RATIONAL FUNCTIONS

Question 1 (***+)

The curves C_1 and C_2 have respective equations

$$y = \frac{1}{x}, \ x \in \mathbb{R}, \ x \neq 0, \qquad \text{and} \qquad y = \frac{x}{x+2}, \ x \in \mathbb{R}, \ x \neq -2.$$

a) Sketch in the same set of axes the graph of C_1 and the graph of C_2.

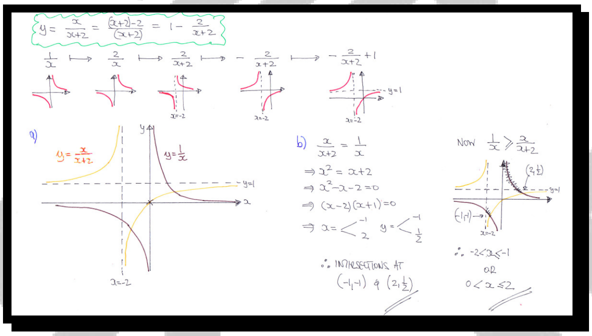
Indicate clearly in the sketch …

- … the equations of any asymptotes.
- … the coordinates of any intersections between each curve with the coordinate axes.

b) Find the coordinates of the points of intersection between the graph of C_1 and C_2, and hence solve the inequality

$$\frac{1}{x} \geq \frac{x}{x+2}.$$

$$\boxed{(-1,-1), \left(2,\frac{1}{2}\right)}, \ \boxed{-2 < x \leq -1 \ \cup \ 0 < x \leq 2}$$



Question 2 (****)

The curve C has equation $y = f(x)$ given by

$$f(x) = \frac{x^2}{(x-1)(x-5)}, \quad x \in \mathbb{R}, \quad x \neq 1, \; x \neq 5.$$

a) State the equation of each of the two vertical asymptotes of the C.

b) Show clearly that $y = 1$ is a horizontal asymptote to the curve.

c) Solve the equation $f(x) = 1$.

The straight line with equation $y = k$, where k is a constant, touches C at two points.

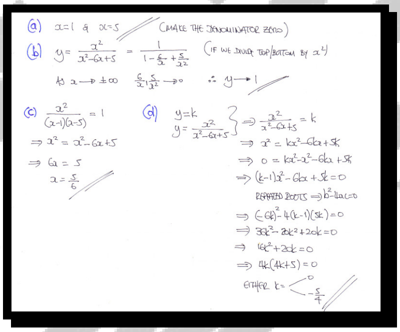
d) Find possible values the of k.

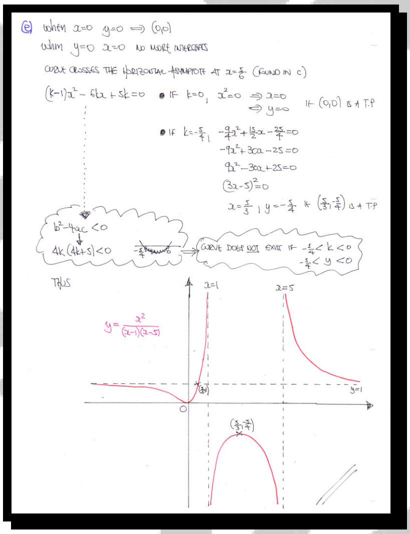
e) Sketch the graph of C.

Indicate clearly in the sketch

- the equations of the asymptotes
- the coordinates of any intersections of C with the coordinate axes.
- the coordinates of any turning points of C.

$\boxed{x = 1, \; 5}$, $\boxed{x = \frac{5}{6}}$, $\boxed{k = -\frac{5}{4}, \; 0}$





Question 3 (****)

The curve C has equation $y = f(x)$ given by

$$f(x) = \frac{x-4}{(x-5)(x-8)}, \quad x \in \mathbb{R}, \quad x \neq 5, \quad x \neq 8.$$

a) State the equation of each of the two vertical asymptotes of the C.

b) Show clearly that $y = 0$ is a horizontal asymptote to the curve.

The straight line with equation $y = k$, where k is a constant, touches C at two points.

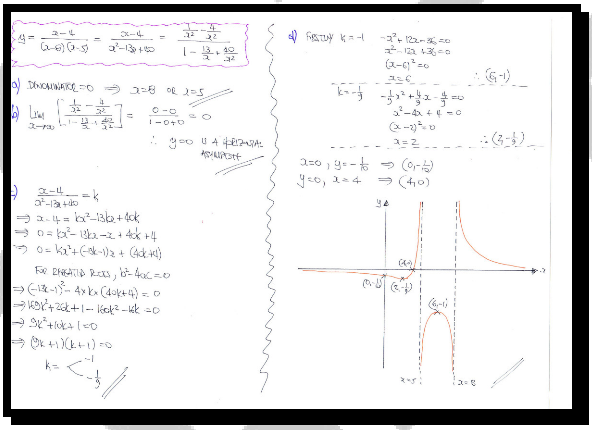
c) Find possible values the of k.

d) Sketch the graph of C.

Indicate clearly in the sketch

- the equations of the asymptotes
- the coordinates of any intersections of C with the coordinate axes.
- the coordinates of any turning points of C.

$\boxed{x = 5,\ 8}$, $\boxed{k = -1,\ -\frac{1}{9}}$



Question 4 (****)

The curves C_1 and C_2 have respective equations

$$y = \frac{x+3}{x}, \ x \in \mathbb{R}, \ x \neq 0, \qquad \text{and} \qquad y = \frac{x}{2-x}, \ x \in \mathbb{R}, \ x \neq 2.$$

a) Sketch in the same set of axes the graph of C_1 and the graph of C_2.

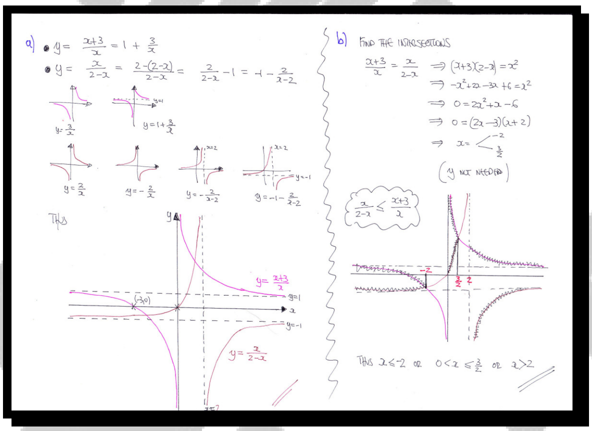
Indicate clearly in the sketch …

- … the equations of any asymptotes.
- … the coordinates of any intersections between each curve with the coordinate axes.

b) Use the sketch of part (a) to solve the inequality

$$\frac{x}{2-x} \leq \frac{x+3}{x}.$$

$$\boxed{x \leq -2 \ \cup \ 0 < x \leq \frac{3}{2} \ \cup \ x > 2}$$



Question 5 (****+)

A curve C has equation

$$h(x) = \frac{f(x)}{g(x)}, \; x \in \mathbb{R}, \; g(x) \neq 0.$$

It is further given that $f(x)$ is a quadratic function and $g(x)$ a linear function.

The asymptotes of C have equations $x = -1$ and $y = x + 2$, and its graph is passing through the point $P(1,5)$.

a) Determine a simplified Cartesian equation for C.

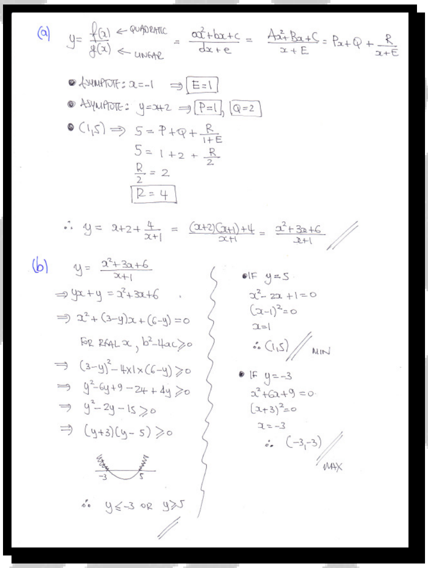
b) Use a discriminant method to find the range of $h(x)$ and hence calculate the coordinates of the stationary points of C.

c) Sketch in separate diagrams, showing all the relevant details including asymptotic behaviour, the graph of …

i. … $y = h(x)$.

ii. … $y^2 = h(x)$.

$$\boxed{h(x) = \frac{x^2+3x+6}{x+1} = x+2+\frac{4}{x+1}}, \; \boxed{h(x) \le -3 \cup h(x) \ge 5}, \; \boxed{(1,5), \; (-3,-3)}$$



(a) $y = \frac{f(x)}{g(x)}$ ← QUADRATIC, ← LINEAR $= \frac{ax^2+bx+c}{dx+e} = \frac{Ax^2+Bx+C}{x+E} = Px+Q+\frac{R}{x+E}$

- ASYMPTOTE: $x=-1 \Rightarrow \boxed{E=1}$
- ASYMPTOTE: $y=x+2 \Rightarrow \boxed{P=1}, \boxed{Q=2}$
- $(1,5) \Rightarrow 5 = P+Q+\frac{R}{1+E}$

$5 = 1+2+\frac{R}{2}$

$\frac{R}{2} = 2$

$\boxed{R=4}$

$\therefore y = x+2+\frac{4}{x+1} = \frac{(x+2)(x+1)+4}{x+1} = \frac{x^2+3x+6}{x+1}$

(b) $y = \frac{x^2+3x+6}{x+1}$

$\Rightarrow yx+y = x^2+3x+6$

$\Rightarrow x^2+(3-y)x+(6-y)=0$

FOR REAL x, $b^2-4ac \geqslant 0$

$\Rightarrow (3-y)^2-4\times1\times(6-y) \geqslant 0$

$\Rightarrow y^2-6y+9-24+4y \geqslant 0$

$\Rightarrow y^2-2y-15 \geqslant 0$

$\Rightarrow (y+3)(y-5) \geqslant 0$

-3, 5

$\therefore y \leqslant -3$ OR $y \geqslant 5$

- IF $y=5$

$x^2-2x+1=0$

$(x-1)^2=0$

$x=1$

$\therefore (1,5)$ MIN

- IF $y=-3$

$x^2+6x+9=0$

$(x+3)^2=0$

$x=-3$

$\therefore (-3,-3)$ MAX