SERIES

79 EXAM QUESTIONS

SUMMATIONS BY FORMULAS

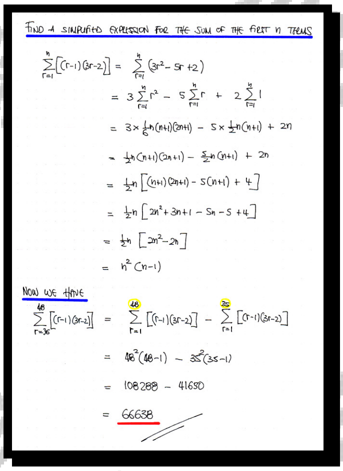
17 BASIC QUESTIONS

Question 1 (**)

Use standard results on summations to find the value of

$$\sum_{r=36}^{48}\left[(r-1)(3r-2)\right].$$

, $\boxed{66638}$



FIND A SIMPLIFIED EXPRESSION FOR THE SUM OF THE FIRST n TERMS

$$\sum_{r=1}^{n}\left[(r-1)(3r-2)\right] = \sum_{r=1}^{n}(3r^2-5r+2)$$
$$= 3\sum_{r=1}^{n}r^2 - 5\sum_{r=1}^{n}r + 2\sum_{r=1}^{n}1$$
$$= 3\times\frac{1}{6}n(n+1)(2n+1) - 5\times\frac{1}{2}n(n+1) + 2n$$
$$= \frac{1}{2}n(n+1)(2n+1) - \frac{5}{2}n(n+1) + 2n$$
$$= \frac{1}{2}n\left[(n+1)(2n+1) - 5(n+1) + 4\right]$$
$$= \frac{1}{2}n\left[2n^2+3n+1-5n-5+4\right]$$
$$= \frac{1}{2}n\left[2n^2-2n\right]$$
$$= n^2(n-1)$$

NOW WE HAVE

$$\sum_{r=36}^{48}\left[(r-1)(3r-2)\right] = \sum_{r=1}^{48}\left[(r-1)(3r-2)\right] - \sum_{r=1}^{35}\left[(r-1)(3r-2)\right]$$
$$= 48^2(48-1) - 35^2(35-1)$$
$$= 108288 - 41650$$
$$= 66638$$

Question 2 (**)

Use standard results on summations to show that

$$\sum_{r=1}^{n} r(r+1)(r+5) = \frac{1}{4}n(n+a)(n+b)(n+c),$$

where a, b, and c are positive integers to be found.

$\boxed{}$, $\boxed{a=1,\ b=2,\ c=7}$

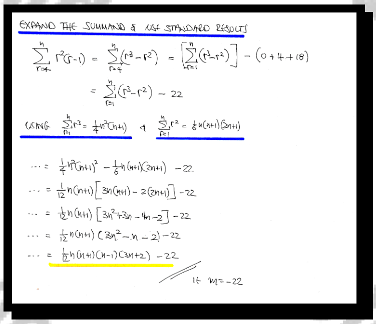
Question 3 (**)

Use standard results on summations to show that

$$\sum_{r=1}^{n}\left[r^2(r-1)\right] = \frac{1}{12}n(n-1)(n+1)(3n+2)+m,$$

where m is an integer to be found.

$\boxed{\quad}$, $\boxed{m=-22}$



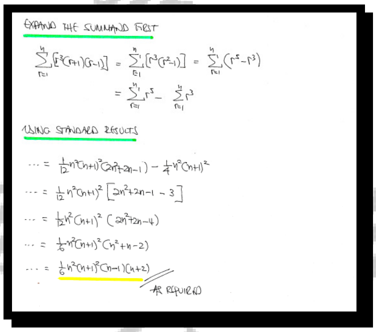
Question 4 **(**)**

Use standard results on summations to show that

$$\sum_{r=1}^{n}\left[r^3(r+1)(r-1)\right] = \frac{1}{6}n^2(n+1)^2(n-1)(n+2).$$

You may assume without proof that $\displaystyle\sum_{r=1}^{n} r^5 = \frac{1}{12}n^2(n+1)^2\left(2n^2+2n-1\right)$

$\boxed{}$, $\boxed{\text{proof}}$



EXPAND THE SUMMAND FIRST

$$\sum_{r=1}^{n}\left[r^3(r+1)(r-1)\right] = \sum_{r=1}^{n}\left[r^3(r^2-1)\right] = \sum_{r=1}^{n}(r^5-r^3)$$

$$= \sum_{r=1}^{n} r^5 - \sum_{r=1}^{n} r^3$$

USING STANDARD RESULTS

$$\ldots = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1) - \frac{1}{4}n^2(n+1)^2$$

$$\ldots = \frac{1}{12}n^2(n+1)^2\left[2n^2+2n-1-3\right]$$

$$\ldots = \frac{1}{12}n^2(n+1)^2(2n^2+2n-4)$$

$$\ldots = \frac{1}{6}n^2(n+1)^2(n^2+n-2)$$

$$\ldots = \frac{1}{6}n^2(n+1)^2(n-1)(n+2)$$

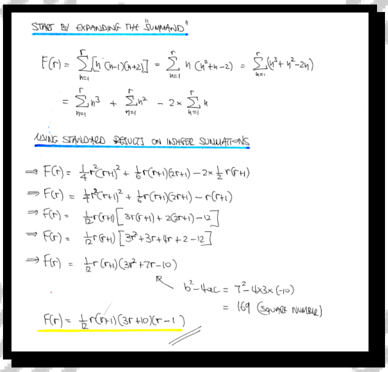
AS REQUIRED

Question 5 (**)

$$F(r) \equiv \sum_{n=1}^{r} \left[n(n-1)(n+2) \right].$$

Use standard results on summations express $F(n)$ in fully factorized from.

[ ], $F(r) = \frac{1}{12} r(r+1)(r-1)(3r+10)$



START BY EXPANDING THE "SUMMAND"

$$F(r) = \sum_{n=1}^{r} [n(n-1)(n+2)] = \sum_{n=1}^{r} n(n^2+n-2) = \sum_{n=1}^{r} (n^3+n^2-2n)$$

$$= \sum_{n=1}^{r} n^3 + \sum_{n=1}^{r} n^2 - 2 \times \sum_{n=1}^{r} n$$

USING STANDARD RESULTS ON INTEGER SUMMATIONS

$$\Rightarrow F(r) = \frac{1}{4}r^2(r+1)^2 + \frac{1}{6}r(r+1)(2r+1) - 2 \times \frac{1}{2}r(r+1)$$

$$\Rightarrow F(r) = \frac{1}{4}r^2(r+1)^2 + \frac{1}{6}r(r+1)(2r+1) - r(r+1)$$

$$\Rightarrow F(r) = \frac{1}{12}r(r+1)\left[3r(r+1) + 2(2r+1) - 12\right]$$

$$\Rightarrow F(r) = \frac{1}{12}r(r+1)\left[3r^2+3r+4r+2-12\right]$$

$$\Rightarrow F(r) = \frac{1}{12}r(r+1)(3r^2+7r-10)$$

$b^2-4ac = 7^2 - 4 \times 3 \times (-10)$
$= 169$ (SQUARE NUMBER)

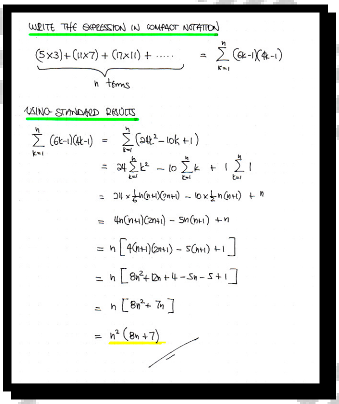
$$F(r) = \frac{1}{12}r(r+1)(3r+10)(r-1)$$

Question 6 (+)**

Find, in fully simplified factorized form, an expression for the sum of the first n terms of the following series.

$$(5\times 3) + (11\times 7) + (17\times 11) + (23\times 15) + \ldots$$

[ ], $n^2(8n+7)$



WRITE THE EXPRESSION IN COMPACT NOTATION

$$\underbrace{(5\times3)+(11\times7)+(17\times11)+\ldots}_{n \text{ terms}} = \sum_{k=1}^{n}(6k-1)(4k-1)$$

USING STANDARD RESULTS

$$\sum_{k=1}^{n}(6k-1)(4k-1) = \sum_{k=1}^{n}(24k^2-10k+1)$$

$$= 24\sum_{k=1}^{n}k^2 - 10\sum_{k=1}^{n}k + 1\sum_{k=1}^{n}1$$

$$= 24 \times \frac{1}{6}n(n+1)(2n+1) - 10 \times \frac{1}{2}n(n+1) + n$$

$$= 4n(n+1)(2n+1) - 5n(n+1) + n$$

$$= n\left[4(n+1)(2n+1) - 5(n+1) + 1\right]$$

$$= n\left[8n^2+12n+4-5n-5+1\right]$$

$$= n\left[8n^2+7n\right]$$

$$= n^2(8n+7)$$

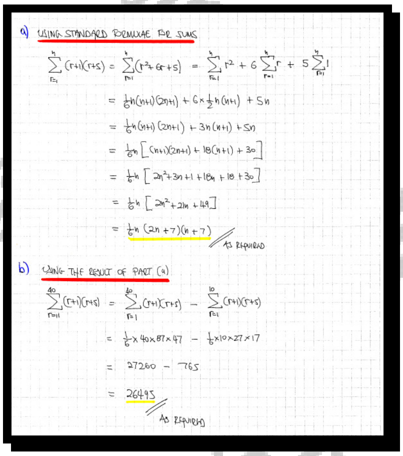
Question 7 **(**+)**

Show by using standard summation results that …

a) … $\displaystyle\sum_{r=1}^{n}(r+1)(r+5)=\frac{1}{6}n(n+7)(2n+7)$.

b) … $\displaystyle\sum_{r=11}^{40}(r+1)(r+5)=26495$.

[ ] , [proof]



a) USING STANDARD FORMULAE FOR SUMS

$\sum_{r=1}^{n}(r+1)(r+5)=\sum_{r=1}^{n}(r^2+6r+5)=\sum_{r=1}^{n}r^2+6\sum_{r=1}^{n}r+5\sum_{r=1}^{n}1$

$=\frac{1}{6}n(n+1)(2n+1)+6\times\frac{1}{2}n(n+1)+5n$

$=\frac{1}{6}n(n+1)(2n+1)+3n(n+1)+5n$

$=\frac{1}{6}n\left[(n+1)(2n+1)+18(n+1)+30\right]$

$=\frac{1}{6}n\left[2n^2+3n+1+18n+18+30\right]$

$=\frac{1}{6}n\left[2n^2+21n+49\right]$

$=\frac{1}{6}n(2n+7)(n+7)$ // AS REQUIRED

b) USING THE RESULT OF PART (a)

$\sum_{r=11}^{40}(r+1)(r+5)=\sum_{r=1}^{40}(r+1)(r+5)-\sum_{r=1}^{10}(r+1)(r+5)$

$=\frac{1}{6}\times40\times87\times47-\frac{1}{6}\times10\times27\times17$

$=27260-765$

$=26495$ // AS REQUIRED

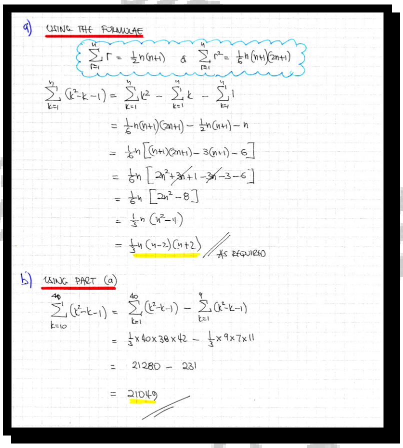
Question 8 **(**+)**

Show by using standard summation results that …

a) … $\displaystyle\sum_{k=1}^{n}\left(k^2-k-1\right)=\frac{1}{3}n(n+2)(n-2)$.

b) … $\displaystyle\sum_{k=10}^{40}\left(k^2-k-1\right)=21049$.

[ ] , [proof]



a) USING THE FORMULAE

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ & $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{k=1}^{n}(k^2-k-1) = \sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k - \sum_{k=1}^{n}1$

$= \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) - n$

$= \frac{1}{6}n\left[(n+1)(2n+1) - 3(n+1) - 6\right]$

$= \frac{1}{6}n\left[2n^2+3n+1-3n-3-6\right]$

$= \frac{1}{6}n\left[2n^2-8\right]$

$= \frac{1}{3}n(n^2-4)$

$= \frac{1}{3}n(n-2)(n+2)$ // AS REQUIRED

b) USING PART (a)

$\sum_{k=10}^{40}(k^2-k-1) = \sum_{k=1}^{40}(k^2-k-1) - \sum_{k=1}^{9}(k^2-k-1)$

$= \frac{1}{3}\times 40\times 38\times 42 - \frac{1}{3}\times 9\times 7\times 11$

$= 21280 - 231$

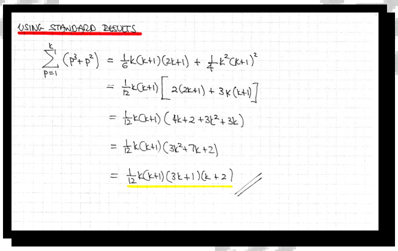
$= 21049$ //

Question 9 (**+)

Find, in fully factorized form, an expression for the sum

$$\sum_{p=1}^{k}\left(p^3+p^2\right).$$

$$\boxed{\tfrac{1}{12}k(k+1)(k+2)(3k+1)}$$

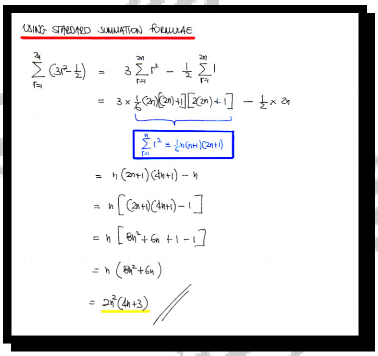


Question 10 (**+)

Find, in fully factorized form, an expression for the sum

$$\sum_{r=1}^{2n}\left(3r^2-\tfrac{1}{2}\right).$$

$$\boxed{2n^2(4n+3)}$$

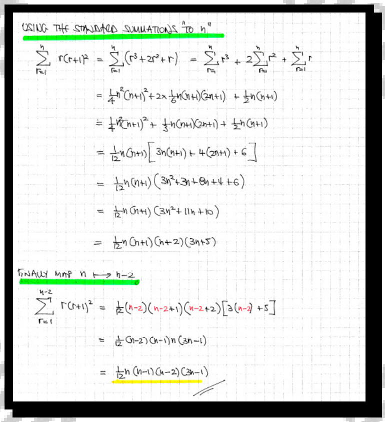


Question 11 **(***)**

Use standard results on summations to show that

$$\sum_{r=1}^{n-2} r(r+1)^2 = \frac{1}{12}n(n-1)(n-2)(3n-1).$$

proof



USING THE STANDARD SUMMATIONS "TO n"

$$\sum_{r=1}^{n} r(r+1)^2 = \sum_{r=1}^{n}(r^3+2r^2+r) = \sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$$

$$= \frac{1}{4}n^2(n+1)^2 + 2\times\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$

$$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{3}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$

$$= \frac{1}{12}n(n+1)\left[3n(n+1) + 4(2n+1) + 6\right]$$

$$= \frac{1}{12}n(n+1)(3n^2+3n+8n+4+6)$$

$$= \frac{1}{12}n(n+1)(3n^2+11n+10)$$

$$= \frac{1}{12}n(n+1)(n+2)(3n+5)$$

FINALLY MAP $n \longmapsto n-2$

$$\sum_{r=1}^{n-2} r(r+1)^2 = \frac{1}{12}(n-2)(n-2+1)(n-2+2)\left[3(n-2)+5\right]$$

$$= \frac{1}{12}(n-2)(n-1)n(3n-1)$$

$$= \frac{1}{12}n(n-1)(n-2)(3n-1)$$

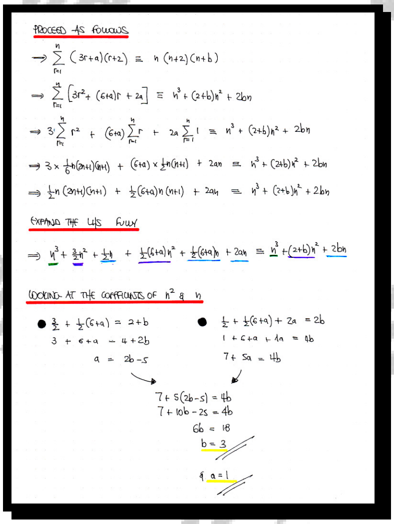
Question 12 (***)

It is given that

$$\sum_{r=1}^{n}\left[(3r+a)(r+2)\right] \equiv n(n+2)(n+b).$$

Determine the values of each of the constants a and b.

$\boxed{}$, $\boxed{a=1}$, $\boxed{b=3}$



PROCEED AS FOLLOWS

$$\Rightarrow \sum_{r=1}^{n}(3r+a)(r+2) \equiv n(n+2)(n+b)$$

$$\Rightarrow \sum_{r=1}^{n}\left[3r^2+(6+a)r+2a\right] \equiv n^3+(2+b)n^2+2bn$$

$$\Rightarrow 3\sum_{r=1}^{n}r^2+(6+a)\sum_{r=1}^{n}r+2a\sum_{r=1}^{n}1 \equiv n^3+(2+b)n^2+2bn$$

$$\Rightarrow 3\times\frac{1}{6}n(2n+1)(n+1)+(6+a)\times\frac{1}{2}n(n+1)+2an \equiv n^3+(2+b)n^2+2bn$$

$$\Rightarrow \frac{1}{2}n(2n+1)(n+1)+\frac{1}{2}(6+a)n(n+1)+2an \equiv n^3+(2+b)n^2+2bn$$

EXPAND THE LHS FULLY

$$\Rightarrow n^3+\frac{3}{2}n^2+\frac{1}{2}n+\frac{1}{2}(6+a)n^2+\frac{1}{2}(6+a)n+2an \equiv n^3+(2+b)n^2+2bn$$

LOOKING AT THE COEFFICIENTS OF n^2 & n

- $\frac{3}{2}+\frac{1}{2}(6+a)=2+b$
  $3+6+a=4+2b$
  $a=2b-5$
- $\frac{1}{2}+\frac{1}{2}(6+a)+2a=2b$
  $1+6+a+4a=4b$
  $7+5a=4b$

$7+5(2b-5)=4b$
$7+10b-25=4b$
$6b=18$
$b=3$

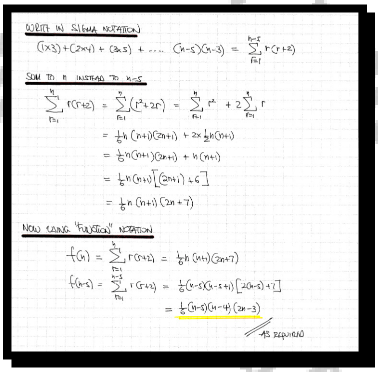
& $a=1$

Question 13 **(***)**

Show clearly that

$$(1\times3)+(2\times4)+(3\times5)+\ldots+(n-5)(n-3) \;=\; \frac{1}{6}(n+6)(2n+11)(n+5).$$

[ ], [proof]

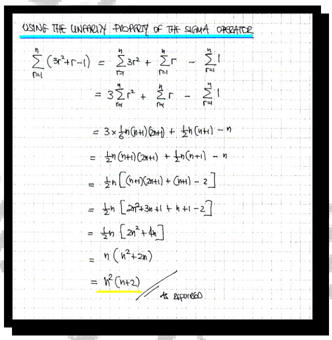


Question 14 **(***)**

Use standard results on summations to show that

$$\sum_{r=1}^{n}\left(3r^2+r-1\right) \equiv n^2(n+2).$$

[ ], [proof]

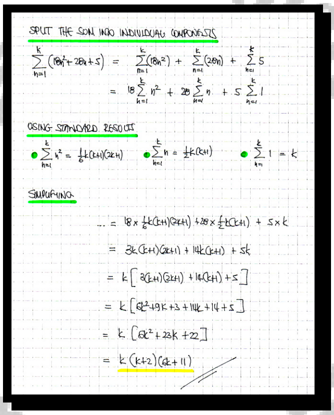


Question 15 (***)

Use standard results on summations to show that

$$\sum_{n=1}^{k}\left(18n^2+28n+5\right) \equiv k(k+2)(6k+11).$$

[ ], [proof]



SPLIT THE SUM INTO INDIVIDUAL COMPONENTS

$\sum_{n=1}^{k}(18n^2+28n+5) = \sum_{n=1}^{k}(18n^2) + \sum_{n=1}^{k}(28n) + \sum_{n=1}^{k}5$

$= 18\sum_{n=1}^{k} n^2 + 28\sum_{n=1}^{k} n + 5\sum_{n=1}^{k}1$

USING STANDARD RESULTS

$\sum_{n=1}^{k} n^2 = \frac{1}{6}k(k+1)(2k+1)$ $\sum_{n=1}^{k} n = \frac{1}{2}k(k+1)$ $\sum_{n=1}^{k}1 = k$

SIMPLIFYING

$\dots = 18\times\frac{1}{6}k(k+1)(2k+1) + 28\times\frac{1}{2}k(k+1) + 5\times k$

$= 3k(k+1)(2k+1) + 14k(k+1) + 5k$

$= k\left[3(k+1)(2k+1) + 14(k+1) + 5\right]$

$= k\left[6k^2+9k+3+14k+14+5\right]$

$= k\left[6k^2+23k+22\right]$

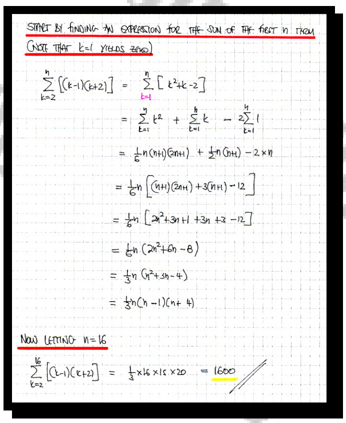
$= k(k+2)(6k+11)$

Question 16 (***)

Use standard results on summations to find the value of the following sum.

$$\sum_{k=2}^{16}\left[(k-1)(k+2)\right].$$

[ ], [1600]



START BY FINDING AN EXPRESSION FOR THE SUM OF THE FIRST n TERMS

(NOTE THAT k=1 YIELDS ZERO)

$\sum_{k=2}^{n}\left[(k-1)(k+2)\right] = \sum_{k=1}^{n}\left[k^2+k-2\right]$

$= \sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k - 2\sum_{k=1}^{n}1$

$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - 2\times n$

$= \frac{1}{6}n\left[(n+1)(2n+1)+3(n+1)-12\right]$

$= \frac{1}{6}n\left[2n^2+3n+1+3n+3-12\right]$

$= \frac{1}{6}n(2n^2+6n-8)$

$= \frac{1}{3}n(n^2+3n-4)$

$= \frac{1}{3}n(n-1)(n+4)$

NOW LETTING n=16

$\sum_{k=2}^{16}\left[(k-1)(k+2)\right] = \frac{1}{3}\times16\times15\times20 = 1600$

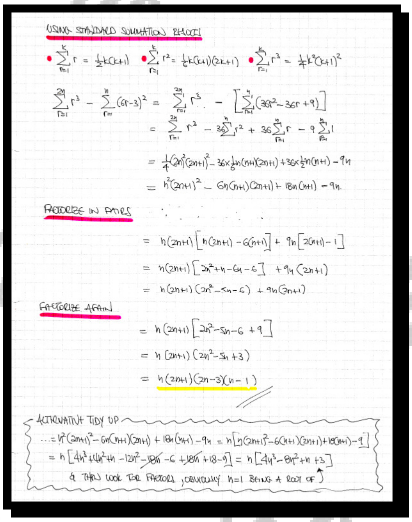
Question 17 (***)

Use standard results on summations to show that

$$\sum_{r=1}^{2n} r^3 - \sum_{r=1}^{n}(6r-3)^2 \equiv f(n),$$

where $f(n)$ is written as a product of 4 linear factors.

$\boxed{}$, $\boxed{f(n) = n(n-1)(2n+1)(2n-3)}$



SUMMATIONS BY FORMULAS

15 STANDARD QUESTIONS

Question 1 (***+)

Find the sum of the first n terms of the series

$$1\cdot2\cdot3+2\cdot3\cdot5+3\cdot4\cdot7+4\cdot5\cdot9+\dots.$$

Express the answer as a product of linear factors.

, proof

START BY WRITTING THE SERIES IN SIGMA NOTATION

$$\underbrace{(1\cdot2\cdot3)+(2\cdot3\cdot5)+(3\cdot4\cdot7)+(4\cdot5\cdot9)+\dots}_{\text{N TERMS}} = \sum_{r=1}^{n}\left[r(r+1)(2r+1)\right]$$

EXPAND & SIMPLIFY

$$\sum_{r=1}^{n}\left[r(r+1)(2r+1)\right] = \sum_{r=1}^{n}\left[2r^3+3r^2+r\right]$$

$$= 2\sum_{r=1}^{n}r^3 + 3\sum_{r=1}^{n}r^2 + \sum_{r=1}^{n}r$$

USING STANDARD RESULT

$$= 2\times\frac{1}{4}n^2(n+1)^2 + 3\times\frac{1}{6}\times n(n+1)(2n+1) + \frac{1}{2}\times n(n+1)$$

$$= \frac{1}{2}n^2(n+1)^2 + \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$

$$= \frac{1}{2}n(n+1)\left[n(n+1) + (2n+1) + 1\right]$$

$$= \frac{1}{2}n(n+1)\left[n^2+3n+2\right]$$

$$= \frac{1}{2}n(n+1)(n+1)(n+2)$$

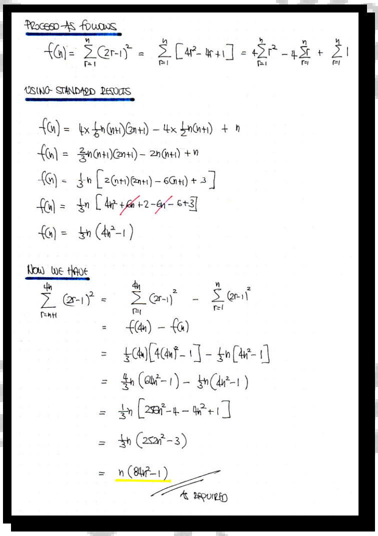
$$= \frac{1}{2}n(n+1)^2(n+2)$$

Question 2 (***+)

By using standard results, show that

$$\sum_{r=n+1}^{4n} (2r-1)^2 \equiv n\left(84n^2-1\right).$$

proof


PROCEED AS FOLLOWS

$f(n) = \sum_{r=1}^{n}(2r-1)^2 = \sum_{r=1}^{n}\left[4r^2-4r+1\right] = 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

USING STANDARD RESULTS

$f(n) = 4\times\frac{1}{6}n(n+1)(2n+1) - 4\times\frac{1}{2}n(n+1) + n$

$f(n) = \frac{2}{3}n(n+1)(2n+1) - 2n(n+1) + n$

$f(n) = \frac{1}{3}n\left[2(n+1)(2n+1) - 6(n+1) + 3\right]$

$f(n) = \frac{1}{3}n\left[4n^2 + 6n + 2 - 6n - 6 + 3\right]$

$f(n) = \frac{1}{3}n\left(4n^2-1\right)$

NOW WE HAVE

$\sum_{r=n+1}^{4n}(2r-1)^2 = \sum_{r=1}^{4n}(2r-1)^2 - \sum_{r=1}^{n}(2r-1)^2$

$= f(4n) - f(n)$

$= \frac{1}{3}(4n)\left[4(4n)^2-1\right] - \frac{1}{3}n\left[4n^2-1\right]$

$= \frac{4}{3}n\left(64n^2-1\right) - \frac{1}{3}n\left(4n^2-1\right)$

$= \frac{1}{3}n\left[256n^2 - 4 - 4n^2 + 1\right]$

$= \frac{1}{3}n\left(252n^2-3\right)$

$= n\left(84n^2-1\right)$

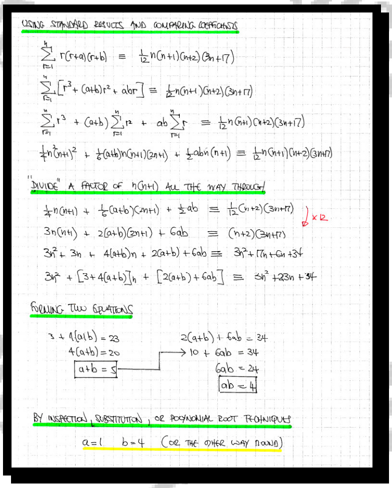
AS REQUIRED

Question 3 **(***+)**

Determine the value of a and the value of b given that

$$\sum_{r=1}^{n} r(r+a)(r+b) \equiv \frac{1}{12}n(n+1)(n+2)(3n+17).$$

	$a=1, \quad b=4$ or the other way round


USING STANDARD RESULTS AND COMPARING COEFFICIENTS

$$\sum_{r=1}^{n} r(r+a)(r+b) \equiv \frac{1}{12}n(n+1)(n+2)(3n+17)$$

$$\sum_{r=1}^{n}\left[r^3+(a+b)r^2+abr\right] \equiv \frac{1}{12}n(n+1)(n+2)(3n+17)$$

$$\sum_{r=1}^{n} r^3 + (a+b)\sum_{r=1}^{n} r^2 + ab\sum_{r=1}^{n} r \equiv \frac{1}{12}n(n+1)(n+2)(3n+17)$$

$$\frac{1}{4}n^2(n+1)^2 + \frac{1}{6}(a+b)n(n+1)(2n+1) + \frac{1}{2}abn(n+1) \equiv \frac{1}{12}n(n+1)(n+2)(3n+17)$$

"DIVIDE" A FACTOR OF $n(n+1)$ ALL THE WAY THROUGH

$$\frac{1}{4}n(n+1) + \frac{1}{6}(a+b)(2n+1) + \frac{1}{2}ab \equiv \frac{1}{12}(n+2)(3n+17) \quad \times 12$$

$$3n(n+1) + 2(a+b)(2n+1) + 6ab \equiv (n+2)(3n+17)$$

$$3n^2 + 3n + 4(a+b)n + 2(a+b) + 6ab \equiv 3n^2 + 17n + 6n + 34$$

$$3n^2 + [3+4(a+b)]n + [2(a+b)+6ab] \equiv 3n^2 + 23n + 34$$

FORMING TWO EQUATIONS

$3 + 4(a+b) = 23$

$4(a+b) = 20$

$a+b = 5$

$2(a+b) + 6ab = 34$

$10 + 6ab = 34$

$6ab = 24$

$ab = 4$

BY INSPECTION, SUBSTITUTION, OR POLYNOMIAL ROOT TECHNIQUES

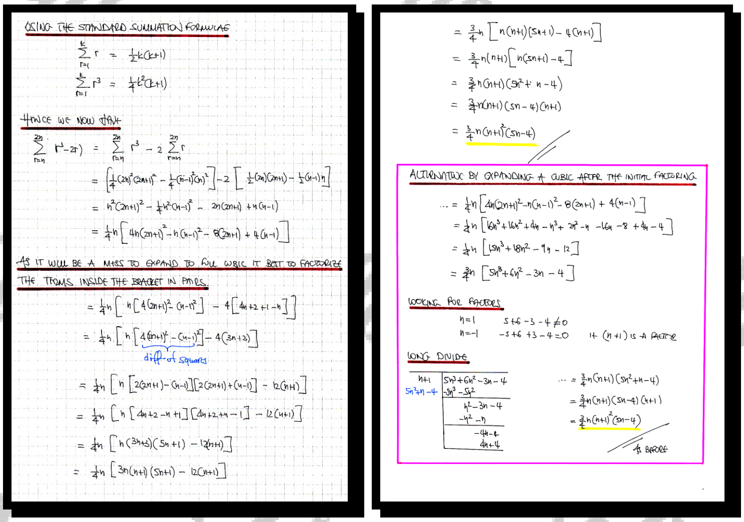
$a=1 \quad b=4$ (OR THE OTHER WAY ROUND)

Question 4 (***+)

Find, in fully factorized form, an expression for the following sum.

$$\sum_{r=n}^{2n}\left(r^3-2r\right).$$

$$\boxed{},\quad \boxed{\sum_{r=n}^{2n}\left(r^3-2r\right)=\frac{3}{4}n(5n-4)(n+1)^2}$$



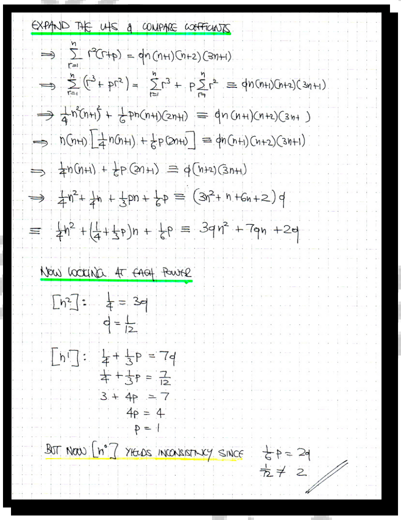
Question 5 **(***+)**

It is thought that for some values of the constants p and q that

$$\sum_{r=1}^{n} r^2(r+p) \equiv n(n+1)(n+2)(3n+q).$$

Use a detailed method to show that there exist no such values of p and q.

[ ] , [proof]



EXPAND THE LHS & COMPARE COEFFICIENTS

$\Rightarrow \sum_{r=1}^{n} r^2(r+p) = qn(n+1)(n+2)(3n+1)$

$\Rightarrow \sum_{r=1}^{n}(r^3+pr^2) = \sum_{r=1}^{n} r^3 + p\sum_{r=1}^{n} r^2 \equiv qn(n+1)(n+2)(3n+1)$

$\Rightarrow \frac{1}{4}n^2(n+1)^2 + \frac{1}{6}pn(n+1)(2n+1) \equiv qn(n+1)(n+2)(3n+\ )$

$\Rightarrow n(n+1)\left[\frac{1}{4}n(n+1) + \frac{1}{6}p(2n+1)\right] \equiv qn(n+1)(n+2)(3n+1)$

$\Rightarrow \frac{1}{4}n(n+1) + \frac{1}{6}p(2n+1) \equiv q(n+2)(3n+1)$

$\Rightarrow \frac{1}{4}n^2 + \frac{1}{4}n + \frac{1}{3}pn + \frac{1}{6}p \equiv (3n^2+n+6n+2)q$

$\equiv \frac{1}{4}n^2 + \left(\frac{1}{4}+\frac{1}{3}p\right)n + \frac{1}{6}p \equiv 3qn^2 + 7qn + 2q$

NOW LOOKING AT EACH POWER

$[n^2]$: $\frac{1}{4} = 3q$

$q = \frac{1}{12}$

$[n^1]$: $\frac{1}{4} + \frac{1}{3}p = 7q$

$\frac{1}{4} + \frac{1}{3}p = \frac{7}{12}$

$3 + 4p = 7$

$4p = 4$

$p = 1$

BUT NOW $[n^0]$ YIELDS INCONSISTENCY SINCE $\frac{1}{6}p = 2q$

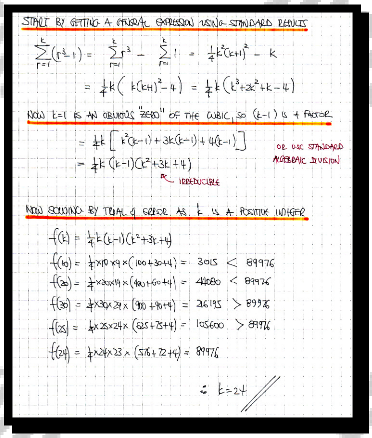
$\frac{1}{6} \neq \frac{2}{12}$ … $\frac{1}{72} \neq 2$

Question 6 (****)

Use standard results on summations to solve the following equation.

$$\sum_{r=1}^{k}\left(r^3-1\right)=89976.$$

$\boxed{}$, $\boxed{k=24}$


START BY GETTING A GENERAL EXPRESSION USING STANDARD RESULTS

$$\sum_{r=1}^{k}(r^3-1)=\sum_{r=1}^{k}r^3-\sum_{r=1}^{k}1=\frac{1}{4}k^2(k+1)^2-k$$

$$=\frac{1}{4}k\left(k(k+1)^2-4\right)=\frac{1}{4}k\left(k^3+2k^2+k-4\right)$$

NOW $k=1$ IS AN OBVIOUS "ZERO" OF THE CUBIC, SO $(k-1)$ IS A FACTOR

$$=\frac{1}{4}k\left[k^2(k-1)+3k(k-1)+4(k-1)\right]$$

$$=\frac{1}{4}k(k-1)(k^2+3k+4)$$

OR USE STANDARD ALGEBRAIC DIVISION

IRREDUCIBLE

NOW SOLVING BY TRIAL & ERROR AS k IS A POSITIVE INTEGER

$f(k)=\frac{1}{4}k(k-1)(k^2+3k+4)$

$f(10)=\frac{1}{4}\times10\times9\times(100+30+4)=3015<89976$

$f(20)=\frac{1}{4}\times20\times19\times(400+60+4)=44080<89976$

$f(30)=\frac{1}{4}\times30\times29\times(900+90+4)=216195>89976$

$f(25)=\frac{1}{4}\times25\times24\times(625+75+4)=105600>89976$

$f(24)=\frac{1}{4}\times24\times23\times(576+72+4)=89976$

$\therefore k=24$

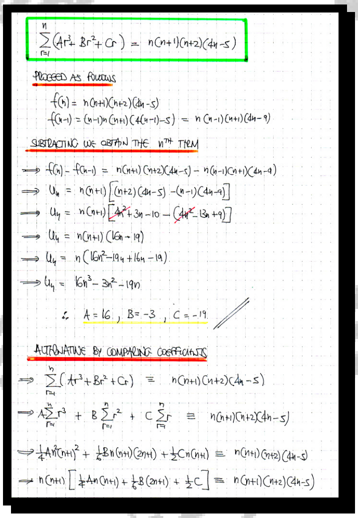
Question 7 (****)

It is given that

$$\sum_{r=1}^{n}\left(Ar^3+Br^2+Cr\right)=n(n+1)(n+2)(4n-5).$$

Use a detailed method to find the value of each of the integer constants, A, B and C.

$\boxed{}$, $\boxed{A=16}$, $\boxed{B=-3}$, $\boxed{C=-19}$

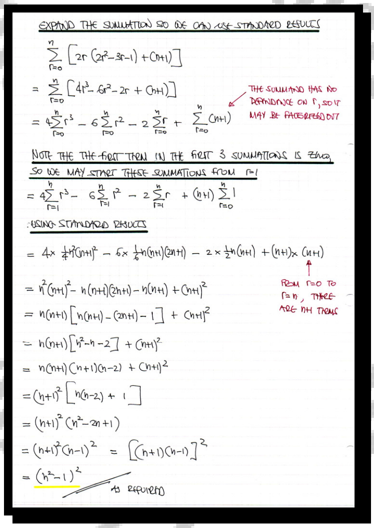


Question 8 (****)

Show by a detailed method that

$$\sum_{r=0}^{n}\left[2r\left(2r^2-3r-1\right)+n+1\right] = \left(n^2-1\right)^2.$$

[ ], [proof]



EXPAND THE SUMMATION SO WE CAN USE STANDARD RESULTS

$$\sum_{r=0}^{n}\left[2r\left(2r^2-3r-1\right)+(n+1)\right]$$

$$= \sum_{r=0}^{n}\left[4r^3-6r^2-2r+(n+1)\right]$$

$$= 4\sum_{r=0}^{n}r^3 - 6\sum_{r=0}^{n}r^2 - 2\sum_{r=0}^{n}r + \sum_{r=0}^{n}(n+1)$$

THE SUMMAND HAS NO DEPENDENCE ON r, SO IT MAY BE FACTORIZED OUT

NOTE THE THE FIRST TERM IN THE FIRST 3 SUMMATIONS IS ZERO, SO WE MAY START THESE SUMMATIONS FROM r=1

$$= 4\sum_{r=1}^{n}r^3 - 6\sum_{r=1}^{n}r^2 - 2\sum_{r=1}^{n}r + (n+1)\sum_{r=0}^{n}1$$

USING STANDARD RESULTS

$$= 4\times\tfrac{1}{4}n^2(n+1)^2 - 6\times\tfrac{1}{6}n(n+1)(2n+1) - 2\times\tfrac{1}{2}n(n+1) + (n+1)\times(n+1)$$

FROM r=0 TO r=n, THERE ARE n+1 TERMS

$$= n^2(n+1)^2 - n(n+1)(2n+1) - n(n+1) + (n+1)^2$$

$$= n(n+1)\left[n(n+1)-(2n+1)-1\right] + (n+1)^2$$

$$= n(n+1)\left[n^2-n-2\right] + (n+1)^2$$

$$= n(n+1)(n+1)(n-2) + (n+1)^2$$

$$= (n+1)^2\left[n(n-2)+1\right]$$

$$= (n+1)^2(n^2-2n+1)$$

$$= (n+1)^2(n-1)^2 = \left[(n+1)(n-1)\right]^2$$

$$= (n^2-1)^2$$ AS REQUIRED

Question 9 **(****)**

The sum, S_n, of the first n terms of a series whose general term is denoted by u_n is given by the following expression.

$$S_n = n^2(n+1)(n+2).$$

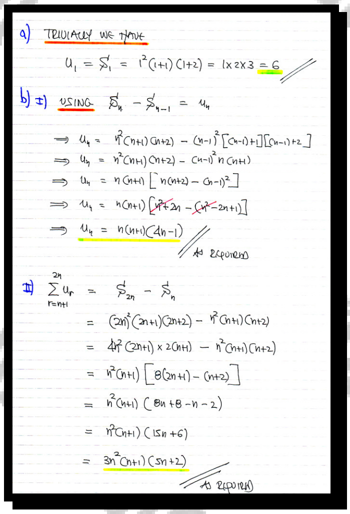
a) Find the first term of the series.

b) Show clearly that …

i. … $u_n = n(n+1)(4n-1)$

ii. … $\displaystyle\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2)$.

$\boxed{}$, $\boxed{u_1 = 6}$


a) TRIVIALLY WE HAVE

$u_1 = S_1 = 1^2(1+1)(1+2) = 1\times2\times3 = 6$

b) I) USING $S_n - S_{n-1} = u_n$

$\Rightarrow u_n = n^2(n+1)(n+2) - (n-1)^2[(n-1)+1][(n-1)+2]$

$\Rightarrow u_n = n^2(n+1)(n+2) - (n-1)^2 n(n+1)$

$\Rightarrow u_n = n(n+1)[n(n+2) - (n-1)^2]$

$\Rightarrow u_n = n(n+1)[n^2+2n - (n^2-2n+1)]$

$\Rightarrow u_n = n(n+1)(4n-1)$ AS REQUIRED

II) $\displaystyle\sum_{r=n+1}^{2n} u_r = S_{2n} - S_n$

$= (2n)^2(2n+1)(2n+2) - n^2(n+1)(n+2)$

$= 4n^2(2n+1)\times2(n+1) - n^2(n+1)(n+2)$

$= n^2(n+1)[8(2n+1) - (n+2)]$

$= n^2(n+1)(8n+8-n-2)$

$= n^2(n+1)(15n+6)$

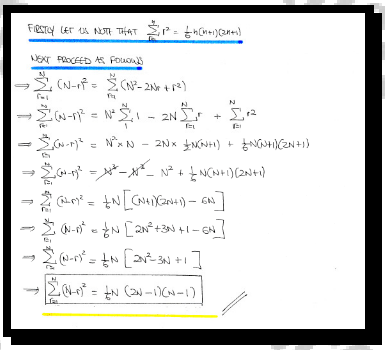
$= 3n^2(n+1)(5n+2)$ AS REQUIRED

Question 10 (****)

Use standard summation results to prove that

$$\sum_{r=1}^{n}(n-r)^2 = \frac{1}{2}n(n-1)(2n-1).$$

proof

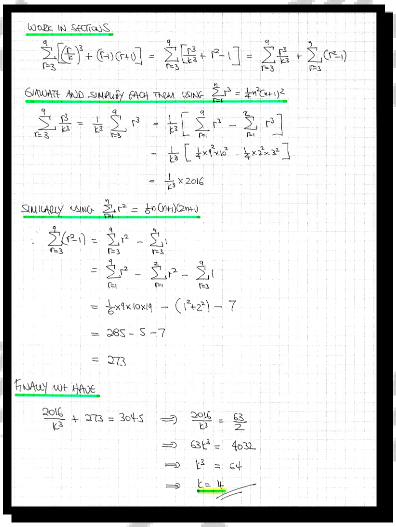


Question 11 (****)

Use standard results on summations to solve the following equation

$$\sum_{r=3}^{9}\left[\left(\frac{r}{k}\right)^3+(r-1)(r+1)\right] = 304.5 .$$

[illegible], $\boxed{k=4}$



WORK IN SECTIONS

$$\sum_{r=3}^{9}\left[\left(\frac{r}{k}\right)^3+(r-1)(r+1)\right] = \sum_{r=3}^{9}\left[\frac{r^3}{k^3}+r^2-1\right] = \sum_{r=3}^{9}\frac{r^3}{k^3}+\sum_{r=3}^{9}(r^2-1)$$

EVALUATE AND SIMPLIFY EACH TERM USING $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

$$\sum_{r=3}^{9}\frac{r^3}{k^3} = \frac{1}{k^3}\sum_{r=3}^{9} r^3 - \frac{1}{k^3}\left[\sum_{r=1}^{9} r^3 - \sum_{r=1}^{2} r^3\right]$$

$$- \frac{1}{k^3}\left[\frac{1}{4}\times 9^2\times 10^2 - \frac{1}{4}\times 2^2\times 3^2\right]$$

$$= \frac{1}{k^3}\times 2016$$

SIMILARLY USING $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$$\sum_{r=3}^{9}(r^2-1) = \sum_{r=3}^{9} r^2 - \sum_{r=3}^{9} 1$$

$$= \sum_{r=1}^{9} r^2 - \sum_{r=1}^{2} r^2 - \sum_{r=3}^{9} 1$$

$$= \frac{1}{6}\times 9\times 10\times 19 - (1^2+2^2) - 7$$

$$= 285 - 5 - 7$$

$$= 273$$

FINALLY WE HAVE

$$\frac{2016}{k^3} + 273 = 304.5 \implies \frac{2016}{k^3} = \frac{63}{2}$$

$$\implies 63k^3 = 4032$$

$$\implies k^3 = 64$$

$$\implies k = 4$$

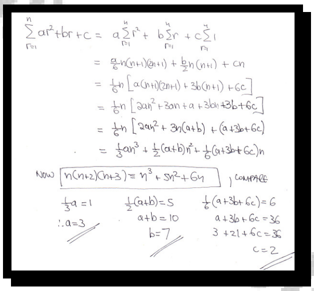
Question 12 **(****)**

$$\sum_{r=1}^{n}\left(ar^2+br+c\right)\equiv n^3+5n^2+6n,$$

where a, b and c are integer constants.

Determine the value of a, b and c.

$\boxed{a=3}$, $\boxed{b=7}$, $\boxed{c=2}$



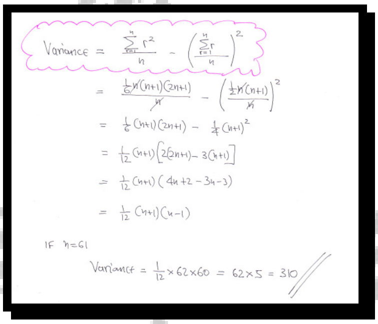
Question 13 **(****)**

The variance $\text{Var}(n)$ of the first n natural numbers is given by

$$\text{Var}(n) = \frac{1}{n}\sum_{r=1}^{n} r^2 - \left[\frac{1}{n}\sum_{r=1}^{n} r\right]^2 .$$

Determine a simplified expression for $\text{Var}(n)$ and hence evaluate $\text{Var}(61)$.

$\boxed{\text{Var}(n) = \frac{1}{12}\left(n^2 - 1\right)}$, $\boxed{\text{Var}(61) = 310}$



$$\text{Variance} = \frac{\sum_{r=1}^{n} r^2}{n} - \left(\frac{\sum_{r=1}^{n} r}{n}\right)^2$$

$$= \frac{\frac{1}{6}n(n+1)(2n+1)}{n} - \left(\frac{\frac{1}{2}n(n+1)}{n}\right)^2$$

$$= \frac{1}{6}(n+1)(2n+1) - \frac{1}{4}(n+1)^2$$

$$= \frac{1}{12}(n+1)\left[2(2n+1) - 3(n+1)\right]$$

$$= \frac{1}{12}(n+1)(4n+2-3n-3)$$

$$= \frac{1}{12}(n+1)(n-1)$$

IF $n = 61$

$$\text{Variance} = \frac{1}{12} \times 62 \times 60 = 62 \times 5 = 310$$

Question 14 (****)

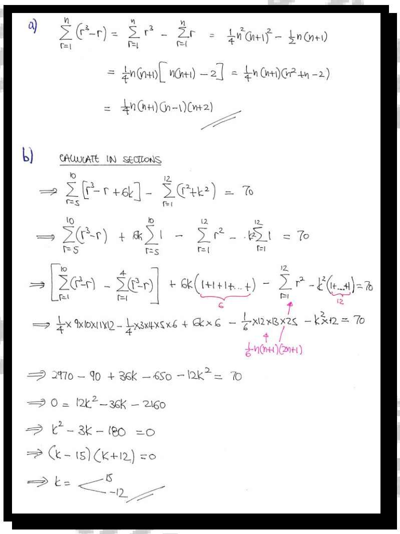
$$f(n) = \sum_{r=1}^{n} \left[r^3 - r \right], \qquad n \in \mathbb{N}.$$

a) Use standard summation results to find a fully factorized expression for $f(n)$.

b) Hence solve the equation

$$\sum_{r=5}^{10} \left[r^3 - r + 6k \right] - \sum_{r=1}^{12} \left[r^2 + k^2 \right] = 70$$

$\boxed{}$, $\boxed{f(n) = \frac{1}{4}n(n-1)(n+1)(n+2)}$, $\boxed{k = -12, \quad k = 15}$


a) $\sum_{r=1}^{n}(r^3-r) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r = \frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1)$

$= \frac{1}{4}n(n+1)\left[n(n+1) - 2\right] = \frac{1}{4}n(n+1)(n^2+n-2)$

$= \frac{1}{4}n(n+1)(n-1)(n+2)$

b) CALCULATE IN SECTIONS

$\Rightarrow \sum_{r=5}^{10}\left[r^3-r+6k\right] - \sum_{r=1}^{12}(r^2+k^2) = 70$

$\Rightarrow \sum_{r=5}^{10}(r^3-r) + 6k\sum_{r=5}^{10}1 - \sum_{r=1}^{12}r^2 - k^2\sum_{r=1}^{12}1 = 70$

$\Rightarrow \left[\sum_{r=1}^{10}(r^3-r) - \sum_{r=1}^{4}(r^3-r)\right] + 6k(1+1+1+\dots+1) - \sum_{r=1}^{12}r^2 - k^2(1+\dots+1) = 70$ (6, 12)

$\Rightarrow \frac{1}{4}\times 9\times 10\times 11\times 12 - \frac{1}{4}\times 3\times 4\times 5\times 6 + 6k\times 6 - \frac{1}{6}\times 12\times 13\times 25 - k^2\times 12 = 70$ ($\frac{1}{6}n(n+1)(2n+1)$)

$\Rightarrow 2970 - 90 + 36k - 650 - 12k^2 = 70$

$\Rightarrow 0 = 12k^2 - 36k - 2160$

$\Rightarrow k^2 - 3k - 180 = 0$

$\Rightarrow (k-15)(k+12) = 0$

$\Rightarrow k = 15, -12$

Question 15 (****)

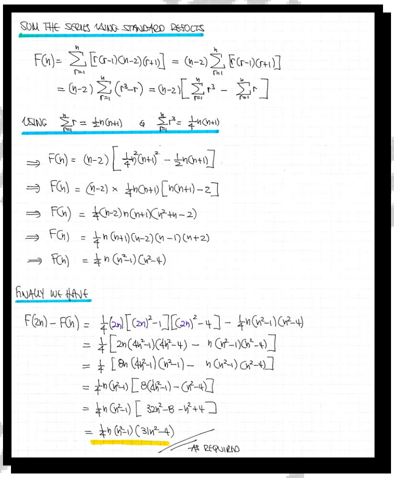
The function $F(n)$ is defined as

$$F(n)=\sum_{r=1}^{n}\left[r(r-1)(n-2)(r+1)\right] \quad n \in \mathbb{N}.$$

Show with detailed workings that

$$F(2n)-F(n)=\frac{1}{2}n\left(n^2-1\right)\left(31n^2-4\right).$$

proof



SUM THE SERIES USING STANDARD RESULTS

$$F(n)=\sum_{r=1}^{n}\left[r(r-1)(n-2)(r+1)\right]=(n-2)\sum_{r=1}^{n}\left[r(r-1)(r+1)\right]$$
$$=(n-2)\sum_{r=1}^{n}(r^3-r)=(n-2)\left[\sum_{r=1}^{n}r^3-\sum_{r=1}^{n}r\right]$$

USING $\sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$ & $\sum_{r=1}^{n}r^3=\frac{1}{4}n(n+1)$

$$\Rightarrow F(n)=(n-2)\left[\frac{1}{4}n^2(n+1)^2-\frac{1}{2}n(n+1)\right]$$
$$\Rightarrow F(n)=(n-2)\times\frac{1}{4}n(n+1)\left[n(n+1)-2\right]$$
$$\Rightarrow F(n)=\frac{1}{4}(n-2)n(n+1)(n^2+n-2)$$
$$\Rightarrow F(n)=\frac{1}{4}n(n+1)(n-2)(n-1)(n+2)$$
$$\Rightarrow F(n)=\frac{1}{4}n(n^2-1)(n^2-4)$$

FINALLY WE HAVE

$$F(2n)-F(n)=\frac{1}{4}(2n)\left[(2n)^2-1\right]\left[(2n)^2-4\right]-\frac{1}{4}n(n^2-1)(n^2-4)$$
$$=\frac{1}{4}\left[2n(4n^2-1)(4n^2-4)-n(n^2-1)(n^2-4)\right]$$
$$=\frac{1}{4}\left[8n(4n^2-1)(n^2-1)-n(n^2-1)(n^2-4)\right]$$
$$=\frac{1}{4}n(n^2-1)\left[8(4n^2-1)-(n^2-4)\right]$$
$$=\frac{1}{4}n(n^2-1)\left[32n^2-8-n^2+4\right]$$
$$=\frac{1}{4}n(n^2-1)(31n^2-4)$$

AS REQUIRED

SUMMATIONS BY FORMULAS

5 HARD QUESTIONS

Question 1 (****+)

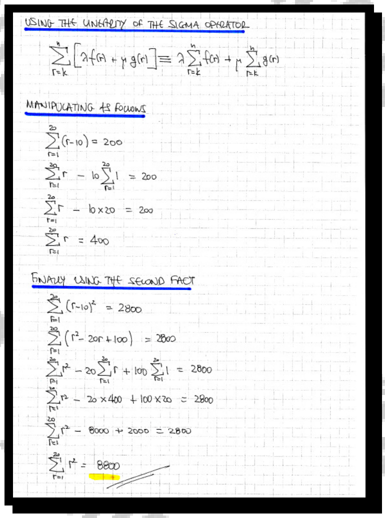
It is given that

$$\sum_{r=1}^{20}(r-10)=200 \quad \text{and} \quad \sum_{r=1}^{20}(r-10)^2=2800.$$

Find the value of

$$\sum_{r=1}^{20} r^2.$$

$$\boxed{}, \quad \boxed{\sum_{r=1}^{20} r^2 = 8800}$$


USING THE LINEARITY OF THE SIGMA OPERATOR

$$\sum_{r=k}^{n}\left[\lambda f(r)+\mu g(r)\right] \equiv \lambda\sum_{r=k}^{n} f(r)+\mu\sum_{r=k}^{n} g(r)$$

MANIPULATING AS FOLLOWS

$$\sum_{r=1}^{20}(r-10)=200$$
$$\sum_{r=1}^{20} r - 10\sum_{r=1}^{20} 1 = 200$$
$$\sum_{r=1}^{20} r - 10\times 20 = 200$$
$$\sum_{r=1}^{20} r = 400$$

FINALLY USING THE SECOND FACT

$$\sum_{r=1}^{20}(r-10)^2 = 2800$$
$$\sum_{r=1}^{20}(r^2-20r+100) = 2800$$
$$\sum_{r=1}^{20} r^2 - 20\sum_{r=1}^{20} r + 100\sum_{r=1}^{20} 1 = 2800$$
$$\sum_{r=1}^{20} r^2 - 20\times 400 + 100\times 20 = 2800$$
$$\sum_{r=1}^{20} r^2 - 8000 + 2000 = 2800$$
$$\sum_{r=1}^{20} r^2 = 8800$$

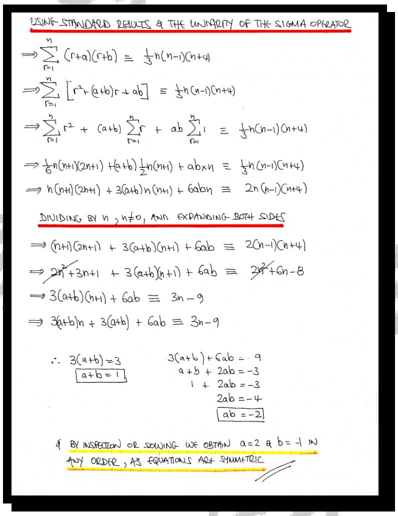
Question 2 **(****+)**

$$\sum_{r=1}^{n}(r+a)(r+b) \equiv \frac{1}{3}n(n-1)(n+4),$$

where a and b are integer constants.

Use a clear algebraic method to determine the value of a and the value of b.

$\boxed{2 \text{ and } -1 \text{ (in any order)}}$


USING STANDARD RESULTS & THE LINEARITY OF THE SIGMA OPERATOR

$\Rightarrow \sum_{r=1}^{n}(r+a)(r+b) \equiv \frac{1}{3}n(n-1)(n+4)$

$\Rightarrow \sum_{r=1}^{n}\left[r^2+(a+b)r+ab\right] \equiv \frac{1}{3}n(n-1)(n+4)$

$\Rightarrow \sum_{r=1}^{n}r^2 + (a+b)\sum_{r=1}^{n}r + ab\sum_{r=1}^{n}1 \equiv \frac{1}{3}n(n-1)(n+4)$

$\Rightarrow \frac{1}{6}n(n+1)(2n+1) + (a+b)\frac{1}{2}n(n+1) + ab \times n \equiv \frac{1}{3}n(n-1)(n+4)$

$\Rightarrow n(n+1)(2n+1) + 3(a+b)n(n+1) + 6abn \equiv 2n(n-1)(n+4)$

DIVIDING BY n, $n \neq 0$, AND EXPANDING BOTH SIDES

$\Rightarrow (n+1)(2n+1) + 3(a+b)(n+1) + 6ab \equiv 2(n-1)(n+4)$

$\Rightarrow 2n^2+3n+1 + 3(a+b)(n+1) + 6ab \equiv 2n^2+6n-8$

$\Rightarrow 3(a+b)(n+1) + 6ab \equiv 3n-9$

$\Rightarrow 3(a+b)n + 3(a+b) + 6ab \equiv 3n-9$

$\therefore 3(a+b)=3$, $\boxed{a+b=1}$

$3(a+b)+6ab=-9$
$a+b+2ab=-3$
$1+2ab=-3$
$2ab=-4$
$\boxed{ab=-2}$

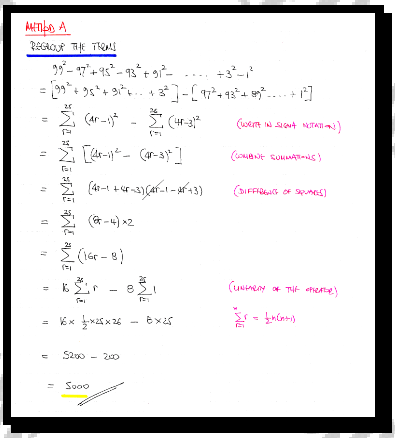
& BY INSPECTION OR SOLVING WE OBTAIN $a=2$ & $b=-1$ IN ANY ORDER, AS EQUATIONS ARE SYMMETRIC

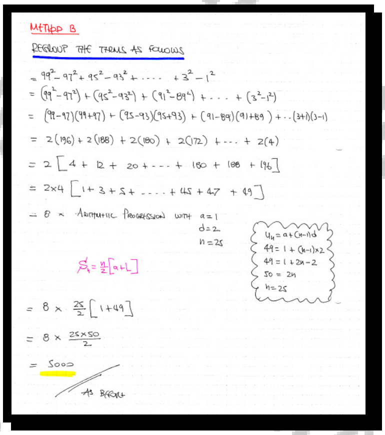
Question 3 **(****+)**

By using an algebraic method, find the value of

$$99^2 - 97^2 + 95^2 - 93^2 + \ldots + 3^2 - 1^2$$

$\boxed{\quad}$, $\boxed{5000}$



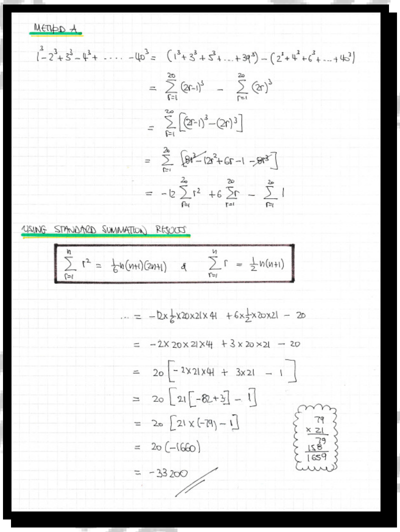


Question 4 (****+)

Show clearly that

$$1^3 - 2^3 + 3^3 - 4^3 + \ldots - 40^3 = -33200 .$$

[ ], [proof]



METHOD A

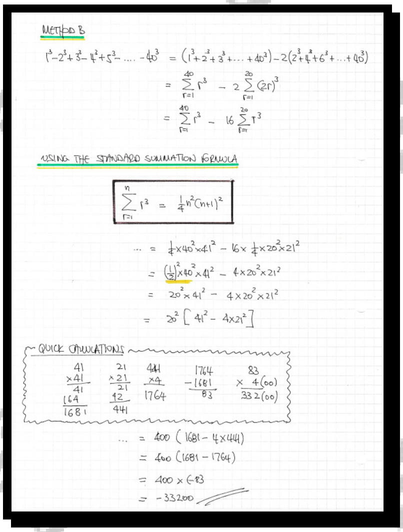
$$1^3 - 2^3 + 3^3 - 4^3 + \ldots - 40^3 = (1^3 + 3^3 + 5^3 + \ldots + 39^3) - (2^3 + 4^3 + 6^3 + \ldots + 40^3)$$
$$= \sum_{r=1}^{20} (2r-1)^3 - \sum_{r=1}^{20} (2r)^3$$
$$= \sum_{r=1}^{20} \left[(2r-1)^3 - (2r)^3 \right]$$
$$= \sum_{r=1}^{20} \left[8r^3 - 12r^2 + 6r - 1 - 8r^3 \right]$$
$$= -12 \sum_{r=1}^{20} r^2 + 6 \sum_{r=1}^{20} r - \sum_{r=1}^{20} 1$$

USING STANDARD SUMMATION RESULTS

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1) \quad \& \quad \sum_{r=1}^{n} r = \frac{1}{2} n(n+1)$$

$$\ldots = -12 \times \frac{1}{6} \times 20 \times 21 \times 41 + 6 \times \frac{1}{2} \times 20 \times 21 - 20$$
$$= -2 \times 20 \times 21 \times 41 + 3 \times 20 \times 21 - 20$$
$$= 20 \left[-2 \times 21 \times 41 + 3 \times 21 - 1 \right]$$
$$= 20 \left[21 [-82 + 3] - 1 \right]$$
$$= 20 \left[21 \times (-79) - 1 \right]$$
$$= 20(-1660)$$
$$= -33200$$

79
× 21
79
158
1659



METHOD B

$$1^3 - 2^3 + 3^3 - 4^3 + 5^3 - \ldots - 40^3 = (1^3 + 2^3 + 3^3 + \ldots + 40^3) - 2(2^3 + 4^3 + 6^3 + \ldots + 40^3)$$
$$= \sum_{r=1}^{40} r^3 - 2 \sum_{r=1}^{20} (2r)^3$$
$$= \sum_{r=1}^{40} r^3 - 16 \sum_{r=1}^{20} r^3$$

USING THE STANDARD SUMMATION FORMULA

$$\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n+1)^2$$

$$\ldots = \frac{1}{4} \times 40^2 \times 41^2 - 16 \times \frac{1}{4} \times 20^2 \times 21^2$$
$$= \left(\frac{1}{2}\right)^2 \times 40^2 \times 41^2 - 4 \times 20^2 \times 21^2$$
$$= 20^2 \times 41^2 - 4 \times 20^2 \times 21^2$$
$$= 20^2 \left[41^2 - 4 \times 21^2 \right]$$

QUICK CALCULATIONS

41 × 41: 41, 164, 1681	21 × 21: 21, 42, 441	441 × 4 = 1764	1764 − 1681 = 83	83 × 4(00) = 332(00)

$$\ldots = 400(1681 - 4 \times 441)$$
$$= 400(1681 - 1764)$$
$$= 400 \times (-83)$$
$$= -33200$$

Question 5 (****+)

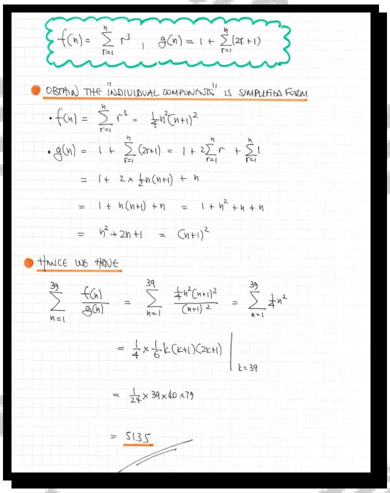
The positive integer functions f and g are defined as

$$f(n) = \sum_{r=1}^{n} r^3 \qquad \text{and} \qquad g(n) = 1 + \sum_{r=1}^{n} (2r+1).$$

Evaluate

$$\sum_{n=1}^{39} \left[\frac{f(n)}{g(n)} \right].$$

$\boxed{}$, $\boxed{5135}$



$f(n) = \sum_{r=1}^{n} r^3$, $g(n) = 1 + \sum_{r=1}^{n}(2r+1)$

- OBTAIN THE "INDIVIDUAL COMPONENTS" IS SIMPLIFIED FORM

$\bullet\ f(n) = \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

$\bullet\ g(n) = 1 + \sum_{r=1}^{n}(2r+1) = 1 + 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$= 1 + 2 \times \frac{1}{2}n(n+1) + n$

$= 1 + n(n+1) + n = 1 + n^2 + n + n$

$= n^2 + 2n + 1 = (n+1)^2$

- HENCE WE HAVE

$\sum_{n=1}^{39} \frac{f(n)}{g(n)} = \sum_{n=1}^{39} \frac{\frac{1}{4}n^2(n+1)^2}{(n+1)^2} = \sum_{n=1}^{39} \frac{1}{4}n^2$

$= \frac{1}{4} \times \frac{1}{6}k(k+1)(2k+1) \Big|_{k=39}$

$= \frac{1}{24} \times 39 \times 40 \times 79$

$= 5135$

SUMMATIONS BY FORMULAS

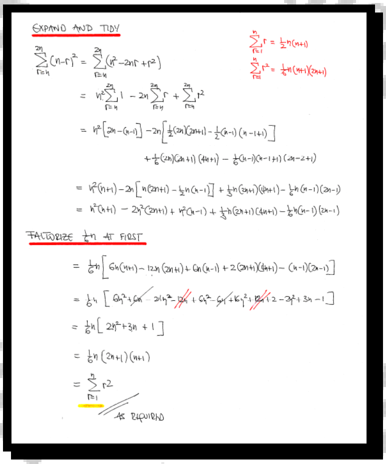
8 ENRICHMENT QUESTIONS

Question 1 (*****)

Use standard summation results to prove that

$$\sum_{r=n}^{2n} (n-r)^2 = \sum_{r=1}^{n} r^2 .$$

, proof



EXPAND AND TIDY

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$$\sum_{r=n}^{2n}(n-r)^2 = \sum_{r=n}^{2n}(n^2-2nr+r^2)$$
$$= n^2\sum_{r=n}^{2n}1 - 2n\sum_{r=n}^{2n}r + \sum_{r=n}^{2n}r^2$$
$$= n^2\left[2n-(n-1)\right] - 2n\left[\tfrac{1}{2}(2n)(2n+1) - \tfrac{1}{2}(n-1)(n-1+1)\right] + \tfrac{1}{6}(2n)(2n+1)(4n+1) - \tfrac{1}{6}(n-1)(n-1+1)(2n-2+1)$$
$$= n^2(n+1) - 2n\left[n(2n+1) - \tfrac{1}{2}n(n-1)\right] + \tfrac{1}{3}n(2n+1)(4n+1) - \tfrac{1}{6}n(n-1)(2n-1)$$
$$= n^2(n+1) - 2n^2(2n+1) + n^2(n-1) + \tfrac{1}{3}n(2n+1)(4n+1) - \tfrac{1}{6}n(n-1)(2n-1)$$

FACTORIZE $\frac{1}{6}n$ AT FIRST

$$= \tfrac{1}{6}n\left[6n(n+1) - 12n(2n+1) + 6n(n-1) + 2(2n+1)(4n+1) - (n-1)(2n-1)\right]$$
$$= \tfrac{1}{6}n\left[6n^2 + 6n - 24n^2 - 12n + 6n^2 - 6n + 16n^2 + 12n + 2 - 2n^2 + 3n - 1\right]$$
$$= \tfrac{1}{6}n\left[2n^2 + 3n + 1\right]$$
$$= \tfrac{1}{6}n(2n+1)(n+1)$$
$$= \sum_{r=1}^{n} r^2$$

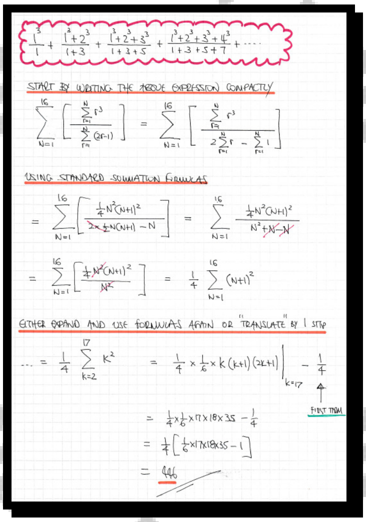
AS REQUIRED

Question 2 (*****)

Find the sum of the first 16 terms of the following series.

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\frac{1^3+2^3+3^3+4^3}{1+3+5+7}+\dots$$

SP-M , 446



$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\frac{1^3+2^3+3^3+4^3}{1+3+5+7}+\dots$$

START BY WRITING THE ABOVE EXPRESSION COMPACTLY

$$\sum_{N=1}^{16}\left[\frac{\sum_{r=1}^{N} r^3}{\sum_{r=1}^{N}(2r-1)}\right] = \sum_{N=1}^{16}\left[\frac{\sum_{r=1}^{N} r^3}{2\sum_{r=1}^{N} r - \sum_{r=1}^{N} 1}\right]$$

USING STANDARD SUMMATION FORMULAS

$$= \sum_{N=1}^{16}\left[\frac{\frac{1}{4}N^2(N+1)^2}{2\times\frac{1}{2}N(N+1)-N}\right] = \sum_{N=1}^{16}\frac{\frac{1}{4}N^2(N+1)^2}{N^2+N-N}$$

$$= \sum_{N=1}^{16}\left[\frac{\frac{1}{4}N^2(N+1)^2}{N^2}\right] = \frac{1}{4}\sum_{N=1}^{16}(N+1)^2$$

EITHER EXPAND AND USE FORMULAS AGAIN OR "TRANSLATE" BY 1 STEP

$$\dots = \frac{1}{4}\sum_{k=2}^{17}k^2 = \frac{1}{4}\times\frac{1}{6}\times k(k+1)(2k+1)\Big|_{k=17} - \frac{1}{4}$$

FIRST TERM

$$= \frac{1}{4}\times\frac{1}{6}\times17\times18\times35-\frac{1}{4}$$

$$= \frac{1}{4}\left[\frac{1}{6}\times17\times18\times35-1\right]$$

$$= 446$$

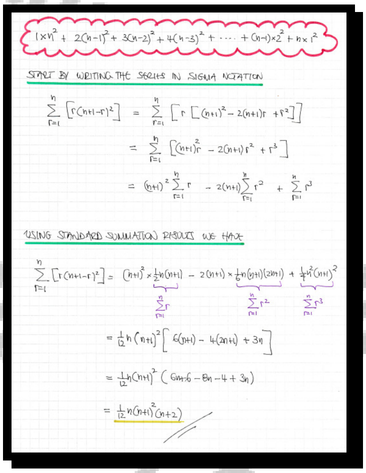
Question 3 (*****)

The function f is defined for $n \in \mathbb{N}$ as

$$f(n) \equiv 1 \times n^2 + 2(n-1)^2 + 3(n-2)^2 + 4(n-3)^2 + \ldots + (n-1) \times 2^2 + n \times 1^2 .$$

Determine a simplified expression for the sum of $f(n)$, giving the final answer in fully factorized form.

$\boxed{}$, $\boxed{f(n) = \frac{1}{12} n(n+2)(n+1)^2}$



Question 4 (*****)

Use an algebraic method justifying each step, to find the greatest value of k, $k \in \mathbb{N}$, which satisfies the following inequality.

$$\sum_{r=k+1}^{80}\left[\frac{r-1}{\log_{8^r}(16)}\right] > 100\,000 .$$

$\boxed{}$, $\boxed{k=48}$

START BY MANIPULATING THE LOGARITHM

$$\frac{1}{\log_{8^r}16} = \log_{16}8^r = r\log_{16}8 = r \times \frac{3}{4}$$

$$\left[\log_a b \equiv \frac{1}{\log_b a}\right] \qquad \text{SINCE } 16^{\frac{3}{4}} = 8$$

SUMMING FROM $r=1$ TO n, TO GET A GENERAL EXPRESSION

$$\sum_{r=1}^{n}\left[\frac{r-1}{\log_{8^r}16}\right] = \sum_{r=1}^{n}\left[\frac{3}{4}r(r-1)\right]$$

$$= \frac{3}{4}\sum_{r=1}^{n}\left[r^2 - r\right]$$

$$= \frac{3}{4}\left[\frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1)\right]$$

$$= \frac{3}{4}\left[\frac{1}{6}n(n+1)\left[(2n+1) - 3\right]\right]$$

$$= \frac{1}{8}n(n+1)(2n-2)$$

$$= \frac{1}{4}n(n+1)(n-1)$$

$$= \frac{1}{4}n(n^2-1)$$

RETURNING TO THE INEQUALITY

$$\Rightarrow \sum_{r=k+1}^{80}\left[\frac{r-1}{\log_{8^r}16}\right] > 100{,}000$$

$$\Rightarrow \frac{1}{4}\left[80(80^2-1)\right] - \frac{1}{4}k(k^2-1) > 100\,000$$

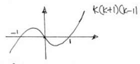
$$\Rightarrow 80(80^2-1) - k(k^2-1) > 400\,000$$

$$\Rightarrow k(k^2-1) - 80(80^2-1) < -400\,000$$

$$\Rightarrow k^3 - k - 511920 < -400000$$

$$\Rightarrow k(k+1)(k-1) < 111920$$

$k \in \mathbb{N}$ BUT IF $k \in \mathbb{R}$



SO $f(k) = k(k+1)(k-1)$ IS INCREASING FOR $k > 1$

BY TRIAL & ERROR NOTING THAT $f(k) \approx k^3$

$$f(40) = 40\times41\times39 = 63960 < 111920$$

$$f(50) = 0\times1\times49 = 124950 > 111920$$

$$f(49) = 49\times50\times48 = 117600 > 111920$$

$$f(48) = 48\times49\times47 = 110544 < 111920$$

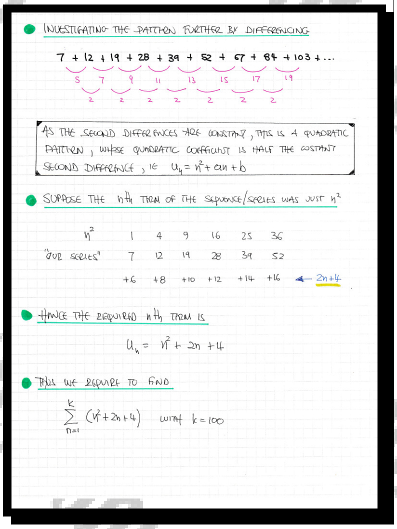
$$\therefore k = 48$$

Question 5 (*****)

Use algebra to find the sum of the first 100 terms of the following sequence.

$$7,\ 12,\ 19,\ 28,\ 39,\ 52,\ 67,\ 84,\ 103,\ \ldots$$

$\boxed{}$, $\boxed{f(n)=\frac{1}{12}n(n+2)(n+1)^2}$


INVESTIGATING THE PATTERN FURTHER BY DIFFERENCING

$7 + 12 + 19 + 28 + 39 + 52 + 67 + 84 + 103 + \ldots$

First differences: 5, 7, 9, 11, 13, 15, 17, 19

Second differences: 2, 2, 2, 2, 2, 2, 2

AS THE SECOND DIFFERENCES ARE CONSTANT, THIS IS A QUADRATIC PATTERN, WHOSE QUADRATIC COEFFICIENT IS HALF THE CONSTANT SECOND DIFFERENCE, IE $u_n = n^2 + an + b$

SUPPOSE THE nth TERM OF THE SEQUENCE/SERIES WAS JUST n^2

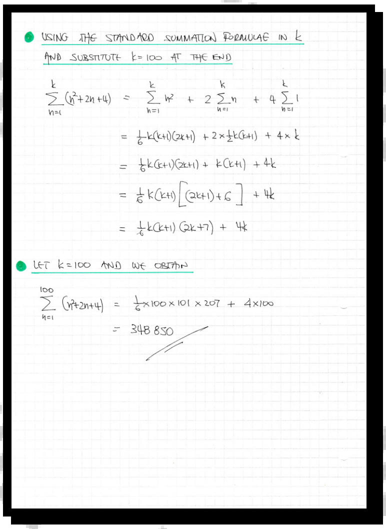
n^2	1	4	9	16	25	36	
"OUR SERIES"	7	12	19	28	39	52	
	+6	+8	+10	+12	+14	+16	← $2n+4$

HENCE THE REQUIRED nth TERM IS

$$u_n = n^2 + 2n + 4$$

THUS WE REQUIRE TO FIND

$$\sum_{n=1}^{k}(n^2+2n+4) \quad \text{WITH } k=100$$


USING THE STANDARD SUMMATION FORMULAE IN k AND SUBSTITUTE $k=100$ AT THE END

$$\sum_{n=1}^{k}(n^2+2n+4) = \sum_{n=1}^{k} n^2 + 2\sum_{n=1}^{k} n + 4\sum_{n=1}^{k} 1$$

$$= \tfrac{1}{6}k(k+1)(2k+1) + 2\times\tfrac{1}{2}k(k+1) + 4\times k$$

$$= \tfrac{1}{6}k(k+1)(2k+1) + k(k+1) + 4k$$

$$= \tfrac{1}{6}k(k+1)\left[(2k+1)+6\right] + 4k$$

$$= \tfrac{1}{6}k(k+1)(2k+7) + 4k$$

LET $k=100$ AND WE OBTAIN

$$\sum_{n=1}^{100}(n^2+2n+4) = \tfrac{1}{6}\times100\times101\times207 + 4\times100$$

$$= 348\,850$$

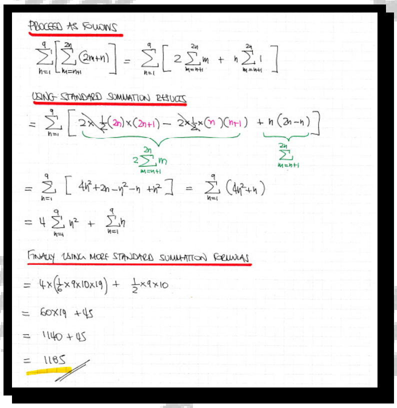
Question 6 (*****)

Evaluate the following expression

$$\sum_{n=1}^{9}\sum_{m=n+1}^{2n}[2m+n].$$

Detailed workings must be shown.

V , , 1185



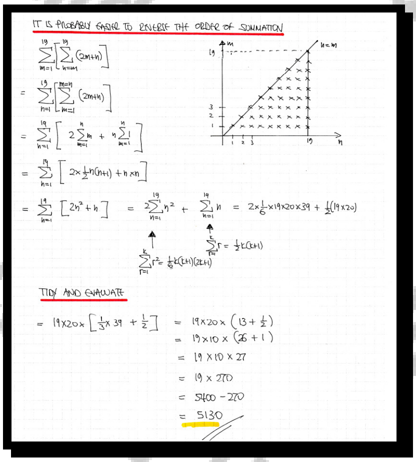
Question 7 (*****)

Evaluate the following expression

$$\sum_{m=1}^{19}\sum_{n=m}^{19}[2m+n].$$

You may find reversing the order of summation useful in this question

V, , 5130



IT IS PROBABLY EASIER TO REVERSE THE ORDER OF SUMMATION

$$\sum_{m=1}^{19}\left[\sum_{n=m}^{19}(2m+n)\right]$$
$$=\sum_{n=1}^{19}\left[\sum_{m=1}^{m=n}(2m+n)\right]$$
$$=\sum_{n=1}^{19}\left[2\sum_{m=1}^{n}m+n\sum_{m=1}^{n}1\right]$$
$$=\sum_{n=1}^{19}\left[2\times\tfrac{1}{2}n(n+1)+n\times n\right]$$
$$=\sum_{n=1}^{19}\left[2n^2+n\right]=2\sum_{n=1}^{19}n^2+\sum_{n=1}^{19}n=2\times\tfrac{1}{6}\times19\times20\times39+\tfrac{1}{2}(19\times20)$$

$\sum_{r=1}^{k}r^2=\frac{1}{6}k(k+1)(2k+1)$, $\sum_{r=1}^{k}r=\frac{1}{2}k(k+1)$

TIDY AND EVALUATE

$$=19\times20\times\left[\tfrac{1}{3}\times39+\tfrac{1}{2}\right]=19\times20\times\left(13+\tfrac{1}{2}\right)=19\times10\times(26+1)=19\times10\times27=19\times270=5400-270=5130$$

Question 8 (*****)

The function f is defined as

$$f(n,y) \equiv \sum_{x=1}^{n} \frac{x^2 y^x}{k}, \quad n \in \mathbb{N}, \ y \in \mathbb{R}$$

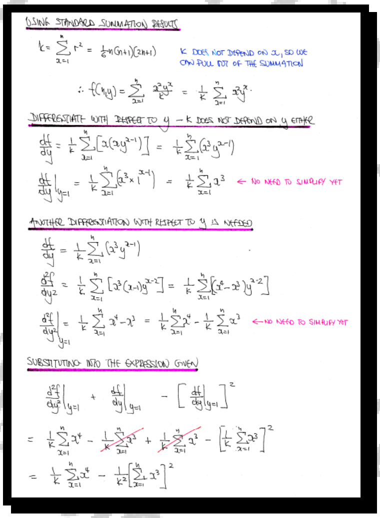
where $k = \sum_{r=1}^{n} r^2$.

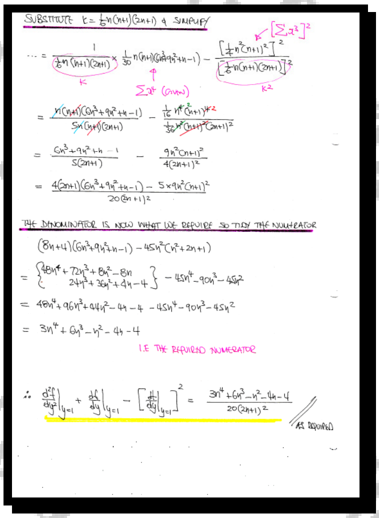
Use standard results on series to show that

$$\left.\frac{d^2 f}{dy^2}\right|_{y=1} + \left.\frac{df}{dy}\right|_{y=1} - \left[\left.\frac{df}{dy}\right|_{y=1}\right]^2 = \frac{3n^4 + 6n^3 - n^2 - 4n - 4}{20(2n+1)^2}.$$

You may assume without proof $\sum_{r=1}^{n} r^4 = \frac{1}{30} n(n+1)\left(6n^3 + 9n^2 + n - 1\right)$.

[illegible], proof





SUMMATIONS

METHOD OF DIFFERENCES

8 BASIC QUESTIONS

Question 1 (**)

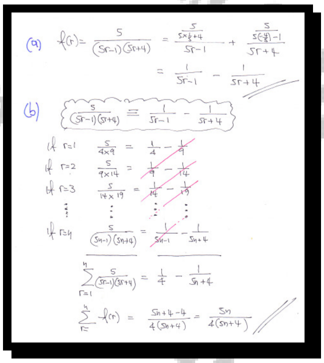
$$f(r) = \frac{5}{(5r-1)(5r+4)}, \quad r \in \mathbb{N}$$

a) Express $f(r)$ into partial fractions

b) Hence show that

$$\sum_{r=1}^{n} f(r) = \frac{5n}{4(5n+4)}.$$

$$\boxed{f(r) = \frac{1}{5r-1} - \frac{1}{5r+4}}$$



Question 2 (**)

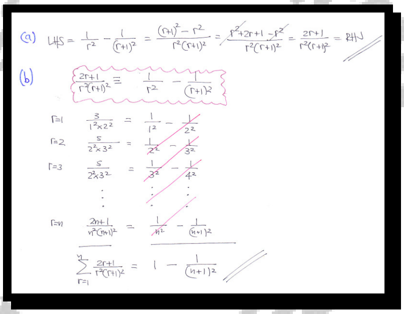
a) Show carefully that

$$\frac{1}{r^2}-\frac{1}{(r+1)^2}=\frac{2r+1}{r^2(r+1)^2}.$$

b) Hence use the method of differences to find

$$\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}.$$

$$\boxed{1-\frac{1}{(n+1)^2}}$$



(a) $\text{LHS}=\frac{1}{r^2}-\frac{1}{(r+1)^2}=\frac{(r+1)^2-r^2}{r^2(r+1)^2}=\frac{r^2+2r+1-r^2}{r^2(r+1)^2}=\frac{2r+1}{r^2(r+1)^2}=\text{RHS}$

(b) $\frac{2r+1}{r^2(r+1)^2}\equiv\frac{1}{r^2}-\frac{1}{(r+1)^2}$

$r=1$ $\quad\frac{3}{1^2\times2^2}=\frac{1}{1^2}-\frac{1}{2^2}$

$r=2$ $\quad\frac{5}{2^2\times3^2}=\frac{1}{2^2}-\frac{1}{3^2}$

$r=3$ $\quad\frac{5}{2^2\times3^2}=\frac{1}{3^2}-\frac{1}{4^2}$

$\vdots$

$r=n$ $\quad\frac{2n+1}{n^2(n+1)^2}=\frac{1}{n^2}-\frac{1}{(n+1)^2}$

$\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2}=1-\frac{1}{(n+1)^2}$

Question 3 (**)

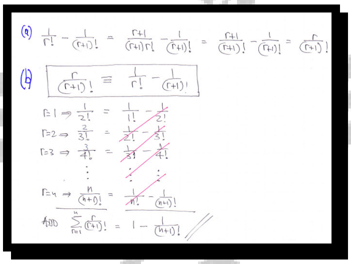
a) Show carefully that

$$\frac{1}{r!}-\frac{1}{(r+1)!}=\frac{r}{(r+1)!}.$$

b) Hence find

$$\sum_{r=1}^{n}\frac{r}{(r+1)!}.$$

$$\boxed{1-\frac{1}{(n+1)!}}$$



Question 4 (+)**

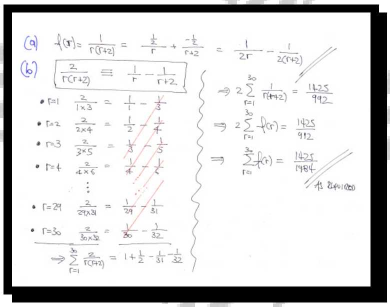
$$f(r) = \frac{1}{r(r+2)}, \quad r \in \mathbb{N}$$

a) Express $f(r)$ into partial fractions.

b) Hence show that

$$\sum_{r=1}^{30} f(r) = \frac{1425}{1984}.$$

$$\boxed{f(r) = \frac{1}{2r} - \frac{1}{2(r+2)}}$$



Question 5 (**+)

$$f(r) = \frac{2}{(r+1)(r+3)}, \quad r \in \mathbb{N}$$

a) Express $f(r)$ into partial fractions

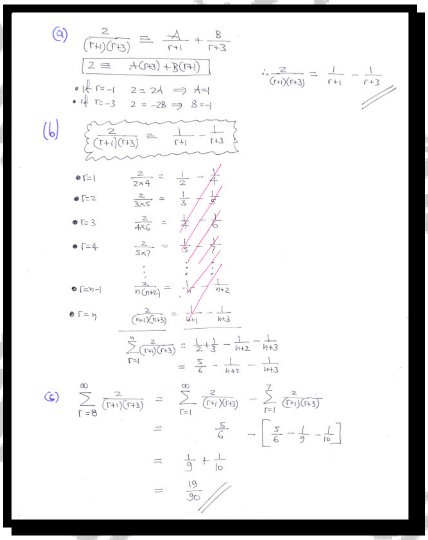
b) Use the method of differences to find

$$\sum_{r=1}^{n} f(r).$$

c) Hence evaluate

$$\sum_{r=8}^{\infty} f(r).$$

$$\boxed{f(r) = \frac{1}{r+1} - \frac{1}{r+3}}, \quad \boxed{\sum_{r=1}^{n} f(r) = \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}}, \quad \boxed{\sum_{r=8}^{\infty} f(r) = \frac{19}{90}}$$


(a) $\frac{2}{(r+1)(r+3)} \equiv \frac{A}{r+1} + \frac{B}{r+3}$

$2 \equiv A(r+3) + B(r+1)$

- if $r=-1$: $2 = 2A \Rightarrow A = 1$
- if $r=-3$: $2 = -2B \Rightarrow B = -1$

$\therefore \frac{2}{(r+1)(r+3)} = \frac{1}{r+1} - \frac{1}{r+3}$

(b) $\frac{2}{(r+1)(r+3)} = \frac{1}{r+1} - \frac{1}{r+3}$

- $r=1$: $\frac{2}{2\times 4} = \frac{1}{2} - \frac{1}{4}$
- $r=2$: $\frac{2}{3\times 5} = \frac{1}{3} - \frac{1}{5}$
- $r=3$: $\frac{2}{4\times 6} = \frac{1}{4} - \frac{1}{6}$
- $r=4$: $\frac{2}{5\times 7} = \frac{1}{5} - \frac{1}{7}$
- $r=n-1$: $\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$
- $r=n$: $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$

$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} = \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}$

(c) $\sum_{r=8}^{\infty} \frac{2}{(r+1)(r+3)} = \sum_{r=1}^{\infty} \frac{2}{(r+1)(r+3)} - \sum_{r=1}^{7} \frac{2}{(r+1)(r+3)} = \frac{5}{6} - \left[\frac{5}{6} - \frac{1}{9} - \frac{1}{10}\right] = \frac{1}{9} + \frac{1}{10} = \frac{19}{90}$

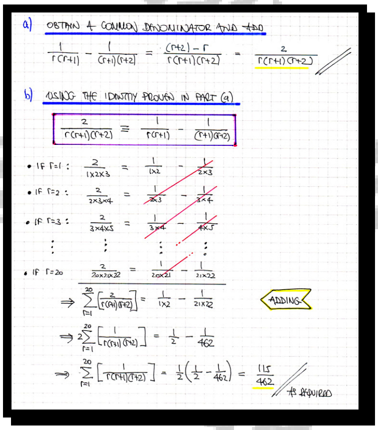
Question 6 (***)

a) Simplify $\dfrac{1}{r(r+1)} - \dfrac{1}{(r+1)(r+2)}$ into a single fraction.

b) Hence show that

$$\sum_{r=1}^{20}\left[\frac{1}{r(r+1)(r+2)}\right] = \frac{115}{462}.$$

proof , $\boxed{\dfrac{2}{r(r+1)(r+2)}}$



Question 7 (***)

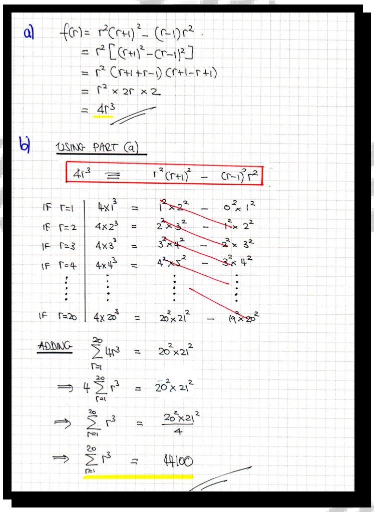
$$f(r) \equiv r^2(r+1)^2 - (r-1)^2 r^2, \; r \in \mathbb{N}.$$

a) Simplify $f(r)$ as far as possible.

b) Use the method of differences to show that

$$\sum_{r=1}^{20} r^3 = 44100.$$

FP4, $f(r) = 4r^3$



Question 8 **(***)**

$$f(r) = \frac{1}{r(r+2)}, \qquad r \in \mathbb{N}.$$

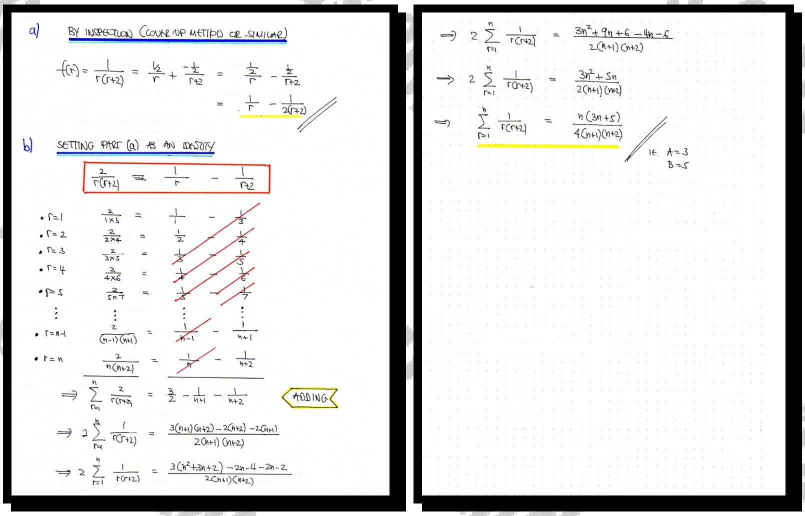
a) Express $f(r)$ in partial fractions.

b) Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} f(r) = \frac{n(An+B)}{4(n+1)(n+2)},$$

where A and B are constants to be found.

$\boxed{}$, $\boxed{A=3}$, $\boxed{B=5}$


a) BY INSPECTION (COVER UP METHOD OR SIMILAR)

$f(r) = \frac{1}{r(r+2)} = \frac{\frac{1}{2}}{r} + \frac{-\frac{1}{2}}{r+2} = \frac{\frac{1}{2}}{r} - \frac{\frac{1}{2}}{r+2} = \frac{1}{2r} - \frac{1}{2(r+2)}$

b) SETTING PART (a) AS AN IDENTITY

$\frac{2}{r(r+2)} \equiv \frac{1}{r} - \frac{1}{r+2}$

- $r=1$: $\frac{2}{1\times3} = \frac{1}{1} - \frac{1}{3}$
- $r=2$: $\frac{2}{2\times4} = \frac{1}{2} - \frac{1}{4}$
- $r=3$: $\frac{2}{3\times5} = \frac{1}{3} - \frac{1}{5}$
- $r=4$: $\frac{2}{4\times6} = \frac{1}{4} - \frac{1}{6}$
- $r=5$: $\frac{2}{5\times7} = \frac{1}{5} - \frac{1}{7}$
- $\vdots$
- $r=n-1$: $\frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}$
- $r=n$: $\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$

$\Rightarrow \sum_{r=1}^{n} \frac{2}{r(r+2)} = \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$ ADDING

$\Rightarrow 2\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{2(n+1)(n+2)}$

$\Rightarrow 2\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3(n^2+3n+2) - 2n - 4 - 2n - 2}{2(n+1)(n+2)}$

$\Rightarrow 2\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3n^2+9n+6-4n-6}{2(n+1)(n+2)}$

$\Rightarrow 2\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3n^2+5n}{2(n+1)(n+2)}$

$\Rightarrow \sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$

i.e. $A=3$, $B=5$

SUMMATIONS

METHOD OF DIFFERENCES

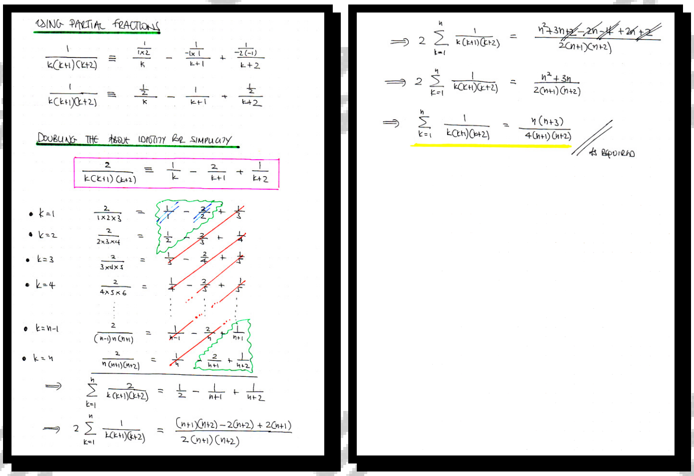
8 STANDARD QUESTIONS

Question 1 (***+)

Use the method of differences to show that

$$\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\ldots+\frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}.$$

[illegible], proof



Question 2 (***+)

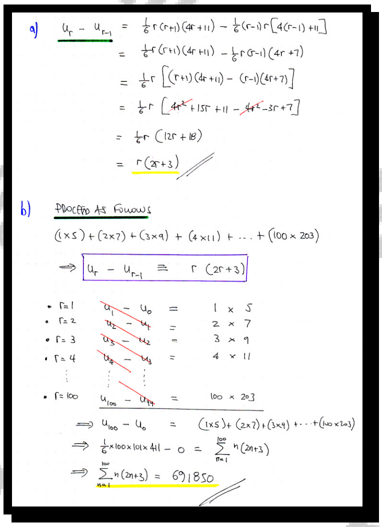
$$u_r = \frac{1}{6} r(r+1)(4r+11), \; r \in \mathbb{N}.$$

a) Simplify $u_r - u_{r-1}$ as far as possible.

b) By using the method of differences, or otherwise, find the sum of the first 100 terms of the following series.

$$(1\times5)+(2\times7)+(3\times9)+(4\times11)+\ldots$$

[ ], $r(2r+3)$, 691850


a) $u_r - u_{r-1} = \frac{1}{6}r(r+1)(4r+11) - \frac{1}{6}(r-1)r[4(r-1)+11]$

$= \frac{1}{6}r(r+1)(4r+11) - \frac{1}{6}r(r-1)(4r+7)$

$= \frac{1}{6}r[(r+1)(4r+11) - (r-1)(4r+7)]$

$= \frac{1}{6}r[4r^2+15r+11-4r^2-3r+7]$

$= \frac{1}{6}r(12r+18)$

$= r(2r+3)$

b) PROCEED AS FOLLOWS

$(1\times5)+(2\times7)+(3\times9)+(4\times11)+\ldots+(100\times203)$

$\Rightarrow u_r - u_{r-1} \equiv r(2r+3)$

- $r=1$: $u_1 - u_0 = 1\times5$
- $r=2$: $u_2 - u_1 = 2\times7$
- $r=3$: $u_3 - u_2 = 3\times9$
- $r=4$: $u_4 - u_3 = 4\times11$
- $\vdots$
- $r=100$: $u_{100} - u_{99} = 100\times203$

$\Rightarrow u_{100} - u_0 = (1\times5)+(2\times7)+(3\times9)+\ldots+(100\times203)$

$\Rightarrow \frac{1}{6}\times100\times101\times411 - 0 = \sum_{n=1}^{100} n(2n+3)$

$\Rightarrow \sum_{n=1}^{100} n(2n+3) = 691850$

Question 3 **(***+)**

$$f(r) = \frac{1}{(r+1)(r-1)}, \quad r \in \mathbb{N}.$$

a) Express $f(r)$ into partial fractions.

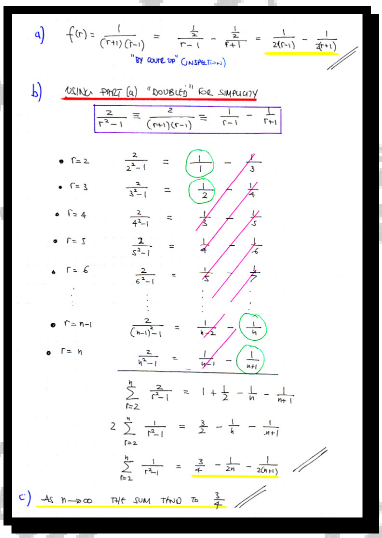
b) Hence show that

$$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}.$$

c) State the value of

$$\sum_{r=2}^{\infty} \frac{1}{r^2-1}$$

$\boxed{}$, $\boxed{f(r) = \frac{1}{2(r-1)} - \frac{1}{2(r+1)}}$, $\boxed{\frac{3}{4}}$



Question 4 (***+)

$$f(r) = \frac{2}{r(r+1)(r+2)}, \quad r \in \mathbb{N}.$$

a) Express $f(r)$ into partial fractions.

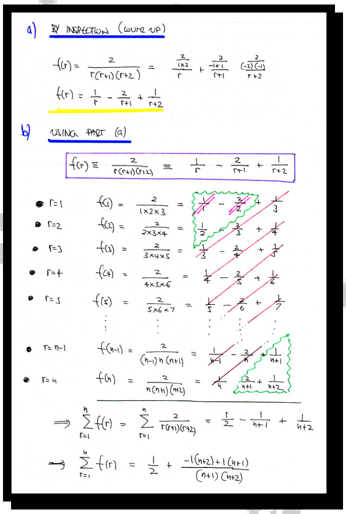
b) Hence show that

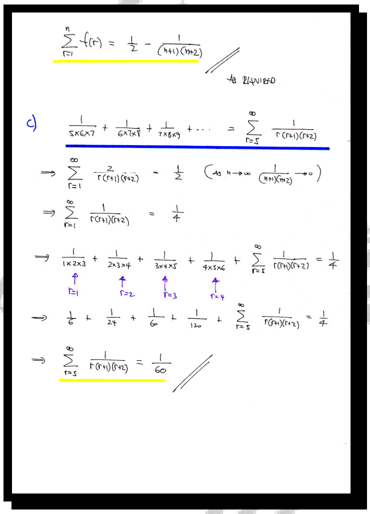
$$\sum_{r=1}^{n} f(r) = \frac{1}{2} - \frac{1}{(n+1)(n+2)}.$$

c) Find the value of the convergent infinite sum

$$\frac{1}{5\times6\times7} + \frac{1}{6\times7\times8} + \frac{1}{7\times8\times9} + \ldots$$

$\boxed{}$, $\boxed{f(r) = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}}$, $\boxed{\frac{1}{60}}$



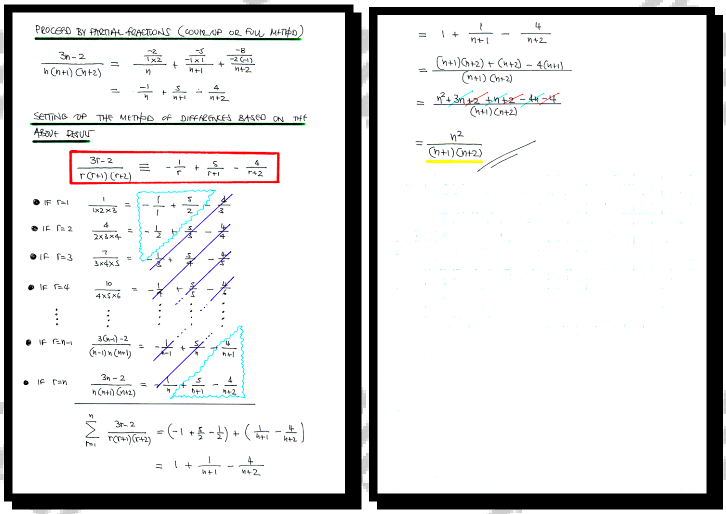


Question 5 (***+)

Use the method of differences to show that

$$\frac{1}{1\times2\times3}+\frac{4}{2\times3\times4}+\frac{7}{3\times4\times5}+\ldots+\frac{3n-2}{n(n+1)(n+2)} = \frac{n^2}{(n+1)(n+2)}$$

[illegible], proof



Question 6 (***+)

It is given that

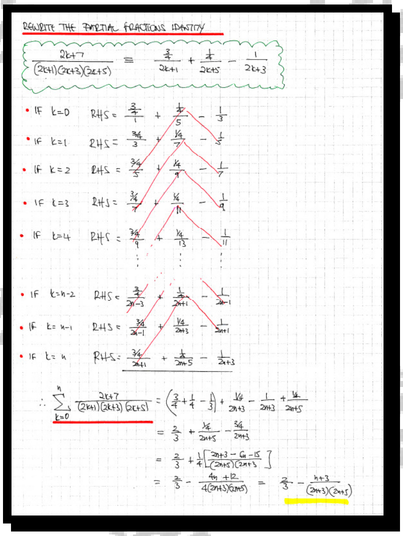
$$\frac{2k+7}{(2k+1)(2k+3)(2k+5)} \equiv \frac{3}{4(2k+1)} - \frac{1}{(2k+3)} + \frac{1}{4(2k+5)}.$$

Use the method of differences to find a simplified expression for

$$\frac{7}{1\times3\times5} + \frac{9}{3\times5\times7} + \frac{11}{5\times7\times9} + \ldots + \frac{2n+7}{(2n+1)(2n+3)(2n+5)}.$$

Give your answer in the form $\frac{2}{3} - f(n)$, where $f(n)$ is a single simplified fraction.

$\boxed{}$, $\boxed{f(n) = -\frac{n+3}{(2n+3)(2n+5)}}$



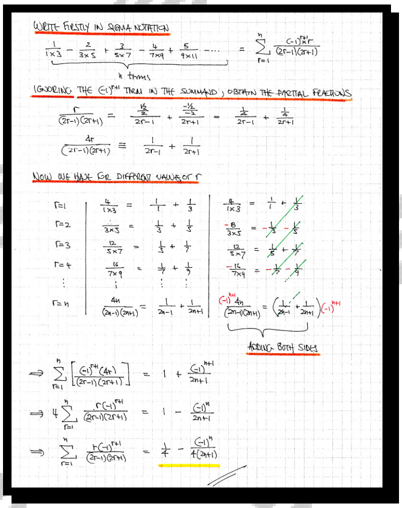
Question 7 (****)

Use the method of differences to find a simplified expression for the first n terms of the following series.

$$\frac{1}{1\times3}+\frac{2}{3\times5}+\frac{3}{5\times7}+\frac{4}{7\times9}+\dots$$

Give your answer in the form $\frac{1}{4}-f(n)$, where $f(n)$ is a single simplified fraction.

$$f(n)=\frac{(-1)^n}{4(2n+1)}$$



Question 8 (****)

$$f(r) = \frac{1}{\sqrt{r+2} + \sqrt{r}}, \quad r \geq 0.$$

a) Rationalize the denominator of $f(r)$.

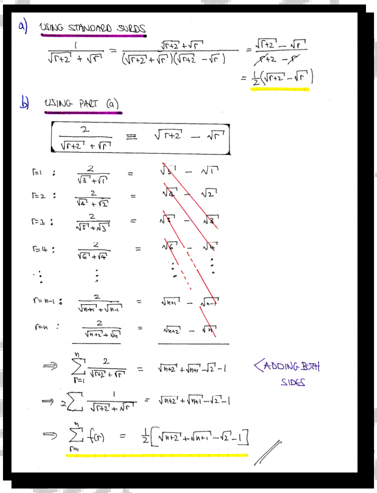
b) Find an expression for

$$\sum_{r=1}^{n} f(r).$$

c) Show clearly that

$$\sum_{r=1}^{48} f(r) = 3 + 2\sqrt{2}$$

$$f(r) = \frac{\sqrt{r+2} - \sqrt{r}}{2}, \quad \sum_{r=1}^{n} f(r) = \frac{1}{2}\left(\sqrt{n+2} + \sqrt{n+1} - \sqrt{2} - 1\right)$$



a) USING STANDARD SURDS

$$\frac{1}{\sqrt{r+2}+\sqrt{r}} = \frac{\sqrt{r+2}+\sqrt{r}}{(\sqrt{r+2}+\sqrt{r})(\sqrt{r+2}-\sqrt{r})} = \frac{\sqrt{r+2}-\sqrt{r}}{r+2-r} = \frac{1}{2}(\sqrt{r+2}-\sqrt{r})$$

b) USING PART (a)

$$\frac{2}{\sqrt{r+2}+\sqrt{r}} \equiv \sqrt{r+2}-\sqrt{r}$$

$r=1$: $\frac{2}{\sqrt{3}+\sqrt{1}} = \sqrt{3}-\sqrt{1}$

$r=2$: $\frac{2}{\sqrt{4}+\sqrt{2}} = \sqrt{4}-\sqrt{2}$

$r=3$: $\frac{2}{\sqrt{5}+\sqrt{3}} = \sqrt{5}-\sqrt{3}$

$r=4$: $\frac{2}{\sqrt{6}+\sqrt{4}} = \sqrt{6}-\sqrt{4}$

$\vdots$

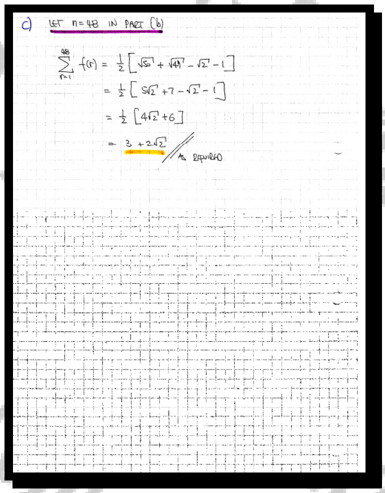
$r=n-1$: $\frac{2}{\sqrt{n+1}+\sqrt{n-1}} = \sqrt{n+1}-\sqrt{n-1}$

$r=n$: $\frac{2}{\sqrt{n+2}+\sqrt{n}} = \sqrt{n+2}-\sqrt{n}$

$$\Rightarrow \sum_{r=1}^{n}\frac{2}{\sqrt{r+2}+\sqrt{r}} = \sqrt{n+2}+\sqrt{n+1}-\sqrt{2}-1$$ ⟨ADDING BOTH SIDES

$$\Rightarrow 2\sum\frac{1}{\sqrt{r+2}+\sqrt{r}} = \sqrt{n+2}+\sqrt{n+1}-\sqrt{2}-1$$

$$\Rightarrow \sum_{r=1}^{n} f(r) = \frac{1}{2}\left[\sqrt{n+2}+\sqrt{n+1}-\sqrt{2}-1\right]$$



c) LET $n=48$ IN PART (b)

$$\sum_{r=1}^{48} f(r) = \frac{1}{2}\left[\sqrt{50}+\sqrt{49}-\sqrt{2}-1\right] = \frac{1}{2}\left[5\sqrt{2}+7-\sqrt{2}-1\right] = \frac{1}{2}\left[4\sqrt{2}+6\right] = 3+2\sqrt{2}$$

AS REQUIRED

SUMMATIONS

METHOD OF DIFFERENCES

3 HARD QUESTIONS

Question 1 (****+)

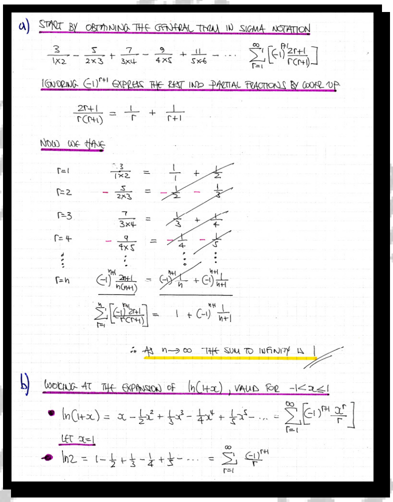
Consider the following infinite convergent series.

$$\frac{3}{1\times 2}-\frac{5}{2\times 3}+\frac{7}{3\times 4}-\frac{9}{4\times 5}+\frac{11}{5\times 6}-\ldots.$$

a) Use the method of differences, to find the sum of this series.

b) Verify the answer of part **(a)** by using a method based on the Maclaurin expansion of $\ln(1+x)$.

V, [ ], [1]



a) START BY OBTAINING THE GENERAL TERM IN SIGMA NOTATION

$$\frac{3}{1\times 2}-\frac{5}{2\times 3}+\frac{7}{3\times 4}-\frac{9}{4\times 5}+\frac{11}{5\times 6}-\ldots \quad \sum_{r=1}^{\infty}\left[(-1)^{r+1}\frac{2r+1}{r(r+1)}\right]$$

IGNORING $(-1)^{r+1}$ EXPRESS THE REST INTO PARTIAL FRACTIONS BY COVER UP

$$\frac{2r+1}{r(r+1)}=\frac{1}{r}+\frac{1}{r+1}$$

NOW WE HAVE

$r=1$: $\frac{3}{1\times 2}=\frac{1}{1}+\frac{1}{2}$

$r=2$: $-\frac{5}{2\times 3}=-\frac{1}{2}-\frac{1}{3}$

$r=3$: $\frac{7}{3\times 4}=\frac{1}{3}+\frac{1}{4}$

$r=4$: $-\frac{9}{4\times 5}=-\frac{1}{4}-\frac{1}{5}$

$\vdots$

$r=n$: $(-1)^{n+1}\frac{2n+1}{n(n+1)}=(-1)^{n+1}\frac{1}{n}+(-1)^{n+1}\frac{1}{n+1}$

$$\sum_{r=1}^{n}\left[\frac{(-1)^{r+1}(2r+1)}{r(r+1)}\right]=1+(-1)^{n+1}\frac{1}{n+1}$$

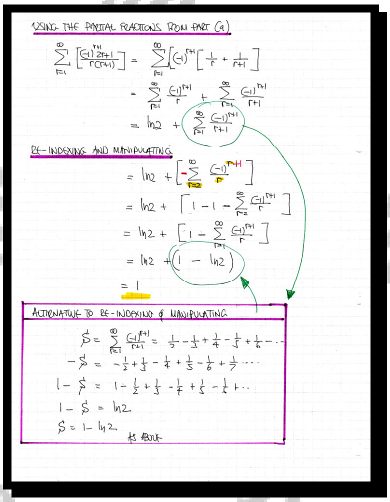
$\therefore$ AS $n\to\infty$ THE SUM TO INFINITY IS 1

b) WORKING AT THE EXPANSION OF $\ln(1+x)$, VALID FOR $-1<x\le 1$

- $\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-\ldots=\sum_{r=1}^{\infty}\left[(-1)^{r+1}\frac{x^r}{r}\right]$

LET $x=1$

- $\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots=\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}$



USING THE PARTIAL FRACTIONS FROM PART (a)

$$\sum_{r=1}^{\infty}\left[\frac{(-1)^{r+1}(2r+1)}{r(r+1)}\right]=\sum_{r=1}^{\infty}(-1)^{r+1}\left[\frac{1}{r}+\frac{1}{r+1}\right]$$

$$=\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}+\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r+1}$$

$$=\ln 2+\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r+1}$$

RE-INDEXING AND MANIPULATING

$$=\ln 2+\left[-\sum_{r=2}^{\infty}\frac{(-1)^{r+1}}{r}\right]$$

$$=\ln 2+\left[1-1-\sum_{r=2}^{\infty}\frac{(-1)^{r+1}}{r}\right]$$

$$=\ln 2+\left[1-\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}\right]$$

$$=\ln 2+(1-\ln 2)$$

$$=1$$

ALTERNATIVE TO RE-INDEXING & MANIPULATING

$S=\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r+1}=\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\ldots$

$-S=-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}\ldots$

$1-S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$

$1-S=\ln 2$

$S=1-\ln 2$ AS ABOVE

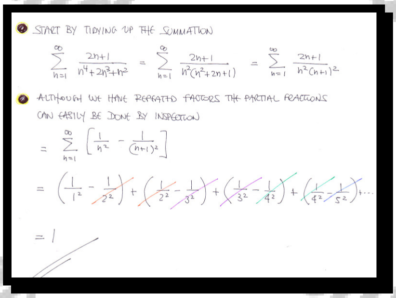
Question 2 **(****+)**

Use partial fractions to sum the following series.

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4+2n^3+n^2}.$$

You may assume that the series converges.

1


START BY TIDYING UP THE SUMMATION

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4+2n^3+n^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n^2+2n+1)} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$$

ALTHOUGH WE HAVE REPEATED FACTORS THE PARTIAL FRACTIONS CAN EASILY BE DONE BY INSPECTION

$$= \sum_{n=1}^{\infty} \left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]$$

$$= \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \left(\frac{1}{4^2} - \frac{1}{5^2}\right) + \dots$$

$$= 1$$

Question 3 (****+)

It is given that

$$f(r) = \frac{6r^4 + 6r^3 - ar^2 - ar + 1}{r(r+1)}, \quad r \in \mathbb{N},$$

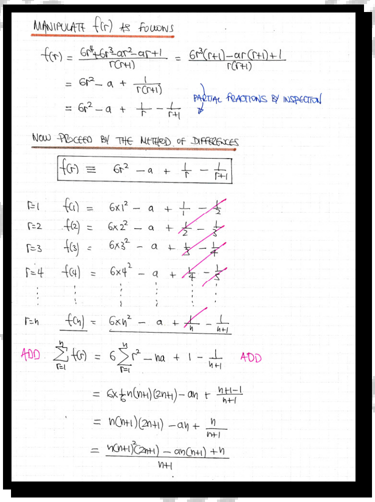
where a is a non zero constant.

It is further given that

$$\sum_{r=1}^{n} f(r) = \frac{n^2(n+2)(2n+1)}{n+1}.$$

Determine the value of a.

$\boxed{}$, $\boxed{a=2}$



MANIPULATE $f(r)$ AS FOLLOWS

$f(r) = \frac{6r^4+6r^3-ar^2-ar+1}{r(r+1)} = \frac{6r^3(r+1)-ar(r+1)+1}{r(r+1)}$

$= 6r^2 - a + \frac{1}{r(r+1)}$

$= 6r^2 - a + \frac{1}{r} - \frac{1}{r+1}$ PARTIAL FRACTIONS BY INSPECTION

NOW PROCEED BY THE METHOD OF DIFFERENCES

$\boxed{f(r) \equiv 6r^2 - a + \frac{1}{r} - \frac{1}{r+1}}$

$r=1 \quad f(1) = 6\times1^2 - a + \frac{1}{1} - \frac{1}{2}$

$r=2 \quad f(2) = 6\times2^2 - a + \frac{1}{2} - \frac{1}{3}$

$r=3 \quad f(3) = 6\times3^2 - a + \frac{1}{3} - \frac{1}{4}$

$r=4 \quad f(4) = 6\times4^2 - a + \frac{1}{4} - \frac{1}{5}$

$\vdots$

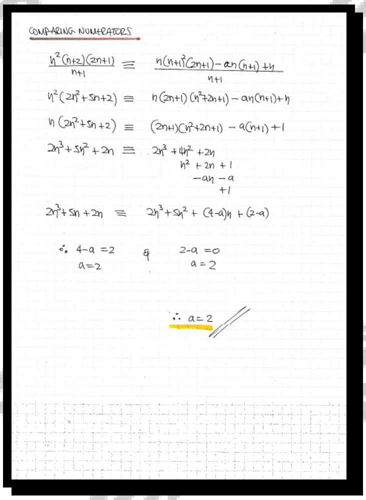
$r=n \quad f(n) = 6\times n^2 - a + \frac{1}{n} - \frac{1}{n+1}$

ADD $\sum_{r=1}^{n} f(r) = 6\sum_{r=1}^{n} r^2 - na + 1 - \frac{1}{n+1}$ ADD

$= 6\times\frac{1}{6}n(n+1)(2n+1) - an + \frac{n+1-1}{n+1}$

$= n(n+1)(2n+1) - an + \frac{n}{n+1}$

$= \frac{n(n+1)^2(2n+1) - an(n+1) + n}{n+1}$



COMPARING NUMERATORS

$\frac{n^2(n+2)(2n+1)}{n+1} \equiv \frac{n(n+1)^2(2n+1) - an(n+1) + n}{n+1}$

$n^2(2n^2+5n+2) \equiv n(2n+1)(n^2+2n+1) - an(n+1) + n$

$n(2n^2+5n+2) \equiv (2n+1)(n^2+2n+1) - a(n+1) + 1$

$2n^3 + 5n^2 + 2n \equiv 2n^3 + 4n^2 + 2n$
$\qquad n^2 + 2n + 1$
$\qquad -an - a$
$\qquad +1$

$2n^3 + 5n + 2n \equiv 2n^3 + 5n^2 + (4-a)n + (2-a)$

$\therefore 4-a = 2 \quad \& \quad 2-a=0$
$a=2 \qquad a=2$

$\therefore a=2$

SUMMATIONS

METHOD OF DIFFERENCES

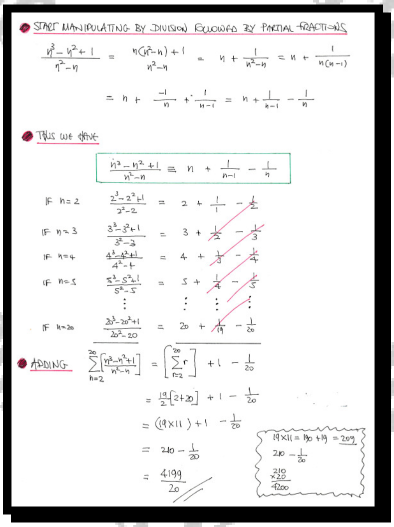
15 ENRICHMENT QUESTIONS

Question 1 (*****)

Determine the exact value of the following sum.

$$\sum_{n=2}^{20}\left[\frac{n^3-n^2+1}{n^2-n}\right].$$

[illegible], $\boxed{\frac{4199}{20}}$



START MANIPULATING BY DIVISION FOLLOWED BY PARTIAL FRACTIONS

$$\frac{n^3-n^2+1}{n^2-n} = \frac{n(n^2-n)+1}{n^2-n} = n + \frac{1}{n^2-n} = n + \frac{1}{n(n-1)}$$

$$= n + \frac{-1}{n} + \frac{1}{n-1} = n + \frac{1}{n-1} - \frac{1}{n}$$

THUS WE HAVE

$$\boxed{\frac{n^3-n^2+1}{n^2-n} \equiv n + \frac{1}{n-1} - \frac{1}{n}}$$

IF $n=2$: $\frac{2^3-2^2+1}{2^2-2} = 2 + \frac{1}{1} - \frac{1}{2}$

IF $n=3$: $\frac{3^3-3^2+1}{3^2-3} = 3 + \frac{1}{2} - \frac{1}{3}$

IF $n=4$: $\frac{4^3-4^2+1}{4^2-4} = 4 + \frac{1}{3} - \frac{1}{4}$

IF $n=5$: $\frac{5^3-5^2+1}{5^2-5} = 5 + \frac{1}{4} - \frac{1}{5}$

$\vdots$

IF $n=20$: $\frac{20^3-20^2+1}{20^2-20} = 20 + \frac{1}{19} - \frac{1}{20}$

ADDING

$$\sum_{n=2}^{20}\left[\frac{n^3-n^2+1}{n^2-n}\right] = \left[\sum_{r=2}^{20} r\right] + 1 - \frac{1}{20}$$

$$= \frac{19}{2}[2+20] + 1 - \frac{1}{20}$$

$$= (19\times 11) + 1 - \frac{1}{20}$$

$$= 210 - \frac{1}{20}$$

$$= \frac{4199}{20}$$

$19\times 11 = 190 + 19 = 209$

$210 - \frac{1}{20}$

$210 \times 20 = 4200$

Question 2 (*****)

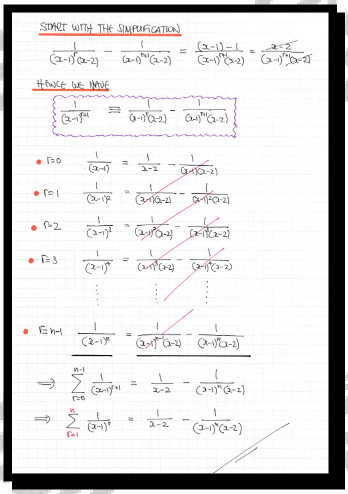
$$f(x,n)=\sum_{r=1}^{n}\left[\frac{1}{(x-1)^{r}}\right],\quad x\in\mathbb{R},\ n\in\mathbb{N}.$$

By observing the simplification of

$$\frac{1}{(x-2)(x-1)^{r}}-\frac{1}{(x-2)(x-1)^{r+1}}$$

find a simplified expression for $f(x,n)$.

SPX-F, $\boxed{f(x,n)=\frac{1}{x-2}-\frac{1}{(x-2)(x-1)^{n}}}$

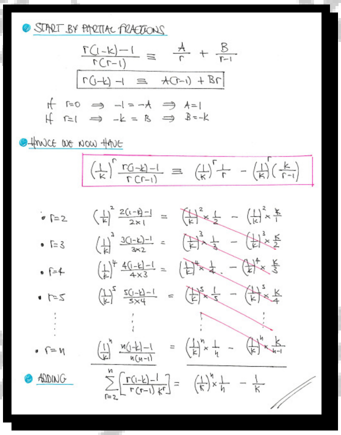


Question 3 (*****)

Determine, in terms of k and n, a simplified expression for

$$\sum_{r=2}^{n}\left[\frac{r(1-k)-1}{r(r-1)k^{r}}\right].$$

$$\boxed{}\ ,\ \boxed{\frac{1}{n}\left(\frac{1}{k}\right)^{n}-\frac{1}{k}}$$

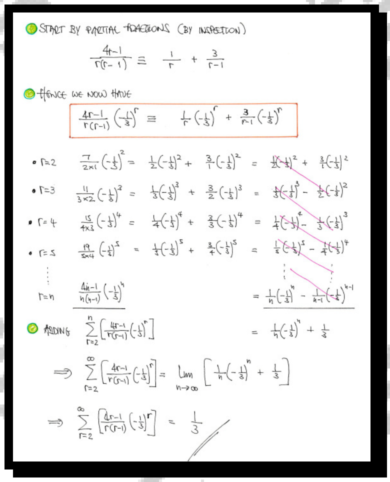


Question 4 (*****)

Determine the value of the following infinite convergent sum.

$$\sum_{r=2}^{\infty}\left[\frac{4r-1}{r(r-1)}\left(-\frac{1}{3}\right)^{r}\right].$$

SP , $\boxed{\frac{1}{3}}$



START BY PARTIAL FRACTIONS (BY INSPECTION)

$$\frac{4r-1}{r(r-1)} \equiv \frac{1}{r} + \frac{3}{r-1}$$

HENCE WE NOW HAVE

$$\boxed{\frac{4r-1}{r(r-1)}\left(-\frac{1}{3}\right)^{r} \equiv \frac{1}{r}\left(-\frac{1}{3}\right)^{r} + \frac{3}{r-1}\left(-\frac{1}{3}\right)^{r}}$$

- $r=2$: $\frac{7}{2\times1}\left(-\frac{1}{3}\right)^{2} = \frac{1}{2}\left(-\frac{1}{3}\right)^{2} + \frac{3}{1}\left(-\frac{1}{3}\right)^{2} = \frac{1}{2}\left(-\frac{1}{3}\right)^{2} + \frac{3}{1}\left(-\frac{1}{3}\right)^{2}$
- $r=3$: $\frac{11}{3\times2}\left(-\frac{1}{3}\right)^{3} = \frac{1}{3}\left(-\frac{1}{3}\right)^{3} + \frac{3}{2}\left(-\frac{1}{3}\right)^{3} = \frac{1}{3}\left(-\frac{1}{3}\right)^{3} - \frac{1}{2}\left(-\frac{1}{3}\right)^{2}$
- $r=4$: $\frac{15}{4\times3}\left(-\frac{1}{3}\right)^{4} = \frac{1}{4}\left(-\frac{1}{3}\right)^{4} + \frac{3}{3}\left(-\frac{1}{3}\right)^{4} = \frac{1}{4}\left(-\frac{1}{3}\right)^{4} - \frac{1}{3}\left(-\frac{1}{3}\right)^{3}$
- $r=5$: $\frac{19}{5\times4}\left(-\frac{1}{3}\right)^{5} = \frac{1}{5}\left(-\frac{1}{3}\right)^{5} + \frac{3}{4}\left(-\frac{1}{3}\right)^{5} = \frac{1}{5}\left(-\frac{1}{3}\right)^{5} - \frac{1}{4}\left(-\frac{1}{3}\right)^{4}$

$\vdots$

$r=n$: $\frac{4n-1}{n(n-1)}\left(-\frac{1}{3}\right)^{n} = \frac{1}{n}\left(-\frac{1}{3}\right)^{n} - \frac{1}{n-1}\left(-\frac{1}{3}\right)^{n-1}$

ADDING $\sum_{r=2}^{n}\left[\frac{4r-1}{r(r-1)}\left(-\frac{1}{3}\right)^{r}\right] = \frac{1}{n}\left(-\frac{1}{3}\right)^{n} + \frac{1}{3}$

$$\Rightarrow \sum_{r=2}^{\infty}\left[\frac{4r-1}{r(r-1)}\left(-\frac{1}{3}\right)^{r}\right] = \lim_{n\to\infty}\left[\frac{1}{n}\left(-\frac{1}{3}\right)^{n} + \frac{1}{3}\right]$$

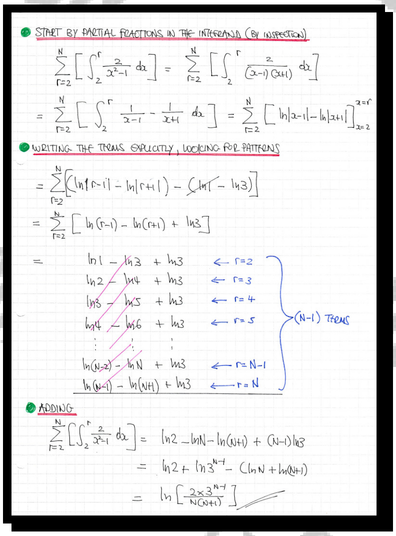
$$\Rightarrow \sum_{r=2}^{\infty}\left[\frac{4r-1}{r(r-1)}\left(-\frac{1}{3}\right)^{r}\right] = \frac{1}{3}$$

Question 5 (*****)

Determine a simplified expression, in the form $\ln\left[f(n)\right]$, for the following sum.

$$\sum_{r=2}^{N}\left[\int_{2}^{r}\frac{2}{x^{2}-1}\,dx\right].$$

$\boxed{}$, $\boxed{\ln\left[\frac{2\times 3^{N-1}}{N(N+1)}\right]}$



START BY PARTIAL FRACTIONS IN THE INTEGRAND (BY INSPECTION)

$$\sum_{r=2}^{N}\left[\int_{2}^{r}\frac{2}{x^{2}-1}\,dx\right]=\sum_{r=2}^{N}\left[\int_{2}^{r}\frac{2}{(x-1)(x+1)}\,dx\right]$$

$$=\sum_{r=2}^{N}\left[\int_{2}^{r}\frac{1}{x-1}-\frac{1}{x+1}\,dx\right]=\sum_{r=2}^{N}\Big[\ln|x-1|-\ln|x+1|\Big]_{x=2}^{x=r}$$

WRITING THE TERMS EXPLICITLY, LOOKING FOR PATTERNS

$$=\sum_{r=2}^{N}\Big[\big(\ln|r-1|-\ln|r+1|\big)-\big(\ln 1-\ln 3\big)\Big]$$

$$=\sum_{r=2}^{N}\Big[\ln(r-1)-\ln(r+1)+\ln 3\Big]$$

$$\begin{aligned}
= \quad & \ln 1-\ln 3+\ln 3 && \leftarrow r=2\\
& \ln 2-\ln 4+\ln 3 && \leftarrow r=3\\
& \ln 3-\ln 5+\ln 3 && \leftarrow r=4\\
& \ln 4-\ln 6+\ln 3 && \leftarrow r=5\\
& \vdots \\
& \ln(N-2)-\ln N+\ln 3 && \leftarrow r=N-1\\
& \ln(N-1)-\ln(N+1)+\ln 3 && \leftarrow r=N
\end{aligned}$$

(N−1) TERMS

ADDING

$$\sum_{r=2}^{N}\left[\int_{2}^{r}\frac{2}{x^{2}-1}\,dx\right]=\ln 2-\ln N-\ln(N+1)+(N-1)\ln 3$$

$$=\ln 2+\ln 3^{N-1}-\big(\ln N+\ln(N+1)\big)$$

$$=\ln\left[\frac{2\times 3^{N-1}}{N(N+1)}\right]$$

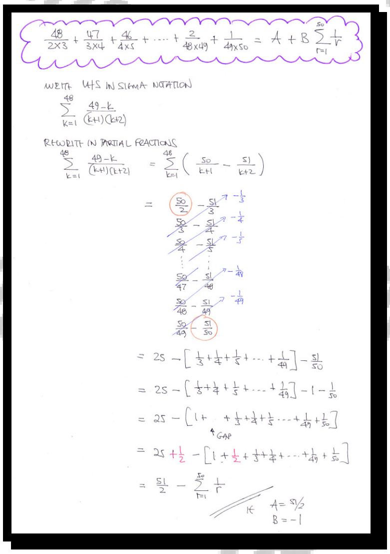
Question 6 (*****)

Show, by a detailed method, that

$$\frac{48}{2\times3}+\frac{47}{3\times4}+\frac{46}{4\times5}\ ...+\frac{2}{48\times49}+\frac{1}{49\times50}=A+B\sum_{r=1}^{50}\frac{1}{r},$$

where A and B are constants to be found.

$\boxed{\quad}$, $\boxed{A=\frac{51}{2},\quad B=-1}$

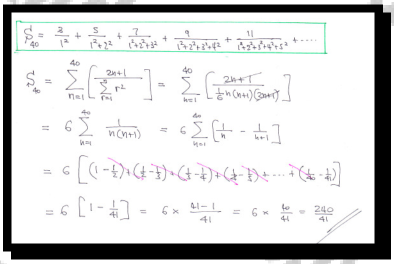


Question 7 **(*****)**

$$\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\frac{9}{1^2+2^2+3^2+4^2}+\frac{11}{1^2+2^2+3^2+4^2+5^2}+\dots,$$

Show, by a detailed method, that the sum of the first 40 terms of this series shown above is $\dfrac{240}{41}$.

[ ] , [proof]



Question 8 (*****)

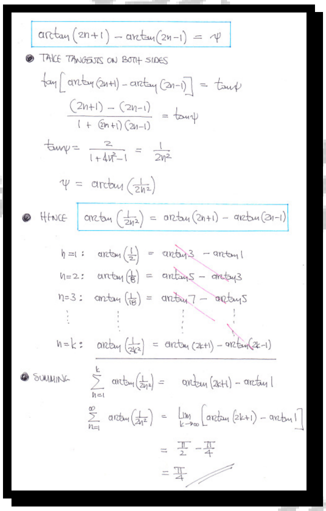
By considering the simplification of

$$\arctan(2n+1) - \arctan(2n-1),$$

determine the exact value of

$$\sum_{n=1}^{\infty}\left[\arctan\left(\frac{1}{2n^2}\right)\right].$$

$\boxed{}$, $\boxed{\frac{\pi}{4}}$



$\arctan(2n+1) - \arctan(2n-1) = \psi$

- TAKE TANGENTS ON BOTH SIDES

$\tan[\arctan(2n+1) - \arctan(2n-1)] = \tan\psi$

$\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)} = \tan\psi$

$\tan\psi = \frac{2}{1+4n^2-1} = \frac{1}{2n^2}$

$\psi = \arctan\left(\frac{1}{2n^2}\right)$

- HENCE $\arctan\left(\frac{1}{2n^2}\right) = \arctan(2n+1) - \arctan(2n-1)$

$n=1$: $\arctan\left(\frac{1}{2}\right) = \arctan 3 - \arctan 1$

$n=2$: $\arctan\left(\frac{1}{8}\right) = \arctan 5 - \arctan 3$

$n=3$: $\arctan\left(\frac{1}{18}\right) = \arctan 7 - \arctan 5$

$\vdots$

$n=k$: $\arctan\left(\frac{1}{2k^2}\right) = \arctan(2k+1) - \arctan(2k-1)$

- SUMMING $\sum_{n=1}^{k}\arctan\left(\frac{1}{2n^2}\right) = \arctan(2k+1) - \arctan 1$

$\sum_{n=1}^{\infty}\arctan\left(\frac{1}{2n^2}\right) = \lim_{k\to\infty}\left[\arctan(2k+1) - \arctan 1\right]$

$= \frac{\pi}{2} - \frac{\pi}{4}$

$= \frac{\pi}{4}$

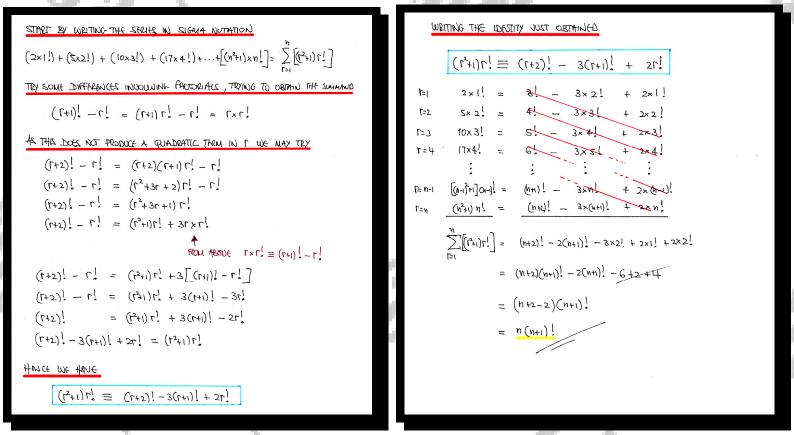
Question 9 (*****)

$$S_n = (2\times1!) + (5\times2!) + (10\times3!) + (17\times4!) + \ldots + (n^2+1)n!$$

Use an appropriate method to show that

$$S_n = n(n+1)!$$

[ ], proof


START BY WRITING THE SERIES IN SIGMA NOTATION

$$(2\times1!) + (5\times2!) + (10\times3!) + (17\times4!) + \ldots + [(n^2+1)\times n!] = \sum_{r=1}^{n}[(r^2+1)r!]$$

TRY SOME DIFFERENCES INVOLVING FACTORIALS, TRYING TO OBTAIN THE SUMMAND

$$(r+1)! - r! = (r+1)r! - r! = r\times r!$$

AS THIS DOES NOT PRODUCE A QUADRATIC TERM IN r WE MAY TRY

$$(r+2)! - r! = (r+2)(r+1)r! - r!$$
$$(r+2)! - r! = (r^2+3r+2)r! - r!$$
$$(r+2)! - r! = (r^2+3r+1)r!$$
$$(r+2)! - r! = (r^2+1)r! + 3r\times r!$$

FROM ABOVE $r\times r! \equiv (r+1)! - r!$

$$(r+2)! - r! = (r^2+1)r! + 3[(r+1)! - r!]$$
$$(r+2)! - r! = (r^2+1)r! + 3(r+1)! - 3r!$$
$$(r+2)! = (r^2+1)r! + 3(r+1)! - 2r!$$
$$(r+2)! - 3(r+1)! + 2r! = (r^2+1)r!$$

HENCE WE HAVE

$$(r^2+1)r! \equiv (r+2)! - 3(r+1)! + 2r!$$

WRITING THE IDENTITY JUST OBTAINED

$$(r^2+1)r! \equiv (r+2)! - 3(r+1)! + 2r!$$

$r=1$: $2\times1! = 3! - 3\times2! + 2\times1!$

$r=2$: $5\times2! = 4! - 3\times3! + 2\times2!$

$r=3$: $10\times3! = 5! - 3\times4! + 2\times3!$

$r=4$: $17\times4! = 6! - 3\times5! + 2\times4!$

$\vdots$

$r=n-1$: $[(n-1)^2+1](n-1)! = (n+1)! - 3\times n! + 2\times(n-1)!$

$r=n$: $(n^2+1)n! = (n+2)! - 3\times(n+1)! + 2\times n!$

$$\sum_{r=1}^{n}[(r^2+1)r!] = (n+2)! - 2(n+1)! - 3\times2! + 2\times1! + 2\times2!$$
$$= (n+2)(n+1)! - 2(n+1)! - 6 + 2 + 4$$
$$= (n+2-2)(n+1)!$$
$$= n(n+1)!$$

Question 10 (*****)

By considering the trigonometric identity for $\tan(A-B)$, with $A=\arctan(n+1)$ and $B=\arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right).$$

You may assume the series converges.

$\boxed{}$, $\boxed{}$, $\boxed{\frac{\pi}{4}}$

CONSIDER THE COMPOUND ANGLE IDENTITY FOR $\tan(A-B)$

$$\Rightarrow \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$$

$$\Rightarrow \tan\big(\arctan(n+1) - \arctan n\big) = \frac{\tan(\arctan(n+1)) - \tan(\arctan n)}{1+\tan(\arctan(n+1))\tan(\arctan n)}$$

$$\Rightarrow \tan\big(\arctan(n+1) - \arctan n\big) = \frac{(n+1)-n}{1+(n+1)\times n}$$

$$\Rightarrow \tan\big(\arctan(n+1) - \arctan n\big) = \frac{1}{n^2+n+1}$$

$$\Rightarrow \arctan\Big[\tan\big(\arctan(n+1) - \arctan n\big)\Big] = \arctan\left(\frac{1}{n^2+n+1}\right)$$

$$\Rightarrow \arctan(n+1) - \arctan n = \arctan\left(\frac{1}{n^2+n+1}\right)$$

REWRITE THE SUMMATION NOW AS

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right) = \sum_{n=1}^{\infty}\big[\arctan(n+1) - \arctan n\big]$$

$$= \sum_{n=1}^{\infty} \arctan(n+1) - \sum_{n=1}^{\infty} \arctan n$$

$$= \lim_{k\to\infty}\left[\sum_{n=1}^{\infty} \arctan(n+1) - \sum_{n=1}^{\infty} \arctan n\right]$$

PROCEED AS FOLLOWS – NOTE THE "ORDER"

$$\ldots \lim_{k\to\infty}\begin{bmatrix} \arctan(k+1) & - & \arctan k \\ + & & + \\ \arctan k & - & \arctan(k-1) \\ + & & + \\ \arctan(k-1) & - & \arctan(k-2) \\ + & & + \\ \vdots & & \vdots \\ + & & + \\ \arctan 3 & - & \arctan 2 \\ \arctan 2 & - & \arctan 1 \end{bmatrix}$$

$$= \lim_{k\to\infty}\big[\arctan(k+1) - \arctan 1\big]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Question 11 (*****)

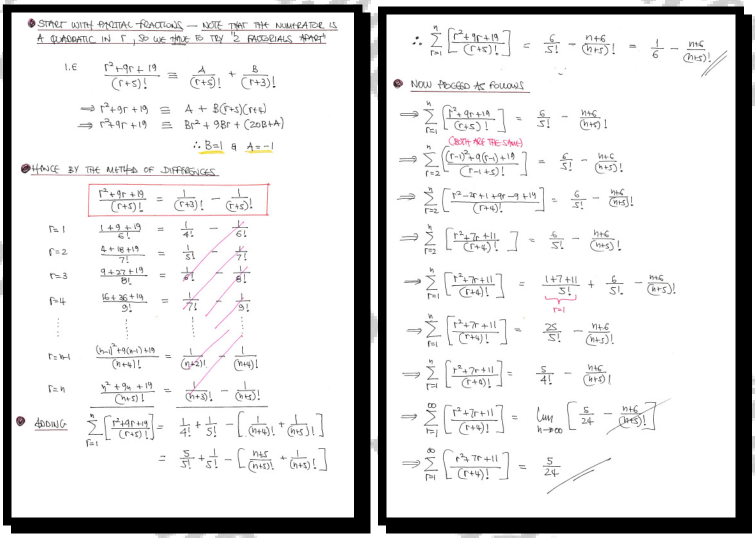
Determine, in terms of n, a simplified expression

$$\sum_{r=1}^{n}\left[\frac{r^2+9r+19}{(r+5)!}\right],$$

and hence, or otherwise, deduce the value of

$$\sum_{r=1}^{\infty}\left[\frac{r^2+7r+11}{(r+4)!}\right].$$

$$\sum_{r=1}^{n}\left[\frac{r^2+9r+19}{(r+5)!}\right]=\frac{1}{6}-\frac{n+5}{(n+5)!}, \quad \sum_{r=1}^{\infty}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\frac{5}{24}$$



START WITH PARTIAL FRACTIONS — NOTE THAT THE NUMERATOR IS A QUADRATIC IN r, SO WE HAVE TO TRY "2 FACTORIALS APART"

I.E $\frac{r^2+9r+19}{(r+5)!} \equiv \frac{A}{(r+5)!}+\frac{B}{(r+3)!}$

$\Rightarrow r^2+9r+19 \equiv A+B(r+5)(r+4)$

$\Rightarrow r^2+9r+19 \equiv Br^2+9Br+(20B+A)$

$\therefore B=1$ & $A=-1$

HENCE BY THE METHOD OF DIFFERENCES

$\frac{r^2+9r+19}{(r+5)!}=\frac{1}{(r+3)!}-\frac{1}{(r+5)!}$

$r=1$ $\quad \frac{1+9+19}{6!}=\frac{1}{4!}-\frac{1}{6!}$

$r=2$ $\quad \frac{4+18+19}{7!}=\frac{1}{5!}-\frac{1}{7!}$

$r=3$ $\quad \frac{9+27+19}{8!}=\frac{1}{6!}-\frac{1}{8!}$

$r=4$ $\quad \frac{16+36+19}{9!}=\frac{1}{7!}-\frac{1}{9!}$

$\vdots$

$r=n-1$ $\quad \frac{(n-1)^2+9(n-1)+19}{(n+4)!}=\frac{1}{(n+2)!}-\frac{1}{(n+4)!}$

$r=n$ $\quad \frac{n^2+9n+19}{(n+5)!}=\frac{1}{(n+3)!}-\frac{1}{(n+5)!}$

ADDING $\sum_{r=1}^{n}\left[\frac{r^2+9r+19}{(r+5)!}\right]=\frac{1}{4!}+\frac{1}{5!}-\left[\frac{1}{(n+4)!}+\frac{1}{(n+5)!}\right]$

$=\frac{5}{5!}+\frac{1}{5!}-\left[\frac{n+5}{(n+5)!}+\frac{1}{(n+5)!}\right]$

$\therefore \sum_{r=1}^{n}\left[\frac{r^2+9r+19}{(r+5)!}\right]=\frac{6}{5!}-\frac{n+6}{(n+5)!}=\frac{1}{6}-\frac{n+6}{(n+5)!}$

NOW PROCEED AS FOLLOWS

$\Rightarrow \sum_{r=1}^{n}\left[\frac{r^2+9r+19}{(r+5)!}\right]=\frac{6}{5!}-\frac{n+6}{(n+5)!}$

(BOTH ARE THE SAME)

$\Rightarrow \sum_{r=2}^{n}\left[\frac{(r-1)^2+9(r-1)+19}{(r-1+5)!}\right]=\frac{6}{5!}-\frac{n+6}{(n+5)!}$

$\Rightarrow \sum_{r=2}^{n}\left[\frac{r^2-2r+1+9r-9+19}{(r+4)!}\right]=\frac{6}{5!}-\frac{n+6}{(n+5)!}$

$\Rightarrow \sum_{r=2}^{n}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\frac{6}{5!}-\frac{n+6}{(n+5)!}$

$\Rightarrow \sum_{r=1}^{n}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\underbrace{\frac{1+7+11}{5!}}_{r=1}+\frac{6}{5!}-\frac{n+6}{(n+5)!}$

$\Rightarrow \sum_{r=1}^{n}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\frac{25}{5!}-\frac{n+6}{(n+5)!}$

$\Rightarrow \sum_{r=1}^{n}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\frac{5}{4!}-\frac{n+6}{(n+5)!}$

$\Rightarrow \sum_{r=1}^{\infty}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\lim_{n\to\infty}\left[\frac{5}{24}-\frac{n+6}{(n+5)!}\right]$

$\Rightarrow \sum_{r=1}^{\infty}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\frac{5}{24}$

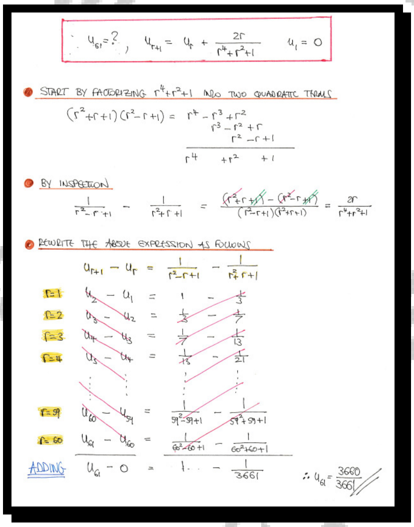
Question 12 (*****)

A sequence is defined as

$$u_{r+1} = u_r + \frac{2r}{r^4 + r^2 + 1}, \quad u_1 = 0, \quad r \in \mathbb{N}.$$

Determine the exact value of u_{61}.

$\boxed{}$, $\boxed{u_{61} = \frac{3660}{3661}}$



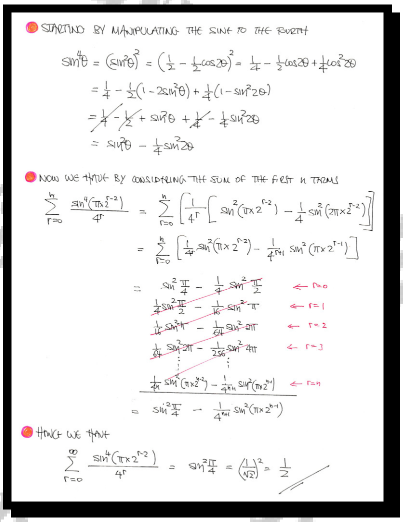
Question 13 (*****)

Find the value of

$$\sum_{r=0}^{\infty}\left[\frac{\sin^4\left(\pi\times 2^{r-2}\right)}{4^r}\right].$$

Hint: Express $\sin^4\theta$ in terms of $\sin^2\theta$ and $\sin^2 2\theta$ only.

$\boxed{}$, $\boxed{\frac{1}{2}}$


STARTING BY MANIPULATING THE SINE TO THE FOURTH

$$\sin^4\theta = (\sin^2\theta)^2 = \left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\right)^2 = \tfrac{1}{4}-\tfrac{1}{2}\cos 2\theta+\tfrac{1}{4}\cos^2 2\theta$$
$$= \tfrac{1}{4}-\tfrac{1}{2}(1-2\sin^2\theta)+\tfrac{1}{4}(1-\sin^2 2\theta)$$
$$= \tfrac{1}{4}-\tfrac{1}{2}+\sin^2\theta+\tfrac{1}{4}-\tfrac{1}{4}\sin^2 2\theta$$
$$= \sin^2\theta-\tfrac{1}{4}\sin^2 2\theta$$

NOW WE HAVE BY CONSIDERING THE SUM OF THE FIRST n TERMS

$$\sum_{r=0}^{n}\frac{\sin^4(\pi\times 2^{r-2})}{4^r} = \sum_{r=0}^{n}\left[\frac{1}{4^r}\left[\sin^2(\pi\times 2^{r-2})-\frac{1}{4}\sin^2(2\pi\times 2^{r-2})\right]\right]$$
$$= \sum_{r=0}^{n}\left[\frac{1}{4^r}\sin^2(\pi\times 2^{r-2})-\frac{1}{4^{r+1}}\sin^2(\pi\times 2^{r-1})\right]$$

$$= \sin^2\frac{\pi}{4}-\frac{1}{4}\sin^2\frac{\pi}{2} \quad \leftarrow r=0$$
$$\frac{1}{4}\sin^2\frac{\pi}{2}-\frac{1}{16}\sin^2\pi \quad \leftarrow r=1$$
$$\frac{1}{16}\sin^2\pi-\frac{1}{64}\sin^2 2\pi \quad \leftarrow r=2$$
$$\frac{1}{64}\sin^2 2\pi-\frac{1}{256}\sin^2 4\pi \quad \leftarrow r=3$$
$$\vdots$$
$$\frac{1}{4^n}\sin^2(\pi\times 2^{n-2})-\frac{1}{4^{n+1}}\sin^2(\pi\times 2^{n-1}) \quad \leftarrow r=n$$
$$= \sin^2\frac{\pi}{4}-\frac{1}{4^{n+1}}\sin^2(\pi\times 2^{n-1})$$

HENCE WE HAVE

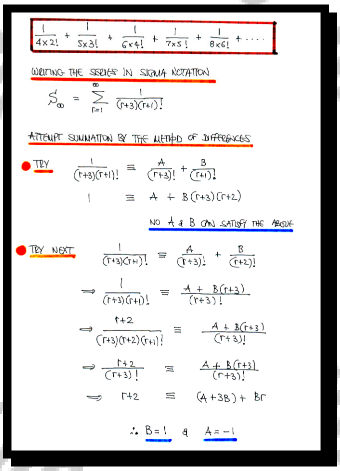
$$\sum_{r=0}^{\infty}\frac{\sin^4(\pi\times 2^{r-2})}{4^r} = \sin^2\frac{\pi}{4} = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

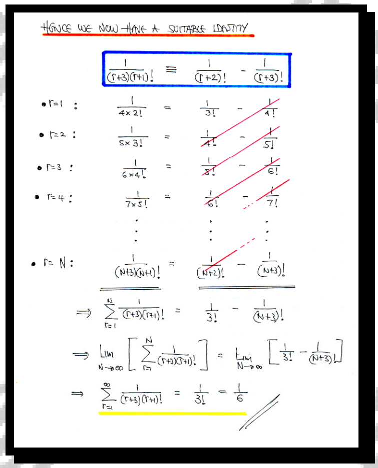
Question 14 (*****)

Find the sum to infinity of the following convergent series.

$$\frac{1}{4\times 2!}+\frac{1}{5\times 3!}+\frac{1}{6\times 4!}+\frac{1}{7\times 5!}+\frac{1}{8\times 6!}+\ldots$$

P2, $\boxed{\frac{1}{6}}$





Question 15 (*****)

Evaluate the following expression

$$\sum_{k=1}^{\infty}\left[\sum_{r=1}^{k} r\right]^{-1}.$$

SP , 2

- REWRITE FOR SIMPLICITY AS FOLLOWS

$$\sum_{k=1}^{\infty}\left[\sum_{r=1}^{k} r\right]^{-1} = \sum_{k=1}^{\infty}\left[\frac{1}{\sum_{r=1}^{k} r}\right]$$

$$= \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots$$

- INTRODUCE A FINITE LIMIT FOR THE SUMMATION, SAY n

$$= \lim_{n\to\infty}\left[1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots + \frac{1}{1+2+3+\dots+n}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{\frac{1}{2}r(r+1)}\right]$$

$$= 2\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{r(r+1)}\right]$$

- SPLIT INTO TWO FRACTIONS BY INSPECTION

$$= 2\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{r} - \frac{1}{r+1}\right]$$

$$= 2\lim_{n\to\infty}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$= 2\lim_{n\to\infty}\left[1-\frac{1}{n+1}\right]$$

$$= 2$$