GENERAL PROOF

Question 1 **(**)**

$$f(n) = n^2 + n + 2, \; n \in \mathbb{N}.$$

Show that $f(n)$ is always even.

proof

PROCEED AS FOLLOWS

$f(n) = n^2+n+2 = n(n+1)+2$

NOW $n(n+1)$ IS THE PRODUCT OF 2 CONSECUTIVE INTEGERS WHICH MUST BE EVEN, AS ONE OF THESE INTEGERS MUST BE EVEN

LET $n(n+1) = 2m$, m BEING AN INTEGER

$\ldots = n(n+1)+2 = 2m+2 = 2(m+2)$

INDEED EVEN

Question 2 **(**)**

Prove that when the square of a positive odd integer is divided by 4 the remainder is always 1.

proof

LET THE ODD POSITIVE INTEGER BE $2n+1$, $n=0,1,2,3,4,\ldots$

$(2n+1)^2 = 4n^2+4n+1 = 4(n^2+n)+1$

$= 4m+1$ (WHERE $m = n^2+n$)

IE LEAVES REMAINDER 1, WHEN DIVIDED BY 4

Question 3 **(**)**

Show that $a^3 - a + 1$ is odd for all positive integer values of a.

proof

METHOD A

- $a^3 - a + 1 = a(a^2-1) + 1 = a(a+1)(a-1) + 1$
- AS $a(a+1)(a-1)$ CONTAINS CONSECUTIVE INTEGERS, AT LEAST ONE OF THEM WILL BE EVEN, SO $a(a+1)(a-1)$ WILL BE EVEN FOR ALL $a \in \mathbb{N}$
- HENCE $a(a+1)(a-1)+1$ WILL BE ODD FOR ALL $a \in \mathbb{N}$

METHOD B (BY EXHAUSTION)

- LET a BE EVEN, $a = 2n$

$$(2n)^3 - 2n + 1 = 8n^3 - 2n + 1 = 2(4n^3 - n) + 1$$
$$= 2m + 1$$
$\therefore$ ODD

- LET a BE ODD, $a = 2n+1$

$$(2n+1)^3 - (2n+1) + 1 = 8n^3 + 12n^2 + 6n + 1 - 2n - 1 + 1$$
$$= 8n^3 + 12n^2 + 4n + 1$$
$$= 2[4n^3 + 6n^2 + n] + 1$$
$$= 2m + 1$$
$\therefore$ ODD

HENCE $a^3 - a + 1$ IS ODD FOR $a \in \mathbb{N}$

Question 4 **(**)**

Prove that the square of a positive integer can never be of the form $3k+2$, $k \in \mathbb{N}$.

proof

PROOF BY EXHAUSTION

"THE SQUARE OF ANY INTEGER CAN NEVER BE OF THE FORM $3k+2$, $k \in \mathbb{N}$"

THE NUMBER TO BE SQUARED, SAY a, CAN TAKE ONE OF THE FOLLOWING 3 FORMS

$$a = 3m, \quad a = 3m+1, \quad a = 3m+2, \quad m \in \mathbb{N}$$

- IF $a = 3m \Rightarrow a^2 = 9m^2 = 3(3m^2) = 3k$, $k \in \mathbb{N}$
- IF $a = 3m+1 \Rightarrow a^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1 = 3k+1$, $k \in \mathbb{N}$
- IF $a = 3m+2 \Rightarrow a^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1 = 3k+1$, $k \in \mathbb{N}$

$\therefore$ SQUARING ANY INTEGER ONLY PRODUCES INTEGERS OF THE FORM $3k$ OR $3k+1$, $k \in \mathbb{N}$

$\therefore$ IT IS NOT POSSIBLE TO HAVE A SQUARE NUMBER OF THE FORM $3k+2$, $k \in \mathbb{N}$

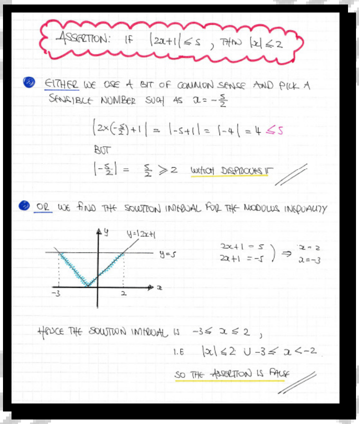
Question 5 (+)**

It is asserted that

$$|2x+1| \le 5 \;\Rightarrow\; |x| \le 2 .$$

Disprove this assertion by a **counter-example**.

[ ], proof

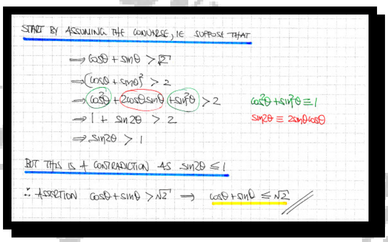


Question 6 (+)**

Prove by **contradiction** that for all real θ

$$\cos\theta + \sin\theta \le \sqrt{2} .$$

[ ], proof



Question 7 (**+)

Prove by contradiction that if p and q are positive integers, then

$$\frac{p}{q}+\frac{q}{p} \geq 2 .$$

MP2-E , proof

SUPPOSE THAT IF p & q WERE POSITIVE INTEGERS

$$\frac{p}{q}+\frac{q}{p}<2$$

THEN PROCEED AS FOLLOWS

$$\Rightarrow \frac{p^2+q^2}{pq}<2$$

$$\Rightarrow p^2+q^2<2pq \qquad (pq>0)$$

$$\Rightarrow p^2-2pq+q^2<0$$

$$\Rightarrow (p-q)^2<0$$

THIS IS A CONTRADICTION AS A SQUARED QUANTITY IS NEGATIVE

$$\therefore \frac{p}{q}+\frac{q}{p} \geqslant 2$$

Question 8 (***)

$$f(n)=5^{2n}-1, \; n \in \mathbb{N} .$$

Without using proof by induction, show that $f(n)$ is a multiple of 8.

, proof

MANIPULATING THE DIFFERENCE OF SQUARES $a^2-b^2 \equiv (a-b)(a+b)$

$$\Rightarrow f(n) = 5^{2n}-1$$
$$= (5^n)^2-1^2$$
$$= (5^n-1)(5^n+1)$$

NOW CONSIDER THE FOLLOWING ARGUMENT

$\Rightarrow 5^n$ IS AN ODD INTEGER AS IT IS A POWER OF 5

I.E. 5, 25, 125, 625, 3125, 15625, . . .

$\Rightarrow 5^n+1$ & 5^n-1 ARE BOTH EVEN

BUT FURTHER TO THIS 5^n-1 & 5^n+1 ARE TWO CONSECUTIVE EVEN NUMBERS, SO ONE OF THEM WILL BE A MULTIPLE OF 4

LET $5^n-1 = 2a \quad a \in \mathbb{N}$

$5^n+1 = 4b \quad b \in \mathbb{N}$

(OR THE OTHER WAY ROUND)

HENCE WE OBTAIN

$$f(n) = (5^n-1)(5^n+1) = 2a \times 4b = 8ab$$

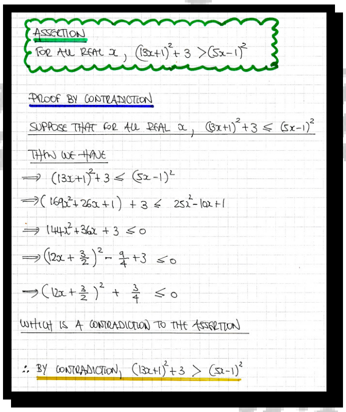
INDEED A MULTIPLE OF 8

Question 9 (***)

Prove by **contradiction** that for all real x

$$(13x+1)^2+3>(5x-1)^2.$$

$\boxed{}$, $\boxed{\text{proof}}$



ASSERTION

FOR ALL REAL x, $(13x+1)^2+3>(5x-1)^2$

PROOF BY CONTRADICTION

SUPPOSE THAT FOR ALL REAL x, $(13x+1)^2+3 \leqslant (5x-1)^2$

THEN WE HAVE

$\Rightarrow (13x+1)^2+3 \leqslant (5x-1)^2$

$\Rightarrow (169x^2+26x+1)+3 \leqslant 25x^2-10x+1$

$\Rightarrow 144x^2+36x+3 \leqslant 0$

$\Rightarrow \left(12x+\frac{3}{2}\right)^2-\frac{9}{4}+3 \leqslant 0$

$\Rightarrow \left(12x+\frac{3}{2}\right)^2+\frac{3}{4} \leqslant 0$

WHICH IS A CONTRADICTION TO THE ASSERTION

$\therefore$ BY CONTRADICTION, $(13x+1)^2+3>(5x-1)^2$

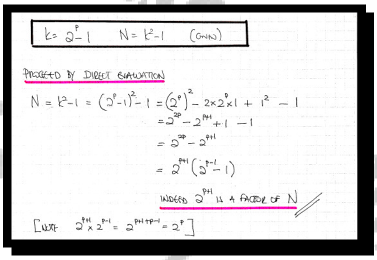
Question 10 (***)

It is given that

$$N=k^2-1 \quad \text{and} \quad k=2^p-1, \quad p \in \mathbb{N}.$$

Use direct proof to show that 2^{p+1} is a factor of N.

$\boxed{}$, $\boxed{\text{proof}}$



$\boxed{k=2^p-1 \quad N=k^2-1 \quad (\text{GIVEN})}$

PROCEED BY DIRECT EVALUATION

$N=k^2-1=(2^p-1)^2-1=(2^p)^2-2\times 2^p\times 1+1^2-1$

$=2^{2p}-2^{p+1}+1-1$

$=2^{2p}-2^{p+1}$

$=2^{p+1}(2^{p-1}-1)$

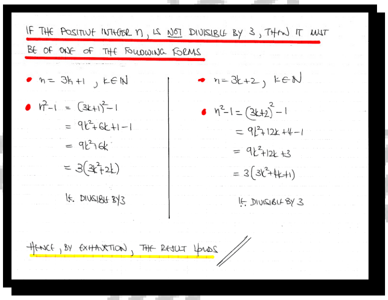
INDEED 2^{p+1} IS A FACTOR OF N

[NOTE $2^{p+1}\times 2^{p-1}=2^{p+1+p-1}=2^p$]

Question 11 (***)

Prove by exhaustion that if n is a positive integer that is **not** divisible by 3, then $n^2 - 1$ is divisible by 3.

[ ], [proof]



IF THE POSITIVE INTEGER n, IS NOT DIVISIBLE BY 3, THEN IT MUST BE OF ONE OF THE FOLLOWING FORMS

- $n = 3k+1, \ k \in \mathbb{N}$
- $n^2 - 1 = (3k+1)^2 - 1 = 9k^2 + 6k + 1 - 1 = 9k^2 + 6k = 3(3k^2 + 2k)$

IE. DIVISIBLE BY 3

- $n = 3k+2, \ k \in \mathbb{N}$
- $n^2 - 1 = (3k+2)^2 - 1 = 9k^2 + 12k + 4 - 1 = 9k^2 + 12k + 3 = 3(3k^2 + 4k + 1)$

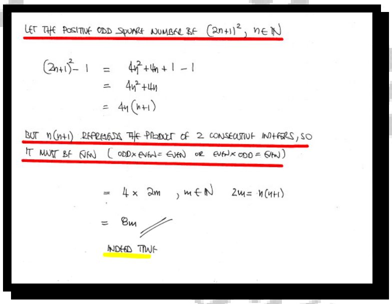
IE. DIVISIBLE BY 3

HENCE, BY EXHAUSTION, THE RESULT HOLDS

Question 12 (***)

Prove that if we subtract 1 from a positive odd square number, the answer is always divisible by 8.

[ ], [proof]



LET THE POSITIVE ODD SQUARE NUMBER BE $(2n+1)^2, \ n \in \mathbb{N}$

$$(2n+1)^2 - 1 = 4n^2 + 4n + 1 - 1 = 4n^2 + 4n = 4n(n+1)$$

BUT $n(n+1)$ REPRESENTS THE PRODUCT OF 2 CONSECUTIVE INTEGERS, SO IT MUST BE EVEN (ODD × EVEN = EVEN OR EVEN × ODD = EVEN)

$$= 4 \times 2m, \ m \in \mathbb{N} \qquad 2m = n(n+1)$$

$$= 8m$$

INDEED TRUE

Question 13 (***+)

Given that $k > 0$, use algebra to show that

$$\frac{k+1}{\sqrt{k}} \geq 2.$$

proof

CONSIDER THE EXPANSION OF $(\sqrt{k}-1)^2$

$$\Rightarrow (\sqrt{k}-1)^2 \geq 0$$
$$\Rightarrow (\sqrt{k})^2 - 2 \times 1 \times \sqrt{k} + 1^2 \geq 0$$
$$\Rightarrow k - 2\sqrt{k} + 1 \geq 0$$
$$\Rightarrow k + 1 \geq 2\sqrt{k}$$

AS $\sqrt{k} > 0$ WE MAY DIVIDE IT

$$\Rightarrow \frac{k+1}{\sqrt{k}} \geq 2$$

AS REQUIRED

ALTERNATIVE BY DIFFERENTIATION

FIRSTLY LET US NOTE THAT AS k GETS LARGER, THE WHOLE EXPRESSION GETS LARGER WITHOUT BOUND, SO ANY STATIONARY POINT WILL BE AN ABSOLUTE MINIMUM

$$\text{e.g. } \lim_{k \to \infty}\left(\frac{k+1}{\sqrt{k}}\right) = \lim_{k \to \infty}\left(\sqrt{k} + \frac{1}{\sqrt{k}}\right)$$

$$y = \frac{k+1}{\sqrt{k}} = \frac{k}{k^{\frac{1}{2}}} + \frac{1}{k^{\frac{1}{2}}} = k^{\frac{1}{2}} + k^{-\frac{1}{2}}$$
$$\frac{dy}{dk} = \frac{1}{2}k^{-\frac{1}{2}} - \frac{1}{2}k^{-\frac{3}{2}}$$

SOLVING FOR ZERO, TO LOOK FOR MINIMUM

$$0 = \frac{1}{2}k^{-\frac{1}{2}} - \frac{1}{2}k^{-\frac{3}{2}}$$
$$\Rightarrow \frac{1}{2}k^{-\frac{1}{2}} = \frac{1}{2}k^{-\frac{3}{2}}$$
$$\Rightarrow k^{-\frac{1}{2}} = k^{-\frac{3}{2}}$$
$$\Rightarrow \frac{1}{k^{\frac{1}{2}}} = \frac{1}{k^{3/2}}$$
$$\Rightarrow \frac{k^{3/2}}{k^{1/2}} = 1$$

AS $k > 0$, WE MAY DIVIDE

$$\Rightarrow k = 1$$

$$\therefore \left(\frac{k+1}{\sqrt{k}}\right)_{\text{MIN}} = \frac{1+1}{\sqrt{1}} = \frac{2}{1} = 2$$

AS REQUIRED

BEST METHOD IS PROOF BY CONTRADICTION

SUPPOSE THAT $\frac{k+1}{\sqrt{k}} < 2$

$$\Rightarrow \left(\frac{k+1}{\sqrt{k}}\right)^2 < 4$$
$$\Rightarrow \frac{(k+1)^2}{k} < 4$$
$$\Rightarrow (k+1)^2 < 4k \qquad (k>0)$$
$$\Rightarrow k^2 + 2k + 1 < 4k$$
$$\Rightarrow k^2 - 2k + 1 < 0$$
$$\Rightarrow (k-1)^2 < 0$$

WHICH IS A CONTRADICTION

$$\therefore \frac{k+1}{\sqrt{k}} \geq 2$$

Question 14 (***)

Prove by the method of **contradiction** that there are no integers n and m which satisfy the following equation.

$$3n + 21m = 137$$

proof

SUPPOSE THAT THERE EXIST INTEGERS m & n SO THAT

$$3n + 21m = 137$$

THEN WE HAVE

$$3(n + 7m) = 137$$
$$n + 7m = \frac{137}{3}$$
$$n + 7m = 45\tfrac{2}{3}$$

BUT n IS AN INTEGER AND $7m$ MUST ALSO BE AN INTEGER, SO $n + 7m$ HAS TO BE AN INTEGER & NOT $45\frac{2}{3}$

THIS IS A CONTRADICTION, SO THE ASSERTION $3n + 21m = 137$ CAN BE SATISFIED BY INTEGERS IS FALSE

Question 15 (***)

Use the method of **proof by contradiction** to show that if x then

$$\left| x + \frac{1}{x} \right| \geq 2 .$$

proof

NEED TO PROVE THAT FOR ALL REAL NUMBERS $\left|x+\frac{1}{x}\right| \geqslant 2$

SUPPOSE THE OPPOSITE

$$\left|x+\frac{1}{x}\right|<2$$

SQUARING BOTH SIDES

$$\left|x+\frac{1}{x}\right|^2 < 4$$
$$\left(x+\frac{1}{x}\right)^2 < 4$$
$$x^2+2+\frac{1}{x^2} < 4$$
$$x^2-2+\frac{1}{x^2} < 0$$
$$\left(x-\frac{1}{x}\right)^2 < 0$$

BUT THIS IS A CONTRADICTION AS NO REAL QUANTITY SQUARED CAN BE NEGATIVE, AND THEREFORE THE ORIGINAL ASSUMPTION IS FALSE

$$\therefore \left|x+\frac{1}{x}\right| \geqslant 2 \ //$$

Question 16 (***)

Prove that the sum of two even consecutive powers of 2 is always a multiple of 20.

proof

WORKING AS FOLLOWS

LET THE CONSECUTIVE EVEN POWERS OF 2 BE 2^{2n} & 2^{2n+2}

$$\Longrightarrow \quad 2^{2n}+2^{2n+2} = 2^{2n}+2^{2n}\times 2^2$$
$$= 2^{2n}+4\times 2^{2n}$$
$$= 5\times 2^{2n}$$
$$= 5\times (2^2)^n$$
$$= 5\times 4^n$$

NOW 4^n IS A MULTIPLE OF 4, AS A POWER OF 4; SAY $4^n = 4k$ FOR SOME POSITIVE INTEGER k

$$\ldots = 5\times 4k$$
$$= 20k$$

INDEED A MULTIPLE OF 20

Question 17 (***+)

Prove by the method of **contradiction** that there are no integers a and b which satisfy the following equation.

$$a^2 - 8b = 7$$

[illegible], proof

SUPPOSE THAT THERE EXIST INTEGERS a & b SO THAT

$$a^2 - 8b = 7$$

THEN WE HAVE

$$a^2 = 8b + 7$$

AS THE R.H.S IS ODD (MULTIPLE OF 8 + 7), IMPLIES THAT a^2 IS ALSO ODD, AND THEREFORE a MUST ALSO BE ODD

LET $a = 2n+1$ WHICH IS ODD FOR n BEING AN INTEGER

$$\Rightarrow (2n+1)^2 = 8b + 7$$
$$\Rightarrow 4n^2 + 4n + 1 = 8b + 7$$
$$\Rightarrow 4n^2 + 4n - 8b = 6$$
$$\Rightarrow 2n^2 + 2n - 4b = 3$$
$$\Rightarrow 2(n^2 + n - 2b) = 3$$
$$\Rightarrow n^2 + n - 2b = \frac{3}{2}$$

BUT THE LHS HAS TO BE AN INTEGER, WHILE THE R.H.S IS NOT

$\therefore$ THIS IS A CONTRADICTION TO THE ASSERTION THAT THERE EXIST INTEGERS a & b WHICH SATISFIES $a^2 - 8b = 7$

Question 18 (***+)

Use **proof by exhaustion** to show that if $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then

$$m^2 - n^2 \neq 102 .$$

[illegible], proof

ASSERTION: $m^2 - n^2 \neq 102$ IF $m \in \mathbb{N}, n \in \mathbb{N}$

PROOF BY EXHAUSTION

REWRITE THE LHS AS A DIFFERENCE OF SQUARES

$$f(m,n) = m^2 - n^2 = (m+n)(m-n)$$

SUPPOSE THAT

(I) BOTH m, n ARE EVEN $\Rightarrow$ $m+n$ AND $m-n$ WILL ALSO BE EVEN

$\Rightarrow \begin{pmatrix} m+n = 2\alpha \\ m-n = 2\beta \end{pmatrix}$ $\alpha, \beta \in \mathbb{N}$

$\Rightarrow f(m,n) = (2\alpha)(2\beta) = 4\alpha\beta$

$\Rightarrow$ $f(m,n)$ DIVIDES BY 4 BUT 102 DOES NOT

(II) BOTH m, n ARE ODD $\Rightarrow$ $m+n$ AND $m-n$ WILL BE

$\Rightarrow$ BY IDENTICAL ARGUMENT AS IN (I) THIS IS NOT POSSIBLE

(III) IF m IS ODD & n IS EVEN (OR THE OTHER ROUND), THEN BOTH $m+n$ AND $m-n$ WILL BE ODD

$\Rightarrow \begin{pmatrix} m+n = 2\lambda+1 \\ m-n = 2\mu+1 \end{pmatrix}$ $\lambda, \mu \in \mathbb{N}$

$\Rightarrow f(m,n) = (2\lambda+1)(2\mu+1)$

$\Rightarrow f(m,n) = 2\lambda + 2\mu + 4\lambda\mu + 1$

$\Rightarrow f(m,n) = 2[2\lambda\mu + \lambda + \mu] + 1$

$\Rightarrow$ $f(m,n)$ IS ODD BUT 102 IS NOT

HENCE WE EXHAUSTED ALL THE POSSIBILITIES AND ALL OF THE POSSIBLE SCENARIOS CANNOT PRODUCE 102

$\therefore$ $m^2 - n^2 \neq 102$ IF $m \in \mathbb{N}$ & $n \in \mathbb{N}$

Question 19 (***+)

Use a **calculus method** to prove that if $x \in \mathbb{R}$, $x > 0$, then

$$x^4 + x^{-4} \geq 2 .$$

proof

ASSERTION: IF $x \in \mathbb{R}, x > 0$ THEN $x^4 + x^{-4} \geqslant 2$

PROOF BY CALCULUS

- LET $f(x) = x^4 + x^{-4} = x^4 + \frac{1}{x^4}$, $x \in \mathbb{R}, x > 0$

AS $x \to +\infty$, $f(x) \to +\infty$ $\left(f(x) \sim x^4\right)$

AS $x \to 0^+$, $f(x) \to +\infty$ $\left(f(x) \sim \frac{1}{x^4}\right)$

- LOOK FOR STATIONARY VALUES

$\Rightarrow f'(x) = 4x^3 - 4x^{-5} = 4x^{-5}(x^8 - 1)$

SOLVING FOR ZERO

$\Rightarrow \frac{4}{x^5}(x^8 - 1) = 0$

$\Rightarrow x^8 - 1 = 0 \quad \left(\frac{4}{x^5} \neq 0\right)$

$\Rightarrow x = \pm 1$ ONLY REAL SOLUTIONS

$\Rightarrow x = 1$ ONLY POSITIVE REAL SOLUTION

$\therefore f(1) = 1^4 + 1^{-4} = 2$

- AS $f(x)$ TENDS TO INFINITY AS $x \to \infty$ OR $x \to 0^+$, THEN $(1,2)$ IS MORE THAN A LOCAL MINIMUM, IT A "PROPER" MINIMUM

$\Rightarrow f(x) \geqslant 2$ WHICH IMPLIES $x^4 + x^{-4} \geqslant 2$, $x \in \mathbb{R}, x > 0$

PROOF WITHOUT CALCULUS

START FROM THE FACT THAT ANY SQUARED EXPRESSION IS NON NEGATIVE

$\Rightarrow (x^4 - 1)^2 \geqslant 0$

$\Rightarrow x^8 - 2x^4 + 1 \geqslant 0$

$\Rightarrow x^8 + 1 \geqslant 2x^4$

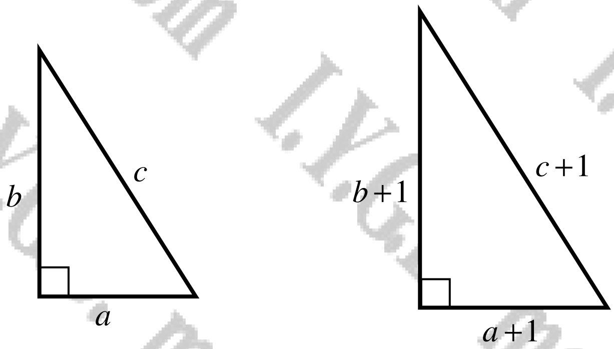
AS $x > 0$ WE MAY SAFELY DIVIDE THE INEQUALITY

$\Rightarrow \frac{x^8 + 1}{x^4} \geqslant 2$

$\Rightarrow x^4 + x^{-4} \geqslant 2$

AS REQUIRED

Question 20 **(***+)**



The figure above shows two right angled triangles.

- The triangle, on the left section of the figure, has side lengths of

$$a, \ b \text{ and } c,$$

  where c is the length of its hypotenuse.

- The triangle, on the right section of the figure, has side lengths of

$$a+1, \quad b+1 \quad \text{and} \quad c+1,$$

  where $c+1$ is the length of its hypotenuse.

Show that a, b and c cannot all be integers.

$\boxed{}$, $\boxed{\text{proof}}$

BY PYTHAGORAS ON THE TRIANGLE ON THE "LEFT"

$$\Rightarrow a^2 + b^2 = c^2$$
$$\Rightarrow a^2 + b^2 - c^2 = 0$$

BY PYTHAGORAS ON THE TRIANGLE ON THE "RIGHT"

$$\Rightarrow (a+1)^2 + (b+1)^2 = (c+1)^2$$
$$\Rightarrow a^2 + 2a + 1 + b^2 + 2b \cancel{+1} = c^2 + 2c \cancel{+1}$$
$$\Rightarrow (a^2 + b^2 - c^2) + 2a + 2b + 1 = 2c$$
$$\Rightarrow 0 + 2(a+b) + 1 = 2c$$
$$\Rightarrow 2(a+b) + 1 = 2c$$

L.H.S. WILL BE ODD IF a & b ARE BOTH INTEGERS

R.H.S WILL BE EVEN IF c IS AN INTEGER

HENCE NOT ALL OF a, b & c ARE INTEGERS

Question 21 (***+)

It is given that $x \in \mathbb{R}$ and $y \in \mathbb{R}$ such that $x + y = 1$.

Prove that

$$x^2 + y = y^2 + x .$$

$\boxed{\phantom{\text{proof}}}$, $\boxed{\text{proof}}$

METHOD A

DEFINE A FUNCTION f & MANIPULATE IT AS FOLLOWS

$$\Rightarrow f(x,y) = x^2 - y^2 + y - x$$
$$\Rightarrow f(x,y) = (x^2 - y^2) - (x - y)$$
$$\Rightarrow f(x,y) = (x-y)(x+y) - (x-y)$$
$$\Rightarrow f(x,y) = (x-y)(x+y-1)$$

BUT WE ARE GIVEN THAT $x+y=1$

$$\Rightarrow f(x,y) = (x-y)(1-1)$$
$$\Rightarrow f(x,y) = 0$$

FOR ALL x & y SUCH THAT $x+y=1$

$$\Rightarrow x^2 - y^2 + y - x = 0$$
$$\Rightarrow x^2 + y = y^2 + x$$

AS REQUIRED

METHOD B

FIRSTLY IF $x = y = \frac{1}{2}$

$$x^2 + y = \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{2} = \tfrac{1}{4} + \tfrac{1}{2} = \tfrac{3}{4}$$
$$y^2 + x = \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{2} = \tfrac{1}{4} + \tfrac{1}{2} = \tfrac{3}{4}$$

I.E THE RESULT HOLDS IF $x = y = \frac{1}{2}$

NEXT SUPPOSE $x \neq y$ SO $x - y \neq 0$

$$\Rightarrow x + y = 1$$
$$\Rightarrow (x-y)(x+y) = 1(x-y)$$
$$\Rightarrow x^2 - y^2 = x - y$$
$$\Rightarrow x^2 + y^2 = y^2 + x$$

AS REQUIRED FOR ALL x & y

Question 22 (***+)

It is given that a and b are positive odd integers, with $a > b$.

Use **proof by contradiction** to show that if $a+b$ is a multiple of 4, then $a-b$ **cannot** be a multiple of 4.

proof

ARGUE AS FOLLOWS — LET a & b BE ODD POSITIVE INTEGERS

$a+b$ IS A MULTIPLE OF 4, SO $a+b = 4m$, $m \in \mathbb{N}$

SUPPOSE NOW $a-b$ IS A MULTIPLE OF 4

$a-b = 4n$, $n \in \mathbb{N}$

ADDING THE EQUATIONS

$$\left.\begin{array}{l} a+b = 4m \\ a-b = 4n \end{array}\right\} \Rightarrow \begin{array}{l} 2a = 4(m+n) \\ a = 2(m+n) \end{array}$$

$\therefore$ a MUST BE EVEN

THIS IS A CONTRADICTION THAT a IS ODD

$\therefore$ $a-b$ CANNOT BE A MULTIPLE OF 4

Question 23 (***+)

Prove by **contradiction** that $\log_{10} 5$ is an irrational number.

proof

PROOF BY CONTRADICTION

SUPPOSE THAT $\log_{10} 5$ IS RATIONAL

$\Rightarrow \log_{10} 5 = \frac{a}{b}$ WHERE a & b ARE POSITIVE INTEGERS, WITHOUT COMMON FACTORS

$\Rightarrow 10^{\log_{10} 5} = 10^{\frac{a}{b}}$

$\Rightarrow 5 = 10^{\frac{a}{b}}$

$\Rightarrow (5)^b = \left(10^{\frac{a}{b}}\right)^b$

$\Rightarrow 5^b = 10^a$

BUT THIS IS A CONTRADICTION AS POWERS OF 5 ARE ODD $(5, 25, 125, 625, 3125 \ldots)$ AND THE POWERS OF 10 ARE EVEN $(10, 100, 1000, 10000, \ldots)$

$\Rightarrow$ THE ASSERTION THAT $\log_{10} 5$ IS RATIONAL IS FALSE

$\Rightarrow$ $\log_{10} 5$ IS IRRATIONAL

Question 24 (****)

Let $a \in \mathbb{N}$ with $\frac{1}{5}a \notin \mathbb{N}$.

a) Show that the remainder of the division of a^2 by 5 is either 1 or 4.

b) Given further that $b \in \mathbb{N}$ with $\frac{1}{5}b \notin \mathbb{N}$, deduce that $\frac{1}{5}\left(a^4 - b^4\right) \in \mathbb{N}$.

$\boxed{}$, $\boxed{\text{proof}}$

a) IF a IS NOT DIVISIBLE BY 5, THEN IT CAN ONLY BE OF THE FORMS

$a = 5n+1,\ a = 5n+2,\ a = 5n+3,\ a = 5n+4 \qquad n \in \mathbb{N}$

HENCE WE HAVE BY EXHAUSTION

$a^2 = (5n+1)^2 = 25n^2 + 10n + 1 = 5(5n^2+2n)+1 = \text{"}5k+1\text{"}$

$a^2 = (5n+2)^2 = 25n^2 + 20n + 4 = 5(5n^2+4n)+4 = \text{"}5k+4\text{"}$

$a^2 = (5n+3)^2 = 25n^2 + 30n + 9 = 5(5n^2+6n+1)+4 = \text{"}5k+4\text{"}$

$a^2 = (5n+4)^2 = 25n^2 + 40n + 16 = 5(5n^2+8n+3)+1 = \text{"}5k+1\text{"}$

$\therefore$ THE ONLY POSSIBLE REMAINDERS ARE EITHER 1 OR 4

b) AGAIN BY EXHAUSTION WE HAVE

- $a^2 = 5k+1$ OR $5k+4$
- $b^2 = 5l+1$ OR $5l+4$

$\Big\}\ k \in \mathbb{N},\ l \in \mathbb{N},\ k > l$

- $a^4 - b^4 = (5k+1)^2 - (5l+1)^2 = 25k^2 + 10k + 1 - 25l^2 - 10l - 1$
  $= 25k^2 - 25l^2 + 10k - 10l = 5(5k^2 - 5l^2 + 2k - 2l)$
- $a^4 - b^4 = (5k+1)^2 - (5l+4)^2 = 25k^2 + 10k + 1 - 25l^2 - 40l - 16$
  $= 25k^2 - 25l^2 + 10k - 40l - 15 = 5(5k^2 - 5l^2 + 2k - 8l - 3)$
- $a^4 - b^4 = (5k+4)^2 - (5l+1)^2 = 25k^2 + 40k + 16 - 25l^2 - 10l - 1$
  $= 25k^2 - 25l^2 + 40k - 10l + 15 = 5(5k^2 - 5l^2 + 8k - 2l + 3)$
- $a^4 - b^4 = (5k+4)^2 - (5l+4)^2 = 25k^2 + 40k + 16 - 25l^2 - 40l - 16$
  $= 25k^2 - 25l^2 + 40k - 40l = 5(5k^2 - 5l^2 + 8k + 8l)$

HENCE IF a AND b ARE NOT DIVISIBLE BY 5, THEN $a^4 - b^4$ WILL BE DIVISIBLE BY 5

Question 25 (****)

It is asserted that

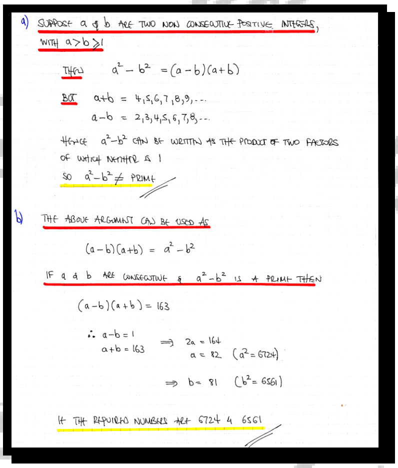
" The difference of the squares of two non consecutive positive integers can never be a prime number ".

a) Prove the validity of the above assertion.

The difference between two consecutive square numbers is 163.

b) Given further that 163 is a prime number find the above mentioned consecutive square numbers.

$\boxed{}$, $\boxed{6561,\ 6724}$



a) SUPPOSE a & b ARE TWO NON CONSECUTIVE POSITIVE INTEGERS, WITH $a > b \geqslant 1$

THEN $a^2 - b^2 = (a-b)(a+b)$

BUT $a+b = 4, 5, 6, 7, 8, 9, \ldots$

$a-b = 2, 3, 4, 5, 6, 7, 8, \ldots$

HENCE a^2-b^2 CAN BE WRITTEN AS THE PRODUCT OF TWO FACTORS OF WHICH NEITHER IS 1

SO $a^2-b^2 \neq$ PRIME

b) THE ABOVE ARGUMENT CAN BE USED AS

$(a-b)(a+b) = a^2-b^2$

IF a & b ARE CONSECUTIVE & a^2-b^2 IS A PRIME THEN

$(a-b)(a+b) = 163$

$\therefore \ a-b = 1$, $a+b = 163$ $\Rightarrow$ $2a = 164$, $a = 82$ $(a^2 = 6724)$

$\Rightarrow b = 81$ $(b^2 = 6561)$

∴ THE REQUIRED NUMBERS ARE 6724 & 6561

Question 26 (****)

By considering $\left(\sqrt{2}\right)^{\sqrt{2}}$, or otherwise, prove that an irrational number raised to the power of an irrational number **can be** a rational number.

$\boxed{}$, $\boxed{\text{proof}}$

CONSIDER $\sqrt{2}^{\sqrt{2}}$

NOW THERE ARE 2 CASES TO CONSIDER

- $\sqrt{2}^{\sqrt{2}}$ = RATIONAL

IF THIS IS TRUE THEN WE FOUND AN IRRATIONAL NUMBER WHICH WHEN RAISED TO THE POWER OF AN IRRATIONAL NUMBER GIVES RATIONAL ....

OR

- $\sqrt{2}^{\sqrt{2}}$ = IRRATIONAL

$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = (\text{IRRATIONAL})^{\sqrt{2}}$

$\sqrt{2}^{2} = (\text{IRRATIONAL})^{\sqrt{2}}$

$2 = (\text{IRRATIONAL})^{\sqrt{2}}$

AGAIN WE FOUND THAT AN IRRATIONAL NUMBER RAISED TO THE POWER OF AN IRRATIONAL NUMBER ($\sqrt{2}$) CAN GIVE A RATIONAL NUMBER

$\therefore$ AN IRRATIONAL NUMBER RAISED TO THE POWER OF AN IRRATIONAL NUMBER CAN PRODUCE A RATIONAL NUMBER

Question 27 (****)

It is given that

$$a^2 + b^2 = c^2, \quad a \in \mathbb{N}, \ b \in \mathbb{N}.$$

Show that a and b cannot both be odd.

$\boxed{}$, $\boxed{\text{proof}}$

$\boxed{a^2 + b^2 = c^2 \qquad a, b, c \in \mathbb{N}}$

- SUPPOSE THAT BOTH a & b ARE ODD

$a = 2m \pm 1$

$b = 2n \pm 1$

- THEN IN THE LHS WE OBTAIN

$\Rightarrow a^2 + b^2 = c^2$

$\Rightarrow (2m \pm 1)^2 + (2n \pm 1)^2 = c^2$

$\Rightarrow 4m^2 \pm 4m + 1 + 4n^2 \pm 4n + 1 = c^2$

$\Rightarrow 4(m^2 + n^2) \pm 4(m + n) + 2 = c^2$

$\Rightarrow 2\left[2(m^2 + n^2) \pm 2(m + n) + 1\right] = c^2$

- SO THE LHS IS EVEN $\Rightarrow c^2$ IS EVEN

$\Rightarrow c$ IS EVEN

$\Rightarrow c = 2p, \ p \in \mathbb{N}$

- SUBSTITUTE INTO THE EQUATION

$\Rightarrow 2\left[2(m^2 + n^2) \pm 2(m + n) + 1\right] = (2p)^2$

$\Rightarrow 2\left[2(m^2 + n^2) \pm 2(m + n) + 1\right] = 4p^2$

$\Rightarrow 2\left[m^2 + n^2\right] \pm 2(m + n) + 1 = 2p^2$

$\Rightarrow 2\left[(m^2 + n^2) \pm (m + n)\right] + 1 = 2p^2$

- WE FOUND THAT IF OUR ORIGINAL ASSUMPTION WAS VALID THAT AN ODD NUMBER (LHS) = EVEN NUMBER (RHS)

$\therefore$ BOTH CANNOT BE ODD

Question 28 (****)

Given that $k \in \mathbb{N}$, use algebra to prove that

$$\frac{2k+2}{2k+3} > \frac{2k}{2k+1}.$$

proof

DEFINE A FUNCTION AS

$$f(k) = \frac{2k+2}{2k+3} - \frac{2k}{2k+1} = \frac{(2k+2)(2k+1) - 2k(2k+3)}{(2k+1)(2k+3)}$$

$$= \frac{4k^2+6k+2-4k^2-6k}{(2k+1)(2k+3)} = \frac{2}{(2k+1)(2k+3)}$$

NOW AS $k \in \mathbb{N}$, $2k+1>0$

$$2k+3>0$$
$$(2k+1)(2k+3)>0$$
$$\frac{1}{(2k+1)(2k+3)}>0$$
$$\frac{2}{(2k+1)(2k+3)}>0$$
$$f(k)>0$$
$$\frac{2k+2}{2k+3} - \frac{2k}{2k+1}>0$$
$$\frac{2k+2}{2k+3} > \frac{2k}{2k+1} \quad //$$

ALTERNATIVE APPROACH

$$\frac{2k+2}{2k+3} = \frac{2k+3-1}{2k+3} = 1 - \frac{1}{2k+3}$$
$$\frac{2k}{2k+1} = \frac{2k+1-1}{2k+1} = 1 - \frac{1}{2k+1}$$

NOW PROCEED AS FOLLOWS

IF $k \in \mathbb{N}$ $\quad 2k+3 > 2k+1$

$$\frac{1}{2k+3} < \frac{1}{2k+1}$$
$$-\frac{1}{2k+3} > -\frac{1}{2k+1}$$
$$1-\frac{1}{2k+3} > 1-\frac{1}{2k+1}$$
$$\frac{2k+2}{2k+3} > \frac{2k}{2k+1} \quad //$$

AS BEFORE

Question 29 (****)

$$f(a) = a^3 + 5a, \quad a \in \mathbb{N}.$$

Without using proof by induction, show that $f(a)$ is a multiple of 6.

proof

PROCEED AS FOLLOWS

$$a^3 + 5a = a^3 - a + 6a$$
$$= a(a^2-1) + 6a$$
$$= a(a-1)(a+1) + 6a$$
$$= (a-1)a(a+1) + 6a$$

NOW $(a-1)a(a+1)$ REPRESENTS 3 CONSECUTIVE INTEGERS

- AT LEAST ONE OF THESE IS EVEN (DIVISIBLE BY 2)
- ONE OF THEM IS A MULTIPLE OF 3

HENCE THE EXPRESSION $(a-1)a(a+1)$ IS DIVISIBLE BY 6

FINALLY WE HAVE

$$a^3 + 5a = \dots (a-1)a(a+1) + 6a$$
$$= 6b + 6a \quad \text{, FOR SOME INTEGER } b$$
$$= 6(b+a)$$

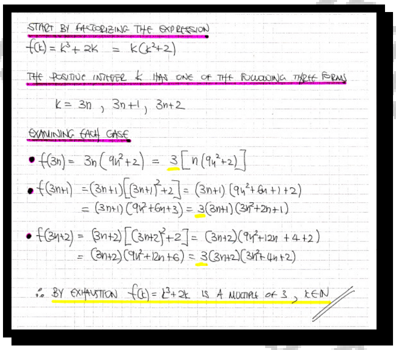
INDEED DIVISIBLE BY 6

Question 30 (****)

$$f(k) = k^3 + 2k, \quad k \in \mathbb{N}.$$

Without using proof by induction, show that $f(k)$ is always a multiple of 3.

$\boxed{}$, $\boxed{\text{proof}}$



START BY FACTORIZING THE EXPRESSION

$f(k) = k^3 + 2k = k(k^2+2)$

THE POSITIVE INTEGER k HAS ONE OF THE FOLLOWING THREE FORMS

$k = 3n, \ 3n+1, \ 3n+2$

EXAMINING EACH CASE

- $f(3n) = 3n(9n^2+2) = 3[n(9n^2+2)]$
- $f(3n+1) = (3n+1)[(3n+1)^2+2] = (3n+1)(9n^2+6n+1+2)$
  $= (3n+1)(9n^2+6n+3) = 3(3n+1)(3n^2+2n+1)$
- $f(3n+2) = (3n+2)[(3n+2)^2+2] = (3n+2)(9n^2+12n+4+2)$
  $= (3n+2)(9n^2+12n+6) = 3(3n+2)(3n^2+4n+2)$

$\therefore$ BY EXHAUSTION $f(k) = k^3+2k$ IS A MULTIPLE OF 3, $k \in \mathbb{N}$

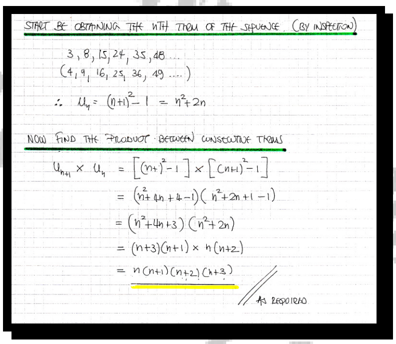
Question 31 (****)

Consider the following sequence

$$3, 8, 15, 24, 35, 48, \ldots$$

Prove that the product of any two consecutive terms of the above sequence can be written as the product of 4 consecutive integers.

$\boxed{}$, $\boxed{\text{proof}}$



START BE OBTAINING THE NTH TERM OF THE SEQUENCE (BY INSPECTION)

$3, 8, 15, 24, 35, 48 \ldots$
$(4, 9, 16, 25, 36, 49 \ldots)$

$\therefore \ u_n = (n+1)^2 - 1 = n^2 + 2n$

NOW FIND THE PRODUCT BETWEEN CONSECUTIVE TERMS

$u_{n+1} \times u_n = [(n+2)^2-1] \times [(n+1)^2-1]$
$= (n^2+4n+4-1)(n^2+2n+1-1)$
$= (n^2+4n+3)(n^2+2n)$
$= (n+3)(n+1) \times n(n+2)$
$= n(n+1)(n+2)(n+3)$

AS REQUIRED

Question 32 **(****)**

Prove that if 1 is added to the product of any 4 consecutive positive integers, the resulting number will always be a square number.

$\boxed{}$, $\boxed{\text{proof}}$

LET THE FOUR CONSECUTIVE POSITIVE INTEGERS ARE $n, n+1, n+2, n+3$

THEN WE HAVE

$$\sqrt{n(n+1)(n+2)(n+3)+1} = \sqrt{(n^2+n)(n^2+5n+6)+1}$$
$$= \sqrt{\begin{matrix} n^4+5n^3+6n^2 \\ \quad n^3+5n^2+6n+1 \end{matrix}}$$
$$= \sqrt{n^4+6n^3+11n^2+6n+1}$$

NOW THIS MUST BE A PERFECT SQUARE

$$n^4+6n^3+11n^2+6n+1 \equiv (n^2+An+1)^2$$
$$\equiv n^4+A^2n^2+1+2n^2+2An^3+2An$$
$$\equiv n^4+2An^3+(A^2+2)n^2+2An+1$$

$$\therefore A=3$$

$$\therefore \sqrt{n(n+1)(n+2)(n+3)+1} = \sqrt{(n^2+3n+1)^2} = n^2+3n+1$$

AS REQUIRED

NOTE MODS ARE NOT REALLY NEEDED HERE

Question 33 **(****+)**

Show that for all positive real numbers a and b

$$a^3+b^3 \geq a^2b+ab^2.$$

$\boxed{}$, $\boxed{\text{proof}}$

$\boxed{\text{ASSERTION} \quad a^3+b^3 \geqslant a^2b+ab^2, \quad a,b \text{ POSITIVE}}$

- DEFINE THE FUNCTION

$$\boxed{f(a,b) = a^3+b^3-a^2b-ab^2}$$

- USING THE SUM OF CUBES IDENTITY

$\{A^3+B^3 \equiv (A+B)(A^2-AB+B^2)\}$

$\Rightarrow f(a,b) = (a+b)(a^2-ab+b^2) - ab(a+b)$
$\Rightarrow f(a,b) = (a+b)[a^2-ab+b^2-ab]$
$\Rightarrow f(a,b) = (a+b)(a^2-2ab+b^2)$
$\Rightarrow f(a,b) = (a+b)(a-b)^2$

- AS $a,b>0$ & $(a-b)^2 \geqslant 0$, $f(a,b) \geqslant 0$

$\Rightarrow (a+b)(a-b)^2 \geqslant 0$
$\Rightarrow a^3+b^3-a^2b-ab^2 \geqslant 0$
$\Rightarrow a^3+b^3 \geqslant a^2b+ab^2$

ALTERNATIVE VARIATION SET AS A CONTRADICTION TYPE PROOF

SUPPOSE THAT $a^3+b^3 < a^2b+ab^2$ $\quad a,b$ POSITIVE

$\Rightarrow a^3+b^3-a^2b+ab^2<0$
$\Rightarrow (a+b)(a^2-ab+b^2) - ab(a+b)<0$
$\Rightarrow (a+b)(a^2-ab+b^2-ab)<0$
$\Rightarrow (a+b)(a^2-2ab+b^2)<0$
$\Rightarrow (a+b)(a-b)^2<0$

WHICH IS A CONTRADICTION AS $(a+b)>0$ $(a-b)^2 \geqslant 0$

$\therefore a^3+b^3 \geqslant a^2b+ab^2$

Question 34 (****+)

Show clearly that for all real numbers α, β and γ

$$\alpha^2 + \beta^2 + \gamma^2 \geq \alpha\beta + \beta\gamma + \gamma\alpha .$$

[ ], [proof]

• STARTING FROM $\boxed{(\alpha-\beta)^2 \geqslant 0}$

$\alpha^2 - 2\alpha\beta + \beta^2 \geqslant 0$

$$\left.\begin{aligned} &\alpha^2+\beta^2 \geqslant 2\alpha\beta \\ &\alpha^2+\gamma^2 \geqslant 2\alpha\gamma \\ &\beta^2+\gamma^2 \geqslant 2\beta\gamma \end{aligned}\right\} \text{ADDING THESE 3 INEQUALITIES}$$

• SIMILARLY :

$\Longrightarrow 2\alpha^2+2\beta^2+2\gamma^2 \geqslant 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha$

$\Longrightarrow \alpha^2+\beta^2+\gamma^2 \geqslant \alpha\beta + \beta\gamma + \gamma\alpha$ //

ALTERNATIVE BY THE AM-GM INEQUALITY

"AM $\geqslant$ GM"

$\frac{A+B}{2} \geqslant \sqrt[2]{AB}$

$\frac{A^2+2AB+B^2}{4} \geqslant AB$

$A^2+2AB+B^2 \geqslant 4AB$

$\boxed{A^2+B^2 \geqslant 2AB}$

HENCE $\left.\begin{aligned} &\alpha^2+\beta^2 \geqslant 2\alpha\beta \\ &\beta^2+\gamma^2 \geqslant 2\beta\gamma \\ &\gamma^2+\alpha^2 \geqslant 2\alpha\gamma \end{aligned}\right\} \Longrightarrow$ ADDING $2(\alpha^2+\beta^2+\gamma^2) \geqslant 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$\alpha^2+\beta^2+\gamma^2 \geqslant \alpha\beta+\beta\gamma+\gamma\alpha$ //

(NOTE THE TECHNICALITY THAT IN THE AM-GM INEQUALITY $AB \geqslant 0$)

Question 35 (****+)

Show, without using proof by induction, that the sum of cubes of any 3 consecutive positive integers is a multiple of 9.

[ ], [proof]

LET THE THREE CONSECUTIVE INTEGERS BE

$k-1, \; k, \; k+1$

CUBING & ADDING GIVES

$f(k) = (k-1)^3 + k^3 + (k+1)^3$

$= (k^3 \cancel{-3k^2} + 3k \cancel{-1}) + k^3 + (k^3 \cancel{+3k^2} + 3k \cancel{+1})$

$= 3k^3 + 6k$

$= 3k(k^2+2)$

NOW k CAN TAKE ONE OF THE FOLLOWING 3 FORMS

$k = 3n, \; 3n+1, \; 3n+2$

EXAMINING EACH CASE

- $f(k) = f(3n) = 3(3n)[(3n)^2+2] = 9n(9n^2+2)$
- $f(k) = f(3n+1) = 3(3n+1)[(3n+1)^2+2] = 3(3n+1)(9n^2+6n+1+2)$
  $= 3(3n+1)(9n^2+6n+3) = 9(3n+1)(3n^2+2n+1)$
- $f(k) = f(3n+2) = 3(3n+2)[(3n+2)^2+2] = 3(3n+2)[9n^2+12n+4+2]$
  $= 3(3n+2)(9n^2+12n+6) = 9(3n+2)(3n^2+4n+2)$

$\therefore$ THE SUM OF CUBES OF ANY 3 CONSECUTIVE POSITIVE INTEGERS WILL BE A MULTIPLE OF 9 //

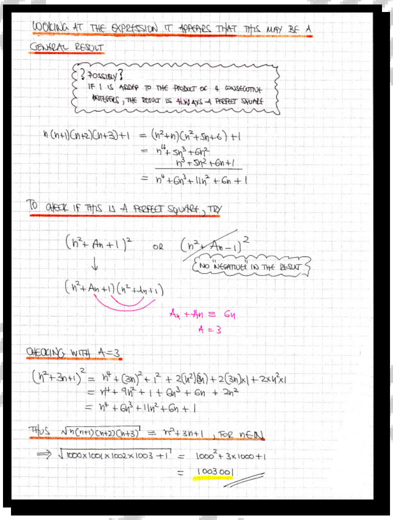
Question 36 (****+)

Use a **detailed method** to show that

$$\sqrt{1000 \times 1001 \times 1002 \times 1003 + 1} = 1003001$$

*You may **NOT** use a calculating aid in this question.*

$\boxed{\text{proof}}$



LOOKING AT THE EXPRESSION IT APPEARS THAT THIS MAY BE A GENERAL RESULT

? POSSIBLY ?
IF 1 IS ADDED TO THE PRODUCT OF 4 CONSECUTIVE INTEGERS, THE RESULT IS ALWAYS A PERFECT SQUARE

$$\begin{aligned} n(n+1)(n+2)(n+3)+1 &= (n^2+n)(n^2+5n+6)+1 \\ &= n^4+5n^3+6n^2 \\ &\quad\;\; n^3+5n^2+6n+1 \\ &= n^4+6n^3+11n^2+6n+1 \end{aligned}$$

TO CHECK IF THIS IS A PERFECT SQUARE, TRY

$(n^2+An+1)^2$ OR $\cancel{(n^2+An-1)^2}$

NO "NEGATIVES" IN THE RESULT

$$\downarrow$$

$(n^2+An+1)(n^2+An+1)$

$An + An = 6n$

$A = 3$

CHECKING WITH $A=3$

$$\begin{aligned} (n^2+3n+1)^2 &= n^4+(3n)^2+1^2+2(n^2)(3n)+2(3n)\times1+2\times n^2\times1 \\ &= n^4+9n^2+1+6n^3+6n+2n^2 \\ &= n^4+6n^3+11n^2+6n+1 \end{aligned}$$

THUS $\sqrt{n(n+1)(n+2)(n+3)} \equiv n^2+3n+1$, FOR $n \in \mathbb{N}$

$$\Rightarrow \sqrt{1000\times1001\times1002\times1003+1} = 1000^2+3\times1000+1$$
$$= 1003001$$

Question 37 (*****)

Show that the square of an odd positive integer greater than 1 is of the form

$$8T+1,$$

where T is a triangular number.

proof

ASSERTION: THE SQUARE OF AN ODD POSITIVE INTEGER IS ALWAYS OF THE FORM $8T+1$, WHERE "T" IS A TRIANGLE NUMBER

PROOF BY EXHAUSTION

- LET n BE ODD

$\Rightarrow n = 2m+1$

$\Rightarrow 2n+1 = 2(2m+1)+1$

$\Rightarrow 2n+1 = 4m+3$

e.g. $7 = (4\times1)+3$
$35 = (4\times8)+3$
$47 = (4\times11)+3$
ETC

- LET n BE EVEN

$\Rightarrow n = 2m$

$\Rightarrow 2n+1 = 2(2m)+1$

$\Rightarrow 2n+1 = 4m+1$

e.g. $5 = (4\times1)+1$
$21 = (4\times5)+1$
$33 = (4\times8)+1$
ETC

- SQUARING THE ODD NUMBER IN EACH CASE YIELDS

$(2n+1)^2 = (4m+3)^2$
$= 16m^2+24m+9$
$= 8(2m^2+3m+1)+1$
$= 8(2m+1)(m+1)+1$

$(2n+1)^2 = (4m+1)^2$
$= 16m^2+8m+1$
$= 8m(2m+1)+1$

I.E IN BOTH CASES THE OUTCOME IS OF THE FORM $8f(m)+1$

NOW TO PROVE THAT $f(m)$ IS A TRIANGLE NUMBER

- TRIANGLE NUMBERS ARE $1, 3, 6, 10, 15, 21, 28, 36, \ldots$

- $1,\ 6,\ 15,\ 28$ (first differences $5,\ 9,\ 13$; second differences $4,\ 4$)

$u_n = 2n^2+an+b$

$2n^2$: $2, 8, 18, 32, \ldots$
HERE: $1, 6, 15, 28$
$-1, -2, -3, -4$

$\therefore u_n = 2n^2-n$

$\Rightarrow u_n = n(2n-1)$

$n \mapsto m+1$

$\Rightarrow u_m = (m+1)[2(m+1)-1]$

$\Rightarrow u_m = (m+1)(2m+1)$

WHICH WE OBTAINED

- $3,\ 10,\ 21,\ 36$ (first differences $7,\ 11,\ 15$; second differences $4,\ 4$)

$u_n = 2n^2+\alpha n+\beta$

$2n^2$: $2, 8, 18, 28$
HERE: $3, 10, 21, 36$
$+1, +2, +3, +4$

$\therefore u_n = 2n^2+n$

$\Rightarrow u_n = n(2n+1)$

OR

$\Rightarrow u_m = m(2m+1)$

WHICH WE OBTAINED

EVERY SQUARE OF AN ODD NATURAL NUMBER GREATER THAN 3, IS OF THE FORM $8T+1$, WHERE T IS A TRIANGLE NUMBER

Question 38 (*****)

It is given that

$$f(m,n) \equiv 2m\left(m^2 + 3n^2\right),$$

where m and n are distinct positive integers, with $m > n$.

By using the expansion of $(A \pm B)^3$, prove that $f(m,n)$ can always be written as the sum of two cubes.

$\boxed{}$, $\boxed{\text{proof}}$

STARTING WITH THE IDENTITY SUGGESTED, WRITTEN IN m & n

$$(m+n)^3 = m^3 + 3m^2n + 3mn^2 + n^3$$
$$(m-n)^3 = m^3 - 3m^2n + 3mn^2 - n^3$$

$$(m+n)^3 + (m-n)^3 = 2m^3 + 6mn^2 \quad \leftarrow \text{ADDING}$$

HENCE WE HAVE

$$\Rightarrow 2m^3 + 6mn^2 = (m+n)^3 + (m-n)^3$$

$$\Rightarrow 2m\left(m^2 + 3n^2\right) = (m+n)^3 + (m-n)^3$$

BOTH CUBE NUMBERS SUMMED

Question 39 (*****)

It is given that

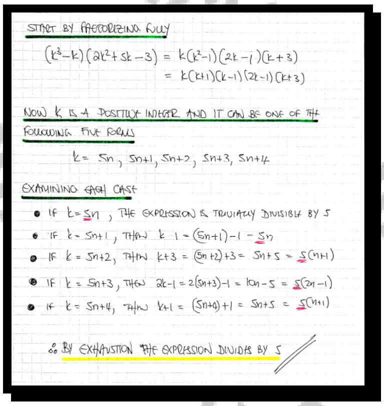
$$f(k) \equiv \left(k^3 - k\right)\left(2k^2 + 5k - 3\right),$$

where k is a positive integer.

Prove that $f(k)$ is divisible by 5.

*You may **not** use proof by induction in this question.*

$\boxed{}$, $\boxed{\text{proof}}$



START BY FACTORIZING FULLY

$$(k^3-k)(2k^2+5k-3) = k(k^2-1)(2k-1)(k+3)$$
$$= k(k+1)(k-1)(2k-1)(k+3)$$

NOW k IS A POSITIVE INTEGER AND IT CAN BE ONE OF THE FOLLOWING FIVE FORMS

$$k = 5n,\ 5n+1,\ 5n+2,\ 5n+3,\ 5n+4$$

EXAMINING EACH CASE

- IF $k = 5n$, THE EXPRESSION IS TRIVIALLY DIVISIBLE BY 5
- IF $k = 5n+1$, THEN $k-1 = (5n+1)-1 = 5n$
- IF $k = 5n+2$, THEN $k+3 = (5n+2)+3 = 5n+5 = 5(n+1)$
- IF $k = 5n+3$, THEN $2k-1 = 2(5n+3)-1 = 10n-5 = 5(2n-1)$
- IF $k = 5n+4$, THEN $k+1 = (5n+4)+1 = 5n+5 = 5(n+1)$

$\therefore$ BY EXHAUSTION THE EXPRESSION DIVIDES BY 5

Question 40 (*****)

Prove that for all real numbers, a and b,

$$\sqrt{a^2+b^2} \le \frac{\sqrt{4a^2+b^2} + \sqrt{a^2+4b^2}}{3}.$$

SPZ M, proof

USING VECTOR TECHNIQUES

LET $\underline{p} = 2a\underline{i} + b\underline{j}$, $\underline{q} = a\underline{i} + 2b\underline{j}$ $\implies \underline{p}+\underline{q} = 3a\underline{i} + 3b\underline{j}$

BY THE TRIANGLE INEQUALITY

$\implies |\underline{p}+\underline{q}| \le |\underline{p}| + |\underline{q}|$

$\implies |3a\underline{i}+3b\underline{j}| \le |2a\underline{i}+b\underline{j}| + |a\underline{i}+2b\underline{j}|$

$\implies \sqrt{9a^2+9b^2} \le \sqrt{4a^2+b^2} + \sqrt{a^2+4b^2}$

$\implies 3\sqrt{a^2+b^2} \le \sqrt{4a^2+b^2} + \sqrt{a^2+4b^2}$

$\implies \sqrt{a^2+b^2} \le \dfrac{\sqrt{4a^2+b^2} + \sqrt{a^2+4b^2}}{3}$ // AS REQUIRED

Question 41 (*****)

Show that for all positive real numbers a and b

$$a^3 + 2b^3 \geq 3ab^2 .$$

proof

METHOD A

ASSERTION $a^3+2b^3 \geqslant 3ab^2$ FOR ALL POSITIVE a & b

- CONSIDER THE FUNCTION

$$f(a,b) = a^3 + 2b^3 - 3ab^2$$

IT WOULD SUFFICE TO SHOW THAT $f(a,b) \geqslant 0$ FOR ALL POSITIVE a & b

- BY INSPECTION $(a-b)$ IS A FACTOR, SINCE

$$f(a,a) = a^3 + 2a^3 - 3a^3 = 0$$

- FACTORIZE BY INSPECTION

$$f(a,b) = (a-b)(a^2 + kab - 2b^2)$$

$-a^2b$

ka^2b

$$ka^2b - a^2b = 0$$
$$k - 1 = 0$$
$$k = 1$$

$$f(a,b) = (a-b)(a^2+ab-2b^2)$$
$$f(a,b) = (a-b)(a+2b)(a-b)$$
$$f(a,b) = (a-b)^2(a+2b)$$

- AS $a,b>0$ $\quad a+2b>0$

$$(a-b)^2 \geqslant 0$$

THUS $(a-b)^2(a+2b) \geqslant 0$

$$a^3+2b^3-3ab \geqslant 0$$
$$a^3+2b^3 \geqslant 3ab$$

AS REQUIRED

METHOD B

BY THE AM–GM INEQUALITY

$$\sum_{i=1}^{n} \frac{a_i}{n} \geqslant \prod_{i=1}^{n} (a_i)^{\frac{1}{n}}$$

- APPLYING THE AM–GM INEQUALITY WITH 3 QUANTITIES

$$\frac{A+B+C}{3} \geqslant \sqrt[3]{A \times B \times C}$$

- LET $A = a^3$, $B = b^3$, $C = b^3$

- THEN

$$\frac{a^3+b^3+b^3}{3} \geqslant \sqrt[3]{a^3 \times b^3 \times b^3}$$
$$\frac{a^3+2b^3}{3} \geqslant (a^3b^6)^{\frac{1}{3}}$$
$$\frac{a^3+2b^3}{3} \geqslant ab^2$$
$$a^3+2b^3 \geqslant 3ab^2$$

Question 42 (*****)

It is given that x, a and b are positive real numbers, with $a > b$ and $x^2 > ab$.

Use proof by contradiction to show that

$$\frac{x+a}{\sqrt{x^2+a^2}} - \frac{x+b}{\sqrt{x^2+b^2}} > 0.$$

$\boxed{}$, $\boxed{\text{proof}}$

ASSERTION $\quad \dfrac{x+a}{\sqrt{x^2+a^2}} - \dfrac{x+b}{\sqrt{x^2+b^2}} > 0 \quad$ $x, a, b > 0$; $a > b$; $x^2 > ab$

SUPPOSE THAT

$$\frac{x+a}{\sqrt{x^2+a^2}} - \frac{x+b}{\sqrt{x^2+b^2}} \leqslant 0$$

$$\Rightarrow \frac{x+a}{\sqrt{x^2+a^2}} \leqslant \frac{x+b}{\sqrt{x^2+b^2}}$$

AS BOTH SIDES ARE POSITIVE WE MAY SQUARE THE INEQUALITY

$$\Rightarrow \frac{x^2+2ax+a^2}{x^2+a^2} \leqslant \frac{x^2+2bx+b^2}{x^2+b^2}$$

$$\Rightarrow 1 + \frac{2ax}{x^2+a^2} \leqslant 1 + \frac{2bx}{x^2+b^2}$$

$$\Rightarrow \frac{2ax}{x^2+a^2} \leqslant \frac{2bx}{x^2+b^2}$$

AS $2x > 0$ WE MAY ALSO DIVIDE IT

$$\Rightarrow \frac{a}{x^2+a^2} \leqslant \frac{b}{x^2+b^2}$$

AS THE DENOMINATORS ARE POSITIVE WE MAY MULTIPLY ACROSS

$$\Rightarrow a(x^2+b^2) \leqslant b(x^2+a^2)$$

$$\Rightarrow ax^2 + ab^2 \leqslant bx^2 + ba^2$$

$$\Rightarrow ax^2 - bx^2 + ab^2 - a^2b \leqslant 0$$

$$\Rightarrow x^2(a-b) - ab(a-b) \leqslant 0$$

$$\Rightarrow (a-b)(x^2-ab) \leqslant 0$$

- BUT $a > b \Rightarrow a-b > 0$
- ALSO $x^2 > ab \Rightarrow x^2 - ab > 0$

$\Big\} \Rightarrow (a-b)(x^2-ab) > 0$

HENCE BY CONTRADICTION

$$\frac{x+a}{\sqrt{x^2+a^2}} - \frac{x+b}{\sqrt{x^2+b^2}} > 0$$

Question 43 (*****)

Prove that the sum of the squares of two distinct positive integers, when doubled, it can be written as the sum of two distinct square numbers

proof

AS THIS MAY NOT BE OBVIOUS WHERE TO START WE LOOK FOR THE PROOF BY LOOKING DIRECTLY AT THE NUMBER PATTERNS

$2(1^2+2^2) = 10 = 1^2+3^2$
$2(1^2+3^2) = 20 = 2^2+4^2$
$2(1^2+4^2) = 34 = 3^2+5^2$
$2(1^2+5^2) = 52 = 4^2+6^2$

$2(2^2+3^2) = 26 = 1^2+5^2$
$2(2^2+4^2) = 40 = 2^2+6^2$
$2(2^2+5^2) = 58 = 3^2+7^2$
$2(2^2+6^2) = 80 = 4^2+8^2$

$2(3^2+4^2) = 50 = 1^2+7^2$
$2(3^2+5^2) = 68 = 2^2+8^2$
$2(3^2+6^2) = 90 = 3^2+9^2$
$2(3^2+7^2) = 116 = 4^2+10^2$

WE MAY CARRY ON A BIT MORE, IF NOT EVIDENT WHAT/WHY BUT HERE IS THE ALGEBRAIC PROOF, FOR $n \in \mathbb{N}, m \in \mathbb{N}$, $n \neq m$

$$2(n^2+m^2) = 2n^2+2m^2 = n^2+m^2+n^2+m^2$$
$$= (n^2+2nm+m^2)+(n^2-2nm+m^2)$$
$$= (n+m)^2+(n-m)^2$$

Question 44 (*****)

The *Rational Zero Theorem* asserts that if the polynomial

$$f(x) \equiv a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$$

has integer coefficients, then **every rational zero** of $f(x)$ has the form pq, where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

Use this result to show that $\sin\left(\frac{\pi}{18}\right)$ is irrational.

$\boxed{}$, $\boxed{\text{proof}}$

SIN $\frac{\pi}{18}$ = SIN10° SO WE CAN RELATE IT TO SIN $\frac{\pi}{6}$ VIA THE TRIPLE ANGLE FORMULA — EASIER TO USE TRIG THAN COMPLEX NUMBERS

$$\begin{aligned}\sin 3\theta = \sin(2\theta+\theta) &= \sin 2\theta\cos\theta + \cos 2\theta \sin\theta \\ &= (2\sin\theta\cos\theta)\cos\theta + (1-2\sin^2\theta)\sin\theta \\ &= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin\theta \\ &= 2\sin\theta(1-\sin\theta) + \sin\theta - 2\sin^3\theta \\ &= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

THUS WE NOW HAVE

$\Rightarrow \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow 4\sin^3\theta - 3\sin\theta + \sin 3\theta = 0$

LET $\theta = \frac{\pi}{18}$

$\Rightarrow 4\sin^3\frac{\pi}{18} - 3\sin\frac{\pi}{18} + \sin\frac{\pi}{6} = 0$

$\Rightarrow 4\sin^3\frac{\pi}{18} - 3\sin\frac{\pi}{18} + \frac{1}{2} = 0$

$\Rightarrow 8\sin^3\frac{\pi}{18} - 6\sin\frac{\pi}{18} + 1 = 0$

$\Rightarrow 8x^3 - 6x + 1 = 0$ WHERE $\sin\frac{\pi}{18} = x$

THE POSSIBLE RATIONAL ZEROS OF THIS POLYNOMIAL WITH INTEGER COEFFICIENTS ARE

± 1 OR $\pm\frac{1}{8}, \pm\frac{1}{4} \pm\frac{1}{2}$

$\left\{\pm\frac{1}{1}\right\}$

$f(x) = 8x^3 - 6x + 1$

$f(1) = 8 - 6 + 1 \neq 0$

$f(-1) = -8 + 6 + 1 \neq 0$

$f(\frac{1}{2}) = 1 - 3 + 1 \neq 0$

$f(-\frac{1}{2}) = -1 + 3 + 1 \neq 0$

$f(\frac{1}{4}) = \frac{1}{8} - \frac{3}{2} + 1 \neq 0$

$f(-\frac{1}{4}) = -\frac{1}{8} + \frac{3}{2} + 1 \neq 0$

$f(\frac{1}{8}) = \frac{1}{64} - \frac{3}{4} + 1 \neq 0$

$f(-\frac{1}{8}) = -\frac{1}{64} + \frac{3}{4} + 1 \neq 0$

$\therefore$ NO RATIONAL ZEROS

$\therefore$ x IS NOT RATIONAL

$\therefore$ $\sin\frac{\pi}{18}$ IS NOT RATIONAL

$\therefore$ $\sin\frac{\pi}{18}$ IS IRRATIONAL

Question 45 (*****)

By using the definition of e as an infinite convergent series, prove **by contradiction** that e is irrational.

proof

SUPPOSE THAT e IS RATIONAL, I.E $\frac{p}{q}=e$, WHERE p & q ARE POSITIVE INTEGERS, $q>1$

THEN WE HAVE

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots+\frac{1}{q!}+\frac{1}{(q+1)!}+\dots$$

$$\frac{p}{q} = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots+\frac{1}{q!}+\frac{1}{(q+1)!}+\frac{1}{(q+2)!}+\frac{1}{(q+3)!}+\dots$$

$$\frac{p}{q}-1-\frac{1}{1!}-\frac{1}{2!}-\frac{1}{3!}-\frac{1}{4!}-\dots-\frac{1}{q!} = \frac{1}{(q+1)!}+\frac{1}{(q+2)!}+\frac{1}{(q+3)!}+\dots$$

MULTIPLY BOTH SIDES BY $q!$

$$q!\left[\frac{p}{q}-1-\frac{1}{1!}-\frac{1}{2!}-\frac{1}{3!}-\frac{1}{4!}-\dots\frac{1}{q!}\right] = q!\left[\frac{1}{(q+1)!}+\frac{1}{(q+2)!}+\frac{1}{(q+3)!}+\dots\right]$$

$$\underbrace{p(q-1)!-q!-\frac{q!}{1!}-\frac{q!}{2!}-\frac{q!}{3!}-\frac{q!}{4!}-\dots 1}_{\text{INTEGER}} = \underbrace{\frac{1}{q+1}+\frac{1}{(q+1)(q+2)}+\frac{1}{(q+1)(q+2)(q+3)}+\dots}_{\text{POSITIVE}}$$

THUS SO FAR WE KNOW THAT BOTH SIDES ARE A POSITIVE INTEGER

BUT ON THE RHS WE HAVE

$$\frac{1}{q+1}+\frac{1}{(q+1)(q+2)}+\frac{1}{(q+1)(q+2)(q+3)}+\frac{1}{(q+1)(q+2)(q+3)(q+4)}+\dots$$

$$< \frac{1}{q+1}+\frac{1}{(q+2)(q+1)}+\frac{1}{(q+3)(q+2)}+\frac{1}{(q+4)(q+3)}+\dots$$

$$= \frac{1}{q+1}+\left(\frac{1}{q+1}-\cancel{\frac{1}{q+2}}\right)+\left(\cancel{\frac{1}{q+2}}-\cancel{\frac{1}{q+3}}\right)+\left(\cancel{\frac{1}{q+3}}-\cancel{\frac{1}{q+4}}\right)+\dots$$

$$= \frac{2}{q+1} < 1$$

AS q IS A POSITIVE INTEGER GREATER THAN 1

SO THE "LHS" = INTEGER & RHS = POSITIVE BUT LESS THAN 1

THIS IS A CONTRADICTION, SO e IS IRRATIONAL