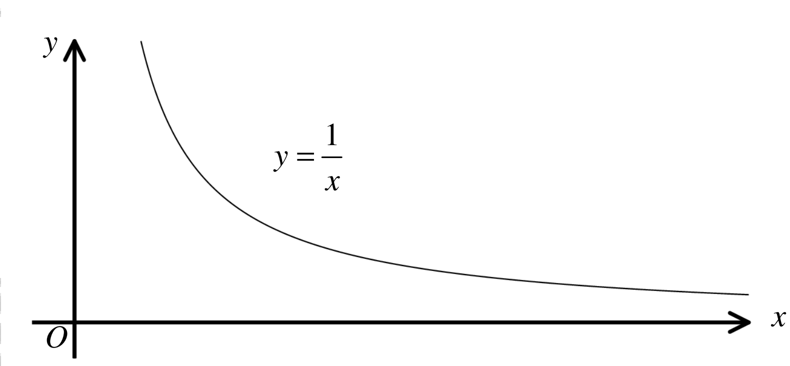
SERIES and INTEGRALS

Question 1 (***)

The figure below shows the curve C with equation $y = e^{-\frac{1}{x}}$, $0 < x \leq 1$.



a) By using a two different sets of rectangles of unit width under and above the graph of C, show that

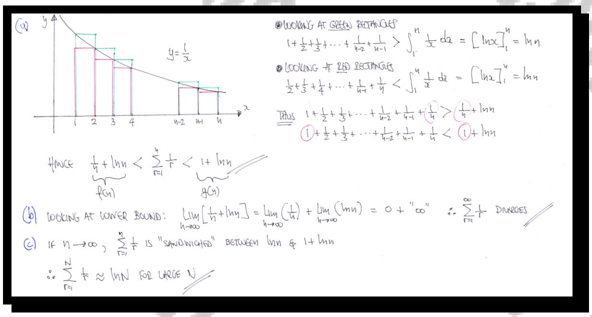
$$f(n) < \sum_{r=1}^{n} \frac{1}{r} < g(n),$$

where $f(n)$ and $g(n)$ are functions involving natural logarithms.

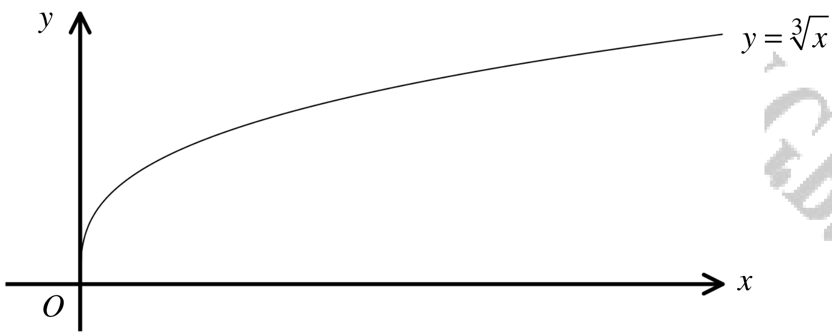
b) Determine whether $\sum_{r=1}^{\infty} \frac{1}{r}$ exists.

c) Write down an approximation for $\sum_{r=1}^{N} \frac{1}{r}$ if N is very large.

$\boxed{f(n) = \frac{1}{n} + \ln n}$, $\boxed{g(n) = 1 + \ln n}$, $\boxed{\sum_{r=1}^{\infty} \frac{1}{r} \text{ diverges}}$, $\boxed{\text{as } n \to \infty, \ \sum_{r=1}^{N} \frac{1}{r} \approx \ln N}$



Question 2 (***)



The figure above shows the curve C with equation $y = \sqrt[3]{x}$, $x \geq 0$.

a) By using two different sets of rectangles of unit width under and above the graph of C, show that

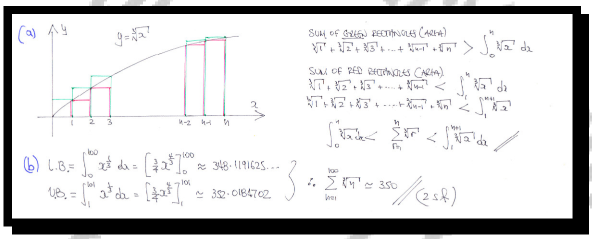
$$\int_a^b \sqrt[3]{x}\ dx < \sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{n} < \int_c^d \sqrt[3]{x} ,$$

stating the limits in the integrals.

b) Hence show that

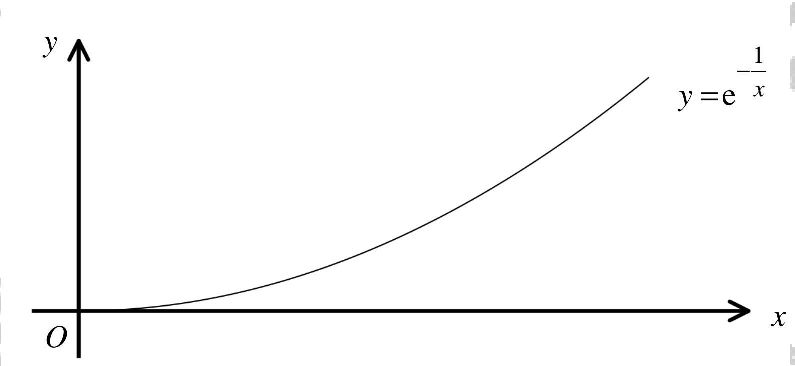
$$\sum_{n=1}^{100} \sqrt[3]{n} \approx 350 .$$

$\boxed{a = 0,\ b = n,\ c = 1,\ d = n+1}$



Question 3 (***+)

The figure below shows the curve C with equation $y = \mathrm{e}^{-\frac{1}{x}}$, $0 < x \le 1$.



a) By using two different sets of rectangles of width $\frac{1}{n}$ under and above the graph of C, show that

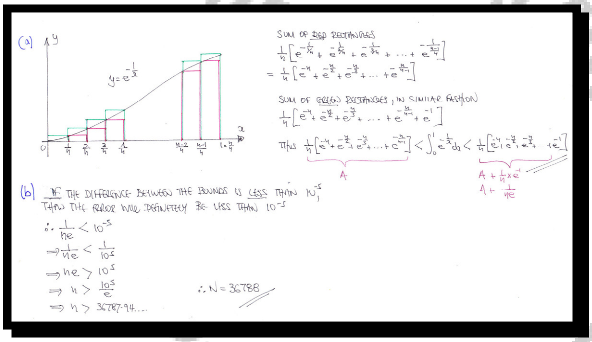
$$A < \int_0^1 \mathrm{e}^{-\frac{1}{x}}\,dx < A + \frac{1}{2\mathrm{e}},$$

where A is an exact finite series involving exponentials.

The above expression is to be used to approximate the area under C for $0 < x \le 1$.
When $n \ge N$, the error is less than 10^{-5}.

b) Determine the least possible value of N.

$$\boxed{A = \frac{1}{n}\left[\mathrm{e}^{-n} + \mathrm{e}^{-\frac{1}{2}n} + \mathrm{e}^{-\frac{1}{3}n} + \mathrm{e}^{-\frac{1}{4}n} + \ldots + \mathrm{e}^{-\frac{n}{n-1}}\right]}, \quad \boxed{N = 36788}$$



(a)

SUM OF RED RECTANGLES

$\frac{1}{n}\left[e^{-\frac{1}{1/n}} + e^{-\frac{1}{2/n}} + e^{-\frac{1}{3/n}} + \ldots + e^{-\frac{1}{\frac{n-1}{n}}}\right]$

$= \frac{1}{n}\left[e^{-n} + e^{-\frac{n}{2}} + e^{-\frac{n}{3}} + \ldots + e^{-\frac{n}{n-1}}\right]$

SUM OF GREEN RECTANGLES, IN SIMILAR FASHION

$\frac{1}{n}\left[e^{-n} + e^{-\frac{n}{2}} + e^{-\frac{n}{3}} + \ldots + e^{-\frac{n}{n-1}} + e^{-1}\right]$

THUS $\underbrace{\frac{1}{n}\left[e^{-n} + e^{-\frac{n}{2}} + e^{-\frac{n}{3}} + \ldots + e^{-\frac{n}{n-1}}\right]}_{A} < \int_0^1 e^{-\frac{1}{x}}\,dx < \underbrace{\frac{1}{n}\left[e^{-n} + e^{-\frac{n}{2}} + e^{-\frac{n}{3}} + \ldots + e^{-1}\right]}_{A + \frac{1}{n} \times e^{-1}}$

$A + \frac{1}{ne}$

(b) IF THE DIFFERENCE BETWEEN THE BOUNDS IS LESS THAN 10^{-5},
THEN THE ERROR WILL DEFINITELY BE LESS THAN 10^{-5}

$\therefore \frac{1}{ne} < 10^{-5}$

$\Rightarrow \frac{1}{ne} < \frac{1}{10^5}$

$\Rightarrow ne > 10^5$

$\Rightarrow n > \frac{10^5}{e}$

$\Rightarrow n > 36787.94\ldots$

$\therefore N = 36788$

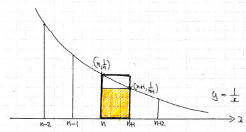
Question 4 (***+)

By considering the area of two different rectangles of unit width under and above the graph of $y = \dfrac{1}{x}$, show that

$$\left(1+\frac{1}{n}\right)^{n} < \mathrm{e} < \left(1+\frac{1}{n}\right)^{n+1}$$

$\boxed{}$, $\boxed{\text{proof}}$

LOOKING AT THE DIAGRAM BELOW, WITH $n =$ INTEGER



THE AREA UNDER THE CURVE BETWEEN n & $n+1$ IS

- GREATER THAN THE YELLOW RECTANGLE OF AREA $1 \times \frac{1}{n+1}$
- LESS THAN THE "BOLD" RECTANGLE OF AREA $1 \times \frac{1}{n}$

HENCE

$$\frac{1}{n+1} < \int_{n}^{n+1} \frac{1}{x}\,dx < \frac{1}{n}$$

$$\frac{1}{n+1} < \Big[\ln x\Big]_{n}^{n+1} < \frac{1}{n}$$

$$\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}$$

$$\frac{1}{n+1} < \ln\left(\frac{n+1}{n}\right) < \frac{1}{n}$$

$$\mathrm{e}^{\frac{1}{n+1}} < \frac{n+1}{n} < \mathrm{e}^{\frac{1}{n}}$$

$$\mathrm{e}^{\frac{1}{n+1}} < 1 + \frac{1}{n} < \mathrm{e}^{\frac{1}{n}}$$

DEALING WITH EACH PART OF THE INEQUALITY SEPARATELY

$$1 + \frac{1}{n} > \mathrm{e}^{\frac{1}{n+1}} \qquad\qquad 1 + \frac{1}{n} < \mathrm{e}^{\frac{1}{n}}$$

$$\left(1 + \frac{1}{n}\right)^{n+1} > \left(\mathrm{e}^{\frac{1}{n+1}}\right)^{n+1} \qquad\qquad \left(1 + \frac{1}{n}\right)^{n} < \left(\mathrm{e}^{\frac{1}{n}}\right)^{n}$$

$$\left(1 + \frac{1}{n}\right)^{n+1} > \mathrm{e} \qquad\qquad \left(1 + \frac{1}{n}\right)^{n} < \mathrm{e}$$

COMBINING WE OBTAIN

$$\left(1 + \frac{1}{n}\right)^{n} < \mathrm{e} < \left(1 + \frac{1}{n}\right)^{n+1}$$

Question 5 (*****)

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx .$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a "right (upper) Riemann sum"

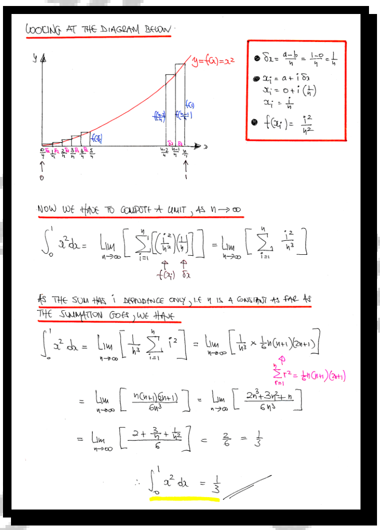
$$I(a,b) = \lim_{n \to \infty}\left[\sum_{i=1}^{n}\left[f(x_i)\,\delta x\right]\right],$$

where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the "right (upper) Riemann sum" definition, and with the aid of a diagram where appropriate, show clearly that

$$\int_0^1 x^2\ dx = \frac{1}{3}.$$

$\boxed{}$, $\boxed{\text{proof}}$


LOOKING AT THE DIAGRAM BELOW

- $\delta x = \frac{a-b}{n} = \frac{1-0}{n} = \frac{1}{n}$
- $x_i = a + i\,\delta x$
  $x_i = 0 + i\left(\frac{1}{n}\right)$
  $x_i = \frac{i}{n}$
- $f(x_i) = \frac{i^2}{n^2}$

NOW WE HAVE TO COMPUTE A LIMIT, AS $n \to \infty$

$$\int_0^1 x^2\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[\underbrace{\left(\frac{i^2}{n^2}\right)}_{f(x_i)}\underbrace{\left(\frac{1}{n}\right)}_{\delta x}\right]\right] = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{i^2}{n^3}\right]$$

AS THE SUM HAS i DEPENDENCE ONLY, I.E n IS A CONSTANT AS FAR AS THE SUMMATION GOES, WE HAVE

$$\int_0^1 x^2\,dx = \lim_{n\to\infty}\left[\frac{1}{n^3}\sum_{i=1}^{n} i^2\right] = \lim_{n\to\infty}\left[\frac{1}{n^3}\times\frac{1}{6}n(n+1)(2n+1)\right]$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$= \lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6n^3}\right] = \lim_{n\to\infty}\left[\frac{2n^3+3n^2+n}{6n^3}\right]$$

$$= \lim_{n\to\infty}\left[\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}\right] = \frac{2}{6} = \frac{1}{3}$$

$$\therefore \int_0^1 x^2\,dx = \frac{1}{3}$$

Question 6 (*****)

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx .$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a "right (upper) Riemann sum"

$$I(a,b) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f(x_i)\,\delta x\right]\right],$$

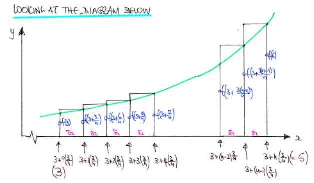
where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the "right (upper) Riemann sum" definition, and with the aid of a diagram where appropriate, show clearly that

$$\int_3^6 x^2\ dx = 63 .$$

$\boxed{}$, $\boxed{\text{proof}}$

LOOKING AT THE DIAGRAM BELOW



- $\delta x = \frac{b-a}{n} = \frac{6-3}{n} = \frac{3}{n}$
- $x_i = a + i\delta x = 3 + i \times \frac{3}{n} = 3 + \frac{3i}{n}$
- $f(x_i) = \left(3+\frac{3i}{n}\right)^2 = 9\left(1+\frac{i}{n}\right)^2 = 9\left(\frac{n+i}{n}\right)^2 = \frac{9}{n^2}(n^2+2ni+i^2)$

USING THE RIEMANN SUM LIMIT

$$\int_3^6 x^2\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[\underbrace{\left[\tfrac{9}{n^2}(n^2+2ni+i^2)\right]}_{f(x_i)}\underbrace{\left(\tfrac{3}{n}\right)}_{\delta x}\right]\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[\frac{27}{n^3}(n^2+2ni+i^2)\right]\right]$$

$$= \lim_{n\to\infty}\left[\frac{27}{n^3}\sum_{i=1}^{n}(n^2+2ni+i^2)\right]$$

AS THE SUMMATION RUNS IN i, NOT IN n

SPLIT THE LIMIT & THE SUM INTO THREE TERMS & TIDY FURTHER

$$\int_3^6 x^2\,dx = \lim_{n\to\infty}\left[\frac{27}{n^3}\sum_{i=1}^{n}n^2\right] + \lim_{n\to\infty}\left[\frac{27}{n^3}\sum_{i=1}^{n}2ni\right] + \lim_{n\to\infty}\left[\frac{27}{n^3}\sum_{i=1}^{n}i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{27}{n}\sum_{i=1}^{n}1\right] + \lim_{n\to\infty}\left[\frac{54}{n^2}\sum_{i=1}^{n}i\right] + \lim_{n\to\infty}\left[\frac{27}{n^3}\sum_{i=1}^{n}i^2\right]$$

n $\qquad \frac{1}{2}n(n+1)$ $\qquad \frac{1}{6}n(n+1)(2n+1)$

NOW USING STANDARD SUMMATION FORMULAE

$$\int_3^6 x^2\,dx = \lim_{n\to\infty}\left[\frac{27}{n}\times n\right] + \lim_{n\to\infty}\left[\frac{54}{n^2}\times\frac{1}{2}n(n+1)\right] + \lim_{n\to\infty}\left[\frac{27}{n^3}\times\frac{1}{6}n(n+1)(2n+1)\right]$$

$$= \lim_{n\to\infty}[27] + \lim_{n\to\infty}\left[\frac{27(n+1)}{n}\right] + \lim_{n\to\infty}\left[\frac{9(n+1)(2n+1)}{2n^2}\right]$$

$$= 27 + 27\lim_{n\to\infty}\left[\frac{n+1}{n}\right] + \frac{9}{2}\lim_{n\to\infty}\left[\frac{2n^2+3n+1}{n^2}\right]$$

$$= 27 + 27\lim_{n\to\infty}\left[\frac{1+\frac{1}{n}}{1}\right] + \frac{9}{2}\lim_{n\to\infty}\left[\frac{2+\frac{3}{n}+\frac{1}{n^2}}{1}\right]$$

$$= 27 + 27\lim_{n\to\infty}\left[1+\frac{1}{n}\right] + \frac{9}{2}\lim_{n\to\infty}\left[2+\frac{3}{n}+\frac{1}{n^2}\right]$$

$$= 27 + 27\times 1 + \frac{9}{2}\times 2$$

$$= 63$$

$$\therefore \int_3^6 x^2\,dx = 63$$

AS REQUIRED

Question 7 (*****)

A curve has equation $y = f(x)$.

The finite region R is bounded by the curve, the x axis and the straight lines with equations $x = a$ and $x = b$, and hence the area of R is given by

$$I(a,b) = \int_a^b f(x)\ dx.$$

The area of R is also given by the limiting value of the sum of the areas of rectangles of width δx and height $f(x_i)$, known as a "right (upper) Riemann sum"

$$I(a,b) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f(x_i)\,\delta x\right]\right],$$

where $\delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\delta x$.

Using the "right (upper) Riemann sum" definition, and with the aid of a diagram where appropriate, show clearly that

$$\lim_{n\to\infty}\left[\sqrt[n]{\frac{n!}{n^n}}\right] = \frac{1}{\mathrm{e}}.$$

$\boxed{}$, $\boxed{\text{proof}}$

LET THE VALUE OF THE LIMIT BE "L"

$$\Rightarrow L = \lim_{n\to\infty}\left[\sqrt[n]{\frac{n!}{n^n}}\right] = \lim_{n\to\infty}\left[\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}\right]$$

TAKING NATURAL LOGARITHMS WE OBTAIN

$$\Rightarrow \ln L = \lim_{n\to\infty}\left[\ln\left[\frac{n!}{n^n}\right]^{\frac{1}{n}}\right]$$

$$\Rightarrow \ln L = \lim_{n\to\infty}\left[\frac{1}{n}\ln\left(\frac{n!}{n^n}\right)\right]$$

$$\Rightarrow \ln L = \lim_{n\to\infty}\left[\frac{1}{n}\ln\left(\frac{n(n-1)(n-2)(n-3)\dots}{n\cdot n\cdot n\cdot n}\ \frac{4\cdot 3\cdot 2\cdot 1}{n\cdot n\cdot n\cdot n}\right)\right]$$

$$\Rightarrow \ln L = \lim_{n\to\infty}\left[\frac{1}{n}\left[\ln\left(\frac{n}{n}\right)+\ln\left(\frac{n-1}{n}\right)+\ln\left(\frac{n-2}{n}\right)+\dots+\ln\left(\frac{3}{n}\right)+\ln\left(\frac{2}{n}\right)+\ln\left(\frac{1}{n}\right)\right]\right]$$

REWRITE BACKWARDS & MULTIPLY INSIDE THE LIMIT

$$\Rightarrow \ln L = \lim_{n\to\infty}\left[\frac{1}{n}\ln\left(\frac{1}{n}\right)+\frac{1}{n}\ln\left(\frac{2}{n}\right)+\frac{1}{n}\ln\left(\frac{3}{n}\right)+\dots+\frac{1}{n}\ln\left(\frac{n-1}{n}\right)+\frac{1}{n}\ln\left(\frac{n}{n}\right)\right]$$

COMPARE WITH THE GENERAL RIEMANN SUM

$$\ln\left(\frac{i}{n}\right) = f(a + i\delta x) \quad \& \quad \delta x = \frac{b-a}{n} = \frac{1}{n}$$

$$\ln\left(\frac{i}{n}\right) = f\left(0 + \frac{i}{n}\right)$$

$$\therefore a = 0 \ \& \ b = 1 \text{ with } f(x) = \ln x$$

THUS WE NOW HAVE

$$\Rightarrow \ln L = \int_0^1 \ln x\ dx$$

NOW EITHER STATE THE INTEGRAL AS A STANDARD RESULT OR CARRY OUT A SIMPLE INTEGRATION BY PARTS OR INSPECTION

- $\frac{d}{dx}(x\ln x) = 1\times\ln x + x\left(\frac{1}{x}\right) = \ln x + 1$
- $\frac{d}{dx}(-x) = -1$

$$\therefore \frac{d}{dx}(x\ln x - x) = (\ln x + 1) - 1 = \ln x$$

$$\therefore x\ln x - x + C = \int \ln x\ dx$$

RETURNING TO THE PROBLEM WE HAVE

$$\Rightarrow \ln L = \int_0^1 \ln x\ dx = \left[x\ln x - x\right]_0^1$$

$\ln x \to -\infty$

BUT $\lim_{x\to 0}[x\ln x] = 0$ AS $x \to 0$ FASTER THAN $\ln x \to -\infty$

$$\Rightarrow \ln L = (0-1) - (0-0)$$

$$\Rightarrow \ln L = -1$$

$$\Rightarrow L = e^{-1}$$

$$\Rightarrow \lim_{n\to\infty}\left[\sqrt[n]{\frac{n!}{n^n}}\right] = \frac{1}{e}$$

AS REQUIRED

Question 8 (*****)

Determine the limit of the following series.

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4}+\ldots+\frac{1}{n+n-2}+\frac{1}{n+n-1}+\frac{1}{n+n}\right].$$

V , $\boxed{\text{S X V}}$, $\boxed{\ln 2}$

AS THIS LOOKS LIKE A RIEMANN SUM, START WITH THE DEFINITION

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\,\delta x\right] \qquad \delta x = \frac{b-a}{n}, \quad x_i = a + i\,\delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f\left(a+i\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)\right]\right]$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+i\frac{b-a}{n}\right)\right]$$

NOW LOOKING AT THE LIMIT OF OUR SERIES & COMPARE

$$\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{n+(n-1)}+\frac{1}{n+n}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{n+i}\right] = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{1+\frac{i}{n}}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^{-1}\right]$$

$b-a=1$ $\quad a=1$ $\quad f(x_i) = x_i^{-1}$

AND BY COMPARISON WE HAVE $a=1$, $b=2$ $f(x)=\frac{1}{x}$

$$= \int_1^2 \frac{1}{x}\,dx = \left[\ln x\right]_1^2 = \ln 2 - \cancel{\ln 1} = \ln 2$$

$$\therefore \lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{n+(n-1)}+\frac{1}{n+n}\right] = \ln 2$$