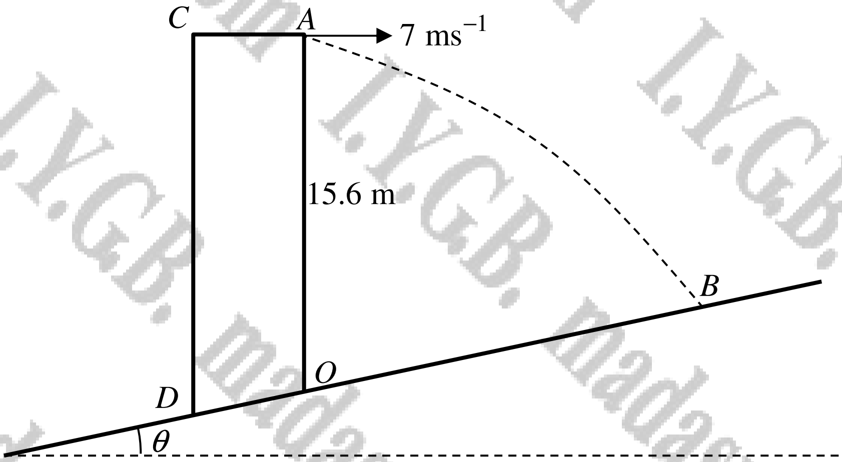
ADVANCED PROJECTILES

GENERAL PROJECTILES

Question 1 (***)



The figure above shows the cross section of a vertical tower $OACD$ standing on a plane inclined at an angle θ to the horizontal, where $\tan\theta = 0.1$.

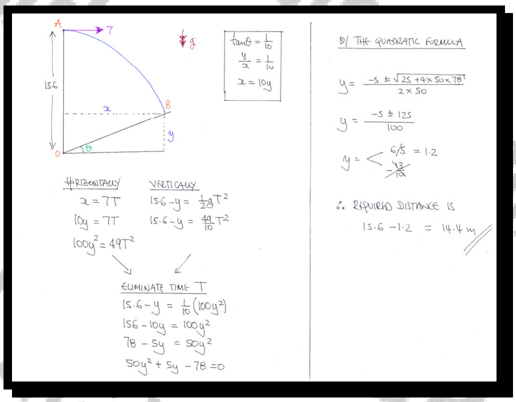
A particle is projected horizontally from A hitting the incline plane at the point B.

The journey of the particle is in a vertical plane containing O, A and B.

Given that $|OA| = 15.6$ m determine the vertical distance through which the particle falls as it travels from A to B.

You may assume that the only force acting on the particle is its weight.

$\boxed{14.4 \text{ m}}$



Question 2 (***)

A particle is projected from a point A on level horizontal ground with speed of U ms^{-1} at an angle of elevation θ.

The particle moves through still air without any resistance, reaching a maximum height H above ground, before it first hits the ground at a point which is R m away from A.

a) Show clearly, in any order, that …

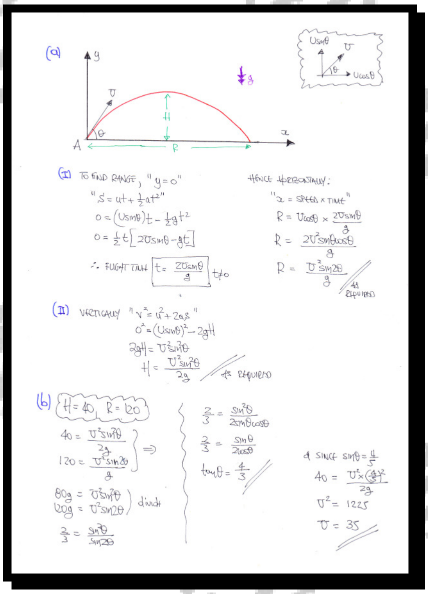
i. … $R = \dfrac{U^2 \sin 2\theta}{g}$.

ii. … $H = \dfrac{U^2 \sin^2 \theta}{2g}$.

It is now given that the particle reaches a greatest height above the ground of 40 m, after travelling a horizontal distance of 60 m from A.

b) Determine in any order the value of U and the value of $\tan\theta$.

$\boxed{U = 35}$, $\boxed{\tan\theta = \frac{4}{3}}$



Question 3 (***)

A particle P is projected from the point A on level horizontal ground with a speed of 21 ms^{-1} at an angle θ to the horizontal. In the subsequent motion P is moving under gravity, without any air resistance.

The particle passes through the point B, t s later. The horizontal and vertical displacement of B from A are 12 m and 2 m, respectively.

a) By considering the horizontal component of the motion of P show

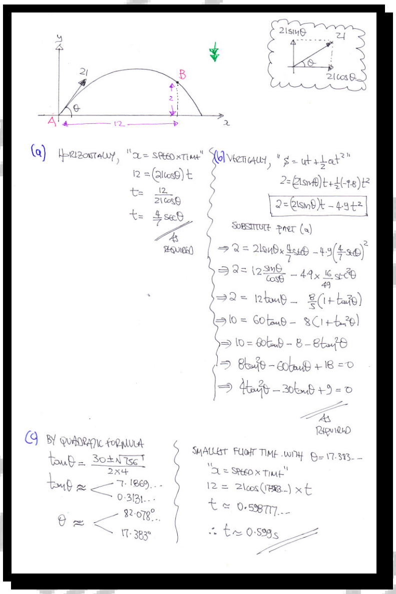
$$t = \frac{4}{7}\sec\theta .$$

b) By considering the vertical component of the motion of P show

$$4\tan^2\theta - 30\tan\theta + 9 = 0 .$$

c) Determine, to three significant figures, the smallest possible flight time of P from A to B.

$\boxed{t \approx 0.599}$



Question 4 (***)

In a golf driving range, a golf ball is struck with a speed of 49 ms^{-1} at an angle of elevation α from a point A, which lies 4.9 m above level horizontal ground.

The ball first strikes the ground at the point B which lies at a horizontal distance of 98 m from A.

The ball is modelled as a particle moving under gravity, without any air resistance.

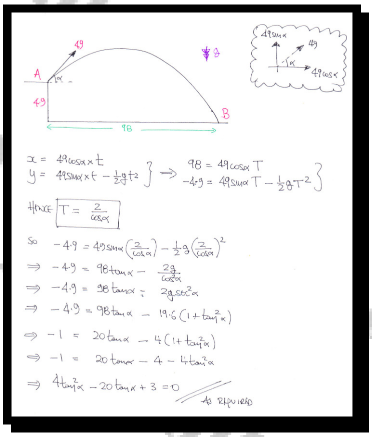
a) Show clearly that

$$4\tan^2 a - 20\tan\alpha + 3 = 0.$$

b) Hence find, to three significant figures, the two possible values of α.

c) Determine, to three significant figures, the smallest possible flight time of the ball from A to B.

$\boxed{\alpha \approx 8.80°, 78.34°}$, $\boxed{t \approx 2.02 \text{ s}}$



$x = 49\cos\alpha \times t$
$y = 49\sin\alpha \times t - \frac{1}{2}gt^2$ $\Rightarrow$ $98 = 49\cos\alpha\, T$, $-4.9 = 49\sin\alpha\, T - \frac{1}{2}gT^2$

HENCE $\boxed{T = \frac{2}{\cos\alpha}}$

SO $-4.9 = 49\sin\alpha\left(\frac{2}{\cos\alpha}\right) - \frac{1}{2}g\left(\frac{2}{\cos\alpha}\right)^2$

$\Rightarrow -4.9 = 98\tan\alpha - \frac{2g}{\cos^2\alpha}$

$\Rightarrow -4.9 = 98\tan\alpha - 2g\sec^2\alpha$

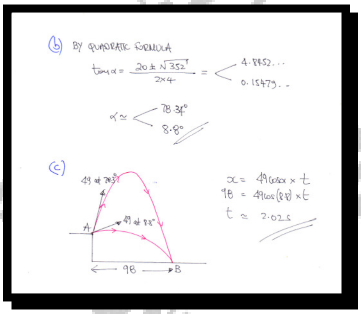
$\Rightarrow -4.9 = 98\tan\alpha - 19.6(1+\tan^2\alpha)$

$\Rightarrow -1 = 20\tan\alpha - 4(1+\tan^2\alpha)$

$\Rightarrow -1 = 20\tan\alpha - 4 - 4\tan^2\alpha$

$\Rightarrow 4\tan^2\alpha - 20\tan\alpha + 3 = 0$

AS REQUIRED



(b) BY QUADRATIC FORMULA

$\tan\alpha = \frac{20 \pm \sqrt{352}}{2\times 4} = 4.8452\ldots$ or $0.15479\ldots$

$\alpha \simeq 78.34°$ or $8.8°$

(c)

$x = 49\cos\alpha \times t$

$98 = 49\cos(8.8) \times t$

$t \simeq 2.02\text{s}$

Question 5 (***+)

A particle is projected with speed $u \text{ ms}^{-1}$ at an angle θ **below** the horizontal, from a point O **above** level horizontal ground. The particle's horizontal and vertical distances from O at time t s after projection, are x m and y m, respectively. The particle is moving under gravity, without any air resistance.

a) Show clearly that

$$y = x\tan\theta + \frac{gx^2}{2u^2\cos^2\theta}.$$

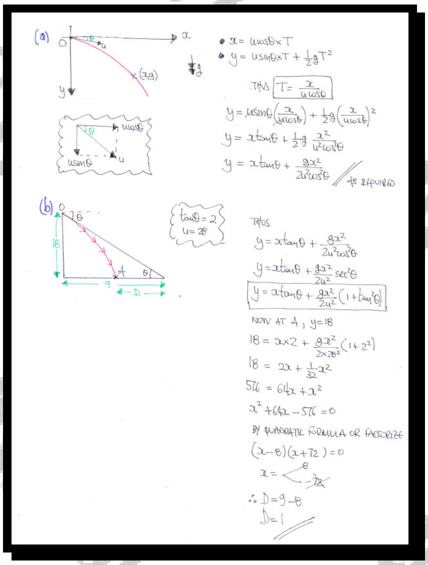
A child is throwing a tennis ball from tower block aiming at a target on the ground.

The ball is thrown from a height of 18 m, with a speed of 28 ms^{-1}, aiming **directly** at the target which is at a horizontal distance of 9 m, from the foot of the block.

The tennis ball lands D m short of the target because of the effect of gravity.

b) Determine the value of D.

$\boxed{D = 1}$



Question 6 (***+)

A fixed origin O is located on level horizontal ground and the vectors $\mathbf{i}$ and $\mathbf{j}$ are unit vectors pointing horizontally and vertically, respectively.

A mortar shell is fired from O with velocity $(U\mathbf{i}+V\mathbf{j})\ \mathrm{ms}^{-1}$, where U and V are positive constants. The shell lands on the enemy target which is located on the same level horizontal ground as O. The highest point on the path of the shell has position vector $(300\mathbf{i}+122.5\mathbf{j})$ m.

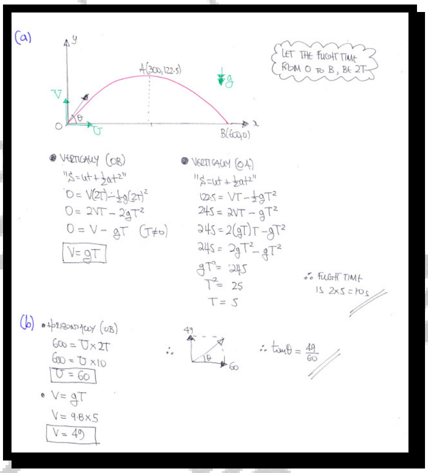
The shell is modelled as a particle moving freely under gravity.

a) Show that the time it takes the shell to hit the target is 10 s.

The shell was projected at an angle of elevation θ.

b) Determine the value of $\tan\theta$.

$$\boxed{\tan\theta = \frac{49}{60}}$$



Question 7 (***)

A cannon fire a shell with a speed of u ms^{-1} at an angle of elevation θ from a point O on level horizontal ground. The shell has horizontal and vertical displacements of x m and y m from O at time t s. The shell is modelled as a particle moving under gravity, without any air resistance.

a) Show clearly that

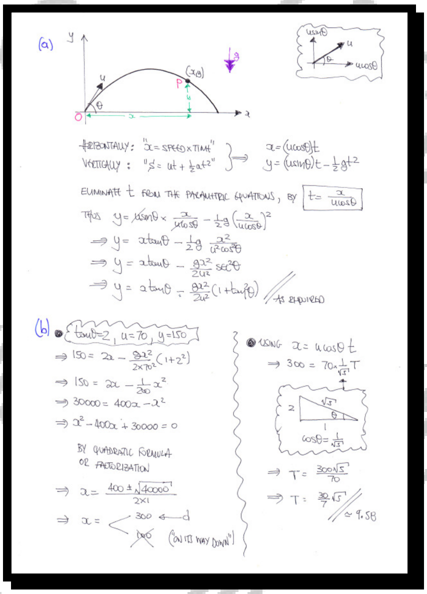
$$y = x\tan\theta - \frac{gx^2}{2u^2}\left(1+\tan^2\theta\right).$$

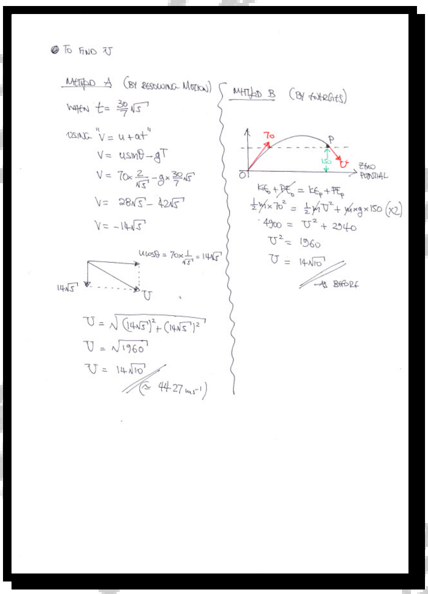
The cannon is aimed at the gate of a fortress which is on a hill at a height of 150 m above the level of the cannon, and a horizontal distance d m from the cannon.

A shell is fired at 70 ms^{-1} at an angle of elevation $\arctan 2$, which hits the gate of the fortress on its way **down**, with a speed U ms^{-1}, T s after it was fired.

b) Determine in any order the value of d, the value of T and the value of U.

$\boxed{d = 300}$, $\boxed{T = \frac{30}{7}\sqrt{5} \approx 9.58}$, $\boxed{U = 14\sqrt{10} \approx 44.27}$





Question 8 (***+)

A footballer sees the goalkeeper off his line and kicks the ball from level horizontal ground with speed U ms^{-1}, at an angle of elevation θ, where $\tan\theta = \frac{5}{12}$.

When the ball was kicked, it was at horizontal distance of 52.8 m from the goal line and perpendicular to it. Consequently a goal is scored as the ball passes just under the horizontal cross bar which stands 2.40 m in vertical height.

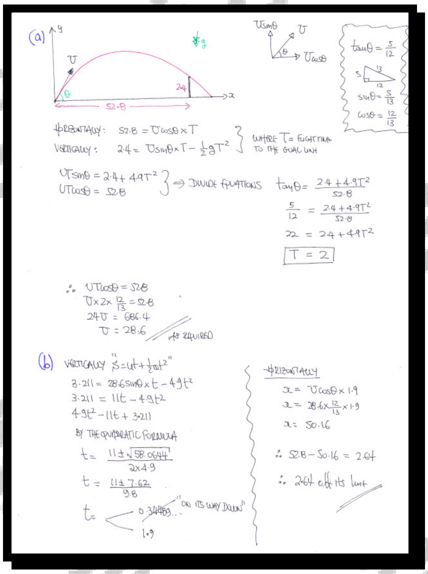
The ball is modelled as a particle moving freely under gravity, whose path lies in a vertical plane perpendicular to the goal line and the cross bar.

a) By considering the horizontal and vertical displacements of the ball, show clearly that $U = 28.6$.

The goalkeeper whose vertical reach is 3.211 m could not prevent the goal.

b) Given that the keeper jumped to save the goal when the ball was on its way down determine the distance of the goalkeeper from his goal line when he jumped for the ball.

$$\boxed{h = 2.64 \text{ m}}$$



(a)

$\tan\theta = \frac{5}{12}$, $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$

HORIZONTALLY: $52.8 = U\cos\theta \times T$

VERTICALLY: $2.4 = U\sin\theta \times T - \frac{1}{2}gT^2$

WHERE T = FLIGHT TIME TO THE GOAL LINE

$UT\sin\theta = 2.4 + 4.9T^2$

$UT\cos\theta = 52.8$

$\Rightarrow$ DIVIDE EQUATIONS $\tan\theta = \frac{2.4 + 4.9T^2}{52.8}$

$\frac{5}{12} = \frac{2.4 + 4.9T^2}{52.8}$

$22 = 2.4 + 4.9T^2$

$\boxed{T = 2}$

$\therefore UT\cos\theta = 52.8$

$U \times 2 \times \frac{12}{13} = 52.8$

$24U = 686.4$

$U = 28.6$ AS REQUIRED

(b) VERTICALLY "$s = ut + \frac{1}{2}at^2$"

$3.211 = 28.6\sin\theta \times t - 4.9t^2$

$3.211 = 11t - 4.9t^2$

$4.9t^2 - 11t + 3.211$

BY THE QUADRATIC FORMULA

$t = \frac{11 \pm \sqrt{58.0644}}{2 \times 4.9}$

$t = \frac{11 \pm 7.62}{9.8}$

$t = \cancel{0.34489...}$ or 1.9 "ON ITS WAY DOWN"

HORIZONTALLY

$x = U\cos\theta \times 1.9$

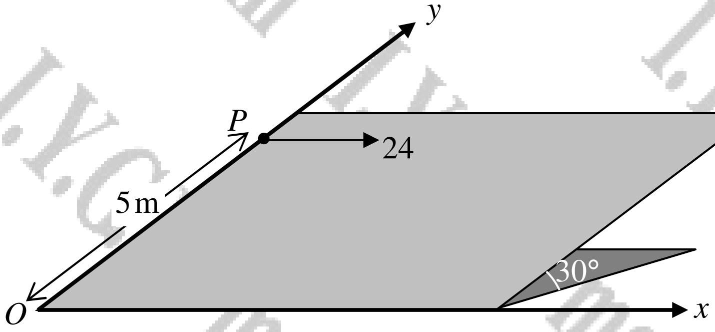
$x = 28.6 \times \frac{12}{13} \times 1.9$

$x = 50.16$

$\therefore 52.8 - 50.16 = 2.64$

$\therefore$ 2.64 off its line

Question 9 (***+)



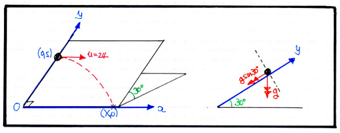
The point O lies at the bottom end of a fixed smooth plane, inclined at $30°$ to the horizontal. A positive y axis is defined up the line of greatest slope of the plane and a positive x axis is defined perpendicular to the y axis through O, as shown in the figure above.

At particle P is projected along the plane with speed 24 ms^{-1} in a direction parallel to the x axis, from the point with coordinates $(0,5)$, relative to O.

P reaches the bottom of the plane at the point $(X,0)$, with speed V, after time T.

Determine in any order the value of X, V and T.

$\boxed{\quad}$, $\boxed{\approx 4.464°}$, $\boxed{\approx 33960 \text{ m}}$, $\boxed{\approx 54'-14''}$



EQUATIONS OF MOTION IN x & y CAN BE SIMPLIFIED, AS THE ACCELERATION IS ZERO IN x & $-g\sin 30°$ IN y

- $\ddot{x} = 0$
- $\dot{x} = 24$
- $x = 24t$
- $\ddot{y} = -\frac{1}{2}g$
- $\dot{y} = -\frac{1}{2}gt$ $(v = \cancel{u} + at)$
- $y = 5 - \frac{1}{4}gt^2$ $(s = s_0 + \cancel{ut} + \frac{1}{2}at^2)$

"JOURNEY TIME" OCCURS WHEN $y = 0$

$\Rightarrow 5 - \frac{1}{4}gt^2 = 0$
$\Rightarrow 20 = gt^2$
$\Rightarrow t = \sqrt{\frac{20}{g}}$
$\Rightarrow t = \frac{10}{7}$ or $T = \frac{10}{7} = 1.43 \text{ s}$

THE VALUE OF X IS SIMPLY

$X = 24T$
$X = 24 \times \frac{10}{7}$
$X = \frac{240}{7}$
$X = 34.3 \text{ m}$

FINALLY THE SPEED

$\dot{x} = 24$ (CONSTANT)

$\dot{y} = -\frac{1}{2}gt$
$\dot{y} = -\frac{1}{2}g \times \frac{10}{7}$
$\dot{y} = 7$ ("DOWNWARDS")

$\Rightarrow$ SPEED $= \sqrt{\dot{x}^2 + \dot{y}^2}$
$= \sqrt{24^2 + 7^2}$
$= 25 \text{ ms}^{-1}$

Question 10 (***+)

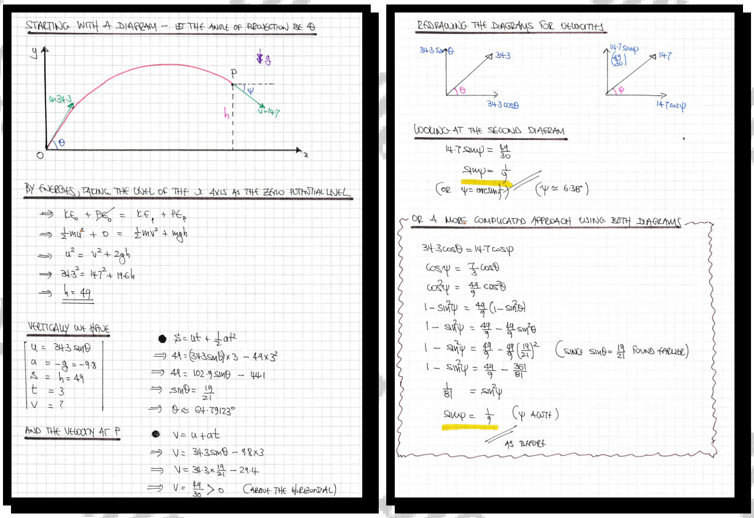
A particle is projected from a point O on level horizontal ground with a speed of 34.3 ms^{-1} at some angle of elevation.

The particle is moving freely under gravity, reaching a greatest height above the ground before it passes through the point P, 3 s after it was projected.

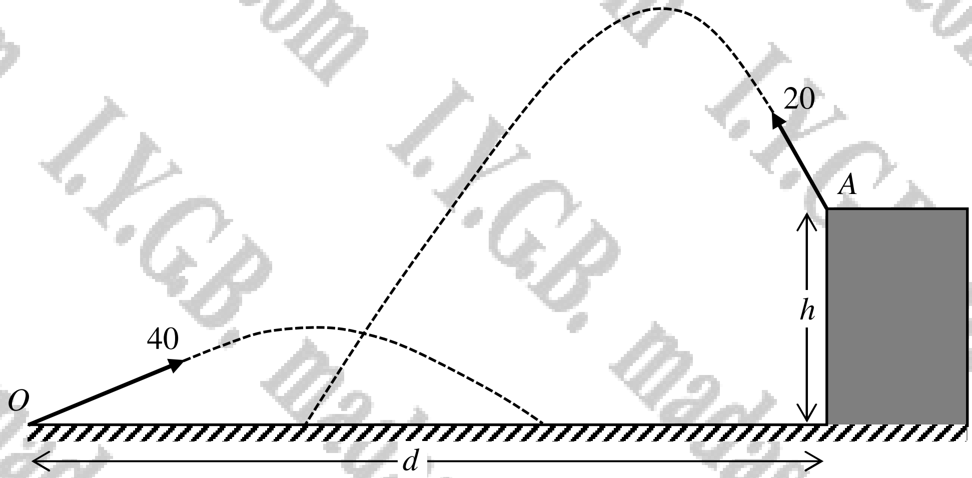
When the particle passes through P it has a speed of 14.7 ms^{-1}, at an angle ψ to the horizontal.

Show that $\psi = \arcsin\left(\frac{1}{9}\right)$, stating further whether this angle is above the horizontal or below the horizontal.

$\boxed{}$, $\boxed{\text{proof}}$



Question 11 (***+)



The point O lies on level horizontal ground and the point A is at a horizontal distance d m away from O and at a height d m above the ground.

A particle is projected from O with speed 40 ms^{-1} at an angle of elevation $\arctan\left(\frac{3}{4}\right)$.

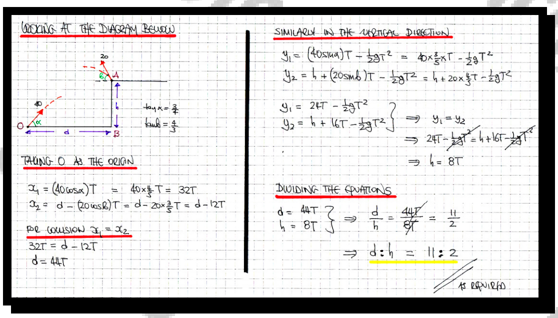
At the same time another particle is projected from A with speed 20 ms^{-1} at an angle of elevation $\arctan\left(\frac{4}{3}\right)$, as shown in the figure above.

The motion of the two particles takes place in the same vertical plane.

Assuming that there is no air resistance present, show that, if the two particles collide during their flights, then

$$d : h = 11 : 2 .$$

$\boxed{}$, $\boxed{\text{proof}}$



LOOKING AT THE DIAGRAM BELOW

$\tan\alpha = \frac{3}{4}$

$\tan\beta = \frac{4}{3}$

TAKING O AS THE ORIGIN

$x_1 = (40\cos\alpha)T = 40 \times \frac{4}{5}T = 32T$

$x_2 = d - (20\cos\beta)T = d - 20 \times \frac{3}{5}T = d - 12T$

FOR COLLISION $x_1 = x_2$

$32T = d - 12T$

$d = 44T$

SIMILARLY IN THE VERTICAL DIRECTION

$y_1 = (40\sin\alpha)T - \frac{1}{2}gT^2 = 40 \times \frac{3}{5} \times T - \frac{1}{2}gT^2$

$y_2 = h + (20\sin\beta)T - \frac{1}{2}gT^2 = h + 20 \times \frac{4}{5}T - \frac{1}{2}gT^2$

$\left.\begin{array}{l} y_1 = 24T - \frac{1}{2}gT^2 \\ y_2 = h + 16T - \frac{1}{2}gT^2 \end{array}\right\} \Rightarrow y_1 = y_2$

$\Rightarrow 24T - \cancel{\frac{1}{2}gT^2} = h + 16T - \cancel{\frac{1}{2}gT^2}$

$\Rightarrow h = 8T$

DIVIDING THE EQUATIONS

$\left.\begin{array}{l} d = 44T \\ h = 8T \end{array}\right\} \Rightarrow \frac{d}{h} = \frac{44\cancel{T}}{8\cancel{T}} = \frac{11}{2}$

$\Rightarrow d : h = 11 : 2$

AS REQUIRED

Question 12 (*+)**

In this question take $g = 10 \text{ ms}^{-2}$.

Two particles, A and B, are projected from the same fixed point O, with the same speed $u \text{ ms}^{-1}$, at angles of elevation θ and 2θ respectively.

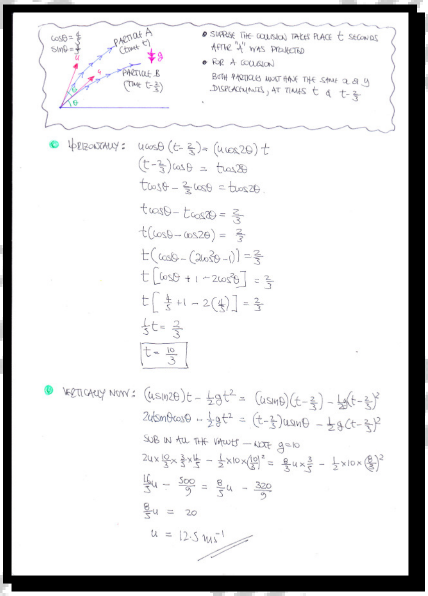
It is further given that …

… B is projected $\frac{2}{3}$ s after A

… $\tan\theta = \frac{3}{4}$.

If A and B collide in the subsequent motion determine the value of u.

$\boxed{u = 12.5}$



• SUPPOSE THE COLLISION TAKES PLACE t SECONDS AFTER "A" WAS PROJECTED

• FOR A COLLISION BOTH PARTICLES MUST HAVE THE SAME x & y DISPLACEMENTS, AT TIMES t & $t - \frac{2}{3}$

HORIZONTALLY: $u\cos\theta\left(t - \frac{2}{3}\right) = (u\cos 2\theta)t$

$\left(t - \frac{2}{3}\right)\cos\theta = t\cos 2\theta$

$t\cos\theta - \frac{2}{3}\cos\theta = t\cos 2\theta$

$t\cos\theta - t\cos 2\theta = \frac{2}{3}$

$t(\cos\theta - \cos 2\theta) = \frac{2}{3}$

$t(\cos\theta - (2\cos^2\theta - 1)) = \frac{2}{3}$

$t\left[\cos\theta + 1 - 2\cos^2\theta\right] = \frac{2}{3}$

$t\left[\frac{4}{5} + 1 - 2\left(\frac{4}{5}\right)\right] = \frac{2}{3}$

$\frac{1}{5}t = \frac{2}{3}$

$\boxed{t = \frac{10}{3}}$

VERTICALLY NOW: $(u\sin 2\theta)t - \frac{1}{2}gt^2 = (u\sin\theta)\left(t - \frac{2}{3}\right) - \frac{1}{2}g\left(t - \frac{2}{3}\right)^2$

$2ut\sin\theta\cos\theta - \frac{1}{2}gt^2 = \left(t - \frac{2}{3}\right)u\sin\theta - \frac{1}{2}g\left(t - \frac{2}{3}\right)^2$

SUB IN ALL THE VALUES — NOTE $g = 10$

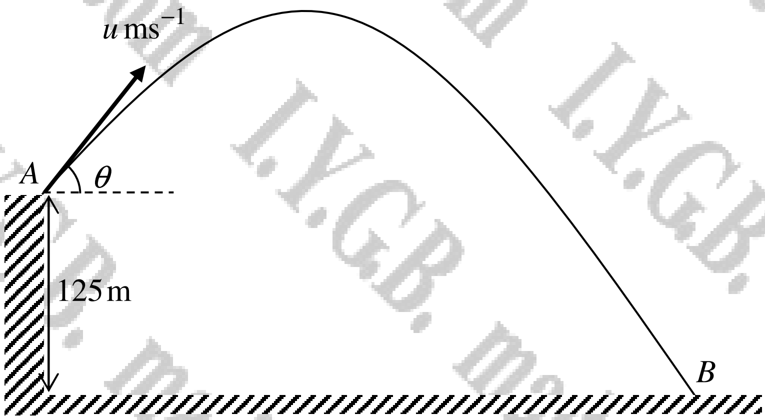
$2u \times \frac{10}{3} \times \frac{3}{5} \times \frac{4}{5} - \frac{1}{2} \times 10 \times \left(\frac{10}{3}\right)^2 = \frac{8}{3}u \times \frac{3}{5} - \frac{1}{2} \times 10 \times \left(\frac{8}{3}\right)^2$

$\frac{16}{5}u - \frac{500}{9} = \frac{8}{5}u - \frac{320}{9}$

$\frac{8}{5}u = 20$

$u = 12.5 \text{ ms}^{-1}$

Question 13 (****)



A particle P is projected with a speed of u ms^{-1} at an angle of elevation θ, from a point A which is 125 m above level horizontal ground. The particle is moving freely under gravity and first strikes the ground at a point B, as shown in the figure above.

It took T s for P to travel from A to B, and the speed of the particle at B is $3u$ ms^{-1}.

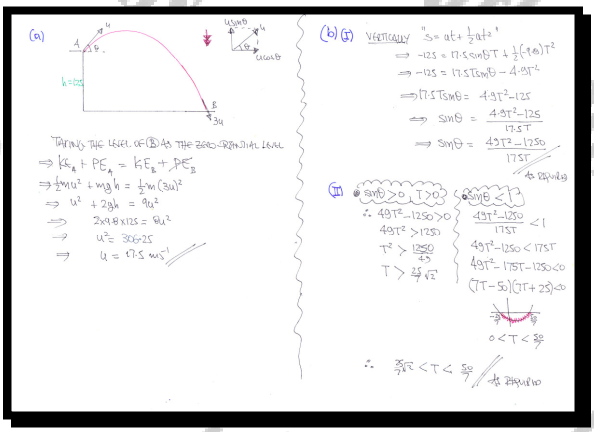
a) Find the value of u.

b) Show clearly that …

i. … $\sin\theta = \dfrac{49T^2 - 1250}{175T}$.

ii. … $\dfrac{25}{7}\sqrt{2} < T < \dfrac{50}{7}$.

$\boxed{u = 17.5}$



Question 14 (****)

Relative to a fixed origin O the unit vectors $\mathbf{i}$ and $\mathbf{j}$ are oriented horizontally and vertically upwards, respectively. The origin lies on level horizontal ground.

A particle is projected with velocity $(7\mathbf{i}+14\mathbf{j})\,\text{ms}^{-1}$ from O and moves freely under gravity passing through the point P with position vector $(x\mathbf{i}+y\mathbf{j})$ m, in time t s.

a) Show clearly that

$$y=\frac{1}{10}x(20-x).$$

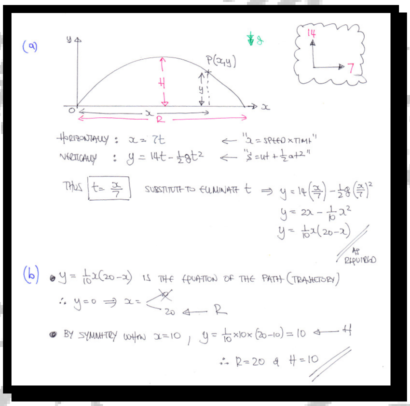
The particle reaches a maximum height of H m above the ground and has a horizontal range of R m.

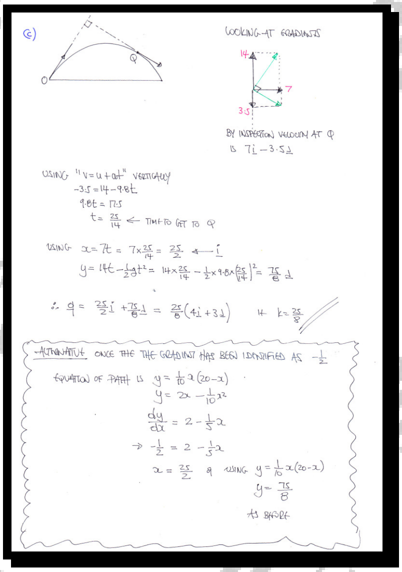
b) Find the values of R and H.

The point Q lies on the particle's trajectory so that its velocity at P is perpendicular to its projection velocity.

c) Show that the position vector of Q is $k(4\mathbf{i}+3\mathbf{j})$ m, where k is an exact constant to be found.

$\boxed{R=20,\ H=10}$, $\boxed{k=\frac{25}{8}}$





Question 15 (****)

In this question take $g = 10 \text{ ms}^{-2}$.

A projectile is fired from a fixed point O with speed u ms^{-1} at an angle of elevation α so that it passes through a point P.

Relative to a Cartesian coordinate system with origin at O the point P has coordinates $\left(10\sqrt{5}, 5\sqrt{5}\right)$.

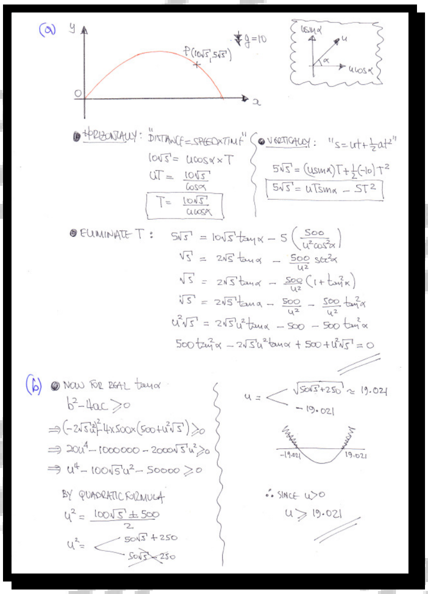
It is assumed that O and P lie in the same vertical plane, and the projectile can be modelled as a particle moving freely under gravity.

a) Show clearly that

$$500\tan^2\alpha - 2\sqrt{5}u^2\tan\alpha + 500 + \sqrt{5}u^2 = 0.$$

b) Hence determine the minimum value of u.

$\boxed{u \geq 19.021...}$


(a)

HORIZONTALLY: "DISTANCE = SPEED × TIME"

$10\sqrt{5} = u\cos\alpha \times T$

$uT = \frac{10\sqrt{5}}{\cos\alpha}$

$\boxed{T = \frac{10\sqrt{5}}{u\cos\alpha}}$

VERTICALLY: "$s = ut + \frac{1}{2}at^2$"

$5\sqrt{5} = (u\sin\alpha)T + \frac{1}{2}(-10)T^2$

$\boxed{5\sqrt{5} = uT\sin\alpha - 5T^2}$

ELIMINATE T:

$5\sqrt{5} = 10\sqrt{5}\tan\alpha - 5\left(\frac{500}{u^2\cos^2\alpha}\right)$

$\sqrt{5} = 2\sqrt{5}\tan\alpha - \frac{500}{u^2}\sec^2\alpha$

$\sqrt{5} = 2\sqrt{5}\tan\alpha - \frac{500}{u^2}(1+\tan^2\alpha)$

$\sqrt{5} = 2\sqrt{5}\tan\alpha - \frac{500}{u^2} - \frac{500}{u^2}\tan^2\alpha$

$u^2\sqrt{5} = 2\sqrt{5}u^2\tan\alpha - 500 - 500\tan^2\alpha$

$500\tan^2\alpha - 2\sqrt{5}u^2\tan\alpha + 500 + u^2\sqrt{5} = 0$

(b)

NOW FOR REAL $\tan\alpha$

$b^2 - 4ac \geqslant 0$

$\Rightarrow (-2\sqrt{5}u^2)^2 - 4 \times 500 \times (500 + u^2\sqrt{5}) \geqslant 0$

$\Rightarrow 20u^4 - 1000000 - 2000\sqrt{5}u^2 \geqslant 0$

$\Rightarrow u^4 - 100\sqrt{5}u^2 - 50000 \geqslant 0$

BY QUADRATIC FORMULA

$u^2 = \frac{100\sqrt{5} \pm 500}{2}$

$u^2 = 50\sqrt{5} + 250$ or $\cancel{50\sqrt{5} - 250}$

$u = \sqrt{50\sqrt{5}+250} \approx 19.021$ or -19.021

$\therefore$ SINCE $u > 0$

$u \geqslant 19.021$

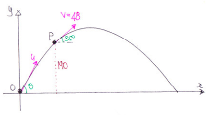
Question 16 (****)

At time $t = 0$, a particle is projected from a point O on level horizontal ground in a non vertical direction.

At some time later the particle is passing through a point P with speed 48 ms^{-1}, at an angle of $35°$ above the horizontal.

Given that P is at a height of 190 m above the ground, determine the time when the particle is **again** at a height of 190 m above the ground

$$\boxed{t \approx 9.744\ldots \text{ s}}$$



BY ENERGY TAKING THE x AXIS AS THE ZERO POTENTIAL LEVEL

$KE_o + \cancel{PE_o} = KE_p + PE_p$

$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgh$

$u^2 = v^2 + 2gh$

$u^2 = 48^2 + 2 \times 9.8 \times 190$

$u^2 = 6028$

$u \simeq 77.64019\ldots$

HORIZONTAL SPEED IS UNCHANGED

$u\cos\theta = V\cos 35$

$\cos\theta = \frac{V\cos 35}{u}$

$\cos\theta = 0.5047\ldots$

$\theta \simeq 59.6847\ldots$

"$s = ut + \frac{1}{2}at^2$"

$190 = (77.64\ldots)\sin(59.68\ldots)t + \frac{1}{2}(-9.8)t^2$

$4.9t^2 - (77.64\ldots)\sin(59.68)t + 190$

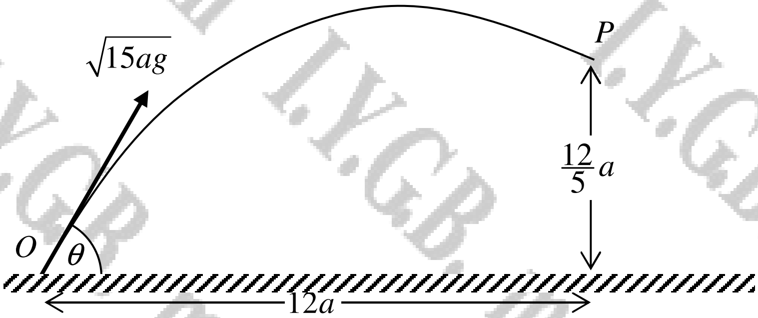
BY QUADRATIC FORMULA

$t = \frac{67.745\ldots \pm \sqrt{797.9977949\ldots}}{9.8}$

$t = \begin{cases} 3.979\ldots \\ 9.744\ldots \end{cases}$

$\therefore t \simeq 9.74$ 3 s.f.

Question 17 (****)



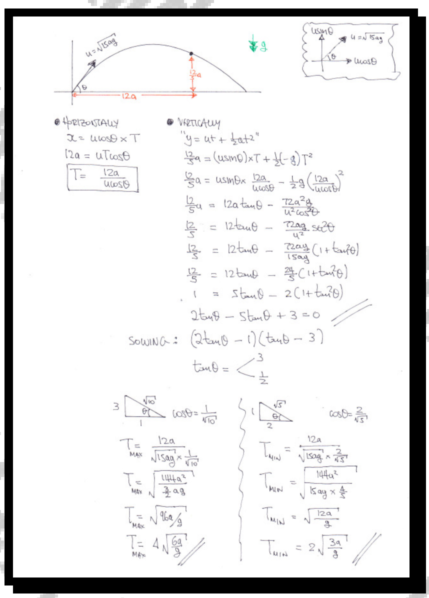
A projectile is fired from a fixed point O with speed $\sqrt{15ag}$ at an angle of elevation θ so that it passes through a point P. Relative to a Cartesian coordinate system with origin at O the point P has coordinates $\left(12a, \frac{12}{5}a\right)$.

It is assumed that O and P lie in the same vertical plane, and the projectile can be modelled as a particle moving freely under gravity.

Show clearly, that the respective minimum and maximum flight times of the projectile from O to P are

$$4\sqrt{\frac{6a}{g}} \quad \text{and} \quad 2\sqrt{\frac{3a}{g}}.$$

$\boxed{\text{proof}}$



Question 18 (****)

A particle P is projected from a point O on level horizontal ground with speed 26 ms^{-1}, at an angle θ to the horizontal.

At the same time, another particle Q is projected horizontally with speed 10 ms^{-1}, from a point A, which lies 78.4 m vertically above O.

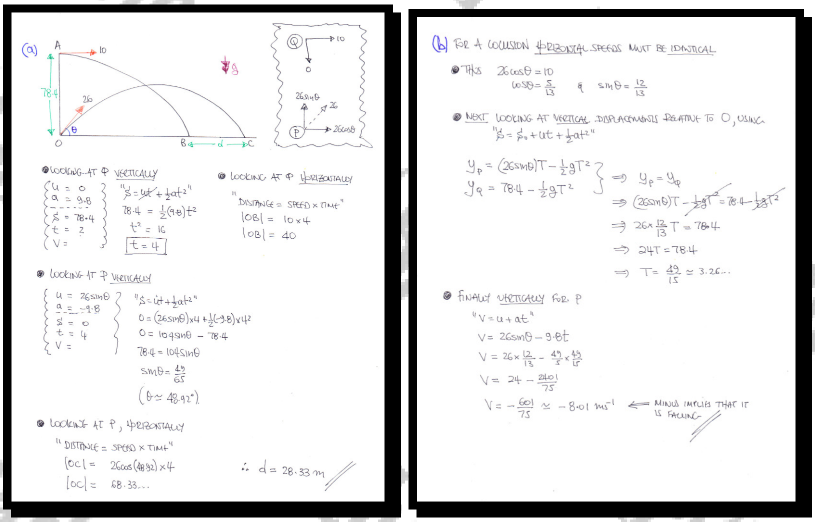
The motion of both particles takes place at the same vertical plane with both particles moving through still air without any resistance.

The particles hit the ground at the same time at two points which are d m apart.

a) Calculate the value of d.

b) Given instead that the particles collide before they reach the ground, determine by detailed calculations whether P is rising or falling immediately before the collision.

$\boxed{d \approx 28.33...}$



Question 19 (****)

Relative to a fixed origin O the unit vectors $\mathbf{i}$ and $\mathbf{j}$ are oriented horizontally and vertically upwards, respectively.

The gravitational acceleration constant g is taken to be $-10\mathbf{j}\ \text{ms}^{-2}$ in this question.

A particle is projected with velocity $(u\mathbf{i}+v\mathbf{j})\ \text{ms}^{-1}$, where u and v are positive constants, from a point P with position vector $105\mathbf{j}$ m.

The particle moves freely under gravity passing through the point Q with position vector $210\mathbf{i}$ m.

a) Show clearly that

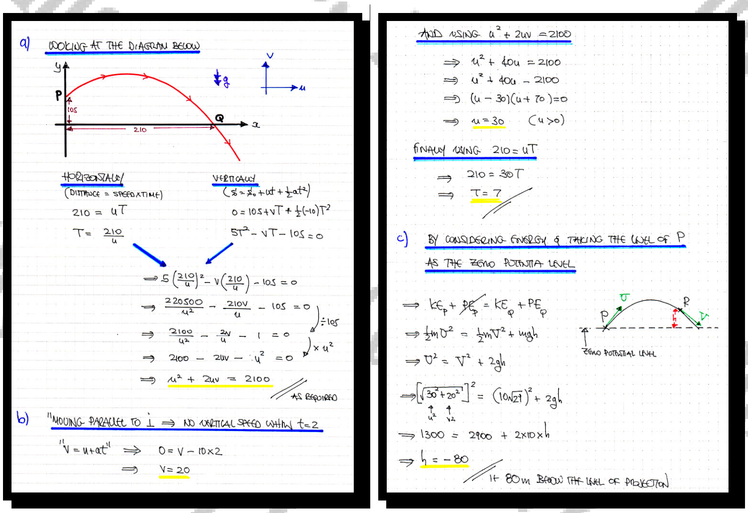
$$u^2+2uv=2100.$$

b) Given that when $t=2$ the particle is moving parallel to $\mathbf{i}$, determine the time it takes the particle to travel from P to Q.

The particle passes through the point R with a speed of $10\sqrt{29}\ \text{ms}^{-1}$.

c) Show R is 80 m below the level of P.

$\boxed{}$, $\boxed{u=30,\ v=20}$, $\boxed{\text{flight time}=7\text{ s}}$



a) LOOKING AT THE DIAGRAM BELOW

HORIZONTALLY (DISTANCE = SPEED×TIME)

$210 = uT$

$T = \frac{210}{u}$

VERTICALLY ($s = s_0 + ut + \frac{1}{2}at^2$)

$0 = 105 + vT + \frac{1}{2}(-10)T^2$

$5T^2 - vT - 105 = 0$

$\Rightarrow 5\left(\frac{210}{u}\right)^2 - v\left(\frac{210}{u}\right) - 105 = 0$

$\Rightarrow \frac{220500}{u^2} - \frac{210v}{u} - 105 = 0 \quad \div 105$

$\Rightarrow \frac{2100}{u^2} - \frac{2v}{u} - 1 = 0 \quad \times u^2$

$\Rightarrow 2100 - 2uv - u^2 = 0$

$\Rightarrow u^2 + 2uv = 2100$

AS REQUIRED

b) "MOVING PARALLEL TO $\mathbf{i}$ $\Rightarrow$ NO VERTICAL SPEED WHEN $t=2$

"$v = u + at$" $\Rightarrow 0 = v - 10\times 2$

$\Rightarrow v = 20$

AND USING $u^2 + 2uv = 2100$

$\Rightarrow u^2 + 40u = 2100$

$\Rightarrow u^2 + 40u - 2100$

$\Rightarrow (u-30)(u+70)=0$

$\Rightarrow u = 30 \quad (u>0)$

FINALLY USING $210 = uT$

$\Rightarrow 210 = 30T$

$\Rightarrow T = 7$

c) BY CONSIDERING ENERGY & TAKING THE LEVEL OF P AS THE ZERO POTENTIAL LEVEL

$\Rightarrow KE_P + \cancel{PE_P} = KE_Q + PE_Q$

$\Rightarrow \frac{1}{2}mU^2 = \frac{1}{2}mV^2 + mgh$

$\Rightarrow U^2 = V^2 + 2gh$

$\Rightarrow \left[\sqrt{30^2+20^2}\right]^2 = (10\sqrt{29})^2 + 2gh$

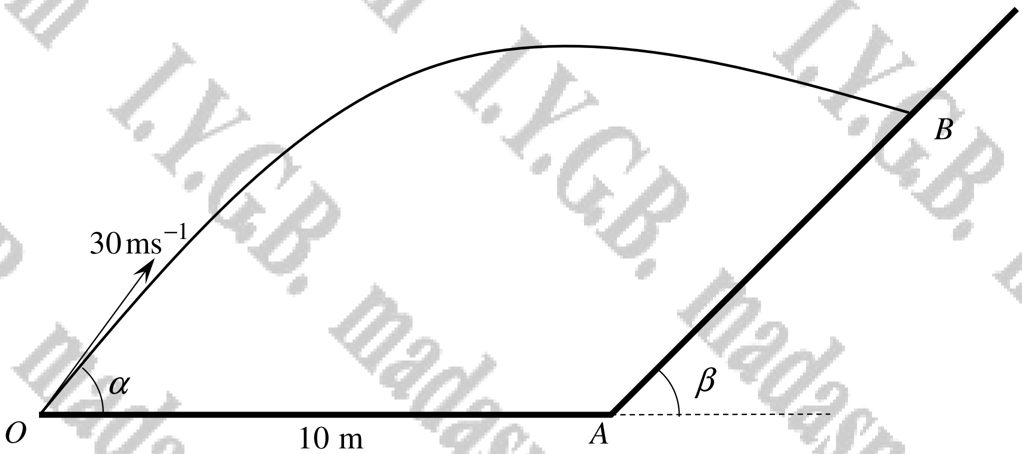
$u^2 \quad v^2$

$\Rightarrow 1300 = 2900 + 2\times10\times h$

$\Rightarrow h = -80$

IE 80 m BELOW THE LEVEL OF PROJECTION

Question 20 (****+)



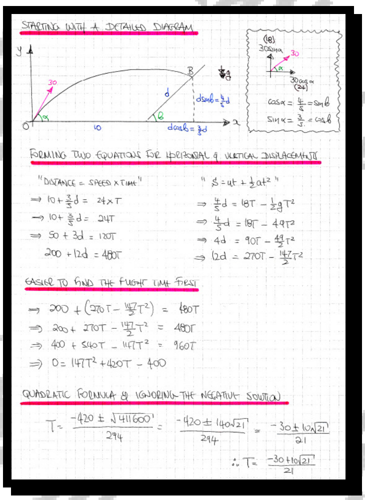
A particle is projected from a point O on level horizontal ground with speed 30 ms^{-1} at an angle of elevation α. The particle is freely moving under gravity, heading towards a plane, inclined at an angle β to the horizontal.

The foot, A, of this incline plane is located at a horizontal distance of 10 m from O, as shown in the figure above.

The particle strikes the incline plane at the point B, so that AB is a line of greatest slope in the same vertical plane which contains O.

Determine the distance AB, given further that $\alpha = \arctan\frac{3}{4}$ and $\beta = \arctan\frac{4}{3}$

M456- , $\boxed{|AB| = \frac{-1550+400\sqrt{21}}{21} \approx 13.48 \text{ m}}$



STARTING WITH A DETAILED DIAGRAM

FORMING TWO EQUATIONS FOR HORIZONTAL & VERTICAL DISPLACEMENTS

"DISTANCE = SPEED × TIME"

$\Rightarrow 10 + \frac{3}{5}d = 24 \times T$

$\Rightarrow 10 + \frac{3}{5}d = 24T$

$\Rightarrow 50 + 3d = 120T$

$200 + 12d = 480T$

"$s = ut + \frac{1}{2}at^2$"

$\Rightarrow \frac{4}{5}d = 18T - \frac{1}{2}gT^2$

$\Rightarrow \frac{4}{5}d = 18T - 4.9T^2$

$\Rightarrow 4d = 90T - \frac{49}{2}T^2$

$\Rightarrow 12d = 270T - \frac{147}{2}T^2$

EASIER TO FIND THE FLIGHT TIME FIRST

$\Rightarrow 200 + \left(270T - \frac{147}{2}T^2\right) = 480T$

$\Rightarrow 200 + 270T - \frac{147}{2}T^2 = 480T$

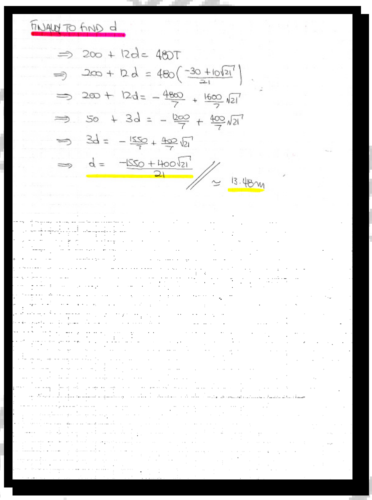
$\Rightarrow 400 + 540T - 147T^2 = 960T$

$\Rightarrow 0 = 147T^2 + 420T - 400$

QUADRATIC FORMULA & IGNORING THE NEGATIVE SOLUTION

$T = \frac{-420 \pm \sqrt{411600}}{294} = \frac{-420 \pm 140\sqrt{21}}{294} = \frac{-30 \pm 10\sqrt{21}}{21}$

$\therefore T = \frac{-30 + 10\sqrt{21}}{21}$



FINALLY TO FIND d

$\Rightarrow 200 + 12d = 480T$

$\Rightarrow 200 + 12d = 480\left(\frac{-30 + 10\sqrt{21}}{21}\right)$

$\Rightarrow 200 + 12d = -\frac{4800}{7} + \frac{1600}{7}\sqrt{21}$

$\Rightarrow 50 + 3d = -\frac{1200}{7} + \frac{400}{7}\sqrt{21}$

$\Rightarrow 3d = -\frac{1550}{7} + \frac{400}{7}\sqrt{21}$

$\Rightarrow d = \frac{-1550 + 400\sqrt{21}}{21} \approx 13.48\text{m}$

Question 21 (****)

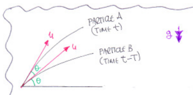
Two particles are projected from the same fixed point, with the same speed u, at angles of elevation θ and 2θ.

If the particles collide in the subsequent motion show that

$$u = \frac{gT(2\cos\theta - 1)(\cos\theta + 1)}{2\sin\theta},$$

where T is the time delay between the projection of the two particles.

proof



SUPPOSE THE COLLISION TAKES PLACE AT TIME t AFTER "A" WAS PROJECTED.

FOR COLLISION BOTH PARTICLES MUST HAVE THE SAME x & y DISPLACEMENT AT TIMES t & $t-T$

HORIZONTALLY: $(u\cos\theta)(t-T) = (u\cos 2\theta)t$

$$\cos\theta(t-T) = (\cos 2\theta)t$$
$$t\cos\theta - T\cos\theta = t\cos 2\theta$$
$$t(\cos\theta - \cos 2\theta) = T\cos\theta$$
$$\boxed{t = \frac{T\cos\theta}{\cos\theta - \cos 2\theta}}$$

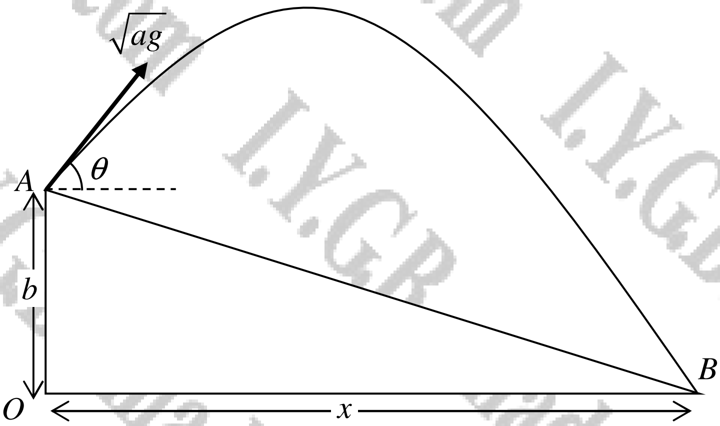
VERTICALLY: $(u\sin 2\theta)t - \frac{1}{2}gt^2 = (u\sin\theta)(t-T) - \frac{1}{2}g(t-T)^2$

$$\Rightarrow ut\sin 2\theta - \frac{1}{2}gt^2 = ut\sin\theta - uT\sin\theta - \frac{1}{2}g(t^2 - 2tT + T^2)$$
$$\Rightarrow ut\sin 2\theta - \cancel{\frac{1}{2}gt^2} = ut\sin\theta - uT\sin\theta - \cancel{\frac{1}{2}gt^2} + gtT - \frac{1}{2}gT^2$$
$$\Rightarrow ut\sin 2\theta - ut\sin\theta + uT\sin\theta = gtT - \frac{1}{2}gT^2$$
$$\Rightarrow u\left[t\sin 2\theta - t\sin\theta + T\sin\theta\right] = \frac{1}{2}gT(2t - T)$$
$$\Rightarrow u\left[t[\sin 2\theta - \sin\theta] + T\sin\theta\right] = \frac{1}{2}gT[2t - T]$$
$$\Rightarrow u\left[\frac{T\cos\theta}{\cos\theta - \cos 2\theta}(\sin 2\theta - \sin\theta) + T\sin\theta\right] = \frac{1}{2}gT\left[\frac{2T\cos\theta}{\cos\theta - \cos 2\theta} - T\right]$$
$$\Rightarrow uT\left[\frac{\cos\theta(\sin 2\theta - \sin\theta)}{\cos\theta - \cos 2\theta} + \sin\theta\right] = \frac{1}{2}gT^2\left[\frac{2\cos\theta}{\cos\theta - \cos 2\theta} - 1\right]$$
$$\Rightarrow u\left[\frac{\cos\theta(\sin 2\theta - \sin\theta)}{\cos\theta - \cos 2\theta} + \sin\theta\right] = \frac{1}{2}gT\left[\frac{2\cos\theta}{\cos\theta - \cos 2\theta} - 1\right]$$

MULTIPLY EQUATION THROUGH BY $\cos\theta - \cos 2\theta \neq 0$

$$\Rightarrow u\left[\cos\theta(\sin 2\theta - \sin\theta) + \sin\theta(\cos\theta - \cos 2\theta)\right] = \frac{1}{2}gT\left[2\cos\theta - (\cos\theta - \cos 2\theta)\right]$$
$$\Rightarrow u\left[\cos\theta\sin 2\theta - \cancel{\cos\theta\sin\theta} + \cancel{\sin\theta\cos\theta} - \sin\theta\cos 2\theta\right] = \frac{1}{2}gT\left[\cos\theta + \cos 2\theta\right]$$
$$\Rightarrow u\left[\sin 2\theta\cos\theta - \cos 2\theta\sin\theta\right] = \frac{1}{2}gT(\cos\theta + \cos 2\theta)$$
$$\Rightarrow u\left[\sin(2\theta - \theta)\right] = \frac{1}{2}gT(\cos\theta + \cos 2\theta)$$
$$\Rightarrow u\sin\theta = \frac{1}{2}gT(\cos\theta + \cos 2\theta)$$
$$\Rightarrow u = \frac{gT(\cos\theta + \cos 2\theta)}{2\sin\theta}$$
$$\Rightarrow u = \frac{gT(2\cos^2\theta - 1 + \cos\theta)}{2\sin\theta}$$
$$\Rightarrow u = \frac{gT(2\cos^2\theta + \cos\theta - 1)}{2\sin\theta}$$
$$\Rightarrow u = \frac{gT(2\cos\theta - 1)(\cos\theta + 1)}{2\sin\theta}$$

Question 22 **(****+)**



A particle is projected from a point B down an incline plane with a speed of $\sqrt{ag}$, where a is a positive constant, at an angle of elevation θ.

The particle is moving freely under gravity and first strikes the ground at a point B. The point O lies vertically below A and at the same horizontal level as B, as shown in the figure above. The plane has constant inclination and the particle moves in a vertical plane which contains the angle of greatest slope of the plane.

a) Show that

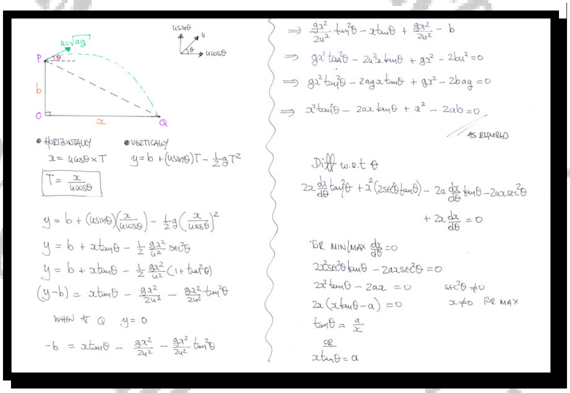
$$x^2 \tan^2\theta - 2ax\tan\theta + x^2 - 2ab = 0,$$

where $OA = b$ and $|OB| = x$.

b) Hence show that the maximum value of x and the corresponding angle of projection θ satisfy

$$x = \sqrt{a(a+2b)} \quad \text{and} \quad \tan\theta = \sqrt{\frac{a}{a+2b}}$$

proof



$u\sin\theta$, u, $u\cos\theta$, $u = \sqrt{ag}$, P, b, O, x, Q

• HORIZONTALLY $x = u\cos\theta \times T$ • VERTICALLY $y = b + (u\sin\theta)T - \frac{1}{2}gT^2$

$\boxed{T = \frac{x}{u\cos\theta}}$

$y = b + (u\sin\theta)\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$

$y = b + x\tan\theta - \frac{1}{2}\frac{gx^2}{u^2}\sec^2\theta$

$y = b + x\tan\theta - \frac{1}{2}\frac{gx^2}{u^2}(1+\tan^2\theta)$

$(y-b) = x\tan\theta - \frac{gx^2}{2u^2} - \frac{gx^2}{2u^2}\tan^2\theta$

WHEN AT Q $y = 0$

$-b = x\tan\theta - \frac{gx^2}{2u^2} - \frac{gx^2}{2u^2}\tan^2\theta$

$\Rightarrow \frac{gx^2}{2u^2}\tan^2\theta - x\tan\theta + \frac{gx^2}{2u^2} - b$

$\Rightarrow gx^2\tan^2\theta - 2u^2x\tan\theta + gx^2 - 2bu^2 = 0$

$\Rightarrow gx^2\tan^2\theta - 2agx\tan\theta + gx^2 - 2bag = 0$

$\Rightarrow x^2\tan^2\theta - 2ax\tan\theta + x^2 - 2ab = 0$

AS REQUIRED

Diff w.r.t θ

$2x\frac{dx}{d\theta}\tan^2\theta + x^2(2\sec^2\theta\tan\theta) - 2a\frac{dx}{d\theta}\tan\theta - 2ax\sec^2\theta + 2x\frac{dx}{d\theta} = 0$

FOR MIN/MAX $\frac{dx}{d\theta} = 0$

$2x^2\sec^2\theta\tan\theta - 2ax\sec^2\theta = 0$

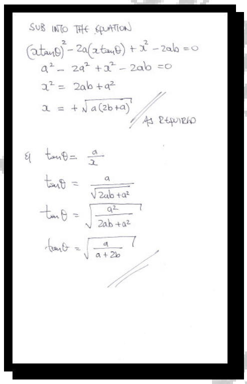
$2x^2\tan\theta - 2ax = 0$ $\quad \sec^2\theta \neq 0$

$2x(x\tan\theta - a) = 0$ $\quad x \neq 0$ FOR MAX

$\tan\theta = \frac{a}{x}$

OR

$x\tan\theta = a$



SUB INTO THE EQUATION

$(x\tan\theta)^2 - 2a(x\tan\theta) + x^2 - 2ab = 0$

$a^2 - 2a^2 + x^2 - 2ab = 0$

$x^2 = 2ab + a^2$

$x = +\sqrt{a(2b+a)}$

AS REQUIRED

& $\tan\theta = \frac{a}{x}$

$\tan\theta = \frac{a}{\sqrt{2ab+a^2}}$

$\tan\theta = \sqrt{\frac{a^2}{2ab+a^2}}$

$\tan\theta = \sqrt{\frac{a}{a+2b}}$

Question 23 (****+)

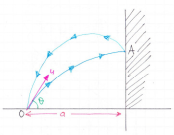
A particle is projected from a point O on level horizontal ground with speed u at an angle θ above the horizontal, towards a smooth vertical wall which is at a horizontal distance a from the point of projection.

The particle moves in a vertical plane perpendicular to the wall and hits the wall before it hits the ground. On impact with the wall the particle rebounds and first strikes the ground at O.

Given that e is the coefficient of restitution, show that

$$u^2 = \frac{ag(e+1)}{e\sin 2\theta}.$$

proof



- LOOKING AT THE JOURNEY OA

<u>HORIZONTALLY</u>

$a = (u\cos\theta)T_1$

$T_1 = \dfrac{a}{u\cos\theta}$

- LOOKING AT THE JOURNEY AO

<u>HORIZONTALLY</u>

REBOUND SPEED $\perp$ WALL $= eu\cos\theta$

$T_2 = \dfrac{a}{eu\cos\theta}$

- LOOKING AT THE ENTIRE JOURNEY IN THE VERTICAL, AS MOMENTUM IS <u>NOT</u> EXCHANGED $\perp$ TO THE WALL

"$s = ut + \frac{1}{2}at^2$"

$0 = (u\sin\theta)T_3 - \frac{1}{2}gT_3^2$

$0 = \frac{1}{2}T_3\left[2u\sin\theta - gT_3\right]$

$T_3 = \dfrac{2u\sin\theta}{g}$ $\quad (T_3 \neq 0)$

- HENCE BY LOOKING AT THE TIMES ALONE

$T_1 + T_2 = T_3$

$\Rightarrow \dfrac{a}{u\cos\theta} + \dfrac{a}{eu\cos\theta} = \dfrac{2u\sin\theta}{g}$

$\Rightarrow \dfrac{ae + a}{eu\cos\theta} = \dfrac{2u\sin\theta}{g}$

$\Rightarrow \dfrac{a(e+1)}{eu\cos\theta} = \dfrac{2u\sin\theta}{g}$

$\Rightarrow ag(e+1) = 2eu^2\cos\theta\sin\theta$

$\Rightarrow eu^2\sin 2\theta = ag(e+1)$

$\Rightarrow u^2 = \dfrac{ag(e+1)}{e\sin 2\theta}$ //

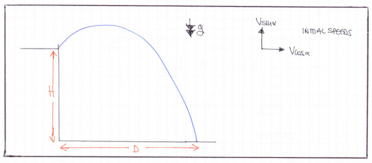
Question 24 (*****)

A particle is projected at an angle α above the horizontal, from a vertical cliff face of height H above level horizontal ground. It first hits the ground at a horizontal distance D, from the bottom of the cliff edge.

Assuming that air resistance can be ignored, show that the greatest height achieved by the particle from the level horizontal ground is

$$H + \frac{D\tan^2\alpha}{4\left(H + D\tan\alpha\right)}.$$

proof



- LET T BE THE TIME TAKEN TO REACH THE MAXIMUM HEIGHT

$$\text{"}v = u + at\text{"} \implies 0 = V\sin\alpha - gT$$
$$\implies T = \frac{V\sin\alpha}{g}$$

- HENCE THE MAXIMUM HEIGHT ABOVE GROUND IS GIVEN BY

$$H + \text{"}ut + \tfrac{1}{2}at^2\text{"} = H + VT\sin\alpha - \tfrac{1}{2}gT^2$$
$$= H + V\left(\frac{V\sin\alpha}{g}\right)\sin\alpha - \tfrac{1}{2}g\left(\frac{V\sin\alpha}{g}\right)^2$$
$$= H + \frac{V^2\sin^2\alpha}{g} - \frac{1}{2}\,\frac{V^2\sin^2\alpha}{g}$$
$$= H + \frac{V^2\sin^2\alpha}{2g}$$

- WE NEED TO GET RID OF V FROM THE ABOVE EXPRESSION — LET τ (TAU) BE THE FLIGHT TIME

HORIZONTALLY

$$D = (V\cos\alpha)\tau$$
$$\tau = \frac{D}{V\cos\alpha}$$

VERTICALLY ($s = ut + \frac{1}{2}at^2$)

$$-H = (V\sin\alpha)\tau - \tfrac{1}{2}g\tau^2$$
$$-H = V\tau\sin\alpha - \tfrac{1}{2}g\tau^2$$

SUBSTITUTE

$$\implies -H = V\left(\frac{D}{V\cos\alpha}\right)\sin\alpha - \tfrac{1}{2}g\left(\frac{D}{V\cos\alpha}\right)^2$$
$$\implies -H = D\tan\alpha - \frac{g}{2}\,\frac{D^2}{V^2\cos^2\alpha}$$
$$\implies \frac{g}{2}\,\frac{D^2}{V^2\cos^2\alpha} = H + D\tan\alpha$$
$$\implies \frac{2V^2\cos^2\alpha}{gD^2} = \frac{1}{H + D\tan\alpha}$$
$$\implies V^2\left(\frac{2\cos^2\alpha}{gD^2}\right) = \frac{1}{H + D\tan\alpha}$$
$$\implies V^2 = \frac{gD^2}{2\cos^2\alpha} \times \frac{1}{H + D\tan\alpha}$$

FINALLY SUBSTITUTE INTO $H + \dfrac{V^2\sin^2\alpha}{2g}$

$$= H + \frac{\cancel{g}D^2}{2\cos^2\alpha} \times \frac{1}{H + D\tan\alpha} \times \frac{\sin^2\alpha}{2\cancel{g}}$$
$$= H + \frac{D\sin^2\alpha}{4\cos^2\alpha\,(H + D\tan\alpha)}$$
$$= H + \frac{D\tan^2\alpha}{4(H + D\tan\alpha)}$$

Question 25 (*****)

A tennis player standing on a level horizontal court serves the ball from a height of 2.25 m above the court. The ball reaches a maximum height of 2.4 m above the court and first hits the court at a horizontal distance of 20 m from the point where the player served the ball. The ball rises for T_1 s and falls for T_2 s .

The ball is modelled as a particle moving through still air without any resistance.

a) Show clearly that

$$\frac{T_2}{T_1} = 4 .$$

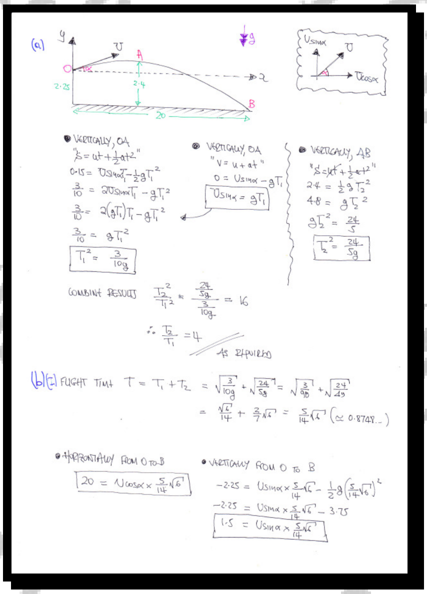
b) Determine the magnitude and direction of the velocity of the ball …

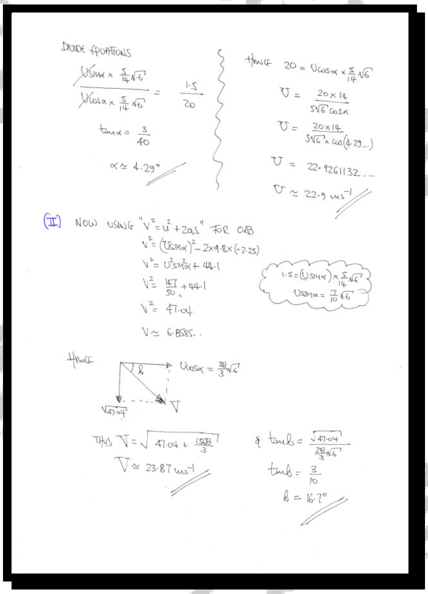
i. … when it was first served.

ii. … as it lands on the court.

$$\boxed{U \approx 22.926...\ \text{ms}^{-1},\ \tan\alpha = \frac{3}{40},\ \alpha \approx 4.289°...},$$

$$\boxed{V \approx 23.8685...\ \text{ms}^{-1},\ \tan\beta = \frac{3}{10},\ \beta \approx 16.699°...}$$





Question 26 (*****)

A particle P is projected from a fixed point O with speed v and at an angle of elevation $\theta°$.

It passes through a point Q which is at a horizontal distance a from O, and a vertical distance h below the level of O.

P is then projected from O with speed v at an angle of depression $(90-\theta)°$ and passes through Q again.

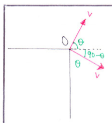
a) Show that

$$v^2 + ag\cot(2\theta) = 0$$

b) Deduce that

$$h + a\tan(2\theta) = 0.$$

proof



BOTH TRAJECTORIES PASS THROUGH THE POINT $(a,-h)$ RELATIVE TO THE "ORIGIN"

FOR PROJECTION AT ANGLE OF ELEVATION θ

$a = (v\cos\theta)t$

$-h = (v\sin\theta)t - \frac{1}{2}gt^2$

SIMILARLY FOR ANGLE OF DEPRESSION $(90-\theta)$

$a = (v\sin\theta)T$

$h = (v\cos\theta)T + \frac{1}{2}gT^2$

a) MANIPULATE EACH PAIR SEPARATELY BY ELIMINATING THE TIMES

$t = \frac{a}{v\cos\theta}$ $\qquad$ $T = \frac{a}{v\sin\theta}$

$-h = (v\sin\theta)\left(\frac{a}{v\cos\theta}\right) - \frac{1}{2}g\left(\frac{a^2}{v^2\cos^2\theta}\right)$ $\qquad$ $h = (v\cos\theta)\frac{a}{v\sin\theta} + \frac{1}{2}g\left(\frac{a^2}{v^2\sin^2\theta}\right)$

$-h = a\tan\theta - \frac{ga^2}{2v^2}\sec^2\theta$ $\qquad$ $h = a\cot\theta + \frac{ga^2}{2v^2}\operatorname{cosec}^2\theta$

$-h = a\tan\theta - \frac{ga^2}{2v^2}(1+\tan^2\theta)$ $\qquad$ $h = a\cot\theta + \frac{ga^2}{2v^2}(1+\cot^2\theta)$

ELIMINATE h

$\Rightarrow a\cot\theta + \frac{ga^2}{2v^2}(1+\cot^2\theta) = \frac{ga^2}{2v^2}(1+\tan^2\theta) - a\tan\theta$

$\Rightarrow 2av^2\cot\theta + ga^2(1+\cot^2\theta) = ga^2(1+\tan^2\theta) - 2av^2\tan\theta$

$\Rightarrow 2\cancel{a}v^2(\cot\theta + \tan\theta) + ga^{\cancel{2}}(1+\cot^2\theta - 1 - \tan^2\theta) = 0$

$\Rightarrow 2v^2(\cot\theta + \tan\theta) + ag(\cot^2\theta - \tan^2\theta) = 0$

$\Rightarrow 2v^2\cancel{(\cot\theta + \tan\theta)} + ag(\cot\theta - \tan\theta)\cancel{(\cot\theta + \tan\theta)} = 0$

$\Rightarrow 2v^2 + ag(\cot\theta - \tan\theta) = 0$

$\Rightarrow 2v^2 + ag\left(\frac{1}{\tan\theta} - \tan\theta\right) = 0$

$\Rightarrow 2v^2 + ag\left(\frac{1-\tan^2\theta}{\tan\theta}\right) = 0$

$\Rightarrow \cancel{2}v^2 + \cancel{2}ag\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) = 0$

$\Rightarrow v^2 + ag\left(\frac{1}{\tan 2\theta}\right) = 0$

$\Rightarrow v^2 + ag\cot 2\theta = 0$

b) NOW $\boxed{-h = a\tan\theta - \frac{ga^2}{2v^2}(1+\tan^2\theta)}$ FROM EARLIER

$-h = a\tan\theta - \frac{ga^2}{2(-ag\cot 2\theta)}(1+\tan^2\theta)$

$0 = h + a\tan\theta + \frac{ga^2(1+\tan^2\theta)}{2ag\cot 2\theta}$

$0 = h + a\left[\tan\theta + \frac{1}{2}(1+\tan^2\theta)\tan 2\theta\right]$

$0 = h + a\left[\tan\theta + \frac{1}{2}(1+\tan^2\theta)\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\right]$

$0 = h + a\left[\tan\theta + \frac{(1+\tan^2\theta)\tan\theta}{1-\tan^2\theta}\right]$

$0 = h + a\left[\frac{\tan\theta(1-\tan^2\theta) + (1+\tan^2\theta)\tan\theta}{1-\tan^2\theta}\right]$

$0 = h + a\left[\frac{\tan\theta - \tan^3\theta + \tan\theta + \tan^3\theta}{1-\tan^2\theta}\right]$

$0 = h + a\left[\frac{2\tan\theta}{1-\tan^2\theta}\right]$

$0 = h + a\tan 2\theta$

PROJECTILES

ON INCLINE PLANES

Question 1 (***)

The point A lies on a smooth plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{5}{12}$.

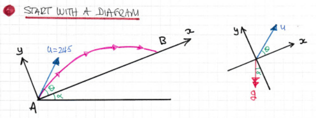
A particle is projected from A, up a line of greatest slope of the plane, with a speed of 24.5 ms^{-1} at an angle of elevation $\alpha + \theta$, where $\tan\theta = \frac{3}{4}$.

The particle is moving freely under gravity and first hits the plane at B.

Given that the coefficient of restitution between the plane and the particle is $\frac{1}{2}\sqrt{3}$, show that the particle first rebounds from B with a speed of 14.7 ms^{-1}.

proof

START WITH A DIAGRAM



$\tan\alpha = \frac{5}{12}$, $\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$, $\tan\theta = \frac{3}{4}$, $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$

DERIVE THE EQUATIONS OF MOTION IN THE ROTATED SET OF AXES (LOOKING AT DIAGRAMS)

- $\ddot{x} = -g\sin\alpha$ • $\ddot{y} = -g\cos\alpha$
- $\dot{x} = -gt\sin\alpha + u\cos\theta$ • $\dot{y} = -gt\cos\alpha + u\sin\theta$
- $x = ut\cos\theta - \frac{1}{2}gt^2\sin\alpha$ • $y = ut\sin\theta - \frac{1}{2}gt^2\cos\alpha$

FIND THE FLIGHT TIME FROM A TO B BY SOLVING $y = 0$

$\Rightarrow 0 = ut\sin\theta - \frac{1}{2}gt^2\cos\alpha$

$\Rightarrow \frac{1}{2}t\left[2u\sin\theta - gt\cos\alpha\right]$

$\Rightarrow t = \frac{2u\sin\theta}{g\cos\alpha} \quad (t \neq 0)$

$\Rightarrow t = \frac{2 \times 24.5 \times 3/5}{9.8 \times \frac{12}{13}}$

$\Rightarrow t = \frac{13}{4} = 3.25$

NEXT WE FIND THE COMPONENTS OF THE VELOCITY PARALLEL AND PERPENDICULAR TO THE PLANE AS THE PARTICLE HITS B

$\Rightarrow \dot{x} = u\cos\theta - gt\sin\alpha$

$\Rightarrow \dot{x} = 24.5 \times \frac{4}{5} - 9.8 \times \frac{13}{4} \times \frac{5}{13}$

$\Rightarrow \dot{x} = 7.35$

AND

$\Rightarrow \dot{y} = u\sin\theta - gt\cos\alpha$

$\Rightarrow \dot{y} = 24.5 \times \frac{3}{5} - 9.8 \times \frac{13}{4} \times \frac{12}{13}$

$\Rightarrow \dot{y} = -14.7$

$\therefore$ THE SPEEDS AFTER THE IMPACT WILL BE $\dot{x} = 7.35$ (UNCHANGED)

$e|\dot{y}| = \frac{\sqrt{3}}{2} \times 14.7 = 7.35\sqrt{3}$

FINALLY THE REBOUND SPEED WILL BE

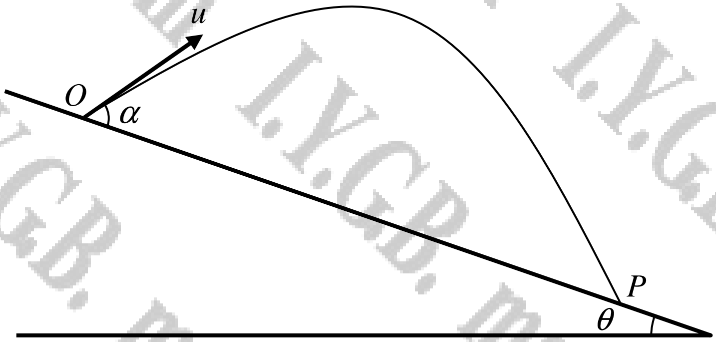
$\Rightarrow$ REBOUND SPEED $= \sqrt{(7.35)^2 + (7.35\sqrt{3})^2}$

$= 7.35\sqrt{1^2 + \sqrt{3}^2}$

$= 7.35 \times 2$

$= 14.7 \text{ ms}^{-1}$ // AS REQUIRED

Question 2 (*+)**



The figure above shows a particle projected from a point O on a plane inclined at an angle θ to the horizontal. The particle is projected down the plane with speed u, at an angle α to a line of greatest slope of the plane.

The particle lands for the first time at the point P, in time T.

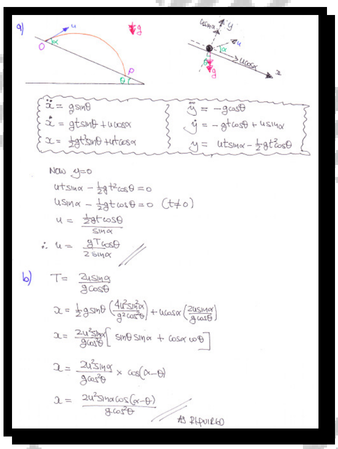
The gravitational acceleration g is assumed constant and air resistance is ignored.

a) Determine an expression connecting u, θ, α and T.

b) Hence show that

$$|OP| = \frac{2u^2}{g\cos^2\theta}\left[\sin\alpha\cos(\alpha-\theta)\right]$$

$$\boxed{2u\sin\alpha = gT\cos\theta}$$



Question 3 (***+)

The point O lies on a plane which is inclined at an angle of $30°$ to the horizontal.

A particle is projected from O, up a line of greatest slope of the plane, with a speed of U ms^{-1}, at an angle ψ **to the plane**.

The particle first strikes the plane at **right angles**, at the point A.

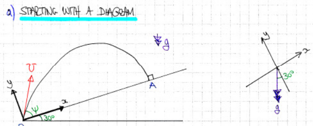
The gravitational acceleration g is assumed constant and air resistance is ignored.

a) Determine the value of $\tan\psi$.

b) Given further that $|OA| = 35$ m, determine the value of U.

$\boxed{\tan\psi = \frac{1}{2}\sqrt{3}}$, $\boxed{U = \frac{49}{2} = 24.5 \text{ ms}^{-1}}$

a) STARTING WITH A DIAGRAM



FORMING SOME EQUATIONS IN COMPONENTS — NOT ALL ARE NEEDED

$\ddot{x} = -g\sin 30 = -\frac{1}{2}g$ $\qquad$ $\ddot{y} = -g\cos 30 = -\frac{\sqrt{3}}{2}g$

$\dot{x} = U\cos\psi - \frac{1}{2}gt$ $\qquad$ $\dot{y} = U\sin\psi - \frac{\sqrt{3}}{2}gt$

$x = Ut\cos\psi - \frac{1}{4}gt^2$ $\qquad$ $y = Ut\sin\psi - \frac{\sqrt{3}}{4}gt^2$

NOW "AT RIGHT ANGLES" $\Rightarrow$ $\dot{x} = 0$ WHEN $y = 0$

$y = 0$

$Ut\sin\psi - \frac{\sqrt{3}}{4}gt^2 = 0$

$U\sin\psi - \frac{\sqrt{3}}{4}gt = 0 \quad (t \neq 0)$

$t = \frac{4U\sin\psi}{\sqrt{3}g}$ (FLIGHT TIME)

$\dot{x} = U\cos\psi - \frac{1}{2}gt$

$0 = U\cos\psi - \frac{1}{2}g\left(\frac{4U\sin\psi}{\sqrt{3}g}\right)$

$0 = U\cos\psi - \frac{2U\sin\psi}{\sqrt{3}}$

$0 = \cos\psi - \frac{2\sin\psi}{\sqrt{3}} \quad (U \neq 0)$

$2\sin\psi = \sqrt{3}\cos\psi$

$\tan\psi = \frac{\sqrt{3}}{2}$

$\left(\sin\psi = \sqrt{\frac{3}{7}}, \cos\psi = \frac{2}{\sqrt{7}}\right)$

$\psi \approx 40.9°$

b) NOW $|OA| = x = 35$ WHEN $t = \frac{4U\sin\psi}{\sqrt{3}g}$

$\Rightarrow x = Ut\cos\psi - \frac{1}{4}gt^2$

$\Rightarrow 35 = U\left(\frac{4U\sin\psi}{\sqrt{3}g}\right)\cos\psi - \frac{1}{4}g\left(\frac{4U\sin\psi}{\sqrt{3}g}\right)^2$

$\Rightarrow 35 = \frac{4U^2\sin\psi\cos\psi}{\sqrt{3}g} - \frac{4U^2\sin^2\psi}{3g}$

$\Rightarrow 35\sqrt{3}g = 4U^2\left(\sin\psi\cos\psi - \sin^2\psi \times \frac{\sqrt{3}}{3}\right)$

$\Rightarrow 35\sqrt{3}g = 4U^2\left[\sqrt{\frac{3}{7}} \times \frac{2}{\sqrt{7}} - \left(\sqrt{\frac{3}{7}}\right)^2 \times \frac{\sqrt{3}}{3}\right]$

$\Rightarrow 35\sqrt{3}g = 4U^2\left[\frac{2}{7}\sqrt{3} - \frac{1}{7}\sqrt{3}\right]$

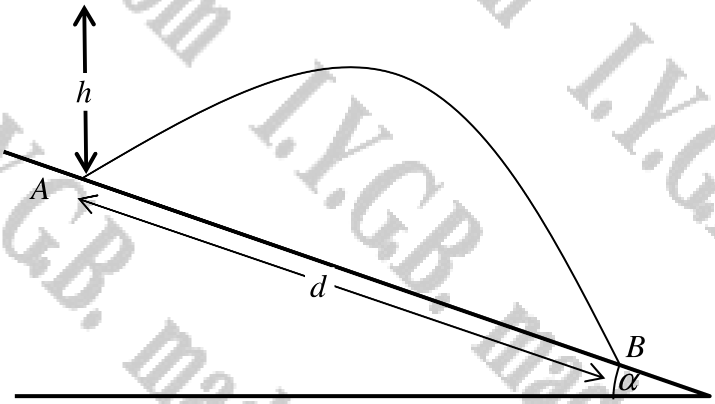
$\Rightarrow 35\cancel{\sqrt{3}}g = 4U^2 \times \frac{1}{7}\cancel{\sqrt{3}}$

$\Rightarrow 35g = \frac{4}{7}U^2$

$\Rightarrow U^2 = 600.25$

$\Rightarrow U = 24.5 \text{ ms}^{-1}$

Question 4 (****)



The figure above shows the path of a particle, released from rest, from a height h above a smooth plane, inclined at an angle α to the horizontal.

The particle strikes the plane at the point A, and rebounds striking the plane for the second time at the point B.

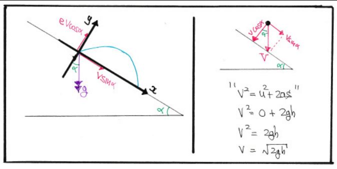
The coefficient of restitution between the plane and the particle is e.

The gravitational acceleration g is assumed constant and air resistance is ignored.

Given that $|AB| = d$, show that

$$d = 4eh(e+1)\sin\alpha .$$

$\boxed{}$, $\boxed{\text{proof}}$



"$V^2 = u^2 + 2as$"
$V^2 = 0 + 2gh$
$V^2 = 2gh$
$V = \sqrt{2gh}$

- START BY COMPUTING THE COMPONENTS OF THE VELOCITY ON IMPACT, PARALLEL & PERPENDICULAR TO THE PLANE (TOP RIGHT). — NO MOMENTUM IS EXCHANGED PARALLEL TO THE PLANE HOWEVER PERPENDICULAR TO THE PLANE THE "REBOUNDING" VELOCITY COMPONENT IS $eV\cos\alpha$ (TOP LEFT)

- DERIVE THE EQUATIONS OF MOTION IN THE CO-ORDINATE SYSTEM SHOWN IN THE ABOVE DIAGRAM

  - $\ddot{x} = g\sin\alpha$
  - $\dot{x} = gt\sin\alpha + V\sin\alpha$
  - $x = \frac{1}{2}gt^2\sin\alpha + Vt\sin\alpha$
  - $\ddot{y} = -g\cos\alpha$
  - $\dot{y} = -gt\cos\alpha + eV\cos\alpha$
  - $y = -\frac{1}{2}gt^2\cos\alpha + eVt\cos\alpha$

- DETERMINE THE FLIGHT TIME TO THE "SECOND" IMPACT, I.E $y=0$

$\Rightarrow 0 = -\frac{1}{2}gt^2\cos\alpha + eVt\cos\alpha$
$\Rightarrow 0 = \frac{1}{2}t\cos\alpha\left[2eV - gt\right]$
$\Rightarrow t = \frac{2eV}{g} \quad (t \neq 0)$

- SUBSTITUTING INTO THE "x-EQUATION" TO FIND THE REQUIRED DISTANCE d

$d = \frac{1}{2}g\left(\frac{2eV}{g}\right)^2\sin\alpha + V\left(\frac{2eV}{g}\right)\sin\alpha$
$d = \frac{2e^2V^2}{g}\sin\alpha + \frac{2eV^2}{g}\sin\alpha$
$d = \left(\frac{2eV^2}{g}\sin\alpha\right)(e+1)$
$d = \frac{2e}{\cancel{g}}(2\cancel{g}h)(\sin\alpha)(e+1)$
$d = 4eh(e+1)\sin\alpha$

$V = \sqrt{2gh}$
$V^2 = 2gh$

Question 5 (****)

The point O lies on a plane which is inclined at an angle θ to the horizontal.

A particle is projected from O, up a line of greatest slope of the plane, with speed of V at an angle of elevation α, along a line of greatest slope of the plane.

The gravitational acceleration g is assumed constant and air resistance is ignored.

The particle lands at a point P on the plane, at time T after projection.

a) Find an expression for T in terms of V, g, θ, and α, and hence show that

$$|OP| = \frac{2V^2 \sin(\alpha - \theta)\cos\alpha}{g\cos^2\theta}.$$

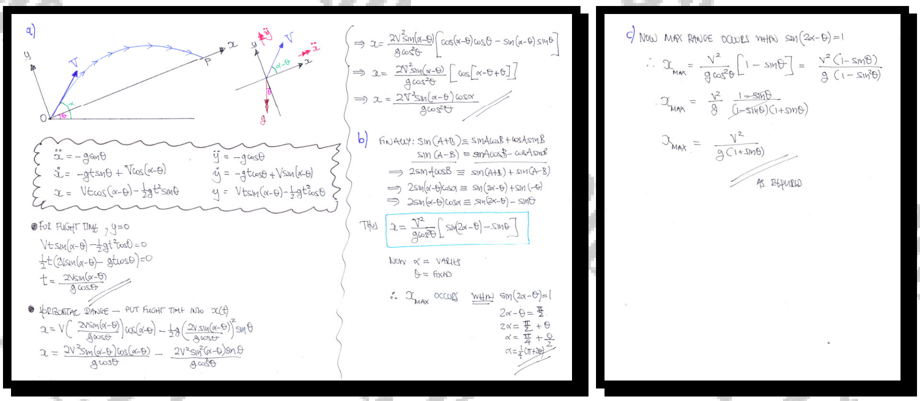
The value of α can vary so that $|OP|$ is greatest.

b) Express α in terms of θ when $|OP|$ is greatest.

c) Show further that greatest value of $|OP|$ is

$$\frac{V^2}{g(1+\sin\theta)}.$$

$$\boxed{T = \frac{2V\sin(\alpha-\theta)}{g\cos\theta}}, \quad \boxed{\alpha = \frac{1}{4}(2\theta + \pi)}$$



Question 6 (****)

The point O lies on a plane which is inclined at an angle α to the horizontal.

A particle is projected from O, up the line of greatest slope of the plane, with speed of V at an angle θ to the line of greatest slope of the plane.

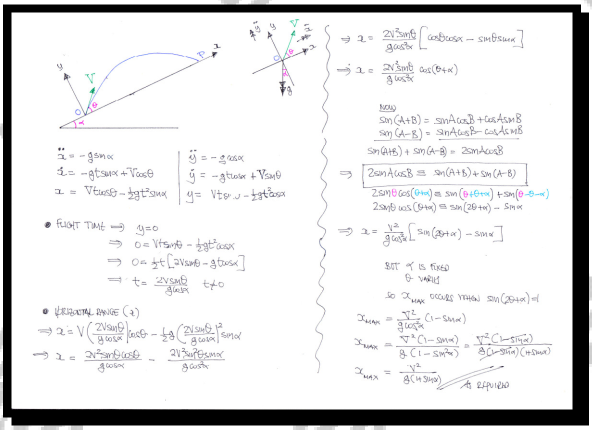
Show that the maximum range of the particle up the plane is

$$\frac{V^2}{g\left(1+\sin\alpha\right)},$$

where g is the gravitational acceleration, assumed constant.

Air resistance is ignored in this question.

proof



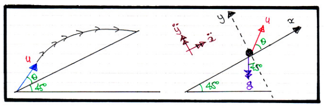
Question 7 (****)

The point O lies on the foot of a fixed plane which is inclined at an angle of 45° to the horizontal. A particle is projected from O, up the line of greatest slope of the plane, with speed of u.

The gravitational acceleration g is assumed constant and air resistance is ignored.

Given that the particle achieves the greatest range up the plane, determine the angle of projection.

M , $\boxed{22.5° \text{ to the plane or } 67.5° \text{ to the horizontal}}$



DETERMINE EQUATIONS FOR DISPLACEMENTS & VELOCITIES IN x & y BY SUCCESSIVE INTEGRATIONS

$\Rightarrow \ddot{x} = -g\sin 45°$
$\Rightarrow \ddot{x} = -g\frac{\sqrt{2}}{2}$
$\Rightarrow \dot{x} = -gt\frac{\sqrt{2}}{2} + u\cos\theta$
$\Rightarrow x = ut\cos\theta - \frac{1}{4}\sqrt{2}gt^2$
$[t=0, x=0, \dot{x} = u\cos\theta]$

$\Rightarrow \ddot{y} = -g\cos 45°$
$\Rightarrow \ddot{y} = -g\frac{\sqrt{2}}{2}$
$\Rightarrow \dot{y} = -gt\frac{\sqrt{2}}{2} + u\sin\theta$
$\Rightarrow y = ut\sin\theta - \frac{1}{4}\sqrt{2}gt^2$
$[t=0, y=0, \dot{y} = u\sin\theta]$

NEXT FIND THE FLIGHT TIME BY SOLVING $y=0$

$\Rightarrow ut\sin\theta - \frac{1}{4}\sqrt{2}gt^2 = 0$
$\Rightarrow \frac{1}{4}t\left[4u\sin\theta - \sqrt{2}gt\right] = 0$
$\Rightarrow t = \cancel{0}$ (INITIALLY) or $\frac{4u\sin\theta}{\sqrt{2}g} = \frac{2\sqrt{2}u\sin\theta}{g}$

NEXT WE FIND THE RANGE UP THE PLANE (x), USING THE FLIGHT TIME

$\Rightarrow x = u\left(\frac{2\sqrt{2}u\sin\theta}{g}\right)\cos\theta - \frac{1}{4}\sqrt{2}g\left(\frac{2\sqrt{2}u\sin\theta}{g}\right)^2$
$\Rightarrow x = \frac{2\sqrt{2}u^2\sin\theta\cos\theta}{g} - \frac{2\sqrt{2}u^2\sin^2\theta}{g}$

$\Rightarrow x = \frac{2\sqrt{2}u^2}{g}\left[\sin\theta\cos\theta - \sin^2\theta\right]$
$\Rightarrow x = \frac{2\sqrt{2}u^2}{g}\left[\frac{1}{2}\sin 2\theta - \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)\right]$
$\Rightarrow x = \frac{2\sqrt{2}u^2}{g}\left[\frac{1}{2}\sin 2\theta + \frac{1}{2}\cos 2\theta - \frac{1}{2}\right]$
$\Rightarrow x = \frac{\sqrt{2}u^2}{g}\left[\sin 2\theta + \cos 2\theta - 1\right]$

MANIPULATE THE TRIGONOMETRIC EXPRESSION DIRECTLY OR BY THE "R-TRANSFORMATION" METHOD

$\Rightarrow x = \frac{\sqrt{2}u^2}{g} \times \sqrt{2} \times \left[\frac{1}{\sqrt{2}}\sin 2\theta + \frac{1}{\sqrt{2}}\cos 2\theta - \frac{1}{\sqrt{2}}\right]$
$\Rightarrow x = \frac{2u^2}{g}\left[\cos 45\sin 2\theta + \sin 45\cos 2\theta - \frac{1}{\sqrt{2}}\right]$
$\Rightarrow x = \frac{2u^2}{g}\left[\sin(2\theta + 45) - \frac{1}{\sqrt{2}}\right]$

TO MAXIMIZE x, WE REQUIRE

$\Rightarrow \sin(2\theta + 45) = 1$
$\Rightarrow 2\theta + 45 = 90$
$\Rightarrow 2\theta = 45$
$\Rightarrow \theta = 22.5$

$\therefore$ PROJECTION ANGLE IS 22.5° TO THE PLANE OR 67.5° TO THE HORIZONTAL

Question 8 **(****)**

The point O lies on a plane which is inclined at an angle of $30°$ to the horizontal.

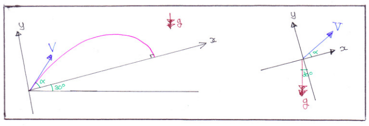
A particle is projected from O, up a line of greatest slope of the plane, with speed of V ms^{-1} at an angle of elevation $(30+\alpha)°$.

The gravitational acceleration g is assumed constant and air resistance is ignored.

The particle first hits the plane at right angles at a point P, 16 s after projection.

Determine the exact value of $\tan\alpha$ and the distance OP.

$\boxed{\tan\alpha = \frac{1}{2}\sqrt{3}}$, $\boxed{|OP| = 64g = 627.2}$



WORKING IN AN OBLIQUE CARTESIAN SYSTEM, WITH x-AXIS PARALLEL TO THE PLANE DERIVE KINEMATIC EQUATIONS IN COMPONENT FORM

$\Rightarrow \ddot{x} = -g\sin 30°$
$\Rightarrow \ddot{x} = -\frac{1}{2}g$
$\Rightarrow \dot{x} = -\frac{1}{2}gt + V\cos\alpha$
$\Rightarrow x = Vt\cos\alpha - \frac{1}{4}gt^2$

$\Rightarrow \ddot{y} = -g\cos 30°$
$\Rightarrow \ddot{y} = -\frac{\sqrt{3}}{2}g$
$\Rightarrow \dot{y} = -\frac{\sqrt{3}}{2}gt + V\sin\alpha$
$\Rightarrow y = Vt\sin\alpha - \frac{\sqrt{3}}{4}gt^2$

NEXT WE WORK THE FLIGHT TIME BY SETTING $y=0$

$\Rightarrow y = 0$
$\Rightarrow 0 = t\left(V\sin\alpha - \frac{\sqrt{3}}{4}gt\right)$
$\Rightarrow t = \frac{4V\sin\alpha}{\sqrt{3}g} \quad (t \neq 0)$

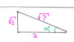
NOW AT THAT INSTANT $\dot{x} = 0$, SO PARTICLE "DROPS" AT RIGHT ANGLES

$\Rightarrow \dot{x} = -\frac{1}{2}gt + V\cos\alpha$
$\Rightarrow 0 = -\frac{1}{2}g\left(\frac{4V\sin\alpha}{\sqrt{3}g}\right) + V\cos\alpha$
$\Rightarrow 0 = -\frac{2\sin\alpha}{\sqrt{3}} + \cos\alpha$
$\Rightarrow 0 = -2\sin\alpha + \sqrt{3}\cos\alpha$
$\Rightarrow 0 = -2\tan\alpha + \sqrt{3}$
$\Rightarrow \tan\alpha = \frac{\sqrt{3}}{2}$

NEXT LOOKING AT

$x = -\frac{1}{4}gt^2 + Vt\cos\alpha \quad \& \quad t = \frac{4V\sin\alpha}{\sqrt{3}g}$

$\tan\alpha = \frac{\sqrt{3}}{2}$
$\sin\alpha = \sqrt{\frac{3}{7}}$
$\cos\alpha = \frac{2}{\sqrt{7}}$



FIRSTLY

$16 = \frac{4V\sqrt{\frac{3}{7}}}{\sqrt{3}g}$
$16\sqrt{3}g = \frac{4V\sqrt{3}}{\sqrt{7}}$
$4V = 16g\sqrt{7}$
$V = g\sqrt{7}$

FINALLY

$x = -\frac{1}{4}g(16)^2 + (4g\sqrt{7}) \times 16 \times \frac{2}{\sqrt{7}}$
$x = -64g + 128g$
$x = 64g$
$x = 627.2$

Question 9 (****)

The point O lies on a plane which is inclined at an angle of $\frac{1}{6}\pi$ to the horizontal.

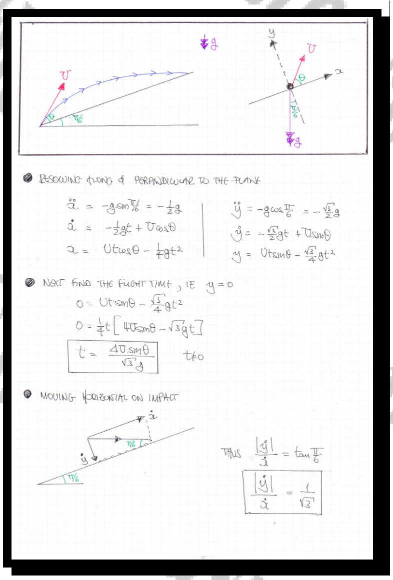
A particle is projected from O, up a line of greatest slope of the plane, with speed of U ms^{-1} at an angle of elevation $\theta + \frac{1}{6}\pi$.

The gravitational acceleration g is assumed constant and air resistance is ignored.

The particle first hits the plane when it is moving horizontally.

Determine the exact value of $\tan\theta$.

$\boxed{\tan\theta = \frac{1}{5}\sqrt{3}}$



- RESOLVING ALONG & PERPENDICULAR TO THE PLANE

$$\ddot{x} = -g\sin\tfrac{\pi}{6} = -\tfrac{1}{2}g \qquad \ddot{y} = -g\cos\tfrac{\pi}{6} = -\tfrac{\sqrt{3}}{2}g$$
$$\dot{x} = -\tfrac{1}{2}gt + U\cos\theta \qquad \dot{y} = -\tfrac{\sqrt{3}}{2}gt + U\sin\theta$$
$$x = Ut\cos\theta - \tfrac{1}{4}gt^2 \qquad y = Ut\sin\theta - \tfrac{\sqrt{3}}{4}gt^2$$

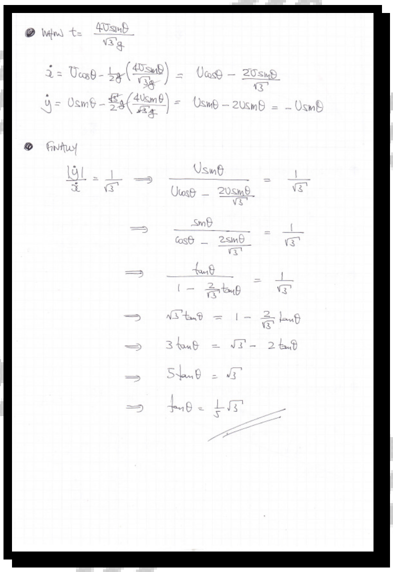
- NEXT FIND THE FLIGHT TIME, IE $y=0$

$$0 = Ut\sin\theta - \tfrac{\sqrt{3}}{4}gt^2$$
$$0 = \tfrac{1}{4}t\left[4U\sin\theta - \sqrt{3}gt\right]$$
$$\boxed{t = \frac{4U\sin\theta}{\sqrt{3}g}} \qquad t \neq 0$$

- MOVING HORIZONTAL ON IMPACT

THUS $\frac{|\dot{y}|}{\dot{x}} = \tan\frac{\pi}{6}$

$$\boxed{\frac{|\dot{y}|}{\dot{x}} = \frac{1}{\sqrt{3}}}$$



- WHEN $t = \frac{4U\sin\theta}{\sqrt{3}g}$

$$\dot{x} = U\cos\theta - \tfrac{1}{2}g\left(\frac{4U\sin\theta}{\sqrt{3}g}\right) = U\cos\theta - \frac{2U\sin\theta}{\sqrt{3}}$$
$$\dot{y} = U\sin\theta - \tfrac{\sqrt{3}}{2}g\left(\frac{4U\sin\theta}{\sqrt{3}g}\right) = U\sin\theta - 2U\sin\theta = -U\sin\theta$$

- FINALLY

$$\frac{|\dot{y}|}{\dot{x}} = \frac{1}{\sqrt{3}} \Longrightarrow \frac{U\sin\theta}{U\cos\theta - \frac{2U\sin\theta}{\sqrt{3}}} = \frac{1}{\sqrt{3}}$$
$$\Longrightarrow \frac{\sin\theta}{\cos\theta - \frac{2\sin\theta}{\sqrt{3}}} = \frac{1}{\sqrt{3}}$$
$$\Longrightarrow \frac{\tan\theta}{1 - \frac{2}{\sqrt{3}}\tan\theta} = \frac{1}{\sqrt{3}}$$
$$\Longrightarrow \sqrt{3}\tan\theta = 1 - \frac{2}{\sqrt{3}}\tan\theta$$
$$\Longrightarrow 3\tan\theta = \sqrt{3} - 2\tan\theta$$
$$\Longrightarrow 5\tan\theta = \sqrt{3}$$
$$\Longrightarrow \tan\theta = \tfrac{1}{5}\sqrt{3}$$

Question 10 (****)

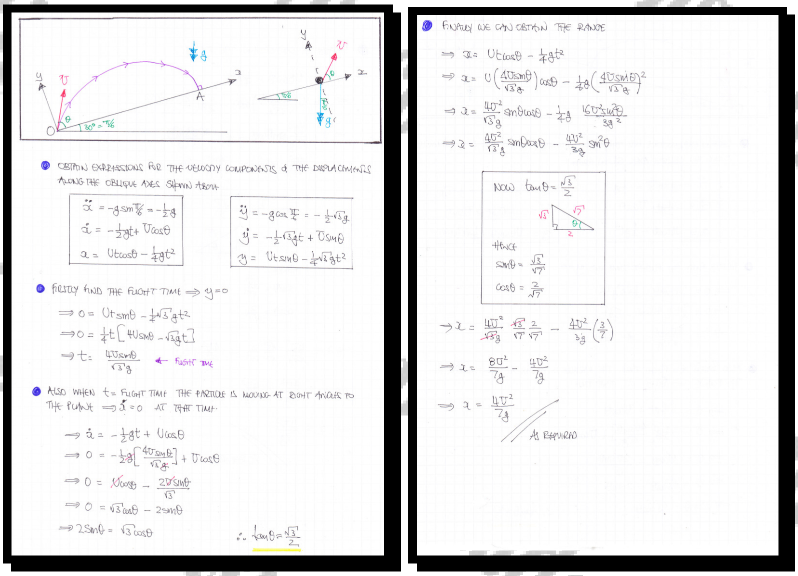
A particle is projected with speed U, from a point O on a plane which is inclined at an angle $\frac{1}{6}\pi$ to the horizontal.

The particle is projected up the plane at an angle θ to the plane and moves in a vertical plane which contains a line of greatest slope to the plane. When the particle first strikes the plane at the point A, it is moving at right angles to the plane.

The gravitational acceleration g is assumed constant and air resistance is ignored.

Show that $|OA| = \dfrac{4U^2}{7g}$.

$\boxed{\text{proof}}$



Question 11 (****)

A particle is projected from a point O on a smooth plane inclined at an angle α to the horizontal.

The particle is projected up the plane with speed u, at an angle β to the plane, and moves in a vertical plane which contains a line of greatest slope of the plane.

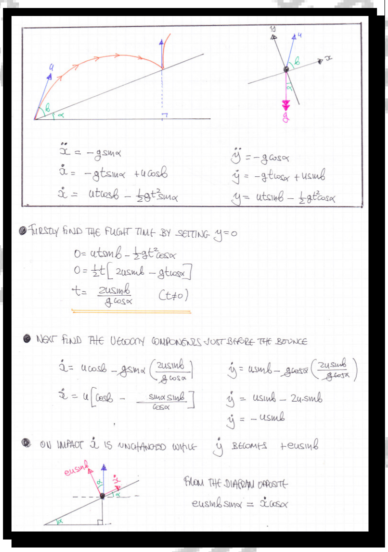
The particle first hits the plane at the point A and rebounds in a vertical direction.

The gravitational acceleration g is assumed constant and air resistance is ignored.

If the coefficient of restitution between the particle and the plane is e, show that

$$\cot\alpha\cot\beta = e + 2.$$

proof



$\ddot{x} = -g\sin\alpha$ $\qquad$ $\ddot{y} = -g\cos\alpha$

$\dot{x} = -gt\sin\alpha + u\cos\beta$ $\qquad$ $\dot{y} = -gt\cos\alpha + u\sin\beta$

$x = ut\cos\beta - \frac{1}{2}gt^2\sin\alpha$ $\qquad$ $y = ut\sin\beta - \frac{1}{2}gt^2\cos\alpha$

- FIRSTLY FIND THE FLIGHT TIME BY SETTING $y=0$

$0 = ut\sin\beta - \frac{1}{2}gt^2\cos\alpha$

$0 = \frac{1}{2}t\left[2u\sin\beta - gt\cos\alpha\right]$

$t = \frac{2u\sin\beta}{g\cos\alpha}$ $\quad (t \neq 0)$

- NEXT FIND THE VELOCITY COMPONENTS JUST BEFORE THE BOUNCE

$\dot{x} = u\cos\beta - g\sin\alpha\left(\frac{2u\sin\beta}{g\cos\alpha}\right)$ $\qquad$ $\dot{y} = u\sin\beta - \cancel{g\cos\alpha}\left(\frac{2u\sin\beta}{\cancel{g\cos\alpha}}\right)$

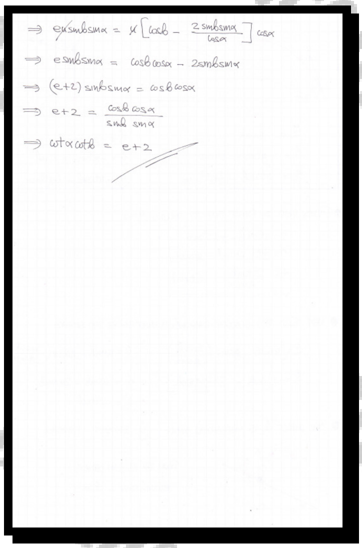
$\dot{x} = u\left[\cos\beta - \frac{\sin\alpha\sin\beta}{\cos\alpha}\right]$ $\qquad$ $\dot{y} = u\sin\beta - 2u\sin\beta$

$\dot{y} = -u\sin\beta$

- ON IMPACT $\dot{x}$ IS UNCHANGED WHILE $\dot{y}$ BECOMES $+eu\sin\beta$

FROM THE DIAGRAM OPPOSITE

$eu\sin\beta\sin\alpha = \dot{x}\cos\alpha$



$\Rightarrow e\cancel{u}\sin\beta\sin\alpha = \cancel{u}\left[\cos\beta - \frac{2\sin\beta\sin\alpha}{\cos\alpha}\right]\cos\alpha$

$\Rightarrow e\sin\beta\sin\alpha = \cos\beta\cos\alpha - 2\sin\beta\sin\alpha$

$\Rightarrow (e+2)\sin\beta\sin\alpha = \cos\beta\cos\alpha$

$\Rightarrow e+2 = \frac{\cos\beta\cos\alpha}{\sin\beta\sin\alpha}$

$\Rightarrow \cot\alpha\cot\beta = e+2$

Question 12 (****+)

The point O lies on a plane which is inclined at an angle of $15°$ to the horizontal.

A particle is projected from O, up a line of greatest slope of the plane, with speed of 30 ms^{-1} at an angle of $75°$ to the horizontal.

The particle first strikes the plane at the point A.

When the particle strikes the plane it rebounds and strikes the plane again at the point B, where B is further up the plane than A.

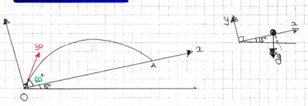
The gravitational acceleration g is assumed constant and air resistance is ignored.

The coefficient of restitution between the particle and the plane is e.

Given further that $\tan 15° = 2 - \sqrt{3}$, show that $e < -1 + \frac{2}{3}\sqrt{3}$.

$\boxed{}$, $\boxed{\text{proof}}$

START WITH A DIAGRAM



DETERMINE THE EQUATIONS OF MOTION IN COMPONENTS

$\Rightarrow \ddot{y} = -g\cos 15$ | $\Rightarrow \ddot{x} = -g\sin 15$

$\Rightarrow \dot{y} = 30\sin 60° - gt\cos 15°$ | $\Rightarrow \dot{x} = 30\cos 60° - gt\sin 15°$

$\Rightarrow y = 30t\sin 60° - \frac{1}{2}gt^2\cos 15$ | $\Rightarrow x = 30t\cos 60° - \frac{1}{2}gt^2\sin 15°$

NEXT OBTAIN THE FLIGHT TIME FROM $y=0$

$\Rightarrow 0 = 30t\sin 60° - \frac{1}{2}gt^2\cos 15°$

$\Rightarrow 30\sin 60° - \frac{1}{2}gt\cos 15° \quad (t \neq 0)$

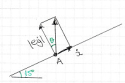
$\Rightarrow t = \frac{60\sin 60°}{g\cos 15°}$

NEXT WE NEED THE VELOCITY COMPONENTS

$\dot{y} = 30\sin 60° - g\left(\frac{60\sin 60°}{g\cos 15°}\right)\cos 15° = -30\sin 60°$

$\dot{x} = 30\cos 60° - g\left(\frac{60\sin 60°}{g\cos 15°}\right)\sin 15° = 30\cos 60° - 60\sin 60°\tan 15$

NEXT THE COLLISION AT A — REBOUNDS WITH SPEED $|e\dot{y}|$



DRAWN AT "CRITICAL" VELOCITY SCENARIO

IN ORDER TO LAND FURTHER UP $\tan\theta > \tan 15$.

$\Rightarrow \tan\theta > \tan 15°$

$\Rightarrow \frac{\dot{x}}{|e\dot{y}|} > \tan 15$

$\Rightarrow \frac{30\cos 60 - 60\sin 60°\tan 15°}{30e\sin 60°} > \tan 15$

$\Rightarrow \frac{30\cos 60 - 60\sin 60\tan 15}{30\sin 60\tan 15} > e$ $e > 0$, $\tan 15 > 0$

$\Rightarrow e < \cot 60\cot 15 - 2$

$\Rightarrow e < \frac{1}{\tan 60\tan 15} - 2$

$\Rightarrow e < \frac{1}{\sqrt{3}(2-\sqrt{3})} - 2$

$\Rightarrow e < \frac{1}{-3+2\sqrt{3}} - 2 = \frac{-3-2\sqrt{3}}{9-12} - 2$

$\Rightarrow e < 1 + \frac{2}{3}\sqrt{3} - 2$

$\Rightarrow e < -1 + \frac{2}{3}\sqrt{3}$ // $e < 0.155$

Question 13 (****+)

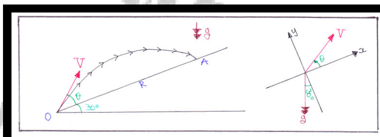
The point O lies on a plane which is inclined at an angle of $30°$ to the horizontal.

A particle is projected from O, up a line of greatest slope of the plane, with speed of V ms^{-1} at an angle θ to the plane.

The gravitational acceleration g is assumed constant and air resistance is ignored.

Show that as θ varies the greatest range of the particle up the plane is achieved when the direction of V bisects the angle between the plane and the upward vertical.

proof



DERIVE THE EQUATIONS OF MOTION IN OBLIQUE COMPONENTS AS SHOWN

- $\ddot{x} = -g\sin 30° = -\frac{1}{2}g$
- $\dot{x} = -\frac{1}{2}gt + V\cos\theta$
- $x = -\frac{1}{4}gt^2 + Vt\cos\theta$
- $\ddot{y} = -g\cos 30 = -\frac{\sqrt{3}}{2}g$
- $\dot{y} = -\frac{\sqrt{3}}{2}gt + V\sin\theta$
- $y = -\frac{\sqrt{3}}{4}gt^2 + Vt\sin\theta$

NEXT FIND THE FLIGHT TIME BY SETTING $y=0$

$$\Rightarrow -\frac{\sqrt{3}}{4}gt^2 + Vt\sin\theta = 0$$
$$\Rightarrow -\frac{1}{4}t\left[g\sqrt{3}t - 4V\sin\theta\right] = 0$$
$$\Rightarrow \boxed{t = \frac{4V\sin\theta}{g\sqrt{3}}} \quad t \neq 0$$

THE RANGE UP THE PLANE IS GIVEN BY

$$\Rightarrow x = -\frac{1}{4}g\left[\frac{4V\sin\theta}{g\sqrt{3}}\right]^2 + V\left[\frac{4V\sin\theta}{g\sqrt{3}}\right]\cos\theta$$
$$\Rightarrow x = -\frac{1}{4}g\left[\frac{16V^2\sin^2\theta}{g^2\times 3} + \frac{4V^2\cos\theta\sin\theta}{\sqrt{3}g}\right]$$
$$\Rightarrow x = \frac{4V^2\cos\theta\sin\theta}{\sqrt{3}g} - \frac{4V^2\sin^2\theta}{3g}$$
$$\Rightarrow x = \frac{4V^2\sin\theta}{g}\left[\frac{\cos\theta}{\sqrt{3}} - \frac{\sin\theta}{3}\right]$$
$$\Rightarrow x = \frac{4V^2\sin\theta}{g}\left[\frac{\sqrt{3}}{3}\cos\theta - \frac{1}{3}\sin\theta\right]$$
$$\Rightarrow x = \frac{4V^2\sin\theta}{3g}\left[\sqrt{3}\cos\theta - \sin\theta\right]$$
$$\Rightarrow x = \frac{8V^2\sin\theta}{3g}\left[\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right]$$
$$\Rightarrow x = \frac{8V^2\sin\theta}{3g}\left[\cos\theta\cos 30° - \sin\theta\sin 30°\right]$$
$$\Rightarrow x = \frac{8V^2\sin\theta}{3g}\cos(\theta+30)$$

NOW MANIPULATE THE TRIGONOMETRIC FUNCTIONS

$$\sin\left[(\theta+30)+\theta\right] = \sin(\theta+30)\cos\theta + \cos(\theta+30)\sin\theta$$
$$\sin\left[(\theta+30)-\theta\right] = \sin(\theta+30)\cos\theta - \cos(\theta+30)\sin\theta$$

$$\Rightarrow \sin(2\theta+30) - \sin 30 = 2\sin\theta\cos(\theta+30)$$
$$\Rightarrow \frac{1}{2}\sin(2\theta+30) - \frac{1}{4} = \sin\theta\cos(\theta+30)$$

RETURNING TO THE EXPRESSION IN x

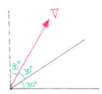
$$\Rightarrow x = \frac{8V^2}{3g}\left[\frac{1}{2}\sin(2\theta+30) - \frac{1}{4}\right]$$
$$\Rightarrow x = \frac{2V^2}{3g}\left[\sin(2\theta+30) - 1\right]$$

TO MAXIMIZE x, $\sin(2\theta+30) = 1$

$2\theta + 30 = 90$

$\theta = 60°$

AS REQUIRED



Question 14 (****+)

The point O lies on a plane which is inclined at an angle θ to the horizontal.

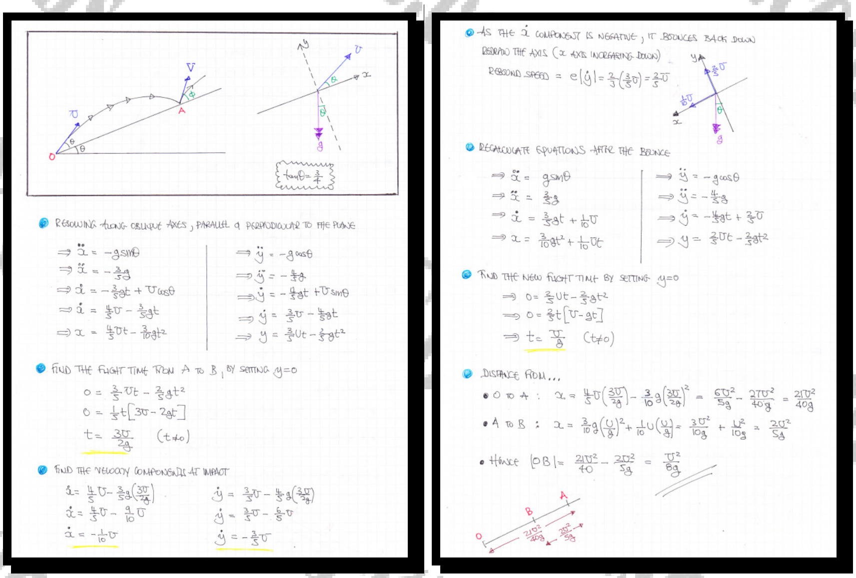
A particle is projected from O, up a line of greatest slope of the plane, with speed of U ms^{-1} at an angle θ to the plane. When the particle hits the plane it rebounds with speed V ms^{-1}. After rebounding the particle first hits the plane at the point B.

The coefficient of restitution between the particle and the plane is $\frac{2}{3}$.

The gravitational acceleration g is assumed constant and air resistance is ignored.

Show that $|OB| = \dfrac{U^2}{8g}$.

proof



Question 15 (****+)

The point O lies on a plane which is inclined at an angle of $45°$ to the horizontal.

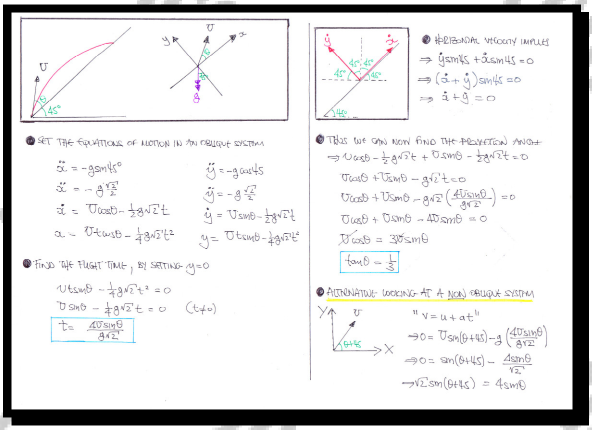
A particle is projected from O, up a line of greatest slope of the plane, with speed of $U \text{ ms}^{-1}$ at an angle θ to the plane. When the particle hits the plane it rebounds with speed $V \text{ ms}^{-1}$.

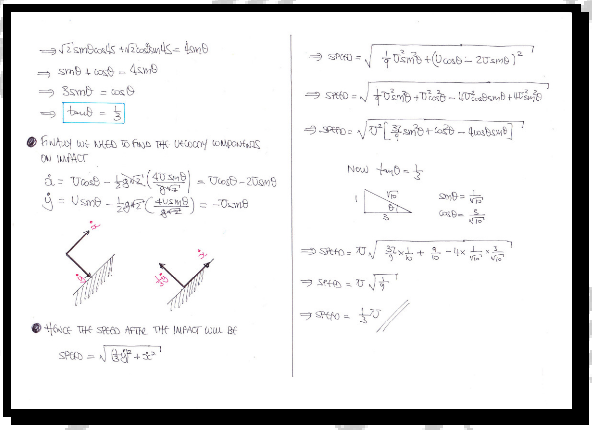
The coefficient of restitution between the article and the plane is $\frac{1}{3}$.

Given further that at the instant when the particle first hits the plane it travels in a horizontal direction show that $V = \frac{1}{3}U$.

The gravitational acceleration g is assumed constant and air resistance is ignored.

proof





Question 16 (*****)

A particle is projected from a point O on a smooth plane inclined at an angle α to the horizontal.

The particle is projected up the plane with speed U, at an angle θ to the plane, and moves in a vertical plane which contains a line of greatest slope of the plane.

The particle first hits the plane at the point A, rebounds and next hits the plane at O.

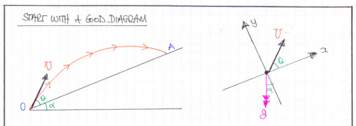
The gravitational acceleration g is assumed constant and air resistance is ignored.

If the coefficient of restitution between the particle and the plane is e, show that

$$\cot\theta = (1+e)\tan\alpha .$$

proof

START WITH A GOOD DIAGRAM



- $\ddot{x} = -g\sin\alpha$
- $\dot{x} = -gt\sin\alpha + U\cos\theta$
- $x = Ut\cos\theta - \frac{1}{2}gt^2\sin\alpha$
- $\ddot{y} = -g\cos\alpha$
- $\dot{y} = -gt\cos\alpha + U\sin\theta$
- $y = Ut\sin\theta - \frac{1}{2}gt^2\cos\alpha$

- FIRSTLY FIND THE FLIGHT TIME

$$y=0 \implies 0 = Ut\sin\theta - \tfrac{1}{2}gt^2\cos\alpha$$
$$0 = \tfrac{1}{2}t\left[2U\sin\theta - gt\cos\alpha\right]$$

$\therefore$ FLIGHT TIME IS $\dfrac{2U\sin\theta}{g\cos\alpha}$

- NEXT FIND THE RANGE UP THE PLANE

$$\implies x = U\left(\frac{2U\sin\theta}{g\cos\alpha}\right)\cos\theta - \frac{1}{2}g\left(\frac{2U\sin\theta}{g\cos\alpha}\right)^2\sin\alpha$$
$$\implies x = \frac{2U^2\sin\theta\cos\theta}{g\cos\alpha} - \frac{2U^2\sin^2\theta}{g\cos^2\alpha}\sin\alpha$$
$$\implies x = \frac{2U^2\sin\theta}{g\cos^2\alpha}\left[\cos\theta\cos\alpha - \sin\theta\sin\alpha\right] \leftarrow \text{COMPOUND}$$
$$\implies x = \frac{2U^2\sin\theta\cos(\theta+\alpha)}{g\cos^2\alpha}$$

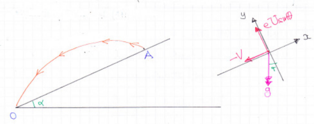
- NEXT FIND THE VELOCITY COMPONENTS AS IT REACHES A

$$\dot{x} = U\cos\theta - g\left(\frac{2U\sin\theta}{g\cos\alpha}\right)\sin\alpha$$
$$\dot{x} = U\cos\theta - 2U\sin\theta\tan\alpha \leftarrow V$$

&

$$\dot{y} = U\sin\theta - g\left(\frac{2U\sin\theta}{g\cos\alpha}\right)\cos\alpha$$
$$\dot{y} = U\sin\theta - 2U\sin\theta$$
$$\dot{y} = -U\sin\theta \leftarrow \text{SAME MAGNITUDE, OPPOSITE DIRECTION TO THAT ON PROJECTION (ENERGY CONSERVATION)}$$

- NEXT WE CONSIDER THE NEXT PART OF THE JOURNEY



- $\ddot{x} = -g\sin\alpha$
- $\dot{x} = -V - gt\sin\alpha$
- $x = -Vt - \frac{1}{2}gt^2\sin\alpha$
- $\ddot{y} = -g\cos\alpha$
- $\dot{y} = eU\sin\theta - gt\cos\alpha$
- $y = eUt\sin\theta - \frac{1}{2}gt^2\cos\alpha$

- AGAIN FIND THE FLIGHT TIME TO THE NEXT BOUNCE

$$y=0 \implies 0 = eUt\sin\theta - \tfrac{1}{2}gt^2\cos\alpha$$
$$0 = \tfrac{1}{2}t\left[2eU\sin\theta - gt\cos\alpha\right]$$

$\therefore$ FLIGHT TIME IS $t = \dfrac{2eU\sin\theta}{g\cos\alpha}$

- IN THAT TIME THE PARTICLE COVERS THE DISTANCE OA AGAIN, WITH ITS NEW $\dot{x}$ VELOCITY

$$|OA| = \frac{2U^2\sin\theta\cos(\theta-\alpha)}{g\cos^2\alpha}$$

"NEW x" $= -V\left(\dfrac{2eU\sin\theta}{g\cos\alpha}\right) - \dfrac{1}{2}g\left(\dfrac{2eU\sin\theta}{g\cos\alpha}\right)^2\sin\alpha$

- FINALLY WE HAVE

$$-\frac{2U^2\sin\theta\cos(\theta-\alpha)}{g\cos^2\alpha} = -\left[2U\sin\theta\tan\alpha - U\cos\theta\right]\left[\frac{2eU\sin\theta}{g\cos\alpha}\right] - \frac{2e^2U^2\sin^2\theta}{g\cos^2\alpha}\sin\alpha$$

- SIMPLIFYING $\left(\div -\dfrac{2U^2\sin\theta}{g\cos\alpha}\right)$

$$\implies \frac{\cos(\theta+\alpha)}{\cos\alpha} = 2e\sin\theta\tan\alpha - e\cos\theta + \frac{e^2\sin\theta}{\cos\alpha}\sin\alpha$$
$$\implies \frac{\cos\theta\cos\alpha - \sin\theta\sin\alpha}{\cos\alpha} = 2e\sin\theta\tan\alpha - e\cos\theta + e^2\sin\theta\tan\alpha$$
$$\implies \cos\theta - \sin\theta\tan\alpha = 2e\sin\theta\tan\alpha - e\cos\theta + e^2\sin\theta\tan\alpha$$
$$\implies e\cos\theta + \cos\theta = e^2\sin\theta\tan\alpha + 2e\sin\theta\tan\alpha + \sin\theta\tan\alpha$$
$$\implies \cos\theta(e+1) = \sin\theta\tan\alpha\,(e^2+2e+1)$$
$$\implies \cot\theta\,(e+1) = \tan\alpha\,(e+1)^2$$
$$\implies \cot\theta = (1+e)\tan\alpha$$

AS REQUIRED

Question 17 (*****)

The point O lies on a plane inclined at an angle θ to the horizontal.

A particle is projected from O, with speed u up and in a direction up the plane, at an angle α to the horizontal.

The particle first strikes the incline plane at the point A.

The motion of the particle takes place in a vertical plane which contains a line of greatest slope of the incline plane.

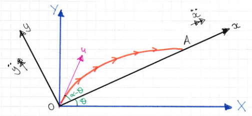
The gravitational acceleration g is assumed constant and air resistance is ignored.

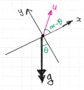
Given that the particle is travelling horizontally as it strikes A, show that

$$\tan\alpha = 2\tan\theta .$$

$\boxed{}$, $\boxed{\text{proof}}$

- STARTING WITH THE USUAL DIAGRAM





- DERIVE EQUATIONS OF MOTION IN THE "ROTATED" AXES

• $\ddot{x} = -g\sin\theta$	• $\ddot{y} = -g\cos\theta$
• $\dot{x} = -gt\sin\theta + u\cos(\alpha-\theta)$	• $\dot{y} = -gt\cos\theta + u\sin(\alpha-\theta)$
• $x = ut\cos(\alpha-\theta) - \frac{1}{2}gt^2\sin\theta$	• $y = ut\sin(\alpha-\theta) - \frac{1}{2}gt^2\cos\theta$

- NEXT FIND THE FLIGHT TIME BY SETTING $y=0$

$$\Rightarrow 0 = ut\sin(\alpha-\theta) - \tfrac{1}{2}gt^2\cos\theta$$

$$\Rightarrow 0 = \tfrac{1}{2}t\left[2u\sin(\alpha-\theta) - gt\cos\theta\right]$$

$$\Rightarrow \boxed{t = \frac{2u\sin(\alpha-\theta)}{g\cos\theta}} \quad (t \neq 0)$$

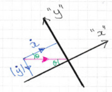
- NOW WE CAN PROCEED IN 2 DISTINCT WAYS

- METHOD A – USING THE EQUATIONS DERIVED

AS THE PARTICLE LANDS

$$\frac{|\dot{y}|}{\dot{x}} = \tan\theta$$

$$|\dot{y}| = \dot{x}\tan\theta$$



- HENCE WE NEED EXPRESSIONS FOR $\dot{x}$ & $\dot{y}$ WHEN $t = \frac{2u\sin(\alpha-\theta)}{g\cos\theta}$

$$\Rightarrow \dot{y} = u\sin(\alpha-\theta) - \cancel{g}\left(\frac{2u\sin(\alpha-\theta)}{\cancel{g}\cancel{\cos\theta}}\right)\cancel{\cos\theta}$$

$$\Rightarrow \dot{y} = -u\sin(\alpha-\theta) \quad \leftarrow \text{"UPWARDS"}$$

$$\Rightarrow |\dot{y}| = +u\sin(\alpha-\theta) \quad \leftarrow \text{"DOWNWARDS"}$$

$$\Rightarrow \boxed{|\dot{y}| = u\sin\alpha\cos\theta - u\cos\alpha\sin\theta}$$

$$\Rightarrow \dot{x} = u\cos(\alpha-\theta) - \cancel{g}\left(\frac{2u\sin(\alpha-\theta)}{\cancel{g}\cos\theta}\right)\sin\theta$$

$$\Rightarrow \dot{x} = \frac{u}{\cos\theta}\left[\cos(\alpha-\theta)\cos\theta - 2\sin(\alpha-\theta)\sin\theta\right]$$

$$\Rightarrow \dot{x} = \frac{u}{\cos\theta}\left[\underbrace{\cos(\alpha-\theta)\cos\theta - \sin(\alpha-\theta)\sin\theta}_{} - \sin(\alpha-\theta)\sin\theta\right]$$

$$\Rightarrow \dot{x} = \frac{u}{\cos\theta}\left[\underbrace{\cos[(\alpha-\theta)+\theta]}_{} - \sin\theta\sin(\alpha-\theta)\right]$$

$$\Rightarrow \dot{x} = \frac{u}{\cos\theta}\left[\cos\alpha - \sin\theta\left[\sin\alpha\cos\theta - \cos\alpha\sin\theta\right]\right]$$

$$\Rightarrow \boxed{\dot{x} = \frac{u}{\cos\theta}\left[\cos\alpha - \sin\alpha\sin\theta\cos\theta + \cos\alpha\sin^2\theta\right]}$$

- NOW WE CAN SUBSTITUTE INTO $|\dot{y}| = \dot{x}\tan\theta$

$$\Rightarrow \cancel{u}\sin\alpha\cos\theta - \cancel{u}\cos\alpha\sin\theta = \frac{\cancel{u}\tan\theta}{\cos\theta}\left[\cos\alpha - \sin\alpha\sin\theta\cos\theta + \cos\alpha\sin^2\theta\right]$$

$$\Rightarrow \sin\alpha\cos\theta - \cos\alpha\sin\theta = \frac{\sin\theta}{\cos^2\theta}\left[\cos\alpha - \sin\alpha\sin\theta\cos\theta + \cos\alpha\sin^2\theta\right]$$

$$\Rightarrow \sin\alpha\cos^3\theta - \cos\alpha\sin\theta\cos^2\theta = \sin\theta\cos\alpha - \sin\alpha\sin^2\theta\cos\theta + \cos\alpha\sin^3\theta$$

DIVIDE THE EQUATION THROUGH BY $\cos\alpha\cos^3\theta \neq 0$

$$\Rightarrow \tan\alpha - \tan\theta = \tan\theta\sec^2\theta - \tan\alpha\tan^2\theta + \tan^3\theta$$

$$\Rightarrow \tan\alpha + \tan\alpha\tan^2\theta = \tan\theta(1+\tan^2\theta) + \tan^3\theta + \tan\theta$$

$$\Rightarrow \tan\alpha + \tan\alpha\tan^2\theta = \tan\theta + \tan^3\theta + \tan^3\theta + \tan\theta$$

$$\Rightarrow \tan\alpha + \tan\alpha\tan^2\theta = 2\tan\theta + 2\tan^3\theta$$

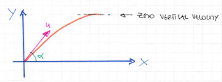
$$\Rightarrow \tan\alpha(1+\tan^2\theta) = 2\tan\theta(1+\tan^2\theta) \qquad 1+\tan^2\theta \neq 0$$

$$\Rightarrow \tan\alpha = 2\tan\theta$$

AS REQUIRED

METHOD B – BY LOOKING AT THE X–Y AXES

LOOKING AT THE X & Y AXES AND THE MOTION OF THE PARTICLE IN THAT FRAME



VERTICALLY USING "$v = u + at$"

$$\Rightarrow v = u\sin\alpha - gt$$

$$\Rightarrow 0 = u\sin\alpha - gt$$

$$\Rightarrow t = \frac{u\sin\alpha}{g}$$

HENCE WE NOW HAVE

$$\Rightarrow \frac{\cancel{u}\sin\alpha}{\cancel{g}} = \frac{2\cancel{u}\sin(\alpha-\theta)}{\cancel{g}\cos\theta} \quad \text{(FROM EARLIER)}$$

$$\Rightarrow \sin\alpha\cos\theta = 2\sin\alpha\cos\theta - 2\cos\alpha\sin\theta$$

$$\Rightarrow 2\cos\alpha\sin\theta = \sin\alpha\cos\theta$$

$$\Rightarrow \frac{2\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \tan\alpha = 2\tan\theta$$

AS BEFORE