ROOTS OF POLYNOMIAL EQUATIONS

QUADRATICS

Question 1 (***)

The quadratic equation

$$x^2 + 2kx + k = 0,$$

where k is a non zero constant, has roots $x = \alpha$ and $x = \beta$.

Find a quadratic equation, in terms of k, whose roots are

$$\frac{\alpha+\beta}{\alpha} \quad \text{and} \quad \frac{\alpha+\beta}{\beta}.$$

$\boxed{x^2 - 4kx + 4k = 0}$

OBTAINING RELATIONSHIPS FOR THE ROOTS OF THE GIVEN QUADRATIC

$x^2 + 2kx + k = 0 \implies \alpha+\beta = -\frac{2k}{1} = -2k$, $\alpha\beta = \frac{k}{1} = k$

PROCEED AS FOLLOWS

$\boxed{A = \frac{\alpha+\beta}{\alpha}, \quad B = \frac{\alpha+\beta}{\beta}}$

- $A+B = \frac{\alpha+\beta}{\alpha} + \frac{\alpha+\beta}{\beta} = \frac{\beta(\alpha+\beta)+\alpha(\alpha+\beta)}{\alpha\beta} = \frac{\alpha\beta+\beta^2+\alpha^2+\alpha\beta}{\alpha\beta} = \frac{\alpha^2+2\alpha\beta+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2}{\alpha\beta} = \frac{(-2k)^2}{k} = \frac{4k^2}{k} = 4k$
- $AB = \frac{\alpha+\beta}{\alpha} \times \frac{\alpha+\beta}{\beta} = \frac{(\alpha+\beta)^2}{\alpha\beta} = \dots 4k$

HENCE THE REQUIRED QUADRATIC WILL BE

$\implies x^2 - (A+B)x + AB = 0$

$\implies x^2 - 4kx + 4k = 0$

Question 2 (***)

The two roots of the quadratic equation

$$x^2 + 2x - 3 = 0$$

are denoted, in the usual notation, as α and β.

Find the quadratic equation, with integer coefficients, whose roots are

$$\alpha^3\beta + 1 \qquad \text{and} \qquad \alpha\beta^3 + 1 .$$

$\boxed{\text{P1 C}}$, $\boxed{x^2 + 28x + 52 = 0}$

OBTAIN RELATIONSHIPS FROM THE GIVEN QUADRATIC

$$x^2 + 2x - 3 = 0 \implies \begin{cases} \alpha + \beta = -\frac{b}{a} = -\frac{2}{1} = -2 \\ \alpha\beta = \frac{c}{a} = -\frac{3}{1} = -3 \end{cases}$$

PROCEED AS FOLLOWS

$$\boxed{\begin{aligned} A &= \alpha^3\beta + 1 \\ B &= \alpha\beta^3 + 1 \end{aligned}}$$

- $A+B = (\alpha^3\beta + 1) + (\alpha\beta^3 + 1) = \alpha^3\beta + \alpha\beta^3 + 2$
  $= \alpha\beta(\alpha^2 + \beta^2) + 2 = \alpha\beta\left[(\alpha+\beta)^2 - 2\alpha\beta\right] + 2$

  $$\boxed{\begin{aligned} (\alpha+\beta)^2 &\equiv \alpha^2 + 2\alpha\beta + \beta^2 \\ \alpha^2 + \beta^2 &\equiv (\alpha+\beta)^2 - 2\alpha\beta \end{aligned}}$$

  $= -3\left[(-2)^2 - 2(-3)\right] + 2 = -28$

- $AB = (\alpha^3\beta + 1)(\alpha\beta^3 + 1) = \alpha^4\beta^4 + \alpha^3\beta + \alpha\beta^3 + 1$
  $= (\alpha\beta)^4 + \alpha\beta(\alpha^2 + \beta^2) + 1 = (\alpha\beta)^4 + \alpha\beta\left[(\alpha+\beta)^2 - 2\alpha\beta\right] + 1$
  $= (-3)^4 - 3\left[(-2)^2 - 2(-3)\right] + 1 = 81 - 30 + 1 = 52$

HENCE THE REQUIRED QUADRATIC WILL BE

$$\implies x^2 - (A+B)x + AB = 0$$
$$\implies x^2 - (-28)x + 52 = 0$$
$$\implies x^2 + 28x + 52 = 0$$

Question 3 (***)

The roots of the quadratic equation

$$x^2 + 2x + 3 = 0$$

are denoted, in the usual notation, as α and β .

Find the quadratic equation, with integer coefficients, whose roots are

$$\alpha - \frac{1}{\beta^2} \qquad \text{and} \qquad \beta - \frac{1}{\alpha^2}.$$

$\boxed{}$, $\boxed{9x^2 + 16x + 34 = 0}$

OBTAIN RELATIONSHIPS FOR THE ROOTS OF THE GIVEN QUADRATIC

$$x^2 + 2x + 3 = 0 \implies \begin{cases} \alpha + \beta = \frac{-2}{1} = -2 & \leftarrow -\frac{b}{a} \\ \alpha\beta = \frac{3}{1} = 3 & \leftarrow \frac{c}{a} \end{cases}$$

PROCEED AS FOLLOWS

$$\boxed{\begin{aligned} A &= \alpha - \frac{1}{\beta^2} \\ B &= \beta - \frac{1}{\alpha^2} \end{aligned}}$$

- $A+B = \left(\alpha - \frac{1}{\beta^2}\right) + \left(\beta - \frac{1}{\alpha^2}\right) = (\alpha+\beta) - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)$

$$= (\alpha+\beta) - \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2} = (\alpha+\beta) - \frac{\alpha^2+\beta^2}{(\alpha\beta)^2}$$

$$= (\alpha+\beta) - \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} = -2 - \frac{(-2)^2 - 2\times 3}{3^2} = -\frac{16}{9}$$

- $AB = \left(\alpha - \frac{1}{\beta^2}\right)\left(\beta - \frac{1}{\alpha^2}\right) = \alpha\beta - \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\alpha^2\beta^2}$

$$= \alpha\beta - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + \frac{1}{(\alpha\beta)^2} = \alpha\beta - \left(\frac{\beta+\alpha}{\alpha\beta}\right) + \frac{1}{(\alpha\beta)^2}$$

$$= 3 - \frac{-2}{3} - \frac{1}{3^2} = 3 + \frac{2}{3} + \frac{1}{9} = \frac{34}{9}$$

HENCE THE REQUIRED EQUATION WILL BE

$$\implies x^2 - (A+B)x + AB = 0$$

$$\implies x^2 - \left(-\frac{16}{9}\right)x + \frac{34}{9} = 0$$

$$\implies 9x^2 + 16x + 34 = 0$$

Question 4 (***)

The roots of the quadratic equation

$$2x^2 - 3x + 5 = 0$$

are denoted by α and β.

Find the quadratic equation, with integer coefficients, whose roots are

$$3\alpha - \beta \qquad \text{and} \qquad 3\beta - \alpha .$$

$\boxed{4x^2 - 12x + 133 = 0}$

LOOKING AT THE QUADRATIC

$2x^2 - 3x + 5 = 0$

- $\alpha + \beta = -\frac{b}{a} = -\frac{-3}{2} = \frac{3}{2}$
- $\alpha\beta = \frac{c}{a} = \frac{5}{2}$

LET $A = 3\alpha - \beta$ & $B = 3\beta - \alpha$

SUM OF ROOTS

$A + B = (3\alpha - \beta) + (3\beta - \alpha) = 2\alpha + 2\beta = 2(\alpha + \beta) = 2 \times \frac{3}{2} = 3$

PRODUCT OF ROOTS

$AB = (3\alpha - \beta)(3\beta - \alpha) = 9\alpha\beta - 3\alpha^2 - 3\beta^2 + \alpha\beta = 10\alpha\beta - 3(\alpha^2 + \beta^2)$

$= 10\alpha\beta - 3\left[(\alpha+\beta)^2 - 2\alpha\beta\right] = 10 \times \frac{5}{2} - 3\left[\left(\frac{3}{2}\right)^2 - 2 \times \frac{5}{2}\right]$

$= 25 + \frac{33}{4}$

$= \frac{133}{4}$

FINALLY WE HAVE

$x^2 - (A+B)x + (AB) = 0$

$x^2 - 3x + \frac{133}{4} = 0$

$4x^2 - 12x + 133 = 0$

Question 5 (***)

The roots of the equation

$$az^2 + bz + c = 0,$$

where a, b and c are real constants, are denoted by α and β.

Given that $b^2 = 2ac \neq 0$, show that $\alpha^2 + \beta^2 = 0$.

$\boxed{\text{proof}}$

$az^2 + bz + c = 0$ & $b^2 = 2ac \neq 0$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(-\frac{b}{a}\right)^2 - 2 \times \frac{c}{a} = \frac{b^2}{a^2} - \frac{2c}{a}$

$= \frac{b^2 - 2ac}{a^2} = 0$

AS REQUIRED

Question 6 **(***)**

The roots of the quadratic equation

$$2x^2 - 8x + 9 = 0$$

are denoted, in the usual notation, as α and β.

Find the quadratic equation, with integer coefficients, whose roots are

$$\alpha^2 - 1 \qquad \text{and} \qquad \beta^2 - 1 .$$

$\boxed{}$, $\boxed{4x^2 - 20x + 57 = 0}$

METHOD A – USING STANDARD ROOTS RELATIONSHIPS

$2x^2 - 8x + 9 = 0$

- $\alpha + \beta = -\frac{b}{a} = -\frac{-8}{2} = 4$
- $\alpha\beta = \frac{c}{a} = \frac{9}{2}$

PROCEED AS FOLLOWS

$A = \alpha^2 - 1$ & $B = \beta^2 - 1$

- $A + B = (\alpha^2 - 1) + (\beta^2 - 1) = \alpha^2 + \beta^2 - 2$
  $= (\alpha + \beta)^2 - 2\alpha\beta - 2 = 4^2 - 2 \times \frac{9}{2} - 2 = 5$
- $AB = (\alpha^2 - 1)(\beta^2 - 1) = \alpha^2\beta^2 - \alpha^2 - \beta^2 + 1 = (\alpha\beta)^2 - (\alpha^2 + \beta^2) + 1$
  $= (\alpha\beta)^2 - \left[(\alpha + \beta)^2 - 2\alpha\beta\right] + 1 = \left(\frac{9}{2}\right)^2 - \left[4^2 - 2 \times \frac{9}{2}\right] + 1$
  $= \frac{81}{4} - 7 + 1 = \frac{57}{4}$

HENCE THE REQUIRED QUADRATIC WILL BE

$\Rightarrow x^2 - (A+B)x + (AB) = 0$

$\Rightarrow x^2 - 5x + \frac{57}{4} = 0$

$\Rightarrow 4x^2 - 20x + 57 = 0$ //

METHOD B – BY "FORCING" A SOLUTION

LET $y = x^2 - 1 \Rightarrow x^2 = y + 1$

$\Rightarrow x = \pm\sqrt{y+1}$

SUBSTITUTE INTO THE QUADRATIC IN x

$\Rightarrow 2\left(\pm\sqrt{y+1}\right)^2 - 8\left(\pm\sqrt{y+1}\right) + 9 = 0$

$\Rightarrow 2(y+1) \pm 8\sqrt{y+1} + 9 = 0$

$\Rightarrow \pm 8\sqrt{y+1} = -9 - 2(y+1)$

$\Rightarrow \pm 8\sqrt{y+1} = -9 - 2y - 2$

$\Rightarrow \pm 8\sqrt{y+1} = -2y - 11$

$\Rightarrow 64(y+1) = (-2y-11)^2$

$\Rightarrow 64y + 64 = 4y^2 + 44y + 121$

$\Rightarrow 0 = 4y^2 - 20y + 57$

OR

$4x^2 - 20x + 57 = 0$ // AS BEFORE

Question 7 (***)

A curve has equation

$$y = 2x^2 + 5x + c,$$

where c is a non zero constant.

Given that the roots of the equation differ by 3, determine the value of c.

$\boxed{}$, $\boxed{c = -\frac{11}{8}}$

LET THE SMALLER ROOT OF THE QUADRATIC BE α

- THE SUM OF THE ROOTS: $\alpha + (\alpha+3) = -\frac{b}{a} = -\frac{5}{2}$

  I.E. $2\alpha + 3 = -\frac{5}{2}$

  $2\alpha = -\frac{11}{2}$

  $\alpha = -\frac{11}{4}$

- THE PRODUCT OF THE ROOTS: $\alpha(\alpha+3) = \frac{c}{a} = \frac{c}{2}$

  I.E. $c = 2\alpha(\alpha+3)$

  $c = 2\left(-\frac{11}{4}\right)\left(-\frac{11}{4}+3\right)$

  $c = -\frac{11}{2} \times \frac{1}{4}$

  $c = -\frac{11}{8}$

ALTERNATIVE – WITHOUT USING DIRECTLY RESULTS ON THE SUM AND PRODUCT OF ROOTS OF A QUADRATIC

- LET THE SMALLER OF THE TWO ROOTS BE α

THEN $2x^2 + 5x + c = 0$

$\Rightarrow x^2 + \frac{5}{2}x + \frac{c}{2} = 0$

$\Rightarrow (x-\alpha)(x-(\alpha+3)) = 0$

$\Rightarrow x^2 - (\alpha+3)x - \alpha x + \alpha(\alpha+3) = 0$

$\Rightarrow x^2 - (2\alpha+3)x + \alpha(\alpha+3) = 0$

- BY COMPARISON WE HAVE

- $\frac{5}{2} = -(2\alpha+3)$ &
- $\frac{c}{2} = \alpha(\alpha+3)$

$\Rightarrow 2\alpha + 3 = -\frac{5}{2}$

$\Rightarrow 4\alpha + 6 = -5$

$\Rightarrow 4\alpha = -11$

$\Rightarrow \alpha = -\frac{11}{4}$

$\Rightarrow c = 2\alpha(\alpha+3)$

$\Rightarrow c = 2\left(-\frac{11}{4}\right)\left(-\frac{11}{4}+3\right)$

$\Rightarrow c = -\frac{11}{2} \times \frac{1}{4}$

$\Rightarrow c = -\frac{11}{8}$

Question 8 (***+)

The roots of the quadratic equation

$$2x^2 - 3x + 5 = 0$$

are denoted by α and β.

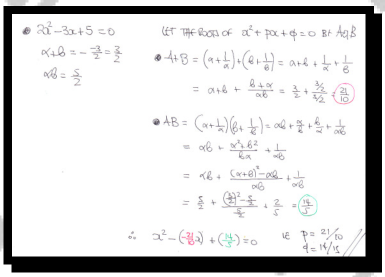
The roots of the quadratic equation

$$x^2 + px + q = 0,$$

where p and q are real constants, are denoted by $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$.

Determine the value of p and the value of q.

$\boxed{p = \frac{21}{10}}$, $\boxed{q = \frac{14}{5}}$


- $2x^2 - 3x + 5 = 0$

$\alpha + \beta = -\frac{-3}{2} = \frac{3}{2}$

$\alpha\beta = \frac{5}{2}$

LET THE ROOTS OF $x^2 + px + q = 0$ BE A & B

- $A + B = \left(\alpha + \frac{1}{\alpha}\right) + \left(\beta + \frac{1}{\beta}\right) = \alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta}$

$= \alpha + \beta + \frac{\beta + \alpha}{\alpha\beta} = \frac{3}{2} + \frac{3/2}{5/2} = \frac{21}{10}$

- $AB = \left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right) = \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}$

$= \alpha\beta + \frac{\alpha^2 + \beta^2}{\beta\alpha} + \frac{1}{\alpha\beta}$

$= \alpha\beta + \frac{(\alpha+\beta)^2 - \alpha\beta}{\alpha\beta} + \frac{1}{\alpha\beta}$

$= \frac{5}{2} + \frac{\left(\frac{3}{2}\right)^2 - \frac{5}{2}}{\frac{5}{2}} + \frac{2}{5} = \frac{14}{5}$

$\therefore \ x^2 - \left(\frac{21}{10}x\right) + \left(\frac{14}{5}\right) = 0$ IE $p = 21/10$, $q = 14/5$

Question 9 (****)

Consider the quadratic equation

$$ax^2 + bx + c = 0,$$

where a, b and c are real constants.

One of the roots of this quadratic equation is double the other.

Show clearly that both roots must be real.

$\boxed{\text{proof}}$

- $ax^2 + bx + c = 0$
- LET THE TWO ROOTS BE α & 2α

$$\left.\begin{array}{l} \alpha + 2\alpha = -\frac{b}{a} \\ \alpha \times 2\alpha = \frac{c}{a} \end{array}\right\} \Rightarrow \left.\begin{array}{l} 3\alpha = -\frac{b}{a} \\ 2\alpha^2 = \frac{c}{a} \end{array}\right\} \Rightarrow \left.\begin{array}{l} 9\alpha^2 = \frac{b^2}{a^2} \\ 2\alpha^2 = \frac{c}{a} \end{array}\right\} \Rightarrow \text{DIVIDE}$$

$$\Rightarrow \frac{9}{2} = \frac{\frac{b^2}{a^2}}{\frac{c}{a}}$$

$$\Rightarrow \frac{9}{2} = \frac{ab^2}{a^2c}$$

$$\Rightarrow \frac{9}{2} = \frac{b^2}{ac}$$

$$\Rightarrow \boxed{b^2 = \frac{9}{2}ac} \text{ OR } \boxed{\frac{1}{2}ac = \frac{1}{9}b^2}$$

NOW THE DISCRIMINANT WILL BE $b^2 - 4ac = \frac{9}{2}ac - 4ac = \frac{1}{2}ac$

$$= \frac{1}{9}b^2 > 0$$

$\therefore$ BOTH ROOTS ARE REAL //

Question 10 (****)

The roots of the quadratic equation

$$x^2 + 2x + 2 = 0$$

are denoted by α and β.

Find the quadratic equation, with integer coefficients, whose roots are

$$\frac{\alpha^2}{\beta} \quad \text{and} \quad \frac{\beta^2}{\alpha}.$$

$$\boxed{x^2 - 2x + 2 = 0}$$

- $A+B = \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\beta\alpha} = \frac{(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)}{\alpha\beta}$

$= \frac{\alpha+\beta}{\alpha\beta} \times \left[\alpha^2+2\alpha\beta+\beta^2-3\alpha\beta\right] = \frac{\alpha+\beta}{\alpha\beta}\left[(\alpha+\beta)^2-3\alpha\beta\right]$

$= \frac{-2}{2}\left[(-2)^2-3\times2\right] = -(4-6) = 2$

- $AB = \frac{\alpha^2}{\beta} \times \frac{\beta^2}{\alpha} = \frac{\alpha^2\beta^2}{\alpha\beta} = \alpha\beta = 2$

$\therefore \ x^2 - (2x) + (2) = 0$

$x^2 - 2x + 2 = 0$

Question 11 (****)

The roots of the quadratic equation

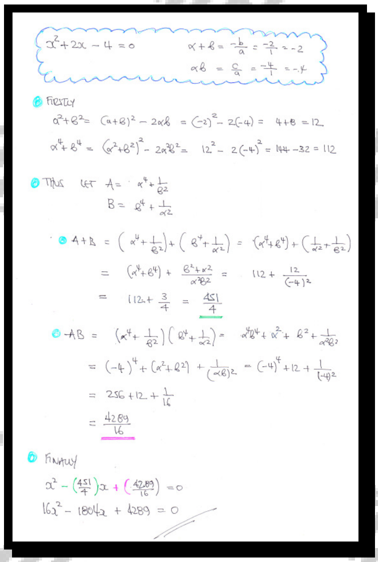
$$x^2 + 2x - 4 = 0$$

are denoted by α and β.

Find the quadratic equation, with integer coefficients, whose roots are

$$\alpha^4 + \frac{1}{\beta^2} \quad \text{and} \quad \beta^4 + \frac{1}{\alpha^2}.$$

$$\boxed{16x^2 - 1804x + 4289 = 0}$$



$x^2 + 2x - 4 = 0$ $\qquad \alpha + \beta = -\frac{b}{a} = \frac{-2}{1} = -2$

$\alpha\beta = \frac{c}{a} = \frac{-4}{1} = -4$

FIRSTLY

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = (-2)^2 - 2(-4) = 4 + 8 = 12$

$\alpha^4 + \beta^4 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2 = 12^2 - 2(-4)^2 = 144 - 32 = 112$

THUS LET $A = \alpha^4 + \frac{1}{\beta^2}$

$B = \beta^4 + \frac{1}{\alpha^2}$

$A + B = \left(\alpha^4 + \frac{1}{\beta^2}\right) + \left(\beta^4 + \frac{1}{\alpha^2}\right) = (\alpha^4 + \beta^4) + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)$

$= (\alpha^4 + \beta^4) + \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2} = 112 + \frac{12}{(-4)^2}$

$= 112 + \frac{3}{4} = \frac{451}{4}$

$AB = \left(\alpha^4 + \frac{1}{\beta^2}\right)\left(\beta^4 + \frac{1}{\alpha^2}\right) = \alpha^4\beta^4 + \alpha^2 + \beta^2 + \frac{1}{\alpha^2\beta^2}$

$= (-4)^4 + (\alpha^2 + \beta^2) + \frac{1}{(\alpha\beta)^2} = (-4)^4 + 12 + \frac{1}{(-4)^2}$

$= 256 + 12 + \frac{1}{16}$

$= \frac{4289}{16}$

FINALLY

$x^2 - \left(\frac{451}{4}\right)x + \left(\frac{4289}{16}\right) = 0$

$16x^2 - 1804x + 4289 = 0$

Question 12 **(****)**

$$x^2 - 4\sqrt{2}\,kx + 2k^4 - 1 = 0.$$

The two roots of the above quadratic equation, where k is a constant, are denoted by α and β.

Given further that $\alpha^2 + \beta^2 = 66$, determine the exact value of $\alpha^3 + \beta^3$.

$$\boxed{\alpha^3 + \beta^3 = 280\sqrt{2}}$$

$x^2 - 4\sqrt{2}kx + 2k^4 - 1 = 0$ & $\alpha^2 + \beta^2 = 66$

FIRSTLY SUM OF ROOTS: $\alpha + \beta = -\frac{b}{a} = 4\sqrt{2}k$

$\alpha\beta = \frac{c}{a} = 2k^4 - 1$

NEXT $(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

$\Rightarrow (4\sqrt{2}k)^2 = 66 + 2(2k^4 - 1)$

$\Rightarrow 32k^2 = 66 + 2(2k^4 - 1)$

$\Rightarrow 16k^2 = 33 + 2k^4 - 1$

$\Rightarrow 0 = 2k^4 - 16k^2 + 32$

$\Rightarrow 0 = k^4 - 8k^2 + 16$

$\Rightarrow 0 = (k^2 - 4)^2$

$\Rightarrow k^2 = 4$

$\Rightarrow \boxed{k = 2}$ $k > 0$

NOW $(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$(\alpha+\beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta)$

$(4\sqrt{2}k)^3 = (\alpha^3 + \beta^3) + 3(2k^4 - 1)(4\sqrt{2}k)$

$(8\sqrt{2})^3 = \alpha^3 + \beta^3 + 3 \times 31 \times 8\sqrt{2}$

$1024\sqrt{2} = \alpha^3 + \beta^3 + 744\sqrt{2}$

$\alpha^3 + \beta^3 = 280\sqrt{2}$

Question 13 (****+)

The quadratic equation

$$ax^2 + bx + c = 0, \; x \in \mathbb{R},$$

where a, b and c are constants, $a \neq 0$, has real roots which differ by 1.

Determine a simplified relationship between a, b and c.

$$\boxed{b^2 - 4ac = a^2}$$

$\boxed{ax^2 + bx + c = 0 \text{ , SOLUTIONS DIFFER BY } 1}$

- LET THE TWO SOLUTIONS BE x_2 & x_1, $x_2 > x_1$

$\Rightarrow x_2 - x_1 = 1$

$\Rightarrow \frac{-b + \sqrt{b^2 - 4ac}}{2a} - \frac{-b - \sqrt{b^2 - 4ac}}{2a} = 1$

$\Rightarrow \frac{\cancel{2}\sqrt{b^2 - 4ac}}{\cancel{2}a} = 1$

$\Rightarrow \sqrt{b^2 - 4ac} = a$

$\Rightarrow b^2 - 4ac = a^2$

ALTERNATIVE APPROACH

- LET THE TWO ROOTS BE α & β, WHERE $\beta = \alpha + 1$

$\left.\begin{array}{l} \alpha + \beta = -\frac{b}{a} \\ \alpha\beta = \frac{c}{a} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \alpha + (\alpha + 1) = -\frac{b}{a} \\ \alpha(\alpha + 1) = \frac{c}{a} \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} 2\alpha + 1 = -\frac{b}{a} \\ \alpha^2 + \alpha = \frac{c}{a} \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2\alpha = -\frac{b}{a} - 1 \\ 4\alpha^2 + 4\alpha = \frac{4c}{a} \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} 4\alpha^2 = \left(-\frac{b}{a} - 1\right)^2 = \left(\frac{b}{a} + 1\right)^2 \\ 4\alpha^2 + 4\alpha = \frac{4c}{a} \end{array}\right\} \Rightarrow \left(\frac{b}{a} + 1\right)^2 + 2\left(-\frac{b}{a} - 1\right) = \frac{4c}{a}$

$\Rightarrow \left(\frac{b + a}{a}\right)^2 - \frac{2(b + a)}{a} = \frac{4c}{a}$

$\Rightarrow \frac{(b + a)^2}{a^2} - \frac{2(b + a)}{a} = \frac{4c}{a}$

$\Rightarrow (b + a)^2 - 2a(b + a) = 4ac$

$\Rightarrow b^2 + \cancel{2ab} + a^2 - \cancel{2ab} - 2a^2 = 4ac$

$\Rightarrow b^2 - a^2 = 4ac$

$\Rightarrow b^2 - 4ac = a^2$ AS BEFORE

Question 14 (****+)

The roots of the quadratic equation

$$x^2 - 3x + 4 = 0$$

are denoted by α and β.

Find the quadratic equation, with integer coefficients, whose roots are

$$\alpha^3 - \beta \quad \text{and} \quad \beta^3 - \alpha .$$

$$\boxed{x^2 + 12x + 99 = 0}$$

START BY OBTAINING THE STANDARD RELATIONSHIPS FOR THE QUADRATIC

$$x^2 - 3x + 4 = 0 \qquad \alpha + \beta = -\frac{b}{a} = -\frac{-3}{1} = 3$$

$$\alpha\beta = \frac{c}{a} = \frac{4}{1} = 4$$

START FORMING THE SUM AND PRODUCT OF THE ROOTS OF THE REQUIRED QUADRATIC AS FOLLOWS

$$\begin{aligned} A &= \alpha^3 - \beta \\ B &= \beta^3 - \alpha \end{aligned}$$

$$A + B = (\alpha^3 - \beta) + (\beta^3 - \alpha) = \alpha^3 + \beta^3 - (\alpha + \beta)$$

NOW

$$\boxed{\begin{aligned} (\alpha+\beta)^3 &= \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 \\ (\alpha+\beta)^3 &= \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta) \\ \alpha^3 + \beta^3 &= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) \end{aligned}}$$

$$\Rightarrow A + B = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) - (\alpha+\beta)$$

$$\Rightarrow A + B = 3^3 - 3\times4\times3 - 3$$

$$\Rightarrow A + B = 27 - 36 - 3$$

$$\Rightarrow A + B = -12$$

$$AB = (\alpha^3 - \beta)(\beta^3 - \alpha) = \alpha^3\beta^3 - \alpha^4 - \beta^4 + \alpha\beta$$

$$\Rightarrow AB = (\alpha\beta)^3 + (\alpha\beta) - (\alpha^4 + \beta^4)$$

NOW

$$\begin{aligned} \alpha^2 + \beta^2 &= (\alpha+\beta)^2 - 2\alpha\beta \\ (\alpha^2)^2 + (\beta^2)^2 &= (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2 \\ \alpha^4 + \beta^4 &= (\alpha^2+\beta^2)^2 - 2(\alpha\beta)^2 \\ \alpha^4 + \beta^4 &= \left[(\alpha+\beta)^2 - 2\alpha\beta\right]^2 - 2(\alpha\beta)^2 \end{aligned}$$

$$\Rightarrow AB = (\alpha\beta)^3 + (\alpha\beta) - \left[\left[(\alpha+\beta)^2 - 2\alpha\beta\right]^2 - 2(\alpha\beta)^2\right]$$

$$\Rightarrow AB = 4^3 + 4 - \left[3^2 - 2\times4\right]^2 + 2\times4^2$$

$$\Rightarrow AB = 64 + 4 - 1 + 32$$

$$\Rightarrow AB = 99$$

HENCE THE REQUIRED QUADRATIC IS

$$x^2 - (-12x) + (+99) = 0$$

$$x^2 + 12x + 99 = 0$$

Question 15 (****+)

$$\frac{1}{x+p}+\frac{1}{x+q}=\frac{1}{r}, \qquad x \neq -p \;\; x \neq -q \, .$$

The roots of the above quadratic equation, where p, q and r are non zero constants, are equal in magnitude but opposite in sign.

Show that the product of these roots is

$$-\tfrac{1}{2}\left[p^2+q^2\right].$$

proof

$$\frac{1}{x+p}+\frac{1}{x+q}=\frac{1}{r}$$

MANIPULATE INTO A 3 TERM QUADRATIC

$$\Rightarrow \frac{x+q+x+p}{(x+p)(x+q)}=\frac{1}{r}$$

$$\Rightarrow 2xr+qr+pr = x^2+px+qx+pq$$

$$\Rightarrow 0 = x^2+(p+q-2r)x+(pq-qr-pr)$$

NOW IF THE QUADRATIC HAS ROOT EQUAL IN MAGNITUDE BUT WITH OPPOSITE SIGNS

$$p+q-2r=0$$

$$\boxed{2r=p+q}$$

HENCE THE PRODUCT OF THE ROOTS WILL BE

$$\begin{aligned} pq-qr-pr &= pq-r(p+q) \\ &= \tfrac{1}{2}\left[2pq-2r(p+q)\right] \\ &= \tfrac{1}{2}\left[2pq-(p+q)(p+q)\right] \\ &= \tfrac{1}{2}\left[2pq-(p^2+2pq+q^2)\right] \\ &= \tfrac{1}{2}\left[-p^2-q^2\right] \\ &= -\tfrac{1}{2}\left[p^2+q^2\right] \end{aligned}$$

AS REQUIRED

Question 16 (****+)

$$2x^2 + kx + 1 = 0\,.$$

The roots of the above equation are α and β, where k is a non zero real constant.

Given further that the following two expressions

$$\frac{\alpha}{\beta\left(1+\alpha^2+\beta^2\right)} \quad \text{and} \quad \frac{\beta}{\alpha\left(1+\alpha^2+\beta^2\right)}$$

are real, finite and distinct, determine the range of the possible values of k.

$$\boxed{|k| > \sqrt{8}}$$

$$\boxed{2x^2 + kx + 1 = 0 \qquad x \in \mathbb{R}}$$

FIRST RECORD SOME STANDARD RESULTS

$$\boxed{\begin{aligned} \alpha+\beta &= -\tfrac{1}{2}k \\ \alpha\beta &= \tfrac{1}{2} \\ \alpha^2+\beta^2 &= (\alpha+\beta)^2 - 2\alpha\beta \\ &= \tfrac{1}{4}k^2 - 1 \end{aligned}}$$

$$A+B = \frac{\alpha}{\beta(1+\alpha^2+\beta^2)} + \frac{\beta}{\alpha(1+\alpha^2+\beta^2)} = \frac{1}{1+\alpha^2+\beta^2}\left[\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right]$$

$$= \frac{\alpha^2+\beta^2}{\alpha\beta} \times \frac{1}{1+\alpha^2+\beta^2} = \frac{\frac{1}{4}k^2-1}{\frac{1}{2}} \times \frac{1}{1+\frac{1}{4}k^2-1}$$

$$= \frac{\frac{1}{4}k^2-1}{\frac{1}{2}} \times \frac{4}{k^2} = \frac{k^2-4}{\frac{1}{2}k^2} = \frac{2k^2-8}{k^2}$$

$$AB = \frac{\alpha}{\beta(1+\alpha^2+\beta^2)} \times \frac{\beta}{\alpha(1+\alpha^2+\beta^2)} = \frac{1}{(1+\alpha^2+\beta^2)^2}$$

$$= \frac{1}{(1+\frac{1}{4}k^2-1)^2} = \frac{1}{\frac{1}{16}k^4} = \frac{16}{k^4}$$

HENCE THE REQUIRED QUADRATIC HAS EQUATION

$$\Rightarrow x^2 - (A+B)x + (AB) = 0$$

$$\Rightarrow x^2 + \frac{8-2k^2}{k^2}x + \frac{16}{k^4} = 0$$

$$\Rightarrow \underline{k^4x^2 + 2k^2(4-k^2)x + 16 = 0}$$

NOW THIS EQUATION HAS DISTINCT REAL FINITE ROOTS

$$b^2 - 4ac > 0 \Rightarrow \left[2k^2(4-k^2)\right]^2 - 4 \times k^4 \times 16 > 0$$

$$\Rightarrow 4k^4(4-k^2)^2 - 64k^4 > 0 \qquad (k^4 \neq 0)$$

$$\Rightarrow (4-k^2)^2 - 16 > 0$$

$$\Rightarrow (4-k^2-4)(4-k^2+4) > 0$$

$$\Rightarrow -k^2(8-k^2) > 0$$

$$\Rightarrow -(8-k^2) > 0 \qquad (k^2 \neq 0)$$

$$\Rightarrow k^2 - 8 > 0$$

$$\Rightarrow k^2 > 8$$

$$\Rightarrow k > \sqrt{8} \quad \underline{\text{OR}} \quad k < -\sqrt{8}$$

Question 17 (*****)

The quadratic equation

$$4x^2 + Px + Q = 0,$$

where P and Q are constants, has roots which differ by 2.

If another quadratic equation has repeated roots which are also the **squares of the roots** of the above given equation, find the value of P and the value of Q.

$\boxed{P=0}$, $\boxed{Q=-4}$

LET THE ROOTS OF THE QUADRATIC BE α & $\alpha+2$

$4x^2+Px+Q=0$

- $\alpha+\alpha+2 = -\frac{P}{4}$
- $\alpha(\alpha+2) = \frac{Q}{4}$

ELIMINATE α BETWEEN THESE RELATIONSHIPS

$$\left\{\begin{array}{l} 2\alpha = -\frac{P}{4}-2 \\ \alpha^2+2\alpha = \frac{Q}{4} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} 4\alpha^2 = \left(-\frac{P}{4}-2\right)^2 \\ 4\alpha^2+8\alpha = Q \end{array}\right\} \Longrightarrow$$

$$\Longrightarrow \left(-\frac{P}{4}-2\right)^2 + 4\left(-\frac{P}{4}-2\right) = Q$$

$$\Longrightarrow \frac{P^2}{16} + \cancel{P} + 4 - \cancel{P} - 8 = Q$$

$$\Longrightarrow Q = \frac{P^2}{16} - 4$$

$$\Longrightarrow 16Q = P^2 - 64 \quad \text{OR} \quad P^2 = 16Q + 64$$

NOW THE NEW QUADRATIC HAS ROOTS WHICH ARE SQUARES OF THE ROOTS OF THIS EQUATION

- $y = x^2$
- $x = \pm\sqrt{y}$

$$\Longrightarrow 4y + P(\pm\sqrt{y}) + Q = 0$$

$$\Longrightarrow 4y + Q = \pm P\sqrt{y}$$

$$\Longrightarrow 16y^2 + 8Qy + Q^2 = P^2 y$$

$$\Longrightarrow 16y^2 + 8Qy + Q^2 = (16Q+64)y$$

$$\Longrightarrow 16y^2 + (-64-8Q)y + Q^2 = 0$$

BUT THIS EQUATION MUST HAVE REPEATED ROOTS

$$b^2 - 4ac = 0$$

$$(-64-8Q)^2 - 4\times16\times Q^2 = 0$$

$$64(8+Q)^2 - 64Q^2 = 0$$

$$(8+Q)^2 - Q^2 = 0$$

$$(8+Q+Q)(8+Q-Q) = 0$$

$$8(8+2Q) = 0$$

$$\therefore Q = -4$$

AND USING $P^2 = 64 + 16Q$ WITH $Q = -4$

$$\therefore P = 0$$

Question 18 (*****)

The quadratic equation

$$x^2 - 4x - 2 = 0,$$

has roots α and β in the usual notation, where $\alpha > \beta$.

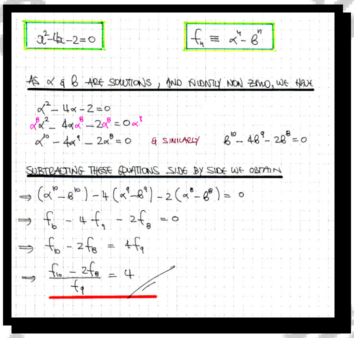
It is further given that

$$f_n \equiv \alpha^n - \beta^n.$$

Determine the value of

$$\frac{f_{10} - 2f_8}{f_9}.$$

$\boxed{\ }$, $\boxed{4}$



$\boxed{x^2-4x-2=0}$ $\boxed{f_n \equiv \alpha^n - \beta^n}$

AS α & β ARE SOLUTIONS, AND EVIDENTLY NON ZERO, WE HAVE

$\alpha^2 - 4\alpha - 2 = 0$

$\alpha^8\alpha^2 - 4\alpha\alpha^8 - 2\alpha^8 = 0\alpha^8$

$\alpha^{10} - 4\alpha^9 - 2\alpha^8 = 0$ & SIMILARLY $\beta^{10} - 4\beta^9 - 2\beta^8 = 0$

SUBTRACTING THESE EQUATIONS SIDE BY SIDE WE OBTAIN

$\Rightarrow (\alpha^{10} - \beta^{10}) - 4(\alpha^9 - \beta^9) - 2(\alpha^8 - \beta^8) = 0$

$\Rightarrow f_{10} - 4f_9 - 2f_8 = 0$

$\Rightarrow f_{10} - 2f_8 = 4f_9$

$\Rightarrow \frac{f_{10} - 2f_8}{f_9} = 4$

Question 19 (*****)

The quadratic equation

$$ax^2 + bx + 1 = 0, \qquad a \neq 0,$$

where a and b are constants, has roots α and β.

Find, in terms of α and β, the roots of the equation

$$x^2 + \left(b^3 - 3ab\right)x + a^3 = 0.$$

$\boxed{\text{SP-X}}$, $\boxed{\dfrac{1}{\alpha^3}, \dfrac{1}{\beta^3}}$

START WITH THE GIVEN EQUATION

" $ax^2 + bx + 1 = 0$ HAS ROOTS α & β "

$$\Longrightarrow \alpha + \beta = -\frac{b}{a} \quad \& \quad \alpha\beta = \frac{1}{a}$$

$$\Longrightarrow \alpha + \beta = -\frac{b}{a} \qquad \frac{1}{\alpha\beta} = a$$

$$\Longrightarrow (\alpha+\beta) \times \frac{1}{\alpha\beta} = -\frac{b}{\cancel{a}} \times \cancel{a}$$

$$\Longrightarrow b = -\frac{\alpha+\beta}{\alpha\beta}$$

NOW LET A & B BE THE ROOTS OF THE EQUATION

$$x^2 + (b^3 - 3ab)x + a^3 = 0$$

THE SUM OF ITS ROOTS ARE

$$A + B = -(b^3 - 3ab) = -b^3 + 3ab$$

$$= -\left(-\frac{\alpha+\beta}{\alpha\beta}\right)^3 + 3\left(\frac{1}{\alpha\beta}\right)\left(-\frac{\alpha+\beta}{\alpha\beta}\right)$$

$$= \left(\frac{\alpha+\beta}{\alpha\beta}\right)^3 - \frac{3(\alpha+\beta)}{\alpha^2\beta^2}$$

$$= \frac{\alpha+\beta}{(\alpha\beta)^2}\left[\frac{(\alpha+\beta)^2}{\alpha\beta} - 3\right]$$

$$= \frac{\alpha+\beta}{\alpha^2\beta^2}\left[\frac{\alpha^2 + 2\alpha\beta + \beta^2}{\alpha\beta} - 3\right]$$

$$= \frac{\alpha+\beta}{\alpha^2\beta^2}\left[\frac{\alpha^2 + 2\alpha\beta + \beta^2 - 3\alpha\beta}{\alpha\beta}\right]$$

$$= \frac{\alpha+\beta}{\alpha^2\beta^2} \times \frac{\alpha^2 - \alpha\beta + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)}{\alpha^3\beta^3} \quad \leftarrow \text{"DIFFERENCE/SUM OF CUBES"}$$

$$= \frac{\alpha^3 + \beta^3}{\alpha^3\beta^3}$$

$$= \frac{1}{\beta^3} + \frac{1}{\alpha^3}$$

SIMILARLY THE PRODUCT OF ITS ROOTS ARE

$$AB = \frac{a^3}{1} = \left(\frac{1}{\alpha\beta}\right)^3 = \frac{1}{\alpha^3\beta^3}$$

BY INSPECTION AS

$$A + B = \frac{1}{\alpha^3} + \frac{1}{\beta^3}$$

$$AB = \frac{1}{\alpha^3} \times \frac{1}{\beta^3}$$

THE REQUIRED ROOTS WILL BE $\frac{1}{\alpha^3}$ & $\frac{1}{\beta^2}$ //

CUBICS

Question 1 (**)

$$x^3 - 6x^2 + 4x + 12 = 0.$$

The three roots of the above cubic are denoted by α, β and γ.

Find the value of …

a) … $\alpha + \beta + \gamma$.

b) … $\alpha^2 + \beta^2 + \gamma^2$.

c) … $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

$\boxed{\alpha + \beta + \gamma = 6}$, $\boxed{\alpha^2 + \beta^2 + \gamma^2 = 28}$, $\boxed{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{3}}$

OBTAIN RELATIONSHIPS FOR ROOTS OF THE GIVEN CUBIC

$x^3 - 6x^2 + 4x + 12 = 0$

- $\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{-6}{1} = 6$
- $\alpha\beta + \beta\gamma + \gamma\alpha = +\frac{c}{a} = \frac{4}{1} = 4$
- $\alpha\beta\gamma = -\frac{d}{a} = -\frac{12}{1} = -12$

a) $\alpha + \beta + \gamma = 6$ (FROM ABOVE)

b) $(\alpha + \beta + \gamma)^2 \equiv \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha$

$\alpha^2 + \beta^2 + \gamma^2 \equiv (\alpha + \beta + \gamma)^2 - 2\alpha\beta - 2\beta\gamma - 2\gamma\alpha$

$\alpha^2 + \beta^2 + \gamma^2 \equiv (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$\alpha^2 + \beta^2 + \gamma^2 = 6^2 - 2 \times 4$

$\alpha^2 + \beta^2 + \gamma^2 = 28$

c) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{4}{-12} = -\frac{1}{3}$

Question 2 (**)

A cubic is given in terms of two constants p and q

$$2x^3 + 7x^2 + px + q = 0 .$$

The three roots of the above cubic are α, $\frac{1}{2}\alpha$ and $(\alpha - 1)$.

Find the value of α, p and q.

$\boxed{\alpha = -1}$, $\boxed{p = 7}$, $\boxed{q = 2}$

$2x^3 + 7x^2 + px + q = 0$

$\alpha + \beta + \gamma = -\frac{7}{2}$
$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{1}{2}p$
$\alpha\beta\gamma = -\frac{1}{2}q$

THUS $\alpha + \frac{\alpha}{2} + (\alpha - 1) = -\frac{7}{2}$
$\frac{5}{2}\alpha - 1 = -\frac{7}{2}$
$\frac{5}{2}\alpha = -\frac{5}{2}$
$\alpha = -1$

NOW
$\frac{1}{2}p = \alpha\beta + \beta\gamma + \gamma\alpha$
$\frac{1}{2}p = \alpha\left(\frac{\alpha}{2}\right) + \left(\frac{\alpha}{2}\right)(\alpha - 1) + (\alpha - 1)\alpha$
$\frac{1}{2}p = \frac{1}{2} + 1 + 2$
$\frac{1}{2}p = \frac{7}{2}$
$p = 7$

AND $-\frac{1}{2}q = \alpha\beta\gamma = \alpha\left(\frac{\alpha}{2}\right)(\alpha - 1)$
$-\frac{1}{2}q = -1$
$q = 2$

Question 3 (**+)

$$x^3 - 2x^2 - 8x + 11 = 0.$$

The roots of the above cubic equation are α, β and γ.

Find a cubic equation, with integer coefficients, whose roots are

$$\alpha + 1, \quad \beta + 1, \quad \gamma + 1.$$

FP1-R, $\boxed{x^3 - 5x^2 - x + 16 = 0}$

METHOD A – USING RELATIONSHIPS OF ROOTS

$x^3 - 2x^2 - 8x + 11 = 0$

- $\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{-2}{1} = 2$
- $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{-8}{1} = -8$
- $\alpha\beta\gamma = -\frac{d}{a} = \frac{-11}{1} = -11$

PROCEED AS FOLLOWS

$A = \alpha + 1$, $B = \beta + 1$, $C = \gamma + 1$

- $A + B + C = (\alpha+1) + (\beta+1) + (\gamma+1) = (\alpha+\beta+\gamma) + 3$
  $= 2 + 3 = 5$
- $AB + BC + CA = (\alpha+1)(\beta+1) + (\beta+1)(\gamma+1) + (\gamma+1)(\alpha+1)$
  $= \alpha\beta + \alpha + \beta + 1$
  $\beta\gamma + \beta + \gamma + 1$
  $\alpha\gamma + \alpha + \gamma + 1$
  $= (\alpha\beta + \beta\gamma + \gamma\alpha) + 2(\alpha+\beta+\gamma) + 3$
  $= -8 + (2 \times 2) + 3$
  $= -1$
- $ABC = (\alpha+1)(\beta+1)(\gamma+1) = (\alpha+1)(\beta\gamma + \beta + \gamma + 1)$
  $= \alpha\beta\gamma + \alpha\beta + \alpha\gamma + \alpha + \beta\gamma + \beta + \gamma + 1$
  $= \alpha\beta\gamma + (\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha+\beta+\gamma) + 1$
  $= -11 - 8 + 2 + 1$
  $= -16$

HENCE THE REQUIRED EQUATION WILL BE

$x^3 - (A+B+C)x^2 + (AB+BC+CA)x - (ABC) = 0$

$x^3 - 5x^2 - x - (-16) = 0$

$x^3 - 5x^2 - x + 16 = 0$

METHOD B – SOLUTION BY "FORCING"

LET $y = x + 1 \Rightarrow x = y - 1$

SUBSTITUTE INTO THE CUBIC

$\Rightarrow x^3 - 2x^2 - 8x + 11 = 0$

$\Rightarrow (y-1)^3 - 2(y-1)^2 - 8(y-1) + 11 = 0$

$\Rightarrow y^3 - 3y^2 + 3y - 1 - 2(y^2 - 2y + 1) - 8y + 8 + 11 = 0$

$\Rightarrow y^3 - 3y^2 + 3y - 1$
$\quad - 2y^2 + 4y - 2$
$\quad - 8y + 19 = 0$

$\Rightarrow y^2 - 5y^2 - y + 16 = 0$

AS BEFORE

Question 4 **(**+)**

The three roots of the cubic equation

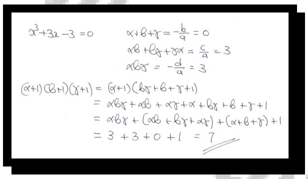
$$x^3 + 3x - 3 = 0$$

are denoted in the usual notation by α, β and γ.

Find the value of

$$(\alpha+1)(\beta+1)(\gamma+1).$$

$\boxed{7}$



Question 5 **(**+)**

The roots of the cubic equation

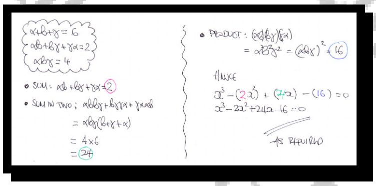
$$x^3 - 6x^2 + 2x - 4 = 0$$

are denoted by α, β and γ.

Show that the equation of the cubic whose roots are $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$ is given by

$$x^3 - 2x^2 + 24x - 16 = 0.$$

$\boxed{\text{proof}}$



Question 6 (+)**

The two roots of the quadratic equation

$$2x^2 - 5x + 8 = 0,$$

are denoted by α and β.

Determine the cubic equation with integer coefficients whose three roots are

$$\alpha^2\beta, \ \alpha\beta^2 \text{ and } \alpha\beta.$$

$$\boxed{x^3 - 14x^2 + 104x - 256 = 0}$$

$2x^2 - 5x + 8 = 0$ $\qquad \alpha + \beta = -\frac{b}{a} = \frac{5}{2}$

$\qquad \alpha\beta = \frac{c}{a} = \frac{8}{2} = 4$

LET THE THREE ROOTS OF THE CUBIC BE A, B & C

- $A+B+C = \alpha^2\beta + \alpha\beta^2 + \alpha\beta = \alpha\beta(\alpha+\beta+1) = 4\left(\frac{5}{2}+1\right) = 14$
- $AB+BC+CA = (\alpha^2\beta \times \alpha\beta^2) + (\alpha\beta^2 \times \alpha\beta) + (\alpha\beta \times \alpha^2\beta)$
  $= \alpha^3\beta^3 + \alpha^2\beta^3 + \alpha^3\beta^2 = \alpha^2\beta^2\left[\alpha\beta + \beta + \alpha\right]$
  $= (\alpha\beta)^2\left[\alpha\beta + (\alpha+\beta)\right] = 16\left[4 + \frac{5}{2}\right] = 104$
- $ABC = \alpha^2\beta \times \alpha\beta^2 \times \alpha\beta = \alpha^4\beta^4 = (\alpha\beta)^4 = 4^4 = 256$

THUS

$x^3 - (14)x^2 + (104)x - (256) = 0$

I.E. $x^3 - 14x^2 + 104x - 256 = 0$

Question 7 **(**+)**

$$x^3 + bx^2 + cx + d = 0,$$

where b, c and d are real constants.

The three roots of the above cubic are denoted by α, β and γ.

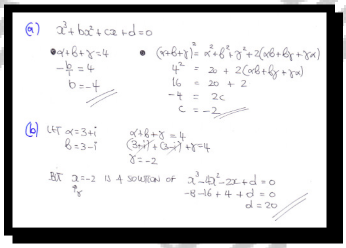
a) Given that

$$\alpha + \beta + \gamma = 4 \quad \text{and} \quad \alpha^2 + \beta^2 + \gamma^2 = 20,$$

find the value of b and the value of c.

b) Given further that $\alpha = 3 + \mathrm{i}$, determine the value of d.

$\boxed{b = -4}$, $\boxed{c = -2}$, $\boxed{d = 20}$


(a) $x^3 + bx^2 + cx + d = 0$

• $\alpha + \beta + \gamma = 4$

$-\frac{b}{1} = 4$

$b = -4$

• $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$4^2 = 20 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$16 = 20 + 2$

$-4 = 2c$

$c = -2$

(b) LET $\alpha = 3 + i$

$\beta = 3 - i$

$\alpha + \beta + \gamma = 4$

$(3 + \cancel{i}) + (3 - \cancel{i}) + \gamma = 4$

$\gamma = -2$

BUT $x = -2$ IS A SOLUTION OF $x^3 - 4x^2 - 2x + d = 0$

$-8 - 16 + 4 + d = 0$

$d = 20$

Question 8 (**+)

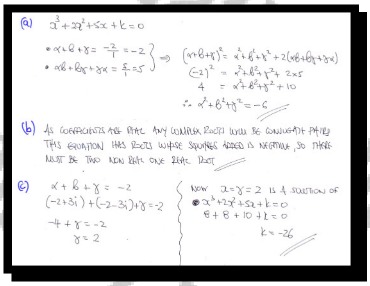
$$x^3 + 2x^2 + 5x + k = 0.$$

The three roots of the above cubic are denoted by α, β and γ, where k is a real constant.

a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$ and hence explain why this cubic has one real root and two non real roots.

b) Given that $x = -2 + 3\text{i}$ is a root of the cubic show that $k = -26$.

$$\boxed{\alpha^2 + \beta^2 + \gamma^2 = -6}$$



(a) $x^3 + 2x^2 + 5x + k = 0$

- $\alpha + \beta + \gamma = -\frac{2}{1} = -2$
- $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{5}{1} = 5$

$\Rightarrow (\alpha+\beta+\gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$(-2)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2 \times 5$

$4 = \alpha^2 + \beta^2 + \gamma^2 + 10$

$\therefore \alpha^2 + \beta^2 + \gamma^2 = -6$

(b) AS COEFFICIENTS ARE REAL ANY COMPLEX ROOTS WILL BE CONJUGATE PAIRS. THIS EQUATION HAS ROOTS WHOSE SQUARES ADDED IS NEGATIVE, SO THERE MUST BE TWO NON REAL ONE REAL ROOT

(c) $\alpha + \beta + \gamma = -2$

$(-2+3i) + (-2-3i) + \gamma = -2$

$-4 + \gamma = -2$

$\gamma = 2$

NOW $x = \gamma = 2$ IS A SOLUTION OF

$x^3 + 2x^2 + 5x + k = 0$

$8 + 8 + 10 + k = 0$

$k = -26$

Question 9 (**+)

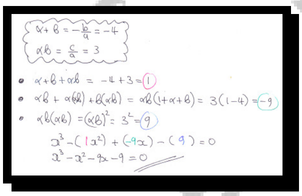
The roots of the quadratic equation

$$x^2 + 4x + 3 = 0$$

are denoted, in the usual notation, as α and β.

Find the cubic equation, with integer coefficients, whose roots are α, β and $\alpha\beta$.

$$\boxed{x^3 + x^2 - 9x - 9 = 0}$$



$\alpha + \beta = -\frac{b}{a} = -4$

$\alpha\beta = \frac{c}{a} = 3$

- $\alpha + \beta + \alpha\beta = -4 + 3 = 1$
- $\alpha\beta + \alpha(\alpha\beta) + \beta(\alpha\beta) = \alpha\beta(1 + \alpha + \beta) = 3(1-4) = -9$
- $\alpha\beta(\alpha\beta) = (\alpha\beta)^2 = 3^2 = 9$

$x^3 - (1x^2) + (-9x) - (9) = 0$

$x^3 - x^2 - 9x - 9 = 0$

Question 10 (***)

$$x^3 - x^2 + 3x + k = 0.$$

The roots of the above cubic equation are denoted by α, β and γ, where k is a real constant.

a) Show that

$$\alpha^2 + \beta^2 + \gamma^2 = -5.$$

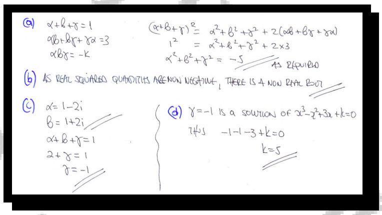
b) Explain why the cubic equation cannot possibly have 3 real roots.

It is further given that $\alpha = 1 - 2\text{i}$.

c) Find the value of β and the value of γ.

d) Show that $k = 5$.

$\boxed{\beta = 1 + 2\text{i}}$, $\boxed{\gamma = -1}$


(a) $\alpha+\beta+\gamma=1$
$\alpha\beta+\beta\gamma+\gamma\alpha=3$
$\alpha\beta\gamma=-k$

$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha)$
$1^2 = \alpha^2+\beta^2+\gamma^2+2\times3$
$\alpha^2+\beta^2+\gamma^2=-5$ AS REQUIRED

(b) AS REAL SQUARED QUANTITIES ARE NON NEGATIVE, THERE IS A NON REAL ROOT

(c) $\alpha=1-2i$
$\beta=1+2i$
$\alpha+\beta+\gamma=1$
$2+\gamma=1$
$\gamma=-1$

(d) $\gamma=-1$ IS A SOLUTION OF $x^3-x^2+3x+k=0$
THUS $-1-1-3+k=0$
$k=5$

Question 11 (***)

The roots of the quadratic equation

$$x^2 + 3x + 3 = 0$$

are denoted by α and β.

Find the cubic equation, with integer coefficients, whose roots are

$$\frac{\alpha}{\beta}, \frac{\beta}{\alpha} \text{ and } \alpha\beta .$$

$\boxed{}$, $\boxed{x^3 - 4x^2 + 4x - 3 = 0}$

STARTING WITH THE QUADRATIC

$$\text{IF } \quad x^2+3x+3=0 \implies \alpha+\beta = -\frac{b}{a} = \frac{-3}{1} = -3$$
$$\implies \alpha\beta = \frac{c}{a} = \frac{3}{1} = 3$$

NOW FORM THE CUBIC AS FOLLOWS – LET THE ROOTS BE A, B & C

- $A+B+C = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \alpha\beta = \frac{\alpha^2+\beta^2}{\alpha\beta} + \alpha\beta = \frac{\alpha^2+\beta^2}{3} + 3$
  $= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{3} + 3 = \frac{(-3)^2 - 2\times 3}{3} + 3 = 4$

- $AB+BC+CA = \frac{\alpha}{\beta}\times\frac{\beta}{\alpha} + \frac{\beta}{\alpha}(\alpha\beta) + \alpha\beta\times\frac{\alpha}{\beta} = 1 + \beta^2 + \alpha^2$
  $= (\alpha+\beta)^2 - 2\alpha\beta + 1 = (-3)^2 - 2\times 3 + 1 = 4$

- $ABC = \frac{\alpha}{\beta}\times\frac{\beta}{\alpha}\times\alpha\beta = \alpha\beta = 3$

FINALLY WE HAVE

$$x^3 - (4x^2) + (4x) - (3) = 0$$
$$x^3 - 4x^2 + 4x - 3 = 0$$

Question 12 (***)

The roots of the cubic equation

$$x^3 + px^2 + 74x + q = 0 ,$$

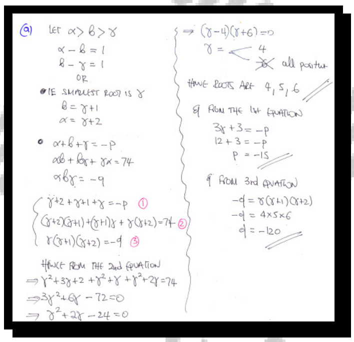
where p and q are constants, form an arithmetic sequence with common difference 1.

Given that all three roots are real and positive find in any order …

a) … the value of p and the value of q.

b) … the roots of the equation.

$\boxed{p = -15,\ q = -120}$, $\boxed{x = 4, 5, 6}$



(a) LET $\alpha > \beta > \gamma$

$\alpha - \beta = 1$
$\beta - \gamma = 1$
OR

• IE SMALLEST ROOT IS γ
$\beta = \gamma + 1$
$\alpha = \gamma + 2$

• $\alpha + \beta + \gamma = -p$
$\alpha\beta + \beta\gamma + \gamma\alpha = 74$
$\alpha\beta\gamma = -q$

$\gamma + 2 + \gamma + 1 + \gamma = -p$ ①
$(\gamma+2)(\gamma+1) + (\gamma+1)\gamma + \gamma(\gamma+2) = 74$ ②
$\gamma(\gamma+1)(\gamma+2) = -q$ ③

HENCE FROM THE 2nd EQUATION
$\Rightarrow \gamma^2 + 3\gamma + 2 + \gamma^2 + \gamma + \gamma^2 + 2\gamma = 74$
$\Rightarrow 3\gamma^2 + 6\gamma - 72 = 0$
$\Rightarrow \gamma^2 + 2\gamma - 24 = 0$
$\Rightarrow (\gamma - 4)(\gamma + 6) = 0$
$\gamma = 4$ or $\cancel{-6}$ all positive

HENCE ROOTS ARE $4, 5, 6$

p FROM THE 1st EQUATION
$3\gamma + 3 = -p$
$12 + 3 = -p$
$p = -15$

q FROM 3rd EQUATION
$-q = \gamma(\gamma+1)(\gamma+2)$
$-q = 4 \times 5 \times 6$
$q = -120$

Question 13 (***)

The roots of the cubic equation

$$x^3 + px^2 + 56x + q = 0,$$

where p and q are constants, form a geometric sequence with common ratio 2.

Given that all three roots are real and positive find in any order …

a) … the value of p and the value of q.

b) … the roots of the equation.

$\boxed{p = -14,\ q = -64}$, $\boxed{x = 2, 4, 8}$

LET ROOTS BE $\alpha, 2\alpha, 4\alpha$

$$\left.\begin{array}{l} \bullet\ \alpha + 2\alpha + 4\alpha = -p \\ \bullet\ 2\alpha^2 + 8\alpha^2 + 4\alpha^2 = 56 \\ \bullet\ 8\alpha^3 = -q \end{array}\right\} \Rightarrow \left.\begin{array}{l} 7\alpha = -p \\ 14\alpha^2 = 56 \\ 8\alpha^3 = -q \end{array}\right\} \text{HENCE } \begin{array}{l} \alpha^2 = 4 \\ \alpha = 2 \quad (\alpha > 0) \end{array}$$

$\therefore$ ROOTS ARE $2, 4, 8$

& $\begin{array}{l} p = -7\alpha \\ q = -8\alpha^3 \end{array} \Rightarrow \begin{array}{l} p = -14 \\ q = -64 \end{array}$

Question 14 (***)

The roots of the cubic equation

$$ax^3 + bx^2 + cx + d = 0,$$

where a, b, c and d are non zero constants, are the first three terms of a geometric sequence with common ratio 2.

Show clearly that

$$4bc = 49ad.$$

$\boxed{\text{proof}}$

LET THE ROOTS BE $\alpha, 2\alpha, 4\alpha$

- $\alpha + 2\alpha + 4\alpha = -\frac{b}{a}$
- $2\alpha^2 + 8\alpha^2 + 4\alpha^2 = \frac{c}{a}$
- $8\alpha^3 = -\frac{d}{a}$

$$\Longrightarrow \quad 7\alpha = -\frac{b}{a}, \quad 14\alpha^2 = \frac{c}{a} \Longrightarrow 98\alpha^3 = -\frac{b}{a} \times \frac{c}{a}$$

$$8\alpha^3 = \frac{d}{a} \quad 8\alpha^3 = -\frac{d}{a}$$

$$\text{It} \quad \frac{98\cancel{\alpha^3}}{8\cancel{\alpha^3}} = \frac{-\frac{bc}{a^2}}{-\frac{d}{a}}$$

$$\frac{49}{4} = \frac{\cancel{a}bc}{\cancel{a^2}d}$$

$$\frac{49}{4} = \frac{bc}{ad}$$

$\therefore 49ad = 4bc$ // AS REQUIRED

Question 15 (***)

$$f(z) = z^3 - (5+\mathrm{i})z^2 + (9+4\mathrm{i})z + k(1+\mathrm{i}), \quad z \in \mathbb{C}, \; k \in \mathbb{R}$$

The roots of the equation $f(z) = 0$ are denoted by α, β and γ.

a) Given that $\alpha = 1+\mathrm{i}$ show that …

i. … $k = -5$.

ii. … $\beta + \gamma = 4$.

b) Hence find the value of β and the value of γ.

$\boxed{\beta = 2+\mathrm{i}}$, $\boxed{\gamma = 2-\mathrm{i}}$

(a) (I) $z^3 - (5+i)z^2 + (9+4i)z + k(1+i) = 0$

• $\alpha = 1+i$ is a root

$(1+i)^3 - (5+i)(1+i)^2 + (9+4i)(1+i) + k(1+i) = 0$

$(1+i)^2 - (5+i)(1+i) + (9+4i) + k = 0$

$(1+2i-1) - (5+5i+i-1) + 9+4i + k = 0$

$\cancel{2i} - 5 \cancel{-6i} + 1 + 9 \cancel{+4i} + k = 0$

$5 + k = 0$

$k = -5$ AS REQUIRED

(II) $\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{-(5+i)}{1} = 5+i$

$\alpha + \beta + \gamma = 5+i$

$1 \cancel{+i} + \beta + \gamma = 5 \cancel{+i}$

$\beta + \gamma = 4$ AS REQUIRED

(b) $\alpha\beta\gamma = -k(1+i)$

$(1+i)\beta\gamma = 5(1+i)$

$\boxed{\beta\gamma = 5}$

& $\beta + \gamma = 4$

$\boxed{\beta = 4 - \gamma}$

THUS $(4-\gamma)\gamma = 5$

$\Rightarrow 4\gamma - \gamma^2 = 5$

$\Rightarrow 0 = \gamma^2 - 4\gamma + 5$

$\Rightarrow (\gamma - 2)^2 - 4 + 5 = 0$

$\Rightarrow (\gamma - 2)^2 = -1$

$\gamma - 2 = \pm i$

$\gamma = 2 \pm i = \begin{cases} 2+i \\ 2-i \end{cases}$

USING $\beta = 4 - \gamma$

$\beta = \begin{cases} 2-i \\ 2+i \end{cases}$

$\therefore \beta$ & $\gamma = 2 \pm i$ IN EITHER ORDER

Question 16 (***)

$$z^3 + pz + q = 0, \ z \in \mathbb{C}, \ p \in \mathbb{R}, \ q \in \mathbb{R}.$$

The roots of the above equation are denoted by α, β and γ.

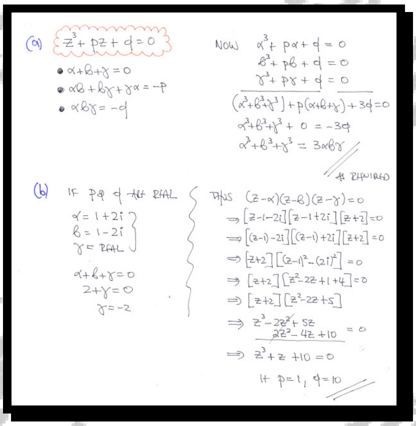
a) Show clearly that

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma.$$

It is further given that $\alpha = 1 + 2\mathrm{i}$.

b) Determine the value of p and the value of q.

$\boxed{p = 1}$, $\boxed{q = 10}$


(a) $z^3 + pz + q = 0$

- $\alpha + \beta + \gamma = 0$
- $\alpha\beta + \beta\gamma + \gamma\alpha = -p$
- $\alpha\beta\gamma = -q$

NOW
$$\alpha^3 + p\alpha + q = 0$$
$$\beta^3 + p\beta + q = 0$$
$$\gamma^3 + p\gamma + q = 0$$
$$(\alpha^3+\beta^3+\gamma^3) + p(\alpha+\beta+\gamma) + 3q = 0$$
$$\alpha^3+\beta^3+\gamma^3 + 0 = -3q$$
$$\alpha^3+\beta^3+\gamma^3 = 3\alpha\beta\gamma$$

AS REQUIRED

(b) IF p & q ARE REAL

$\alpha = 1+2i$
$\beta = 1-2i$
$\gamma =$ REAL

$\alpha + \beta + \gamma = 0$
$2 + \gamma = 0$
$\gamma = -2$

THUS $(z-\alpha)(z-\beta)(z-\gamma) = 0$
$\Rightarrow [z-1-2i][z-1+2i][z+2] = 0$
$\Rightarrow [(z-1)-2i][(z-1)+2i][z+2] = 0$
$\Rightarrow [z+2][(z-1)^2-(2i)^2] = 0$
$\Rightarrow [z+2][z^2-2z+1+4] = 0$
$\Rightarrow [z+2][z^2-2z+5]$
$\Rightarrow z^3 - \cancel{2z^2} + 5z + \cancel{2z^2} - 4z + 10 = 0$
$\Rightarrow z^3 + z + 10 = 0$

$\therefore p = 1, \ q = 10$

Question 17 (***+)

The three solutions of the cubic equation

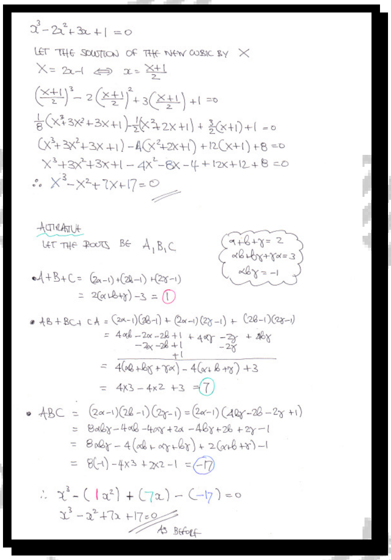
$$x^3 - 2x^2 + 3x + 1 = 0 \quad x \in \mathbb{R},$$

are denoted by α, β and γ.

Find a cubic equation with integer coefficients whose solutions are

$$2\alpha - 1, \ 2\beta - 1 \ \text{and} \ 2\gamma - 1.$$

$$\boxed{x^3 - x^2 + 7x + 17 = 0}$$



$x^3 - 2x^2 + 3x + 1 = 0$

LET THE SOLUTION OF THE NEW CUBIC BY X

$X = 2x - 1 \Longleftrightarrow x = \frac{X+1}{2}$

$\left(\frac{X+1}{2}\right)^3 - 2\left(\frac{X+1}{2}\right)^2 + 3\left(\frac{X+1}{2}\right) + 1 = 0$

$\frac{1}{8}(X^3 + 3X^2 + 3X + 1) - \frac{1}{2}(X^2 + 2X + 1) + \frac{3}{2}(X+1) + 1 = 0$

$(X^3 + 3X^2 + 3X + 1) - 4(X^2 + 2X + 1) + 12(X+1) + 8 = 0$

$X^3 + 3X^2 + 3X + 1 - 4X^2 - 8X - 4 + 12X + 12 + 8 = 0$

$\therefore X^3 - X^2 + 7X + 17 = 0$

ALTERNATIVE

LET THE ROOTS BE A, B, C

$\alpha + \beta + \gamma = 2$
$\alpha\beta + \beta\gamma + \gamma\alpha = 3$
$\alpha\beta\gamma = -1$

- $A + B + C = (2\alpha - 1) + (2\beta - 1) + (2\gamma - 1)$
$= 2(\alpha + \beta + \gamma) - 3 = 1$

- $AB + BC + CA = (2\alpha - 1)(2\beta - 1) + (2\alpha - 1)(2\gamma - 1) + (2\beta - 1)(2\gamma - 1)$
$= 4\alpha\beta - 2\alpha - 2\beta + 1 + 4\alpha\gamma - 2\gamma + 4\beta\gamma$
$\quad - 2\alpha - 2\beta + 1 \quad - 2\gamma$
$\quad + 1$
$= 4(\alpha\beta + \beta\gamma + \gamma\alpha) - 4(\alpha + \beta + \gamma) + 3$
$= 4 \times 3 - 4 \times 2 + 3 = 7$

- $ABC = (2\alpha - 1)(2\beta - 1)(2\gamma - 1) = (2\alpha - 1)(4\beta\gamma - 2\beta - 2\gamma + 1)$
$= 8\alpha\beta\gamma - 4\alpha\beta - 4\alpha\gamma + 2\alpha - 4\beta\gamma + 2\beta + 2\gamma - 1$
$= 8\alpha\beta\gamma - 4(\alpha\beta + \alpha\gamma + \beta\gamma) + 2(\alpha + \beta + \gamma) - 1$
$= 8(-1) - 4 \times 3 + 2 \times 2 - 1 = -17$

$\therefore x^3 - (1x^2) + (7x) - (-17) = 0$

$x^3 - x^2 + 7x + 17 = 0$

AS BEFORE

Question 18 (***+)

The roots of the cubic equation

$$16x^3 - 8x^2 + 4x - 1 = 0 \quad x \in \mathbb{R},$$

are denoted in the usual notation by α, β and γ.

Find a cubic equation, with integer coefficients, whose roots are

$$\tfrac{4}{3}(\alpha - 1), \ \tfrac{4}{3}(\beta - 1) \ \text{ and } \ \tfrac{4}{3}(\gamma - 1).$$

$\boxed{}$, $\boxed{27x^3 + 90x^2 + 108x + 44 = 0}$

- USING A SUBSTITUTION HERE

$$y = \tfrac{4}{3}(x-1)$$
$$3y = 4x - 4$$
$$4x = 3y + 4$$

- MANIPULATE THE CUBIC FOR SIMPLICITY

$\Rightarrow 16x^3 - 8x^2 + 4x - 1 = 0$ $\times 4$

$\Rightarrow 64x^3 - 32x^2 + 16x - 4 = 0$

$\Rightarrow (4x)^3 - 2(4x)^2 + 4(4x) - 4 = 0$

$\Rightarrow (3y+4)^3 - 2(3y+4)^2 + 4(3y+4) - 4 = 0$

NOW:

$$\boxed{\begin{aligned} (A+B)^3 &= A^3 + 3A^2B + 3AB^2 + B^3 \\ (3y+4)^3 &= 27y^3 + 3(3y)^2 \times 4 + 3(3y) \times 16 + 64 \\ &= 27y^3 + 108y^2 + 144y + 64 \end{aligned}}$$

$\Rightarrow 27y^3 + 108y^2 + 144y + 64 - 2(9y^2 + 24y + 16) + 12y + 16 - 4 = 0$

$$\Rightarrow \left.\begin{aligned} 27y^3 + 108y^2 + 144y + 64 \\ -18y^2 - 48y - 32 \\ 12y + 12 \end{aligned}\right\} = 0$$

$\Rightarrow 27y^3 + 90y^2 + 108y + 44 = 0$

Question 19 (***+)

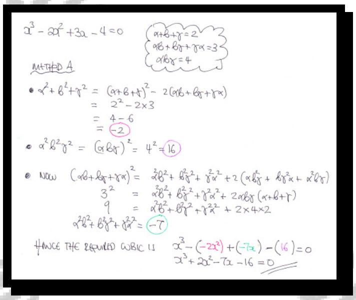
The roots of the equation

$$x^3 - 2x^2 + 3x - 4 = 0,$$

are denoted in the usual notation by α, β and γ.

Find a cubic equation with integers coefficients whose roots are α^2, β^2 and γ^2.

$$\boxed{x^3 + 2x^2 - 7x - 16 = 0}$$



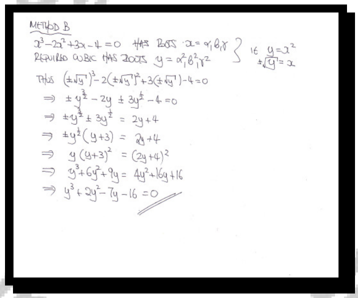
$x^3 - 2x^2 + 3x - 4 = 0$

$\alpha+\beta+\gamma = 2$
$\alpha\beta+\beta\gamma+\gamma\alpha = 3$
$\alpha\beta\gamma = 4$

METHOD A

- $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = 2^2 - 2\times 3 = 4 - 6 = -2$
- $\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = 4^2 = 16$
- NOW $(\alpha\beta+\beta\gamma+\gamma\alpha)^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 + 2(\alpha\beta^2\gamma + \beta\gamma^2\alpha + \alpha^2\beta\gamma)$
  $3^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha+\beta+\gamma)$
  $9 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 + 2\times4\times2$
  $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = -7$

HENCE THE REQUIRED CUBIC IS $x^3 - (-2x^2) + (-7x) - (16) = 0$
$x^3 + 2x^2 - 7x - 16 = 0$



METHOD B

$x^3 - 2x^2 + 3x - 4 = 0$ HAS ROOTS $x = \alpha, \beta, \gamma$
REQUIRED CUBIC HAS ROOTS $y = \alpha^2, \beta^2, \gamma^2$
IE $y = x^2$, $\pm\sqrt{y} = x$

THUS $(\pm\sqrt{y})^3 - 2(\pm\sqrt{y})^2 + 3(\pm\sqrt{y}) - 4 = 0$

$\Rightarrow \pm y^{\frac{3}{2}} - 2y \pm 3y^{\frac{1}{2}} - 4 = 0$

$\Rightarrow \pm y^{\frac{3}{2}} \pm 3y^{\frac{1}{2}} = 2y + 4$

$\Rightarrow \pm y^{\frac{1}{2}}(y+3) = 2y + 4$

$\Rightarrow y(y+3)^2 = (2y+4)^2$

$\Rightarrow y^3 + 6y^2 + 9y = 4y^2 + 16y + 16$

$\Rightarrow y^3 + 2y^2 - 7y - 16 = 0$

Question 20 (***+)

The roots of the equation

$$x^3 - 2x^2 + 3x + 3 = 0$$

are denoted by α, β and γ.

Find the cubic equation with integer coefficients whose roots are

$$\frac{1}{\beta\gamma},\ \frac{1}{\gamma\alpha} \text{ and } \frac{1}{\alpha\beta}.$$

$$\boxed{9x^3 + 6x^2 + 3x - 1 = 0}$$

$$\alpha+\beta+\gamma = 2$$
$$\alpha\beta+\beta\gamma+\gamma\alpha = 3$$
$$\alpha\beta\gamma = -3$$

$$\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}+\frac{1}{\alpha\beta} = \frac{\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2}{\alpha^2\beta^2\gamma^2} = \frac{\alpha\beta\gamma(\alpha+\beta+\gamma)}{(\alpha\beta\gamma)^2}$$
$$= \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{2}{-3}$$

$$\frac{1}{\beta\gamma}\times\frac{1}{\gamma\alpha}+\frac{1}{\gamma\alpha}\times\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}\times\frac{1}{\alpha\beta} = \frac{1}{\alpha\beta\gamma^2}+\frac{1}{\alpha^2\beta\gamma}+\frac{1}{\alpha\beta^2\gamma}$$
$$= \frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha^2\beta^2\gamma^2} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{(\alpha\beta\gamma)^2} = \frac{3}{(-3)^2} = \frac{1}{3}$$

$$\frac{1}{\beta\gamma}\times\frac{1}{\gamma\alpha}\times\frac{1}{\alpha\beta} = \frac{1}{\alpha^2\beta^2\gamma^2} = \frac{1}{(\alpha\beta\gamma)^2} = \frac{1}{(-3)^2} = \frac{1}{9}$$

HENCE

$$x^3 - \left(-\tfrac{2}{3}x^2\right) + \left(\tfrac{1}{3}x\right) - \left(\tfrac{1}{9}\right) = 0$$
$$x^3 + \tfrac{2}{3}x^2 + \tfrac{1}{3}x - \tfrac{1}{9} = 0$$
$$9x^3 + 6x^2 + 3x - 1 = 0$$

Question 21 (***+)

The roots of the equation

$$x^3 + 2kx^2 - 27 = 0,$$

are α, β and $\alpha + \beta$, where k is a real constant.

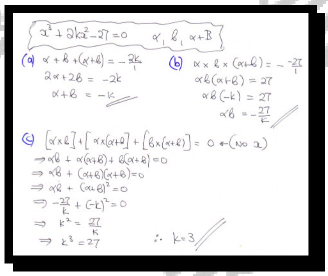
a) Find, in terms of k, the value of …

i. … $\alpha + \beta$

ii. … $\alpha\beta$

b) Use these results to show that $k = 3$.

$\boxed{\alpha + \beta = -k}$, $\boxed{\alpha\beta = -\frac{27}{k}}$



$x^3 + 2kx^2 - 27 = 0 \qquad \alpha, \beta, \alpha+\beta$

(a) $\alpha + \beta + (\alpha+\beta) = -\frac{2k}{1}$

$2\alpha + 2\beta = -2k$

$\alpha + \beta = -k$

(b) $\alpha \times \beta \times (\alpha+\beta) = -\frac{-27}{1}$

$\alpha\beta(\alpha+\beta) = 27$

$\alpha\beta(-k) = 27$

$\alpha\beta = -\frac{27}{k}$

(c) $[\alpha \times \beta] + [\alpha \times (\alpha+\beta)] + [\beta \times (\alpha+\beta)] = 0 \leftarrow$ (NO x)

$\Rightarrow \alpha\beta + \alpha(\alpha+\beta) + \beta(\alpha+\beta) = 0$

$\Rightarrow \alpha\beta + (\alpha+\beta)(\alpha+\beta) = 0$

$\Rightarrow \alpha\beta + (\alpha+\beta)^2 = 0$

$\Rightarrow -\frac{27}{k} + (-k)^2 = 0$

$\Rightarrow k^2 = \frac{27}{k}$

$\Rightarrow k^3 = 27 \qquad \therefore k = 3$

Question 22 **(***+)**

The roots of the equation

$$x^3 + 2x^2 + 3x - 4 = 0$$

are denoted by α, β and γ.

a) Show that that for all w, y and z

$$w^2 + y^2 + z^2 \equiv (w + y + z)^2 - 2(wy + yz + zw).$$

Another cubic equation has roots A, B and C where

$$A = \frac{\beta\gamma}{\alpha}, \; B = \frac{\gamma\alpha}{\beta} \text{ and } C = \frac{\alpha\beta}{\gamma}.$$

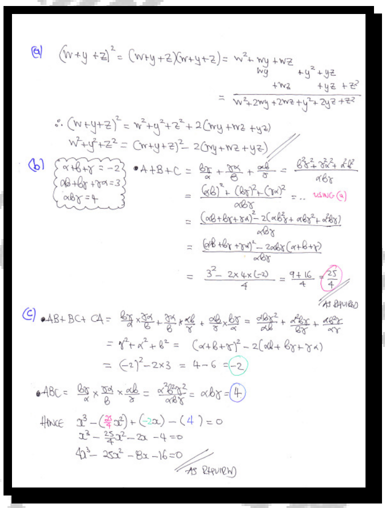
b) Show clearly that

$$A + B + C = \frac{25}{4}.$$

c) Show that the equation of the cubic whose roots are A, B and C is

$$4x^3 - 25x^2 - 8x - 16 = 0.$$

$\boxed{\text{proof}}$



(a) $(w+y+z)^2 = (w+y+z)(w+y+z) = w^2 + wy + wz + wy + y^2 + yz + wz + yz + z^2$
$= w^2 + 2wy + 2wz + y^2 + 2yz + z^2$

$\therefore (w+y+z)^2 = w^2 + y^2 + z^2 + 2(wy + wz + yz)$
$w^2 + y^2 + z^2 = (w+y+z)^2 - 2(wy + wz + yz)$

(b) $\alpha + \beta + \gamma = -2$, $\alpha\beta + \beta\gamma + \gamma\alpha = 3$, $\alpha\beta\gamma = 4$

• $A + B + C = \frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta} + \frac{\alpha\beta}{\gamma} = \frac{\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2}{\alpha\beta\gamma}$
$= \frac{(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2}{\alpha\beta\gamma} = \ldots$ USING (a)
$= \frac{(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta^2\gamma + \alpha\beta\gamma^2 + \alpha^2\beta\gamma)}{\alpha\beta\gamma}$
$= \frac{(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)}{\alpha\beta\gamma}$
$= \frac{3^2 - 2 \times 4 \times (-2)}{4} = \frac{9 + 16}{4} = \frac{25}{4}$

AS REQUIRED

(c) • $AB + BC + CA = \frac{\beta\gamma}{\alpha} \times \frac{\gamma\alpha}{\beta} + \frac{\gamma\alpha}{\beta} \times \frac{\alpha\beta}{\gamma} + \frac{\alpha\beta}{\gamma} \times \frac{\beta\gamma}{\alpha} = \frac{\alpha\beta\gamma^2}{\alpha\beta} + \frac{\alpha^2\beta\gamma}{\beta\gamma} + \frac{\alpha\beta^2\gamma}{\alpha\gamma}$
$= \gamma^2 + \alpha^2 + \beta^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
$= (-2)^2 - 2 \times 3 = 4 - 6 = -2$

• $ABC = \frac{\beta\gamma}{\alpha} \times \frac{\gamma\alpha}{\beta} \times \frac{\alpha\beta}{\gamma} = \frac{\alpha^2\beta^2\gamma^2}{\alpha\beta\gamma} = \alpha\beta\gamma = 4$

HENCE $x^3 - \left(\frac{25}{4}x^2\right) + (-2x) - (4) = 0$
$x^3 - \frac{25}{4}x^2 - 2x - 4 = 0$
$4x^3 - 25x^2 - 8x - 16 = 0$

AS REQUIRED

Question 23 (***+)

The cubic equation

$$2z^3 + kz^2 + 1 = 0 \ , \ z \in \mathbb{C},$$

where k is a non zero constant, is given.

a) If the above cubic has two identical roots, determine the value of k.

b) If **instead** one of the roots is $1+\mathrm{i}$, find the value of k in this case.

$\boxed{}$, $\boxed{k=-3}$, $\boxed{k=-\frac{1}{2}(4+3\mathrm{i})}$

$2z^3 + kz^2 + 1 = 0 \ , \ z \in \mathbb{C}$

a) REPEATED ROOTS CASE

$$\left.\begin{aligned} \alpha+\beta+\gamma &= -\tfrac{k}{2} \\ \alpha\beta+\beta\gamma+\gamma\alpha &= 0 \\ \alpha\beta\gamma &= -\tfrac{1}{2} \end{aligned}\right\}$$

LET $\alpha = \beta$

$$\left.\begin{aligned} 2\alpha+\gamma &= -\tfrac{k}{2} \\ \alpha^2+\alpha\gamma+\gamma\alpha &= 0 \\ \alpha^2\gamma &= -\tfrac{1}{2} \end{aligned}\right\} \Rightarrow \begin{aligned} k &= -4\alpha-2\gamma \quad \text{—(I)} \\ \alpha^2+2\alpha\gamma &= 0 \quad \text{—(II)} \\ \alpha^2\gamma &= -\tfrac{1}{2} \quad \text{—(III)} \end{aligned}$$

LOOKING AT THE "SECOND" EQUATION

$\Rightarrow \alpha^2 + 2\alpha\gamma = 0$

$\Rightarrow \alpha(\alpha+2\gamma) = 0$

$\alpha \neq 0$ BY INSPECTION

$\Rightarrow \alpha = -2\gamma$

SUB INTO THE "THIRD" EQUATION

$\Rightarrow (-2\gamma)^2\gamma = -\frac{1}{2}$

$\Rightarrow 4\gamma^3 = -\frac{1}{2}$

$\Rightarrow \gamma^3 = -\frac{1}{8}$

$\Rightarrow \gamma = -\frac{1}{2}$ & $\alpha = 1$

FINALLY k CAN BE FOUND

$\Rightarrow k = -4\alpha - 2\gamma = -4 + 1 = -3$ $\quad \therefore k = -3$

b) CASE WHERE ONE OF THE ROOTS IS $1+i$

$\Rightarrow z = 1+i$

$\Rightarrow z^2 = (1+i)^2 = 1+2i+i^2 = 1+2i-1 = 2i$

$\Rightarrow z^3 = (1+i)(1+i)^2 = (1+i)\times 2i = 2i-2 = -2+2i$

SUBSTITUTE INTO THE CUBIC

$\Rightarrow 2z^3 + kz^2 + 1 = 0$

$\Rightarrow 2(-2+2i) + k(2i) + 1 = 0$

$\Rightarrow -4+4i+2ki+1 = 0$

$\Rightarrow -3+4i+2ki = 0$ $\quad \times(-i)$

$\Rightarrow +3i+4+2k = 0$

$\Rightarrow 2k = -4-3i$

$\Rightarrow k = -\frac{1}{2}(4+3i)$

Question 24 (***+)

A cubic equation is given below as

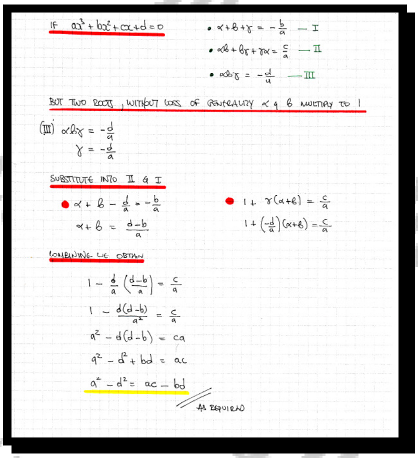
$$ax^3 + bx^2 + cx + d = 0,$$

where a, b, c and d are non zero constants.

Given that the product of two of the three roots of above cubic equation is 1, show that

$$a^2 - d^2 = ac - bd.$$

$\boxed{}$, $\boxed{\text{proof}}$



IF $ax^3+bx^2+cx+d=0$

• $\alpha+\beta+\gamma = -\frac{b}{a}$ — I

• $\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$ — II

• $\alpha\beta\gamma = -\frac{d}{a}$ — III

BUT TWO ROOTS, WITHOUT LOSS OF GENERALITY α & β MULTIPLY TO 1

(III) $\alpha\beta\gamma = -\frac{d}{a}$

$\gamma = -\frac{d}{a}$

SUBSTITUTE INTO II & I

• $\alpha+\beta-\frac{d}{a} = -\frac{b}{a}$

$\alpha+\beta = \frac{d-b}{a}$

• $1+\gamma(\alpha+\beta) = \frac{c}{a}$

$1+\left(-\frac{d}{a}\right)(\alpha+\beta) = \frac{c}{a}$

COMBINING WE OBTAIN

$1-\frac{d}{a}\left(\frac{d-b}{a}\right) = \frac{c}{a}$

$1-\frac{d(d-b)}{a^2} = \frac{c}{a}$

$a^2 - d(d-b) = ca$

$a^2 - d^2 + bd = ac$

$a^2 - d^2 = ac - bd$

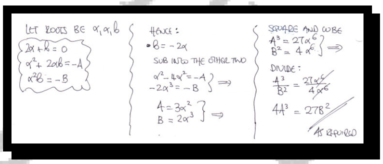
AS REQUIRED

Question 25 (***+)

If the cubic equation $x^3 - Ax + B = 0$, has two equal roots, show that

$$4A^3 = 27B^2.$$

$\boxed{\text{proof}}$



LET ROOTS BE α, α, β

$2\alpha+\beta = 0$

$\alpha^2 + 2\alpha\beta = -A$

$\alpha^2\beta = -B$

HENCE:

• $\beta = -2\alpha$

SUB INTO THE OTHER TWO

$\left.\begin{array}{l}\alpha^2 - 4\alpha^2 = -A \\ -2\alpha^3 = -B\end{array}\right\} \Rightarrow$

$\left.\begin{array}{l}A = 3\alpha^2 \\ B = 2\alpha^3\end{array}\right\} \Rightarrow$

SQUARE AND CUBE

$\left.\begin{array}{l}A^3 = 27\alpha^6 \\ B^2 = 4\alpha^6\end{array}\right\} \Rightarrow$

DIVIDE:

$\frac{A^3}{B^2} = \frac{27\cancel{\alpha^6}}{4\cancel{\alpha^6}}$

$4A^3 = 27B^2$

AS REQUIRED

Question 26 (****)

$$bx^3 + bx^2 + cx + d = 0,$$

where a, b and c are non zero constants.

If the three roots of the above cubic equation are in geometric progression show that

$$b^3 = ca^3.$$

proof

$$\boxed{\begin{aligned} \alpha+\beta+\gamma &= -a \\ \alpha\beta+\beta\gamma+\gamma\alpha &= b \\ \alpha\beta\gamma &= -c \end{aligned}}$$

IF IN GEOMETRIC PROGRESSION, LET WITHOUT LOSS OF GENERALITY BE

$\alpha, \ \alpha r, \ \alpha r^2$ ($r \neq 0$, COMMON RATIO)

HENCE $\alpha + \alpha r + \alpha r^2 = -a \implies \alpha(1+r+r^2) = -a$

$\alpha^2 r + \alpha^2 r^3 + \alpha^2 r^2 = b \implies \alpha^2 r(1+r^2+r) = b$ DIVIDE THESE TWO

& $\boxed{\alpha^3 r^3 = -c}$

$\implies \dfrac{\alpha}{\alpha^2 r} = -\dfrac{a}{b}$

$\implies \dfrac{1}{\alpha r} = -\dfrac{a}{b}$

$\implies \alpha r = -\dfrac{b}{a}$

$\implies \alpha^3 r^3 = -\dfrac{b^3}{a^3}$

$\implies -c = -\dfrac{b^3}{a^3}$

$\therefore \ b^3 = a^3 c$ AS REQUIRED

Question 27 (****)

The three roots of the equation

$$x^3 + 2x^2 + 10x + k = 0,$$

where k is a non zero constant, are in geometric progression.

Determine the value of k.

$\boxed{}$, $\boxed{k = 125}$

- USING THE STANDARD RELATIONSHIPS BETWEEN THE ROOTS AND THE COEFFICIENTS OF A CUBIC

$$x^3 + 2x^2 + 10x + k = 0 \qquad \begin{aligned} \alpha+\beta+\gamma &= -\frac{b}{a} = \frac{-2}{1} = -2 \\ \alpha\beta+\beta\gamma+\gamma\alpha &= \frac{c}{a} = \frac{10}{1} = 10 \\ \alpha\beta\gamma &= -k \end{aligned}$$

- AS THE ROOTS ARE IN GEOMETRIC PROGRESSION

$$\alpha+\beta+\gamma = \alpha + \alpha r + \alpha r^2$$
$$\alpha\beta+\beta\gamma+\gamma\alpha = \alpha(\alpha r) + (\alpha r)(\alpha r^2) + (\alpha r^2)\alpha = \alpha^2 r + \alpha^2 r^3 + \alpha^2 r^2$$
$$\alpha\beta\gamma = \alpha(\alpha r)(\alpha r^2) = \alpha^3 r^3$$

- TIDYING UP THESE EXPRESSIONS

$$\left.\begin{aligned} \alpha + \alpha r + \alpha r^2 &= -2 \\ \alpha^2 r + \alpha^2 r^3 + \alpha^2 r^2 &= 10 \\ \alpha^3 r^3 &= -k \end{aligned}\right\} \Longrightarrow \begin{aligned} \alpha(1+r+r^2) &= -2 \quad &-\mathrm{I} \\ \alpha^2 r(1+r^2+r) &= 10 \quad &-\mathrm{II} \\ k &= -(\alpha r)^3 \quad &-\mathrm{III} \end{aligned}$$

- DIVIDING EQUATIONS I & II

$$\frac{\alpha^2 r\cancel{(1+r+r^2)}}{\alpha\cancel{(1+r+r^2)}} = \frac{10}{-2} \qquad \therefore \quad \alpha r = -5$$

- HENCE EQUATION III GIVES

$$k = -(\alpha r)^3 = -(-5)^3 = 125$$

Question 28 (****)

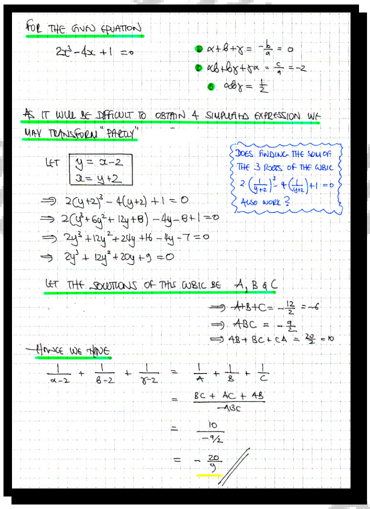
$$2x^3 - 4x + 1 = 0.$$

The cubic equation shown above has three roots, denoted by α, β and γ.

Determine, as an exact simplified fraction, the value of

$$\frac{1}{\alpha-2} + \frac{1}{\beta-2} + \frac{1}{\gamma-2}.$$

$\boxed{}$, $\boxed{-\frac{20}{9}}$



FOR THE GIVEN EQUATION

$2x^3 - 4x + 1 = 0$

- $\alpha+\beta+\gamma = -\frac{b}{a} = 0$
- $\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} = -2$
- $\alpha\beta\gamma = \frac{1}{2}$

AS IT WILL BE DIFFICULT TO OBTAIN A SIMPLIFIED EXPRESSION WE MAY TRANSFORM "PARTLY"

LET $\boxed{\begin{array}{l} y = x-2 \\ x = y+2 \end{array}}$

DOES FINDING THE SUM OF THE 3 ROOTS OF THE CUBIC $2\left(\frac{1}{y+2}\right)^3 - 4\left(\frac{1}{y+2}\right) + 1 = 0$ ALSO WORK?

$\Rightarrow 2(y+2)^3 - 4(y+2) + 1 = 0$

$\Rightarrow 2(y^3+6y^2+12y+8) - 4y - 8 + 1 = 0$

$\Rightarrow 2y^3 + 12y^2 + 24y + 16 - 4y - 7 = 0$

$\Rightarrow 2y^3 + 12y^2 + 20y + 9 = 0$

LET THE SOLUTIONS OF THIS CUBIC BE A, B & C

$\Rightarrow A+B+C = -\frac{12}{2} = -6$

$\Rightarrow ABC = -\frac{9}{2}$

$\Rightarrow AB+BC+CA = \frac{20}{2} = 10$

HENCE WE HAVE

$$\frac{1}{\alpha-2} + \frac{1}{\beta-2} + \frac{1}{\gamma-2} = \frac{1}{A} + \frac{1}{B} + \frac{1}{C} = \frac{BC+AC+AB}{ABC} = \frac{10}{-9/2} = -\frac{20}{9}$$

Question 29 (****)

A cubic equation is given below as

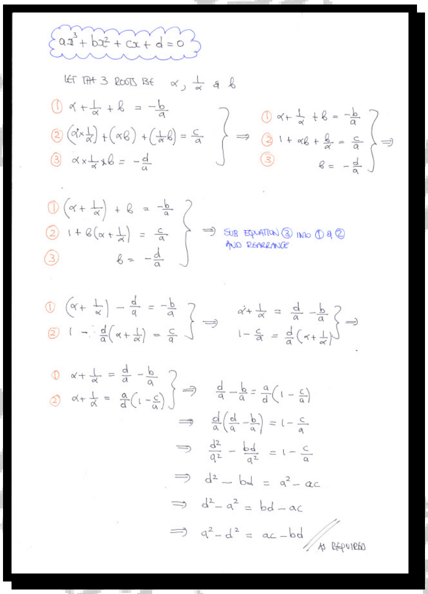
$$ax^3 + bx^2 + cx + d = 0,$$

where a, b, c and d are non zero constants.

Given that two of the three roots of above cubic equation are reciprocals of one another show that

$$a^2 - d^2 = ac - bd.$$

$\boxed{\text{proof}}$



$$ax^3 + bx^2 + cx + d = 0$$

LET THE 3 ROOTS BE α, $\frac{1}{\alpha}$ & β

$$\left.\begin{array}{ll} ① & \alpha + \frac{1}{\alpha} + \beta = -\frac{b}{a} \\ ② & \left(\alpha \times \frac{1}{\alpha}\right) + (\alpha\beta) + \left(\frac{1}{\alpha}\beta\right) = \frac{c}{a} \\ ③ & \alpha \times \frac{1}{\alpha} \times \beta = -\frac{d}{a} \end{array}\right\} \Rightarrow \left.\begin{array}{ll} ① & \alpha + \frac{1}{\alpha} + \beta = -\frac{b}{a} \\ ② & 1 + \alpha\beta + \frac{\beta}{\alpha} = \frac{c}{a} \\ ③ & \beta = -\frac{d}{a} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{ll} ① & \left(\alpha + \frac{1}{\alpha}\right) + \beta = -\frac{b}{a} \\ ② & 1 + \beta\left(\alpha + \frac{1}{\alpha}\right) = \frac{c}{a} \\ ③ & \beta = -\frac{d}{a} \end{array}\right\} \Rightarrow$$ SUB EQUATION ③ INTO ① & ② AND REARRANGE

$$\left.\begin{array}{ll} ① & \left(\alpha + \frac{1}{\alpha}\right) - \frac{d}{a} = -\frac{b}{a} \\ ② & 1 - \frac{d}{a}\left(\alpha + \frac{1}{\alpha}\right) = \frac{c}{a} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \alpha + \frac{1}{\alpha} = \frac{d}{a} - \frac{b}{a} \\ 1 - \frac{c}{a} = \frac{d}{a}\left(\alpha + \frac{1}{\alpha}\right) \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{ll} ① & \alpha + \frac{1}{\alpha} = \frac{d}{a} - \frac{b}{a} \\ ② & \alpha + \frac{1}{\alpha} = \frac{a}{d}\left(1 - \frac{c}{a}\right) \end{array}\right\} \Rightarrow \frac{d}{a} - \frac{b}{a} = \frac{a}{d}\left(1 - \frac{c}{a}\right)$$

$$\Rightarrow \frac{d}{a}\left(\frac{d}{a} - \frac{b}{a}\right) = 1 - \frac{c}{a}$$

$$\Rightarrow \frac{d^2}{a^2} - \frac{bd}{a^2} = 1 - \frac{c}{a}$$

$$\Rightarrow d^2 - bd = a^2 - ac$$

$$\Rightarrow d^2 - a^2 = bd - ac$$

$$\Rightarrow a^2 - d^2 = ac - bd$$

AS REQUIRED

Question 30 (****)

$$x^3 - 2x^2 + kx + 10 = 0\,,\ k \neq 0$$

The roots of the above cubic equation are α, β and γ.

a) Show clearly that

$$\left(\alpha^3 + \beta^3 + \gamma^3\right) - 2\left(\alpha^2 + \beta^2 + \gamma^2\right) + k\left(\alpha + \beta + \gamma\right) + 30 = 0\,.$$

It is given that $\alpha^3 + \beta^3 + \gamma^3 = -4$

b) Show further that $k = -3$.

proof

(a) AS α, β, γ ARE ROOTS $\Rightarrow$

$$\alpha^3 - 2\alpha^2 + k\alpha + 10 = 0$$
$$\beta^3 - 2\beta^2 + k\beta + 10 = 0$$
$$\gamma^3 - 2\gamma^2 + k\gamma + 10 = 0$$

ADD $(\alpha^3+\beta^3+\gamma^3) - 2(\alpha^2+\beta^2+\gamma^2) + k(\alpha+\beta+\gamma) + 30 = 0$

AS REQUIRED

(b) NOW $\alpha+\beta+\gamma = 2$

$\alpha\beta+\beta\gamma+\gamma\alpha = k$

$\alpha\beta\gamma = -10$

$$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2 + 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$4 = \alpha^2+\beta^2+\gamma^2 + 2k$$
$$\boxed{\alpha^2+\beta^2+\gamma^2 = 4-2k}$$

HENCE:

$$(\alpha^3+\beta^3+\gamma^3) - 2(\alpha^2+\beta^2+\gamma^2) + k(\alpha+\beta+\gamma) + 30 = 0$$
$$-4 - 2(4-2k) + k\times 2 + 30 = 0$$
$$-4 - 8 + 4k + 2k + 30 = 0$$
$$6k = -18$$
$$k = -3$$

AS REQUIRED

Question 31 (****)

The three roots of the equation

$$z^3 + pz^2 + qz + r = 0,$$

where p, q and r are constants, are denoted by α, β and γ.

a) Given that

$$\alpha\beta + \beta\gamma + \gamma\alpha = -2 + 3\text{i} \quad \text{and} \quad \alpha^2 + \beta^2 + \gamma^2 = 4 - 6\text{i},$$

determine the value of p and the value of q.

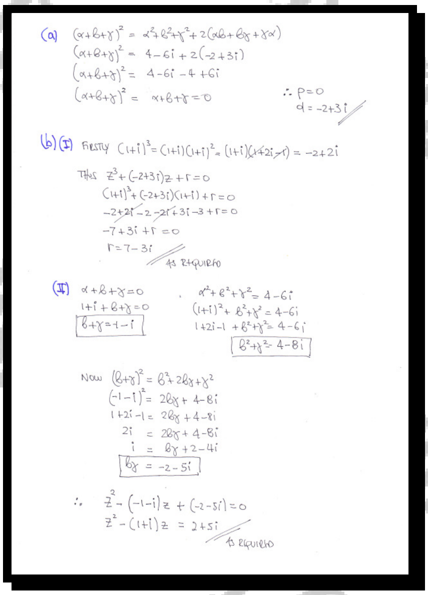
b) Given further that $\alpha = 1 + \text{i}$, show that …

i. … $r = 7 - 3\text{i}$

ii. … β and γ are solutions of the equation

$$z^2 - (1+\text{i})z = 2 + 5\text{i}.$$

$\boxed{p = 0}$, $\boxed{q = -2 + 3\text{i}}$


(a) $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha)$
$(\alpha+\beta+\gamma)^2 = 4-6i+2(-2+3i)$
$(\alpha+\beta+\gamma)^2 = 4-6i-4+6i$
$(\alpha+\beta+\gamma)^2 = \alpha+\beta+\gamma=0$

$\therefore p=0$
$q=-2+3i$

(b)(I) FIRSTLY $(1+i)^3=(1+i)(1+i)^2=(1+i)(\cancel{1}+2i\cancel{-1})=-2+2i$

THUS $z^3+(-2+3i)z+r=0$
$(1+i)^3+(-2+3i)(1+i)+r=0$
$-2+\cancel{2i}-2\cancel{-2i}+3i-3+r=0$
$-7+3i+r=0$
$r=7-3i$
AS REQUIRED

(II) $\alpha+\beta+\gamma=0$
$1+i+\beta+\gamma=0$
$\boxed{\beta+\gamma=-1-i}$

$\alpha^2+\beta^2+\gamma^2=4-6i$
$(1+i)^2+\beta^2+\gamma^2=4-6i$
$1+2i-1+\beta^2+\gamma^2=4-6i$
$\boxed{\beta^2+\gamma^2=4-8i}$

NOW $(\beta+\gamma)^2=\beta^2+2\beta\gamma+\gamma^2$
$(-1-i)^2=2\beta\gamma+4-8i$
$1+2i-1=2\beta\gamma+4-8i$
$2i=2\beta\gamma+4-8i$
$i=\beta\gamma+2-4i$
$\boxed{\beta\gamma=-2-5i}$

$\therefore$ $z^2-(-1-i)z+(-2-5i)=0$
$z^2-(1+i)z=2+5i$
AS REQUIRED

Question 32 (****)

$$z^3 + 2z^2 + k = 0,$$

The roots of the above cubic equation, where k is a non zero constant, are denoted by α, β and γ.

a) Show that …

i. … $\alpha^2 + \beta^2 + \gamma^2 = 4$.

ii. … $\alpha^3 + \beta^3 + \gamma^3 = -8 - 3k$.

It is further given that $\alpha^4 + \beta^4 + \gamma^4 = 4$.

b) Show further that $k = -1$.

c) Determine the value of

$$\alpha^5 + \beta^5 + \gamma^5.$$

$\boxed{}$, $\boxed{\alpha^5 + \beta^5 + \gamma^5 = -4}$

a) LOOKING AT THE CUBIC

$\alpha+\beta+\gamma = -2$
$\alpha\beta+\beta\gamma+\gamma\alpha = 0$
$\alpha\beta\gamma = -k$

I) $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2 + \cancel{(\alpha\beta+\beta\gamma+\gamma\alpha)}$

$\therefore \alpha^2+\beta^2+\gamma^2 = (-2)^2$
$\alpha^2+\beta^2+\gamma^2 = 4$

AS REQUIRED

II) AS α, β, γ ARE ROOTS

$\left.\begin{array}{l}\alpha^3 + 2\alpha^2 + k = 0\\ \beta^3 + 2\beta^2 + k = 0\\ \gamma^3 + 2\gamma^2 + k = 0\end{array}\right\}$ ADDING $\alpha^3+\beta^3+\gamma^3 + 2(\alpha^2+\beta^2+\gamma^2) + 3k = 0$

$\alpha^3+\beta^3+\gamma^3 + 8 + 3k = 0$

$\therefore \alpha^3+\beta^3+\gamma^3 = -8-3k$

AS REQUIRED

b) AS $\alpha\beta\gamma = -k \neq 0$ THEN $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$

$\left.\begin{array}{l}\alpha^3 + 2\alpha^2 + k = 0\\ \beta^3 + 2\beta^2 + k = 0\\ \gamma^3 + 2\gamma^2 + k = 0\end{array}\right\}$ MULTIPLY THROUGH EACH EQUATION BY α, β & γ RESPECTIVELY

$\left.\begin{array}{l}\alpha^4 + 2\alpha^3 + \alpha k = 0\\ \beta^4 + 2\beta^3 + \beta k = 0\\ \gamma^4 + 2\gamma^3 + \gamma k = 0\end{array}\right\}$ ADDING AS BEFORE

$\Rightarrow (\alpha^4+\beta^4+\gamma^4) + 2(\alpha^3+\beta^3+\gamma^3) + (\alpha+\beta+\gamma)k = 0$
$\Rightarrow 8 + 2(-8-3k) + (-2)k = 0$
$\Rightarrow 8 - 16 - 6k - 2k = 0$
$\Rightarrow -8 = -8k$

$\therefore k = -1$

AS REQUIRED

c) USING THE APPROACH OF PART (b)

$\alpha^5 + 2\alpha^4 + k\alpha^2 = 0$
$\beta^5 + 2\beta^4 + k\beta^2 = 0$
$\gamma^5 + 2\gamma^4 + k\gamma^2 = 0$

ADDING & NOTING $k=-1$

$(\alpha^5+\beta^5+\gamma^5) + 2(\alpha^4+\beta^4+\gamma^4) + k(\alpha^2+\beta^2+\gamma^2) = 0$
$\alpha^5+\beta^5+\gamma^5 + 2\times 4 + (-1)\times 4 = 0$

$\therefore \alpha^5+\beta^5+\gamma^5 = -4$

Question 33 (****)

The cubic equation shown below has a real root α.

$$x^3 + kx^2 - 1 = 0,$$

where k is a real constant.

Given that one of the complex roots of the equation is $u + iv$, determine the value of v^2 in terms of α.

$$\boxed{v^2 = \frac{1}{\alpha} - \frac{1}{4\alpha^4}}$$

PROCEED AS FOLLOWS

$x^3 + kx^2 - 1 = 0 \quad , \quad k \in \mathbb{R}$

IF α IS A SOLUTION (REAL OR OTHERWISE)

$\Rightarrow \alpha^3 + k\alpha^2 - 1 = 0$

$\Rightarrow k\alpha^2 = 1 - \alpha^3$

$\Rightarrow k = \frac{1-\alpha^3}{\alpha^2}$

$\Rightarrow k = \frac{1}{\alpha^2} - \alpha$

NOW FROM THE ROOT - COEFFICIENT RELATIONSHIPS

$\Rightarrow \alpha + \beta + \gamma = -\frac{k}{1}$

$\Rightarrow \alpha + (u+iv) + (u-iv) = -\left(\frac{1}{\alpha^2} - \alpha\right)$

AS COEFFICIENTS ARE REAL THE OTHER 2 ROOTS ARE COMPLEX CONJUGATES

$\Rightarrow \alpha + 2u = \alpha - \frac{1}{\alpha^2}$

$\Rightarrow 2u = -\frac{1}{\alpha^2}$

$\Rightarrow \boxed{u = -\frac{1}{2\alpha^2}}$

FINALLY FROM ANOTHER RELATIONSHIP

$\alpha\beta\gamma = -\frac{-1}{1} = 1$

$\alpha(u+iv)(u-iv) = 1$

$\alpha(u^2+v^2) = 1$

$\alpha\left[\frac{1}{4\alpha^4} + v^2\right] = 1$

$v^2 + \frac{1}{4\alpha^4} = \frac{1}{\alpha}$

$\therefore v^2 = \frac{1}{\alpha} - \frac{1}{4\alpha^4}$ //

Question 34 (****)

$$x^3 + 2x + 5 = 0.$$

The cubic equation shown above has three roots, denoted by α, β and γ.

Determine the value of

$$\alpha^4 + \beta^4 + \gamma^4.$$

$\boxed{}$, $\boxed{8}$

METHOD A

- $x^3 + 2x + 5 = 0$

$$\boxed{\begin{aligned}&\alpha+\beta+\gamma=0\\&\alpha\beta+\beta\gamma+\gamma\alpha=2\\&\alpha\beta\gamma=-5\end{aligned}}$$

- $\alpha^4+\beta^4+\gamma^4 = (\alpha^2+\beta^2+\gamma^2)^2 - 2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)$

$$= (\alpha^2+\beta^2+\gamma^2)^2 - 2\left[(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2\right]$$

$$= \left[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)\right]^2 - 2\left[(\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2(\alpha\beta^2\gamma+\beta\gamma^2\alpha+\alpha^2\beta\gamma)\right]$$

$$= \left[\underbrace{\cancel{(\alpha+\beta+\gamma)^2}}_{\text{ZERO}} - 2(\alpha\beta+\beta\gamma+\gamma\alpha)\right]^2 - 2\left[(\alpha\beta+\beta\gamma+\gamma\alpha)^2 - \underbrace{\cancel{2\alpha\beta\gamma(\alpha+\beta+\gamma)}}_{\text{ZERO}}\right]$$

$$= 4(\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)^2$$

$$= 2(\alpha\beta+\beta\gamma+\gamma\alpha)^2$$

$$= 2 \times 2^2$$

$$= 8$$

METHOD B – USING A SUBSTITUTION

LET $x = \sqrt{y} = y^{\frac{1}{2}}$

$$\Rightarrow x^3 + 2x + 5 = 0$$
$$\Rightarrow y^{\frac{3}{2}} + 2y^{\frac{1}{2}} + 5 = 0$$
$$\Rightarrow y^{\frac{3}{2}} + 2y^{\frac{1}{2}} = -5$$
$$\Rightarrow \left(y^{\frac{3}{2}} + 2y^{\frac{1}{2}}\right)^2 = 25$$
$$\Rightarrow y^3 + 4y^2 + 4y = 25$$
$$\Rightarrow y^3 + 4y^2 + 4y - 25 = 0$$

NOW IF THE ROOTS OF THE ABOVE CUBIC ARE A, B, C, THEN $A = \alpha^2$, $B = \beta^2$, $C = \gamma^2$

$$\Rightarrow A+B+C = -4 \quad \leftarrow \alpha^2+\beta^2+\gamma^2$$
$$AB+BC+CA = 4 \quad \leftarrow \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2$$
$$ABC = 25 \quad \leftarrow \alpha^2\beta^2\gamma^2$$

$$\Rightarrow \alpha^4+\beta^4+\gamma^4 = A^2+B^2+C^2$$
$$= (A+B+C)^2 - 2(AB+BC+CA)$$
$$= (-4)^2 - 2\times 4$$
$$= 16 - 8$$
$$= 8$$

Question 35 **(****)**

The three roots of the cubic equation

$$x^3 + 2x - 1 = 0,$$

are denoted by α, β and γ.

Determine the exact value of $\dfrac{1}{\alpha^4} + \dfrac{1}{\beta^4} + \dfrac{1}{\gamma^4}$.

$\boxed{}$, $\boxed{24}$

GET ALL THE THREE VALUES OF THE SUMS

$$x^3 + 0x^2 + 2x - 1 = 0$$

$$\boxed{\begin{aligned} \alpha+\beta+\gamma &= -\tfrac{0}{1} = 0 \\ \alpha\beta+\beta\gamma+\gamma\alpha &= +\tfrac{2}{1} = 2 \\ \alpha\beta\gamma &= -\tfrac{-1}{1} = 1 \end{aligned}}$$

START THE TIDY UP

$$\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} = \frac{\beta^4\gamma^4 + \alpha^4\gamma^4 + \alpha^4\beta^4}{(\alpha\beta\gamma)^4} = \frac{(\beta^2\gamma^2)^2 + (\alpha^2\gamma^2)^2 + (\alpha^2\beta^2)^2}{(\alpha\beta\gamma)^4}$$

NOW USING $A^2 + B^2 + C^2 \equiv (A+B+C)^2 - 2(AB+BC+CA)$

$$\ldots = \frac{(\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2)^2 - 2(\alpha^2\beta^2\gamma^4 + \alpha^4\beta^2\gamma^2 + \alpha^2\beta^4\gamma^2)}{1^4}$$

$$= (\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2)^2 - 2\alpha^2\beta^2\gamma^2(\gamma^2 + \alpha^2 + \beta^2)$$

$$= (\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2)^2 - 2(\alpha\beta\gamma)^2(\alpha^2 + \beta^2 + \gamma^2)$$

$$= \left[(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2\right]^2 - 2 \times 1^2 (\alpha^2 + \beta^2 + \gamma^2)$$

$$= \left[(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2\right]^2 - 2(\alpha^2 + \beta^2 + \gamma^2)$$

REAPPLY THE IDENTITY FROM ABOVE

$$= \left[(\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2(\alpha\beta^2\gamma + \alpha\beta\gamma^2 + \alpha^2\beta\gamma)\right]^2 - 2\left[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)\right]$$

$$= \left[(\alpha\beta+\beta\gamma+\gamma\alpha)^2 - \cancel{2\alpha\beta\gamma(\alpha+\beta+\gamma)}\right]^2 - 2\left[\cancel{(\alpha+\beta+\gamma)^2} - 2(\alpha\beta+\beta\gamma+\gamma\alpha)\right]$$

$$= (\alpha\beta+\beta\gamma+\gamma\alpha)^4 + 4(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$= 2^4 + 4 \times 2$$

$$= 24$$

Question 36 (****+)

The roots of the cubic equation

$$x^3 - 4x^2 + 2x - 5 = 0$$

are denoted in the usual notation by α, β and γ.

Show that the cubic equation whose roots are

$$\frac{\beta\gamma}{\alpha}, \ \frac{\gamma\alpha}{\beta} \text{ and } \frac{\alpha\beta}{\gamma},$$

is given by

$$5x^3 + 36x^2 + 60x - 25 = 0$$

$\boxed{}$, $\boxed{\text{proof}}$

FIRSTLY IN THE USUAL NOTATION FOR THE GIVEN CUBIC

$\alpha + \beta + \gamma = -\frac{b}{a} = 4$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = 2$

$\alpha\beta\gamma = -5$

NOW FOR THE REQUIRED CUBIC

$A + B + C = \frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta} + \frac{\alpha\beta}{\gamma} = \frac{\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2}{\alpha\beta\gamma} = \frac{(\beta\gamma)^2 + (\gamma\alpha)^2 + (\alpha\beta)^2}{\alpha\beta\gamma}$

NOW WE HAVE FOR THE NUMERATOR

$(\beta\gamma + \gamma\alpha + \alpha\beta)^2 = (\beta\gamma)^2 + (\gamma\alpha)^2 + (\alpha\beta)^2 + 2[\beta\gamma^2\alpha + \alpha^2\beta\gamma + \alpha\beta^2\gamma]$

$2^2 = (\beta\gamma)^2 + (\gamma\alpha)^2 + (\alpha\beta)^2 + 2\alpha\beta\gamma(\gamma + \alpha + \beta)$

$4 = (\beta\gamma)^2 + (\gamma\alpha)^2 + (\alpha\beta)^2 + 2(-5)\times 4$

$(\beta\gamma)^2 + (\gamma\alpha)^2 + (\alpha\beta)^2 = -36$

$\therefore A + B + C = \frac{-36}{5}$

NOW THE SUM IN PAIRS

$AB + BC + CA = \frac{\alpha\beta\gamma^2}{\alpha\beta} + \frac{\alpha^2\beta\gamma}{\beta\gamma} + \frac{\alpha\beta^2\gamma}{\alpha\gamma} = \gamma^2 + \alpha^2 + \beta^2$

$= (\alpha + \beta + \gamma)^2 - [2\alpha\beta + 2\beta\gamma + 2\gamma\alpha]$

$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 16 - 2\times 2$

$= 12$

$\therefore AB + BC + CA = 12$

FINALLY THE PRODUCT OF THREE

$ABC = \frac{\alpha^2\beta^2\gamma^2}{\alpha\beta\gamma} = \alpha\beta\gamma = -5$

$\therefore ABC = -5$

HENCE THE REQUIRED CUBIC IS

$x^3 - (A + B + C)x^2 + (AB + BC + CA)x - (ABC) = 0$

$x^3 - \left(-\frac{36}{5}x^2\right) + 12x - 5 = 0$

$x^3 + \frac{36}{5}x^2 + 12x - 5 = 0$

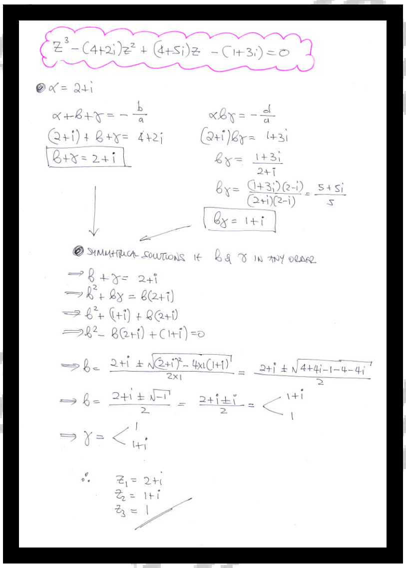
$5x^3 + 36x^2 + 60x - 25 = 0$

Question 37 (****+)

$$z^3 - (4+2i)z^2 + (4+5i)z - (1+3i) = 0, \ z \in \mathbb{C}.$$

Given that one of the solutions of the above cubic equation is $z = 2+i$, find the other two solutions.

$\boxed{}$, $\boxed{z=1}$, $\boxed{z=1+i}$



$z^3 - (4+2i)z^2 + (4+5i)z - (1+3i) = 0$

- $\alpha = 2+i$

$\alpha + \beta + \gamma = -\frac{b}{a}$

$(2+i) + \beta + \gamma = 4+2i$

$\boxed{\beta + \gamma = 2+i}$

$\alpha\beta\gamma = -\frac{d}{a}$

$(2+i)\beta\gamma = 1+3i$

$\beta\gamma = \frac{1+3i}{2+i}$

$\beta\gamma = \frac{(1+3i)(2-i)}{(2+i)(2-i)} = \frac{5+5i}{5}$

$\boxed{\beta\gamma = 1+i}$

- SYMMETRICAL SOLUTIONS IF β & γ IN ANY ORDER

$\Rightarrow \beta + \gamma = 2+i$

$\Rightarrow \beta^2 + \beta\gamma = \beta(2+i)$

$\Rightarrow \beta^2 + (1+i) + \beta(2+i)$

$\Rightarrow \beta^2 - \beta(2+i) + (1+i) = 0$

$\Rightarrow \beta = \frac{2+i \pm \sqrt{(2+i)^2 - 4\times 1(1+i)}}{2\times 1} = \frac{2+i \pm \sqrt{4+4i-1-4-4i}}{2}$

$\Rightarrow \beta = \frac{2+i \pm \sqrt{-1}}{2} = \frac{2+i \pm i}{2} = \left\langle \begin{matrix} 1+i \\ 1 \end{matrix} \right.$

$\Rightarrow \gamma = \left\langle \begin{matrix} 1 \\ 1+i \end{matrix} \right.$

$\therefore \ z_1 = 2+i$

$z_2 = 1+i$

$z_3 = 1$

Question 38 (****+)

The roots of the cubic equation

$$x^3 - 4x^2 - 3x - 2 = 0$$

are denoted in the usual notation by α, β and γ.

Show that the cubic equation whose roots are

$$\alpha + \beta, \quad \beta + \gamma \text{ and } \quad \gamma + \alpha,$$

is given by

$$x^3 - 8x^2 + 13x + 14 = 0$$

, proof

FOR THE CUBIC $x^3 - 4x^2 - 3x - 2 = 0$

$$\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{-4}{1} = 4$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{-3}{1} = -3$$
$$\alpha\beta\gamma = -\frac{d}{a} = -\frac{-2}{1} = 2$$

LET THE THREE ROOTS OF THE REQUIRED CUBIC BE

- $A = \alpha + \beta$
- $B = \beta + \gamma$
- $C = \gamma + \alpha$

- $A + B + C = (\alpha+\beta) + (\beta+\gamma) + (\gamma+\alpha)$
$= 2(\alpha+\beta+\gamma)$
$= 2 \times 4$
$= 8$

- $AB + BC + CA = (\alpha+\beta)(\beta+\gamma) + (\beta+\gamma)(\gamma+\alpha) + (\gamma+\alpha)(\alpha+\beta)$
$= \alpha\beta + \alpha\gamma + \beta^2 + \beta\gamma$
$\quad \alpha\beta + \alpha\gamma + \gamma^2 + \beta\gamma$
$\quad \alpha\beta + \alpha\gamma + \alpha^2 + \beta\gamma$
$= (\alpha^2+\beta^2+\gamma^2) + 3(\alpha\beta + \beta\gamma + \gamma\alpha)$
$= (\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha) + (\alpha\beta + \beta\gamma + \gamma\alpha)$
$= (\alpha+\beta+\gamma)^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)$
$= 4^2 + (-3)$
$= 16 - 3$
$= 13$

- $ABC = (\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$
$= (\alpha+\beta)(\beta\gamma + \alpha\beta + \alpha\gamma + \gamma^2$
$= \alpha\beta\gamma + \alpha^2\beta + \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \alpha\beta^2 + \alpha\beta\gamma + \beta\gamma^2$
$= 2\alpha\beta\gamma + \alpha^2\beta + \alpha\beta^2 + \alpha^2\gamma + \alpha\gamma^2 + \beta\gamma^2 + \beta^2\gamma$
$= 2\alpha\beta\gamma + \alpha\beta(\alpha+\beta) + \alpha\gamma(\alpha+\gamma) + \beta\gamma(\beta+\gamma)$
$= 2\alpha\beta\gamma + \alpha\beta(\alpha+\beta+\gamma) - \alpha\beta\gamma$
$\quad + \alpha\gamma(\alpha+\gamma+\beta) - \alpha\beta\gamma$
$\quad + \beta\gamma(\beta+\gamma+\alpha) - \alpha\beta\gamma$
$= (\alpha\beta + \alpha\gamma + \beta\gamma)(\alpha+\beta+\gamma) - \alpha\beta\gamma$
$= -3 \times 4 - 2$
$= -14$

HENCE THE REQUIRED CUBIC WILL BE

$$x^3 - (A+B+C)x^2 + (AB+BC+CD)x - (ABC) = 0$$
$$\underline{x^3 - 8x^2 + 13x + 14 = 0}$$

AS REQUIRED

Question 39 (*****)

A system of simultaneous equations is given below

$$x + y + z = 1$$
$$x^2 + y^2 + z^2 = 21$$
$$x^3 + y^3 + z^3 = 55.$$

By forming an auxiliary cubic equation find the solution to the above system.

You may find the identity

$$x^3 + y^3 + z^3 \equiv (x+y+z)\left(x^2 + y^2 + z^2 - xy - yz - zx\right) + 3xyz,$$

useful in this question.

SPX-F , $\boxed{x, y, z = -2, -1, 4 \text{ in any order}}$

START BY USING THE IDENTITY $(x+y+z)^2 \equiv \ldots$

$\Rightarrow (x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2xz$

$\Rightarrow 1^2 = 21 + 2(xy+yz+xz)$

$\Rightarrow 2(xy+yz+xz) = -20$

$\Rightarrow (xy+yz+xz) = -10$

USING THE IDENTITY GIVEN

$\Rightarrow x^3+y^3+z^3 = (x+y+z)(x^2+y^2+z^2-xy-yz-zx) + 3xyz$

$\Rightarrow 55 = 1 \times [21-(-10)] + 3xyz$

$\Rightarrow 55 = 31 + 3xyz$

$\Rightarrow 3xyz = 24$

$\Rightarrow xyz = 8$

FORMING A CUBIC IN ANOTHER VARIABLE, SAY a

$\Rightarrow a^3 - (\underbrace{1}_{x+y+z}a^2) + (\underbrace{-10}_{xy+yz+zx}a) - (\underbrace{8}_{xyz}) = 0$

WHERE x, y, z ARE THE SOLUTIONS OF THIS CUBIC IN a

$\Rightarrow a^3 - a^2 - 10a - 8 = 0$

BY INSPECTION, $a=-1$ IS AN OBVIOUS SOLUTION, $(-1)^3-(-1)^2-10(-1)-8=0$

$\Rightarrow a^2(a+1) - 2a(a+1) - 8(a-1) = 0$

$\Rightarrow (a+1)(a^2-2a-8) = 0$

$\Rightarrow (a+1)(a-2)(a-4) = 0$

$\Rightarrow a = -1, 2, 4$

$\therefore x=-1, y=2, z=4$ IN ANY ORDER

Question 40 (*****)

The roots of the cubic equation

$$8x^3 + 12x^2 + 2x - 3 = 0$$

are denoted in the usual notation by α, β and γ.

An integer function S_n, is defined as

$$S_n \equiv (2\alpha+1)^n + (2\beta+1)^n + (2\gamma+1)^n, \quad n \in \mathbb{Z}.$$

Determine the value of S_3 and the value of S_{-2}.

$\boxed{}$, $\boxed{S_3 = 6}$, $\boxed{S_{-2} = 1}$

LOOKING AT THE EXPRESSION TO BE EVALUATED, WE TRY TO FIND ANOTHER CUBIC WHOSE ROOTS ARE $2\alpha+1$, $2\beta+1$, $2\gamma+1$

LET $y = 2x+1$
$2x = y-1$

REWRITE THE CUBIC FOR SIMPLICITY AS

$$\Rightarrow 8x^3 + 12x^2 + 2x - 3 = 0$$
$$\Rightarrow (2x)^3 + 3(2x)^2 + (2x) - 3 = 0$$
$$\Rightarrow (y-1)^3 + 3(y-1)^2 + (y-1) - 3 = 0$$
$$\Rightarrow \left\{\begin{array}{r} y^3 - 3y^2 + 3y - 1 \\ 3y^2 - 6y + 3 \\ y - 4 \end{array}\right\} = 0$$
$$\Rightarrow y^3 - 2y - 2 = 0$$

HENCE WE NOW HAVE

$$S_n = (2\alpha+1)^n + (2\beta+1)^n + (2\gamma+1)^n$$
$$S_n = A^n + B^n + C^n$$

WHERE
- $A = 2\alpha+1$
- $B = 2\beta+1$
- $C = 2\gamma+1$

AND LOOKING AT THE CUBIC IN y
- $A+B+C = 0$
- $AB+BC+CA = -2$
- $ABC = +2$

WE CAN NOW EVALUATE SIMPLER EXPRESSIONS

- $S_3 = A^3 + B^3 + C^3 = 6$

(SEE OPPOSITE)

$$\boxed{y^3 = 2y+2}$$
$$\left\{\begin{array}{l} A^3 = 2A+2 \\ B^3 = 2B+2 \\ C^3 = 2C+2 \end{array}\right\}$$
$$A^3+B^3+C^3 = \cancel{2(A+B+C)} + 6$$
$$A^3+B^3+C^3 = 6$$

- $S_{-2} = A^{-2} + B^{-2} + C^{-2} = \frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} = \frac{A^2B^2 + A^2C^2 + B^2C^2}{A^2B^2C^2}$

$$= \frac{(AB)^2 + (BC)^2 + (CA)^2}{(ABC)^2}$$

$$= \frac{(AB+BC+CA)^2 - 2(A^2BC + AB^2C + ABC^2)}{(ABC)^2}$$

WE USED HERE $(a+b+c)^2 \equiv a^2+b^2+c^2+2(ab+bc+ca)$
$a^2+b^2+c^2 \equiv (a+b+c)^2 - 2(ab+bc+ca)$

$$= \frac{(AB+BC+CA)^2 - 2ABC(A+B+C)}{(ABC)^2}$$

$$= \frac{(-2)^2 - 2\times 2\times 0}{2^2}$$

$$= 1$$

QUARTICS