frac{d^{n+1}y}{dx^{n+1}} + (n+1)\frac{d^n y}{dx^n} = 0$

$(x+3)\frac{d^{n+1}y}{dx^{n+1}} + (n+1)\frac{d^{n+1-1}y}{dx^{n+1-1}} = 0$

- IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$.

SINCE THE RESULT HOLDS FOR $n = 1$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

$n = 1$

$(x+3)\frac{d^1 y}{dx^1} + 1\frac{d^{1-1}y}{dx^{1-1}} = 0$

y

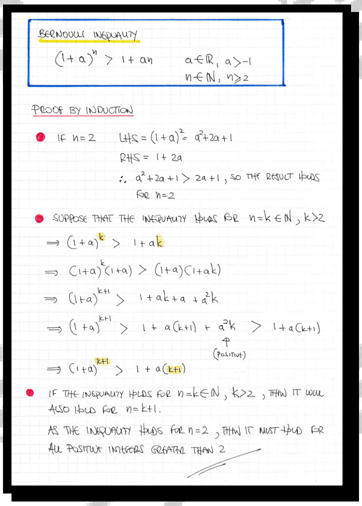
Question 6 (***)

Bernoulli's inequality asserts that if $a \in \mathbb{R}$, $a > -1$ and $n \in \mathbb{N}$, $n \geq 2$, then

$$(1+a)^n > 1+an.$$

Prove, by induction, the validity of Bernoulli's identity.

[ ], proof



BERNOULLI INEQUALITY

$(1+a)^n > 1+an$ $\quad a \in \mathbb{R},\ a>-1$ $\quad n \in \mathbb{N},\ n \geq 2$

PROOF BY INDUCTION

- IF $n=2$ $\quad$ LHS $= (1+a)^2 = a^2+2a+1$
  RHS $= 1+2a$
  $\therefore\ a^2+2a+1 > 2a+1$, SO THE RESULT HOLDS FOR $n=2$
- SUPPOSE THAT THE INEQUALITY HOLDS FOR $n=k \in \mathbb{N}$, $k>2$

$\Rightarrow (1+a)^k > 1+ak$

$\Rightarrow (1+a)^k(1+a) > (1+a)(1+ak)$

$\Rightarrow (1+a)^{k+1} > 1+ak+a+a^2k$

$\Rightarrow (1+a)^{k+1} > 1+a(k+1)+a^2k > 1+a(k+1)$ $\quad$ (a^2k POSITIVE)

$\Rightarrow (1+a)^{k+1} > 1+a(k+1)$

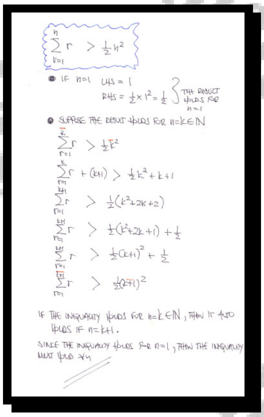
- IF THE INEQUALITY HOLDS FOR $n=k \in \mathbb{N}$, $k>2$, THEN IT WILL ALSO HOLD FOR $n=k+1$.
  AS THE INEQUALITY HOLDS FOR $n=2$, THEN IT MUST HOLD FOR ALL POSITIVE INTEGERS GREATER THAN 2

Question 7 **(***+)**

Prove by induction that

$$\sum_{r=1}^{n} r > \frac{1}{2}n^2, \quad \text{for } n \geq 1,\, n \in \mathbb{N} .$$

proof



$$\sum_{r=1}^{n} r > \frac{1}{2}n^2$$

- IF $n=1$ LHS $= 1$, RHS $= \frac{1}{2} \times 1^2 = \frac{1}{2}$ } THE RESULT HOLDS FOR $n=1$
- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\sum_{r=1}^{k} r > \frac{1}{2}k^2$$
$$\sum_{r=1}^{k} r + (k+1) > \frac{1}{2}k^2 + k + 1$$
$$\sum_{r=1}^{k+1} r > \frac{1}{2}(k^2+2k+2)$$
$$\sum_{r=1}^{k+1} r > \frac{1}{2}(k^2+2k+1) + \frac{1}{2}$$
$$\sum_{r=1}^{k+1} r > \frac{1}{2}(k+1)^2 + \frac{1}{2}$$
$$\sum_{r=1}^{k+1} r > \frac{1}{2}(k+1)^2$$

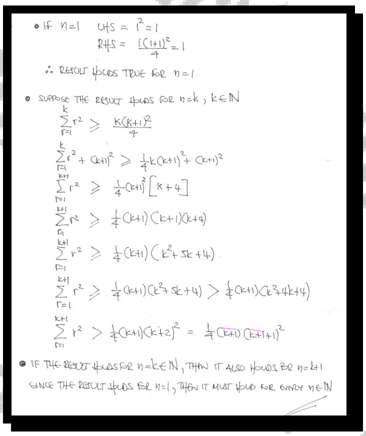
IF THE INEQUALITY HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS IF $n=k+1$.
SINCE THE INEQUALITY HOLDS FOR $n=1$, THEN THE INEQUALITY MUST HOLD $\forall n$

Question 8 **(***+)**

Prove by induction that

$$\sum_{r=1}^{n} r^2 \geq \frac{1}{4}n(n+1)^2, \quad \text{for } n \geq 1,\, n \in \mathbb{N} .$$

proof



- IF $n=1$ LHS $= 1^2 = 1$, RHS $= \frac{1(1+1)^2}{4} = 1$

∴ RESULT HOLDS TRUE FOR $n=1$

- SUPPOSE THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$

$$\sum_{r=1}^{k} r^2 \geq \frac{k(k+1)^2}{4}$$
$$\sum_{r=1}^{k} r^2 + (k+1)^2 \geq \frac{1}{4}k(k+1)^2 + (k+1)^2$$
$$\sum_{r=1}^{k+1} r^2 \geq \frac{1}{4}(k+1)^2\left[k+4\right]$$
$$\sum_{r=1}^{k+1} r^2 \geq \frac{1}{4}(k+1)(k+1)(k+4)$$
$$\sum_{r=1}^{k+1} r^2 \geq \frac{1}{4}(k+1)(k^2+5k+4)$$
$$\sum_{r=1}^{k+1} r^2 \geq \frac{1}{4}(k+1)(k^2+5k+4) > \frac{1}{4}(k+1)(k^2+4k+4)$$
$$\sum_{r=1}^{k+1} r^2 > \frac{1}{4}(k+1)(k+2)^2 = \frac{1}{4}(k+1)(k+1+1)^2$$

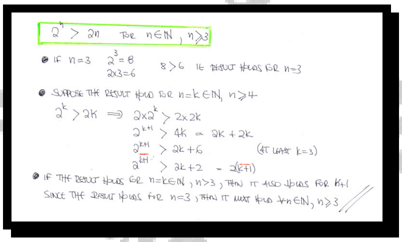
- IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR EVERY $n \in \mathbb{N}$

Question 9 **(****)**

Prove by induction that

$$2^n > 2n, \quad \text{for } n \geq 3, n \in \mathbb{N}.$$

proof

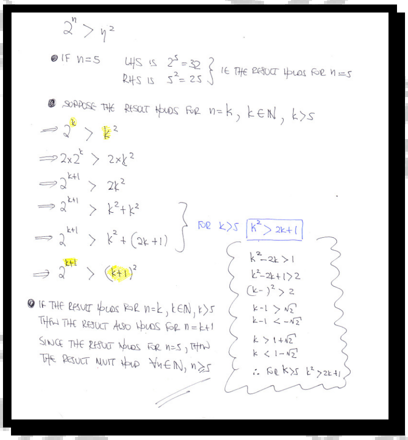


Question 10 **(****)**

Prove by induction that

$$2^n > n^2, \quad \text{for } n \geq 5, n \in \mathbb{N}.$$

proof

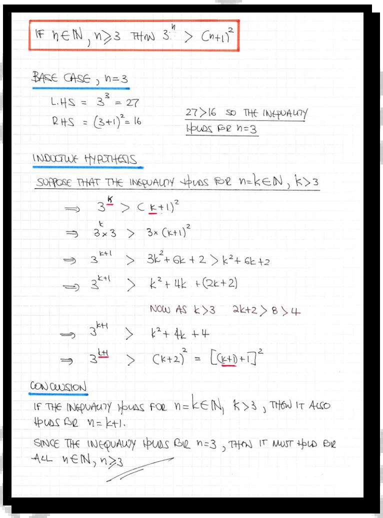


Question 11 (****)

Prove by induction that if $n \in \mathbb{N}$, $n \geq 3$, then

$$3^n > (n+1)^2 .$$

, proof


IF $n \in \mathbb{N}$, $n \geqslant 3$ THEN $3^n > (n+1)^2$

BASE CASE, $n=3$

L.H.S $= 3^3 = 27$

R.H.S $= (3+1)^2 = 16$

$27 > 16$ SO THE INEQUALITY HOLDS FOR $n=3$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE INEQUALITY HOLDS FOR $n=k \in \mathbb{N}$, $k>3$

$\Rightarrow 3^k > (k+1)^2$

$\Rightarrow 3^k \times 3 > 3 \times (k+1)^2$

$\Rightarrow 3^{k+1} > 3k^2 + 6k + 2 > k^2 + 6k + 2$

$\Rightarrow 3^{k+1} > k^2 + 4k + (2k+2)$

NOW AS $k>3$ $2k+2 > 8 > 4$

$\Rightarrow 3^{k+1} > k^2 + 4k + 4$

$\Rightarrow 3^{k+1} > (k+2)^2 = [(k+1)+1]^2$

CONCLUSION

IF THE INEQUALITY HOLDS FOR $n=k \in \mathbb{N}$, $k>3$, THEN IT ALSO HOLDS FOR $n=k+1$.

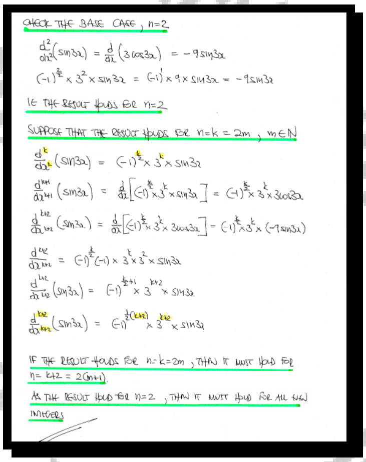
SINCE THE INEQUALITY HOLDS FOR $n=3$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$, $n \geqslant 3$

Question 12 (****)

Prove by induction that for all even natural numbers n

$$\frac{d^n}{dx^n}(\sin 3x) = (-1)^{\frac{n}{2}} \times 3^n \times \sin 3x .$$

, proof


CHECK THE BASE CASE, $n=2$

$\frac{d^2}{dx^2}(\sin 3x) = \frac{d}{dx}(3\cos 3x) = -9\sin 3x$

$(-1)^{\frac{2}{2}} \times 3^2 \times \sin 3x = (-1)^1 \times 9 \times \sin 3x = -9\sin 3x$

IE THE RESULT HOLDS FOR $n=2$

SUPPOSE THAT THE RESULT HOLDS FOR $n=k=2m$, $m \in \mathbb{N}$

$\frac{d^k}{dx^k}(\sin 3x) = (-1)^{\frac{k}{2}} \times 3^k \times \sin 3x$

$\frac{d^{k+1}}{dx^{k+1}}(\sin 3x) = \frac{d}{dx}\left[(-1)^{\frac{k}{2}} \times 3^k \times \sin 3x\right] = (-1)^{\frac{k}{2}} \times 3^k \times 3\cos 3x$

$\frac{d^{k+2}}{dx^{k+2}}(\sin 3x) = \frac{d}{dx}\left[(-1)^{\frac{k}{2}} \times 3^k \times 3\cos 3x\right] = (-1)^{\frac{k}{2}} \times 3^k \times (-9\sin 3x)$

$\frac{d^{k+2}}{dx^{k+2}} = (-1)^{\frac{k}{2}}(-1) \times 3^k \times 3^2 \times \sin 3x$

$\frac{d^{k+2}}{dx^{k+2}}(\sin 3x) = (-1)^{\frac{k}{2}+1} \times 3^{k+2} \times \sin 3x$

$\frac{d^{k+2}}{dx^{k+2}}(\sin 3x) = (-1)^{\frac{1}{2}(k+2)} \times 3^{k+2} \times \sin 3x$

IF THE RESULT HOLDS FOR $n=k=2m$, THEN IT MUST HOLD FOR $n=k+2=2(m+1)$

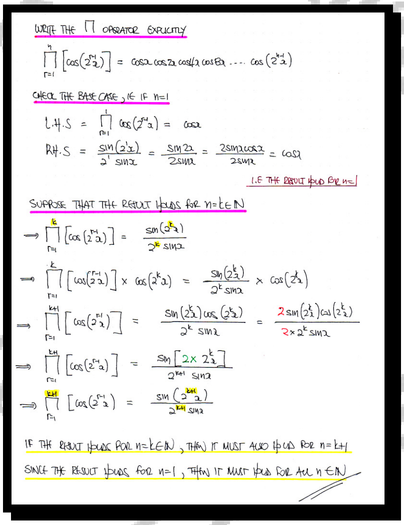
AS THE RESULT HOLD FOR $n=2$, THEN IT MUST HOLD FOR ALL EVEN INTEGERS

Question 13 (****+)

Prove by induction that for $n \geq 1,\ n \in \mathbb{N}$

$$\prod_{r=1}^{n}\left(\cos\left(2^{r-1}x\right)\right)=\frac{\sin\left(2^{n}x\right)}{2^{n}\sin x}.$$

[ ], [proof]


WRITE THE $\prod$ OPERATOR EXPLICITLY

$$\prod_{r=1}^{n}\left[\cos\left(2^{r-1}x\right)\right] = \cos x\cos 2x\cos 4x\cos 8x \dots \cos\left(2^{n-1}x\right)$$

CHECK THE BASE CASE, IE IF $n=1$

$$\text{L.H.S} = \prod_{r=1}^{1}\cos\left(2^{r-1}x\right) = \cos x$$

$$\text{R.H.S} = \frac{\sin\left(2^{1}x\right)}{2^{1}\sin x} = \frac{\sin 2x}{2\sin x} = \frac{2\sin x\cos x}{2\sin x} = \cos x$$

I.E THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n=k\in\mathbb{N}$

$$\Longrightarrow \prod_{r=1}^{k}\left[\cos\left(2^{r-1}x\right)\right] = \frac{\sin\left(2^{k}x\right)}{2^{k}\sin x}$$

$$\Longrightarrow \prod_{r=1}^{k}\left[\cos\left(2^{r-1}x\right)\right]\times\cos\left(2^{k}x\right) = \frac{\sin\left(2^{k}x\right)}{2^{k}\sin x}\times\cos\left(2^{k}x\right)$$

$$\Longrightarrow \prod_{r=1}^{k+1}\left[\cos\left(2^{r-1}x\right)\right] = \frac{\sin\left(2^{k}x\right)\cos\left(2^{k}x\right)}{2^{k}\sin x} = \frac{2\sin\left(2^{k}x\right)\cos\left(2^{k}x\right)}{2\times2^{k}\sin x}$$

$$\Longrightarrow \prod_{r=1}^{k+1}\left[\cos\left(2^{r-1}x\right)\right] = \frac{\sin\left[2\times2^{k}x\right]}{2^{k+1}\sin x}$$

$$\Longrightarrow \prod_{r=1}^{k+1}\left[\cos\left(2^{r-1}x\right)\right] = \frac{\sin\left(2^{k+1}x\right)}{2^{k+1}\sin x}$$

IF THE RESULT HOLDS FOR $n=k\in\mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$

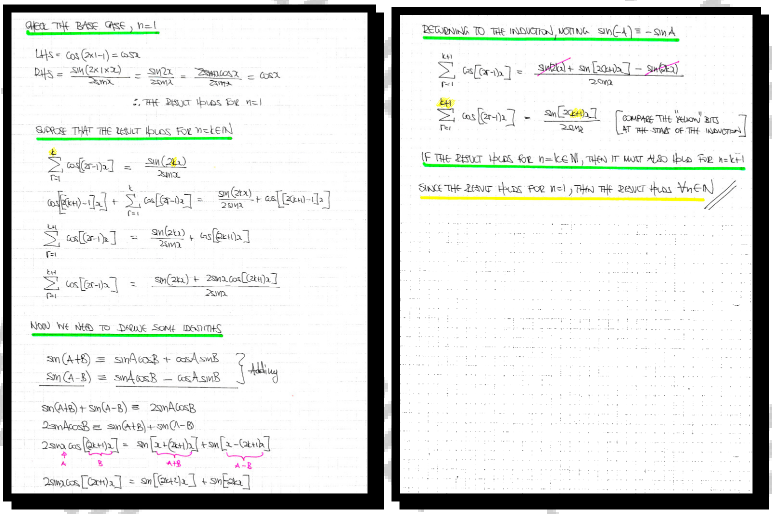
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n\in\mathbb{N}$

Question 14 (****+)

Prove by induction that

$$\cos x+\cos 3x+\cos 5x+...+\cos\left[(2n-1)x\right]\equiv\frac{\sin(2nx)}{2\sin x}.$$

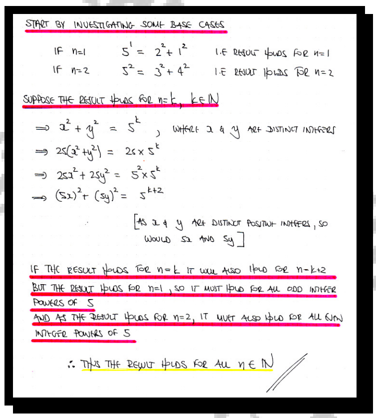
, proof



Question 15 (****+)

Prove by induction that every positive integer power of 5 can be written as the sum of squares of two distinct positive integers.

, proof

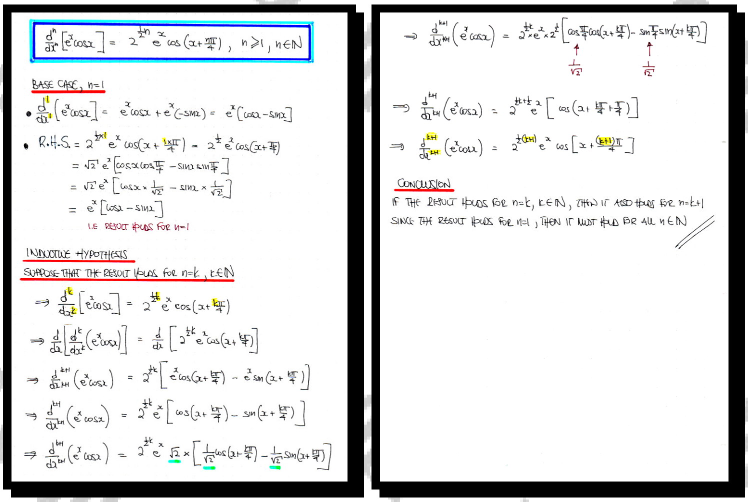


Question 16 (*****)

Prove by induction that

$$\frac{d^n}{dx^n}\left(e^x \cos x\right) = 2^{\frac{1}{2}n}\, e^x \cos\left(x + \frac{n\pi}{4}\right), \quad n \geq 1,\ n \in \mathbb{N} .$$

proof



$$\frac{d^n}{dx^n}\left[e^x \cos x\right] = 2^{\frac{1}{2}n} e^x \cos\left(x+\frac{n\pi}{4}\right), \quad n \geqslant 1,\ n \in \mathbb{N}$$

BASE CASE, n=1

- $\frac{d^1}{dx^1}\left[e^x \cos x\right] = e^x \cos x + e^x(-\sin x) = e^x\left[\cos x - \sin x\right]$
- $\text{R.H.S.} = 2^{\frac{1}{2}\times 1} e^x \cos\left(x + \frac{1\times\pi}{4}\right) = 2^{\frac{1}{2}} e^x \cos\left(x+\frac{\pi}{4}\right)$
  $= \sqrt{2}\, e^x \left[\cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4}\right]$
  $= \sqrt{2}\, e^x \left[\cos x \times \frac{1}{\sqrt{2}} - \sin x \times \frac{1}{\sqrt{2}}\right]$
  $= e^x\left[\cos x - \sin x\right]$

I.E RESULT HOLDS FOR n=1

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR n=k, k∈N

$\Rightarrow \frac{d^k}{dx^k}\left[e^x \cos x\right] = 2^{\frac{1}{2}k} e^x \cos\left(x+\frac{k\pi}{4}\right)$

$\Rightarrow \frac{d}{dx}\left[\frac{d^k}{dx^k}\left(e^x\cos x\right)\right] = \frac{d}{dx}\left[2^{\frac{1}{2}k} e^x \cos\left(x+\frac{k\pi}{4}\right)\right]$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}}\left(e^x \cos x\right) = 2^{\frac{1}{2}k}\left[e^x\cos\left(x+\frac{k\pi}{4}\right) - e^x \sin\left(x+\frac{k\pi}{4}\right)\right]$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}}\left(e^x \cos x\right) = 2^{\frac{1}{2}k} e^x\left[\cos\left(x+\frac{k\pi}{4}\right) - \sin\left(x+\frac{k\pi}{4}\right)\right]$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}}\left(e^x \cos x\right) = 2^{\frac{1}{2}k} e^x \sqrt{2} \times \left[\frac{1}{\sqrt{2}}\cos\left(x+\frac{k\pi}{4}\right) - \frac{1}{\sqrt{2}}\sin\left(x+\frac{k\pi}{4}\right)\right]$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}}\left(e^x \cos x\right) = 2^{\frac{1}{2}k} \times e^x \times 2^{\frac{1}{2}}\left[\cos\frac{\pi}{4}\cos\left(x+\frac{k\pi}{4}\right) - \sin\frac{\pi}{4}\sin\left(x+\frac{k\pi}{4}\right)\right]$

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}}\left(e^x \cos x\right) = 2^{\frac{1}{2}k+\frac{1}{2}} e^x\left[\cos\left(x + \frac{k\pi}{4} + \frac{\pi}{4}\right)\right]$

$\Rightarrow \frac{d^{k+1}}{dx^{k+1}}\left(e^x \cos x\right) = 2^{\frac{1}{2}(k+1)} e^x \cos\left[x + \frac{(k+1)\pi}{4}\right]$

CONCLUSION

IF THE RESULT HOLDS FOR n=k, k∈N, THEN IT ALSO HOLDS FOR n=k+1

SINCE THE RESULT HOLDS FOR n=1, THEN IT MUST HOLD FOR ALL n∈N

Question 17 (*****)

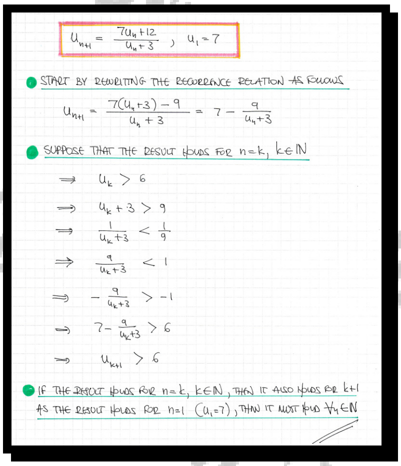
It is given that for $n \in \mathbb{N}$

$$u_{n+1} = \frac{7u_n + 12}{u_n + 3}, \quad u_1 = 7.$$

Prove by induction that

$$u_n > 6.$$

[ ] , [proof]



$u_{n+1} = \frac{7u_n+12}{u_n+3}$, $u_1 = 7$

- START BY REWRITING THE RECURRENCE RELATION AS FOLLOWS

$u_{n+1} = \frac{7(u_n+3)-9}{u_n+3} = 7 - \frac{9}{u_n+3}$

- SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$

$\Rightarrow u_k > 6$

$\Rightarrow u_k + 3 > 9$

$\Rightarrow \frac{1}{u_k+3} < \frac{1}{9}$

$\Rightarrow \frac{9}{u_k+3} < 1$

$\Rightarrow -\frac{9}{u_k+3} > -1$

$\Rightarrow 7 - \frac{9}{u_k+3} > 6$

$\Rightarrow u_{k+1} > 6$

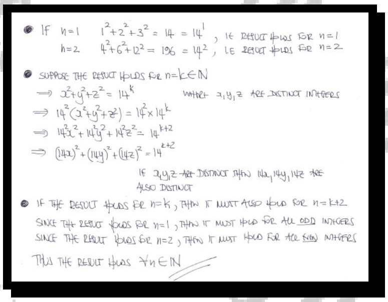
- IF THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $k+1$

AS THE RESULT HOLDS FOR $n=1$ $(u_1=7)$, THEN IT MUST HOLD $\forall n \in \mathbb{N}$

Question 18 (*****)

Prove by induction that every positive integer power of 14 can be written as the sum of squares of three distinct positive integers.

[SPX-I] , [proof]



- IF $n=1$ $1^2+2^2+3^2 = 14 = 14^1$, IE RESULT HOLDS FOR $n=1$

$n=2$ $4^2+6^2+12^2 = 196 = 14^2$, I.E RESULT HOLDS FOR $n=2$

- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$\Rightarrow x^2+y^2+z^2 = 14^k$ WHERE x, y, z ARE DISTINCT INTEGERS

$\Rightarrow 14^2(x^2+y^2+z^2) = 14^2 \times 14^k$

$\Rightarrow 14^2x^2 + 14^2y^2 + 14^2z^2 = 14^{k+2}$

$\Rightarrow (14x)^2 + (14y)^2 + (14z)^2 = 14^{k+2}$

IF x, y, z ARE DISTINCT THEN $14x, 14y, 14z$ ARE ALSO DISTINCT

- IF THE RESULT HOLDS FOR $n=k$, THEN IT MUST ALSO HOLD FOR $n=k+2$

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL ODD INTEGERS

SINCE THE RESULT HOLDS FOR $n=2$, THEN IT MUST HOLD FOR ALL EVEN INTEGERS

THUS THE RESULT HOLDS $\forall n \in \mathbb{N}$

Question 19 (*****)

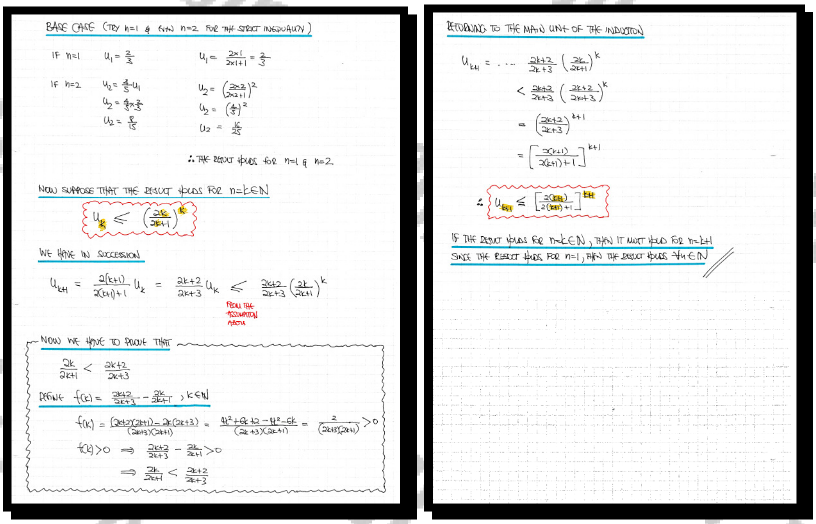
It is given that for $n \in \mathbb{N}$

$$U_n = \frac{2n}{2n+1} U_{n-1}, \quad U_1 = \frac{2}{3}.$$

Prove by induction that

$$U_n \le \left(\frac{2n}{2n+1} \right)^n.$$

[ ] , [proof]



BASE CASE (TRY $n=1$ & EVEN $n=2$ FOR THE STRICT INEQUALITY)

IF $n=1$: $u_1 = \frac{2}{3}$; $u_1 = \frac{2\times1}{2\times1+1} = \frac{2}{3}$

IF $n=2$: $u_2 = \frac{4}{5}u_1$, $u_2 = \frac{4}{5}\times\frac{2}{3}$, $u_2 = \frac{8}{15}$; $u_2 = \left(\frac{2\times2}{2\times2+1}\right)^2$, $u_2 = \left(\frac{4}{5}\right)^2$, $u_2 = \frac{16}{25}$

∴ THE RESULT HOLDS FOR $n=1$ & $n=2$

NOW SUPPOSE THAT THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$u_k \le \left(\frac{2k}{2k+1}\right)^k$$

WE HAVE IN SUCCESSION

$$u_{k+1} = \frac{2(k+1)}{2(k+1)+1}u_k = \frac{2k+2}{2k+3}u_k \le \frac{2k+2}{2k+3}\left(\frac{2k}{2k+1}\right)^k$$

FROM THE ASSUMPTION ABOVE

NOW WE HAVE TO PROVE THAT

$$\frac{2k}{2k+1} < \frac{2k+2}{2k+3}$$

DEFINE $f(k) = \frac{2k+2}{2k+3} - \frac{2k}{2k+1}$, $k \in \mathbb{N}$

$$f(k) = \frac{(2k+2)(2k+1) - 2k(2k+3)}{(2k+3)(2k+1)} = \frac{4k^2+6k+2-4k^2-6k}{(2k+3)(2k+1)} = \frac{2}{(2k+3)(2k+1)} > 0$$

$$f(k) > 0 \Rightarrow \frac{2k+2}{2k+3} - \frac{2k}{2k+1} > 0 \Rightarrow \frac{2k}{2k+1} < \frac{2k+2}{2k+3}$$

RETURNING TO THE MAIN LINE OF THE INDUCTION

$$u_{k+1} = \dots \frac{2k+2}{2k+3}\left(\frac{2k}{2k+1}\right)^k < \frac{2k+2}{2k+3}\left(\frac{2k+2}{2k+3}\right)^k = \left(\frac{2k+2}{2k+3}\right)^{k+1} = \left[\frac{2(k+1)}{2(k+1)+1}\right]^{k+1}$$

$$\therefore u_{k+1} \le \left[\frac{2(k+1)}{2(k+1)+1}\right]^{k+1}$$

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST HOLD FOR $n=k+1$

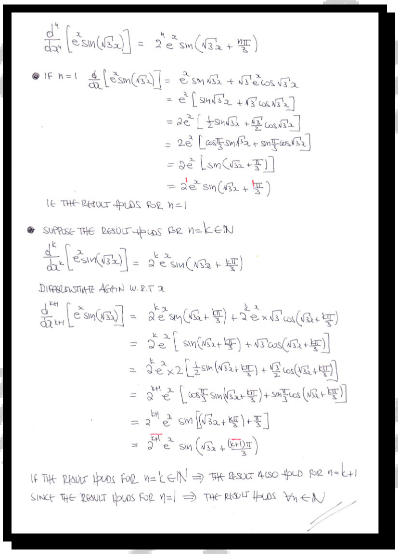
SINCE THE RESULT HOLDS FOR $n=1$, THEN THE RESULT HOLDS $\forall n \in \mathbb{N}$

Question 20 (*****)

Prove by induction that

$$\frac{d^n}{dx^n}\left(e^x \sin\left(\sqrt{3}x\right)\right) = 2^n e^x \sin\left(\sqrt{3}x + \frac{n\pi}{3}\right), \quad n \geq 1, n \in \mathbb{N} .$$

SP P, proof



$$\frac{d^n}{dx^n}\left[e^x \sin(\sqrt{3}x)\right] = 2^n e^x \sin\left(\sqrt{3}x + \frac{n\pi}{3}\right)$$

- IF $n=1$ $\frac{d}{dx}\left[e^x \sin(\sqrt{3}x)\right] = e^x \sin\sqrt{3}x + \sqrt{3}e^x \cos\sqrt{3}x$

$$= e^x\left[\sin\sqrt{3}x + \sqrt{3}\cos\sqrt{3}x\right]$$
$$= 2e^x\left[\tfrac{1}{2}\sin\sqrt{3}x + \tfrac{\sqrt{3}}{2}\cos\sqrt{3}x\right]$$
$$= 2e^x\left[\cos\tfrac{\pi}{3}\sin\sqrt{3}x + \sin\tfrac{\pi}{3}\cos\sqrt{3}x\right]$$
$$= 2e^x\left[\sin\left(\sqrt{3}x + \tfrac{\pi}{3}\right)\right]$$
$$= 2^1 e^x \sin\left(\sqrt{3}x + \tfrac{1\pi}{3}\right)$$

IE THE RESULT HOLDS FOR $n=1$

- SUPPOSE THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\frac{d^k}{dx^k}\left[e^x \sin(\sqrt{3}x)\right] = 2^k e^x \sin\left(\sqrt{3}x + \frac{k\pi}{3}\right)$$

DIFFERENTIATE AGAIN W.R.T x

$$\frac{d^{k+1}}{dx^{k+1}}\left[e^x \sin(\sqrt{3}x)\right] = 2^k e^x \sin\left(\sqrt{3}x + \tfrac{k\pi}{3}\right) + 2^k e^x \times \sqrt{3}\cos\left(\sqrt{3}x + \tfrac{k\pi}{3}\right)$$
$$= 2^k e^x\left[\sin\left(\sqrt{3}x + \tfrac{k\pi}{3}\right) + \sqrt{3}\cos\left(\sqrt{3}x + \tfrac{k\pi}{3}\right)\right]$$
$$= 2^k e^x \times 2\left[\tfrac{1}{2}\sin\left(\sqrt{3}x + \tfrac{k\pi}{3}\right) + \tfrac{\sqrt{3}}{2}\cos\left(\sqrt{3}x + \tfrac{k\pi}{3}\right)\right]$$
$$= 2^{k+1} e^x\left[\cos\tfrac{\pi}{3}\sin\left(\sqrt{3}x + \tfrac{k\pi}{3}\right) + \sin\tfrac{\pi}{3}\cos\left(\sqrt{3}x + \tfrac{k\pi}{3}\right)\right]$$
$$= 2^{k+1} e^x \sin\left[\left(\sqrt{3}x + \tfrac{k\pi}{3}\right) + \tfrac{\pi}{3}\right]$$
$$= 2^{k+1} e^x \sin\left(\sqrt{3}x + \tfrac{(k+1)\pi}{3}\right)$$

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N} \Rightarrow$ THE RESULT ALSO HOLD FOR $n=k+1$

SINCE THE RESULT HOLDS FOR $n=1 \Rightarrow$ THE RESULT HOLDS $\forall n \in \mathbb{N}$

Question 21 (*****)

The function $f(x)$ is defined by

$$f(x) = 2 - \frac{1}{x}, \quad x \in \mathbb{R},\ x \neq 0 .$$

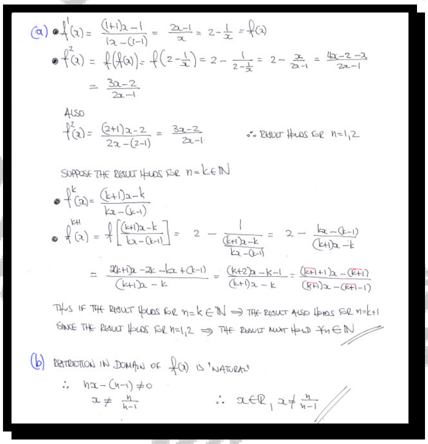
a) Prove that

$$f^{n}(x) = \frac{(n+1)x - n}{nx - (n-1)}, \quad n \geq 1,$$

where $f^{n}(x)$ denotes the n^{th} composition of $f(x)$ by itself.

b) State an expression for the domain of $f^{n}(x)$.

[ ], $x \in \mathbb{R},\ x \neq \frac{n-1}{n}$



(a) • $f^{1}(x) = \frac{(1+1)x - 1}{1x - (1-1)} = \frac{2x-1}{x} = 2 - \frac{1}{x} = f(x)$

• $f^{2}(x) = f(f(x)) = f\left(2 - \frac{1}{x}\right) = 2 - \frac{1}{2-\frac{1}{x}} = 2 - \frac{x}{2x-1} = \frac{4x-2-x}{2x-1}$

$= \frac{3x-2}{2x-1}$

ALSO

$f^{2}(x) = \frac{(2+1)x-2}{2x-(2-1)} = \frac{3x-2}{2x-1}$ ∴ RESULT HOLDS FOR $n=1,2$

SUPPOSE THE RESULT HOLDS FOR $n = k \in \mathbb{N}$

• $f^{k}(x) = \frac{(k+1)x-k}{kx-(k-1)}$

• $f^{k+1}(x) = f\left[\frac{(k+1)x-k}{kx-(k-1)}\right] = 2 - \frac{1}{\frac{(k+1)x-k}{kx-(k-1)}} = 2 - \frac{kx-(k-1)}{(k+1)x-k}$

$= \frac{2(k+1)x - 2k - kx + (k-1)}{(k+1)x - k} = \frac{(k+2)x-k-1}{(k+1)x-k} = \frac{(k+1+1)x-(k+1)}{(k+1)x-(k+1-1)}$

THUS IF THE RESULT HOLDS FOR $n = k \in \mathbb{N} \Rightarrow$ THE RESULT ALSO HOLDS FOR $n = k+1$

SINCE THE RESULT HOLDS FOR $n=1,2 \Rightarrow$ THE RESULT MUST HOLD $\forall n \in \mathbb{N}$

(b) RESTRICTION IN DOMAIN OF $f(x)$ IS 'NATURAL'

∴ $nx - (n-1) \neq 0$

$x \neq \frac{n}{n-1}$ ∴ $x \in \mathbb{R},\ x \neq \frac{n}{n-1}$

Question 22 (*****)

Prove by induction that if $n \in \mathbb{N}$, $n \geq 3$, then

$$n^{n+1} > (n+1)^n,$$

and hence deduce that if $n \in \mathbb{N}$, $n \geq 3$, then

$$\sqrt[n]{n} \;>\; \sqrt[n+1]{n+1}$$

[ ], [proof]

IF $n \in \mathbb{N}$, $n \geq 3$ THEN $n^{n+1} > (n+1)^n$

BASE CASE, $n=3$

L.H.S $= 3^4 = 81$

R.H.S. $= 4^3 = 64$

$81 > 64$, SO THE RESULT HOLDS FOR $n=3$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n=k \geq 3$, $k \in \mathbb{N}$

$$\Rightarrow k^{k+1} > (k+1)^k$$
$$\Rightarrow k^{k+1}(k+1)^{k+2} > (k+1)^k (k+1)^{k+2}$$
$$\Rightarrow k^{k+1}(k+1)^{k+2} > (k+1)^{2k+2}$$
$$\Rightarrow (k+1)^{k+2} > \frac{(k+1)^{2k+2}}{k^{k+1}}$$

NOW WE NEED TO SHOW THAT

$$\frac{(k+1)^{2k+2}}{k^{k+1}} \geq (k+2)^{k+1} \Rightarrow (k+1)^{2k+2} \geq k^{k+1} \cdot (k+2)^{k+1}$$
$$\Rightarrow \left[(k+1)^2\right]^{k+1} \geq \left[k(k+2)\right]^{k+1}$$
$$\Rightarrow (k+1)^2 \geq k(k+2)$$
$$\Rightarrow k^2+2k+1 > k^2+2k$$

WHICH HOLDS

RETURNING TO THE MAIN LINE OF THE INDUCTIVE HYPOTHESIS

- IF $k^{k+1} > (k+1)^k$

  . . . .

- THEN $(k+1)^{k+2} > \frac{(k+1)^{2k+2}}{k^{k+1}} > (k+2)^{k+1}$

I.E $(k+1)^{[k+1+1]} > \left[(k+1)+1\right]^{k+1}$

CONCLUSION

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, WITH $n \geq 3$ THEN IT MUST ALSO HOLD FOR $n=k+1$

AS THE RESULT HOLDS FOR $n=3$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$, WITH $n \geq 3$

FINALLY WE HAVE

$$n^{n+1} > (n+1)^n \qquad n \in \mathbb{N},\ n \geq 3$$
$$\Rightarrow \left(n^{\frac{1}{n}}\right)^{n(n+1)} > \left[(n+1)^{\frac{1}{n+1}}\right]^{(n+1)n}$$
$$\Rightarrow \left[n^{\frac{1}{n}}\right]^{n^2+2n} > \left[(n+1)^{\frac{1}{n+1}}\right]^{n^2+n}$$
$$\Rightarrow \sqrt[n]{n} > \sqrt[n+1]{n+1}$$