WORK and VECTORS

Question 1 (**)

The vectors $\mathbf{i}$ *and* $\mathbf{j}$ *are horizontal unit vectors perpendicular to each other.*

A bead of mass 0.2 kg is threaded on a smooth straight horizontal wire.

The bead is at rest at the point A with position vector $(2\mathbf{i}+5\mathbf{j})$ m.

A single force $(2.6\mathbf{i}-0.1\mathbf{j})$ N acts on the bead and moves it to the point B with position vector $(17\mathbf{i}-5\mathbf{j})$ m.

Find the speed of the bead at B.

$\boxed{\text{M456-C}}$, $\boxed{20 \text{ ms}^{-1}}$

START BY FINDING $\overrightarrow{AB}$

$\overrightarrow{AB} = \underline{b}-\underline{a} = (17\underline{i}-5\underline{j})-(2\underline{i}+5\underline{j}) = 15\underline{i}-10\underline{j}$

FIND THE WORK DONE (IN OR OUT) BY THE FORCE

$W = \underline{F}\cdot\underline{s} = (2.6\underline{i}-0.1\underline{j})\cdot(15\underline{i}-10\underline{j})$

$= 39+1 = 40 \text{ J}$

BY ENERGIES

WORK IN = GAIN IN KINETIC ENERGY

$40 = \frac{1}{2}mv^2$

$40 = \frac{1}{2}\times 0.2\times v^2$

$v^2 = 400$

$v = 20 \text{ ms}^{-1}$

Question 2 (**)

The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to each other.*

A particle P moves from the point A, with position vector $(2\mathbf{i}+4\mathbf{j}+a\mathbf{k})$ m, where a is a constant, to the point B, with position vector $(3\mathbf{i}+a\mathbf{j}+2\mathbf{k})$ m, under the action of a constant force $\mathbf{F}=(4\mathbf{i}+a\mathbf{j}+3\mathbf{k})$ N.

The work done by $\mathbf{F}$, as it moves P from A to B, is 18 J.

Find the possible values of a.

$\boxed{a=-1, \quad a=8}$

- $\overrightarrow{AB} = \underline{b} - \underline{a} = (3,a,2)-(2,4,a) = (1,a-4,2-a)$
- $W = \underline{F} \cdot \overrightarrow{AB}$

$\Rightarrow 18 = (4,a,3)\cdot(1,a-4,2-a)$

$\Rightarrow 18 = 4+a(a-4)+3(2-a)$

$\Rightarrow 18 = 4+a^2-4a+6-3a$

$\Rightarrow 0 = a^2-7a-8$

$\Rightarrow (a+1)(a-8)=0$

$\therefore a = 8, -1$ //

Question 3 (***)

A small bead is threaded on a smooth, straight horizontal wire which passes through the point $A(3,-4)$ and the point $B(5,4)$.

The bead moves under the action of a single horizontal force $\mathbf{F}$ of magnitude 65 N, whose line of action is parallel to the straight line with equation

$$5x-12y=10.$$

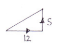
Given that all distances are measured in m, find the work done by $\mathbf{F}$ as it moves the bead from A to B.

$\boxed{320\text{ J}}$

LINE : $5x-12y=10$

$5x-10=12y$

$y=\frac{5}{12}x-\frac{5}{6}$



- $\underline{F}$ IS IN THE DIRECTION $12\underline{i}+5\underline{j}$
- $|12\underline{i}+5\underline{j}|=13$
- HENCE $\underline{F}=5(12\underline{i}+5\underline{j})$

$65 \div 13$

WORK $=\underline{F}\cdot\underline{r}$

$\vec{AB}=\underline{b}-\underline{a}=(5,4)-(3,-4)=(2,8)$

THUS

WORK $=(60\underline{i}+25\underline{j})\cdot(2\underline{i}+8\underline{j})$

$=120+200$

$=320\text{ J}$

Question 4 (***)

The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to each other.*

A particle of mass 5 kg is initially at rest at the point P with position vector $(4\mathbf{i}-\mathbf{j}+7\mathbf{k})$ m when is acted by a force $\mathbf{F}$ which causes it to move to the point Q with position vector $(9\mathbf{i}+9\mathbf{j}-3\mathbf{k})$ m.

It is further given that

$$\mathbf{F}=\left[(\lambda+1)\mathbf{i}+(\lambda+\mu)\mathbf{j}+(1+\lambda-2\mu)\mathbf{k}\right]\text{ N},$$

where λ and μ are scalar constants.

If $\mathbf{F}$ is acting in the direction PQ, determine the speed of the particle as it passes Q.

$\boxed{}$, $\boxed{6\text{ ms}^{-1}}$

START WITH THE VECTOR $\overrightarrow{PQ}$

$\overrightarrow{PQ} = \underline{q} - \underline{p} = (9,9,-3)-(4,-1,7) = (5,10,-10)$

SCALE THIS DIRECTION TO $(1,2,-2)$

$\Longrightarrow (\lambda+1,\lambda+\mu,1+\lambda-\mu)$ PROPORTIONAL TO $(1,2,-2)$

$\Longrightarrow \frac{\lambda+1}{1}=\frac{\lambda+\mu}{2}=\frac{1+\lambda-2\mu}{-2}$

① & ②

$2\lambda+2=\lambda+\mu$

$\lambda=\mu-2$

① & ③

$-2\lambda-2=1+\lambda-2\mu$

$2\mu-3\lambda=3$

$2\mu-3(\mu-2)=3$

$-\mu+6=3$

$3=\mu$

IE $\mu=3$, $\lambda=1$, SO $\underline{F}=(2,4,-4)$

GETTING THE WORK DONE BY THE FORCE

$W=\underline{F}\cdot\underline{r} = (2,4,-4)\cdot(5,10,-10) = 10+40+40 = 90$

FINALLY BY ENERGIES, NOTING IT "STARTS FROM REST"

KE INCREASE = WORK DONE BY $\underline{F}$

$90 = \frac{1}{2}\times5\times v^2$

$v^2=36$

$|\underline{v}| = 6\text{ ms}^{-1}$

Question 5 (****)

The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to each other.*

A particle, of mass 0.5 kg, passes through the point A whose position vector is $(12\mathbf{i}-15\mathbf{j}-2\mathbf{k})\text{ m}$, with speed $U\text{ ms}^{-1}$. The particle is moving due to the action of the following two constant forces, $\mathbf{F}_1$, and $\mathbf{F}_2$.

$$\mathbf{F}_1 = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}\text{ N} \quad \text{and} \quad \mathbf{F}_2 = \begin{pmatrix} k-2 \\ 2k+3 \\ 3k-1 \end{pmatrix}\text{ N},$$

where k is a scalar constant.

Determine the value of U, given further that it passes through the point B, whose position vector is $(-8\mathbf{i}+5\mathbf{j}+2\mathbf{k})\text{ m}$, with speed 29 ms^{-1}.

$\boxed{}$, $\boxed{U = 5\text{ ms}^{-1}}$

$\underline{F}_1 = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$ A$(12,-15,-2)$, SPEED U ms^{-1}

$\underline{F}_2 = \begin{pmatrix} k-2 \\ 2k+3 \\ 3k-1 \end{pmatrix}$ B$(-8,5,2)$, SPEED 29 ms^{-1}

PARTICLE OF MASS, $m = 0.5$ kg

THE RESULTANT OF $\underline{F}_1$ & $\underline{F}_2$ IS $\underline{F}$

$$\underline{F} = \underline{F}_1 + \underline{F}_2 = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} + \begin{pmatrix} k-2 \\ 2k+3 \\ 3k-1 \end{pmatrix} = \begin{pmatrix} k-4 \\ 2k+7 \\ 3k+4 \end{pmatrix}$$

THE VECTOR $\overrightarrow{AB}$ IS GIVEN BY

$$\overrightarrow{AB} = \underline{b} - \underline{a} = (-8,5,2) - (12,-15,-2) = (-20,20,4) = 4(-5,5,1)$$

$\underline{F}$ MUST BE IN THE DIRECTION OF $(-5,5,1)$

$$\Rightarrow \begin{pmatrix} k-4 \\ 2k+7 \\ 3k+4 \end{pmatrix} = \alpha \begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} -5\alpha = k-4 \\ 5\alpha = 2k+7 \end{array}$$

$$\Rightarrow 0 = 3k+3$$

$$\Rightarrow k = -1$$

WORK DONE BY $\underline{F}$ IS GIVEN BY

$$W = \underline{F} \cdot \overrightarrow{AB} = \begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -20 \\ 20 \\ 4 \end{pmatrix} = 100 + 100 + 4 = 204\text{ J}$$

BY ENERGIES

$$\Rightarrow KE_A + \text{WORK} = KE_B$$

$$\Rightarrow \tfrac{1}{2}mu^2 + W = \tfrac{1}{2}mv^2$$

$$\Rightarrow \tfrac{1}{2} \times \tfrac{1}{2} \times U^2 + 204 = \tfrac{1}{2} \times \tfrac{1}{2} \times 29^2$$

$$\Rightarrow \tfrac{1}{4}U^2 + 204 = \tfrac{841}{4}$$

$$\Rightarrow U^2 + 816 = 841$$

$$\Rightarrow U^2 = 25$$

$$\Rightarrow |U| = 5\text{ ms}^{-1}$$

Question6 (****)

The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to each other.*

A particle of mass 2 kg, which is free to move in any direction, is acted by two forces

$$\mathbf{F}_1 = (-5\mathbf{i} + 4\mathbf{j} - 9\mathbf{k}) \text{ N} \quad \text{and} \quad \mathbf{F}_2 = (\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}) \text{ N},$$

where α, β and γ are scalar constants.

These two forces cause the particle to move **directly** from the point P with position vector $(4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$ m to the point Q with position vector $(-4\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ m.

If the respective speeds of the particle at P and Q are 4 ms^{-1} and 14 ms^{-1}, determine the values of α, β and γ.

$\boxed{}$, $\boxed{[\alpha, \beta, \gamma] = [-5, 6, 4]}$

FIND THE RESULTANT FIRST AND THE VECTOR $\overrightarrow{PQ}$

$\underline{F}_1 + \underline{F}_2 = (\alpha - 5, \beta + 4, \gamma - 9)$

$\overrightarrow{PQ} = \underline{q} - \underline{p}$

$= (-4, 5, 1) - (4, -3, 5)$

$= (-8, 8, -4)$

AS THE PARTICLE IS NOT CONSTRAINED TO MOVE ON A FIXED PATH THE RESULTANT MUST BE IN THE DIRECTION $\overrightarrow{PQ}$, SAY $(-2, 2, -1)$

$\therefore \; \alpha - 5 = -2\lambda$

$\beta + 4 = 2\lambda$

$\gamma - 9 = -\lambda$

$\therefore$ RESULTANT $= \underline{R} = (-2\lambda, 2\lambda, -\lambda)$

BY DOTTING THE RESULTANT WITH $\overrightarrow{PQ}$, WE GET THE WORK DONE

$\Rightarrow$ WORK DONE = RESULTANT DOTTED WITH $\overrightarrow{PQ}$

$\Rightarrow W = \underline{R} \cdot \overrightarrow{PQ}$

$\Rightarrow \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = (-2\lambda, 2\lambda, -\lambda) \cdot (-8, 8, -4)$

$\Rightarrow \frac{1}{2} \times 2 \times 14^2 - \frac{1}{2} \times 2 \times 4^2 = 16\lambda + 16\lambda + 4\lambda$

$\Rightarrow 196 - 16 = 36\lambda$

$\Rightarrow \lambda = 5$

$\therefore \; \alpha - 5 = -10$

$\beta + 4 = 10$

$\gamma - 9 = -5$

$\therefore \; \alpha = -5$

$\beta = 6$

$\gamma = 4$ //

Question 7 (****)

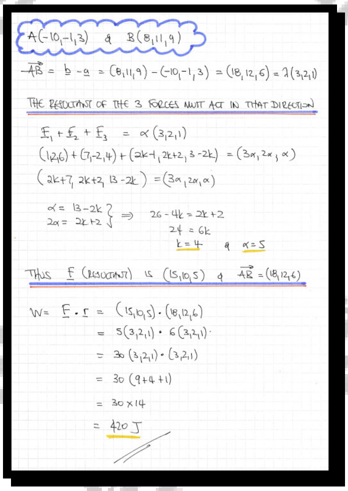
The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to each other.*

A particle P moves from the point A, with position vector $(-10\mathbf{i}-\mathbf{j}+3\mathbf{k})$ m to the point B, with position vector $(8\mathbf{i}+11\mathbf{j}+9\mathbf{k})$ m, under the action of the following three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$.

- $\mathbf{F}_1 = (\mathbf{i}+2\mathbf{j}+6\mathbf{k})$ N.
- $\mathbf{F}_2 = (7\mathbf{i}-2\mathbf{j}+4\mathbf{k})$ N
- $\mathbf{F}_3 = \left[(2k-1)\mathbf{i}+(2k+2)\mathbf{j}+(3-2k)\mathbf{k}\right]$ N, where k is a scalar constant.

Determine the work done by the three forces in moving the particle from A to B.

$\boxed{}$, $\boxed{W = 420 \text{ J}}$



$A(-10,-1,3)$ & $B(8,11,9)$

$\vec{AB} = \underline{b} - \underline{a} = (8,11,9) - (-10,-1,3) = (18,12,6) = 2(3,2,1)$

THE RESULTANT OF THE 3 FORCES MUST ACT IN THAT DIRECTION

$\underline{F}_1 + \underline{F}_2 + \underline{F}_3 = \alpha(3,2,1)$

$(1,2,6) + (7,-2,4) + (2k-1, 2k+2, 3-2k) = (3\alpha, 2\alpha, \alpha)$

$(2k+7, 2k+2, 13-2k) = (3\alpha, 2\alpha, \alpha)$

$\left.\begin{array}{l}\alpha = 13-2k \\ 2\alpha = 2k+2\end{array}\right\} \Rightarrow 26-4k = 2k+2$

$24 = 6k$

$k = 4$ & $\alpha = 5$

THUS $\underline{F}$ (RESULTANT) IS $(15,10,5)$ & $\vec{AB} = (18,12,6)$

$W = \underline{F} \cdot \underline{r} = (15,10,5)\cdot(18,12,6)$

$= 5(3,2,1) \cdot 6(3,2,1)$

$= 30(3,2,1)\cdot(3,2,1)$

$= 30(9+4+1)$

$= 30 \times 14$

$= 420$ J

Question 8 (*****)

The vectors $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are unit vectors mutually perpendicular to each other.*

A particle, of mass 0.5 kg, passes through the point A whose position vector is $(14\mathbf{i}-10\mathbf{j})$ m, with speed 5 ms^{-1}. The particle is moving due to the action of the following two constant forces, $\mathbf{F}_1$, and $\mathbf{F}_2$.

$$\mathbf{F}_1 = \begin{pmatrix} -\lambda \\ 2\lambda \\ \nu \end{pmatrix} \text{N} \quad \text{and} \quad \mathbf{F}_2 = \begin{pmatrix} \mu-2 \\ 2\mu+3 \\ 3\mu-1 \end{pmatrix} \text{N},$$

where λ, μ and ν are scalar constants.

It further given that the particle passes through the point B, whose position vector is $(-6\mathbf{i}+10\mathbf{j}+4\mathbf{k})$ m, with speed 29 ms^{-1}.

Determine the value of each of the constants λ, μ and ν.

$\boxed{}$, $\boxed{\lambda=2}$, $\boxed{\mu=-1}$, $\boxed{\nu=5}$

$\boxed{F_1 = \begin{pmatrix} -\lambda \\ 2\lambda \\ \nu \end{pmatrix} \quad A(14,-10,0), \text{ SPEED } 5 \text{ ms}^{-1} \quad F_2 = \begin{pmatrix} \mu-2 \\ 2\mu+3 \\ 3\mu-1 \end{pmatrix} \quad B(-6,10,4), \text{ SPEED } 29 \text{ ms}^{-1} \quad \text{PARTICLE MASS}, m = 0.5 \text{ kg}}$

THE RESULTANT OF F_1 & F_2 IS F

$$F = F_1 + F_2 = \begin{pmatrix} -\lambda \\ 2\lambda \\ \nu \end{pmatrix} + \begin{pmatrix} \mu-2 \\ 2\mu+3 \\ 3\mu-1 \end{pmatrix} = \begin{pmatrix} -\lambda+\mu-2 \\ 2\lambda+2\mu+3 \\ \nu+3\mu-1 \end{pmatrix}$$

THE VECTOR $\overrightarrow{AB}$ IS GIVEN BY

$$\overrightarrow{AB} = \underline{b} - \underline{a} = (-6,10,4)-(14,-10,0) = (-20,20,4) = 4(-5,5,1)$$

F MUST ACT IN THE DIRECTION OF $\overrightarrow{AB}$

$$\Rightarrow \begin{pmatrix} -\lambda+\mu-2 \\ 2\lambda+2\mu+3 \\ \nu+3\mu-1 \end{pmatrix} = \alpha \begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -5\alpha = -\lambda+\mu-2 \\ 5\alpha = 2\lambda+2\mu+3 \\ \alpha = \nu+3\mu-1 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} -5(\nu+3\mu-1) = -\lambda+\mu-2 \\ 5(\nu+3\mu-1) = 2\lambda+2\mu+3 \end{matrix}$$

$$\Rightarrow \begin{pmatrix} -5\nu-15\mu+5 = -\lambda+\mu-2 \\ 5\nu+15\mu-5 = 2\lambda+2\mu+3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\lambda+16\mu+5\nu = 7 \\ 2\lambda-13\mu-5\nu = -8 \end{pmatrix}$$

ANOTHER EQUATION WILL BE OBTAINED VIA ENERGIES

$$KE_A + W = KE_B$$
$$\tfrac{1}{2}mu^2 + W = \tfrac{1}{2}mv^2$$
$$\tfrac{1}{2}\times\tfrac{1}{2}\times 5^2 + W = \tfrac{1}{2}\times\tfrac{1}{2}\times 29^2$$
$$W = 204$$

NEXT THE WORK DONE BY F IS GIVEN BY

$$\Rightarrow W = F \cdot \overrightarrow{AB} = \begin{pmatrix} -\lambda+\mu-2 \\ 2\lambda+2\mu+3 \\ \nu+3\mu-1 \end{pmatrix} \cdot \begin{pmatrix} -20 \\ 20 \\ 4 \end{pmatrix} = 204$$

$$\Rightarrow \begin{pmatrix} -\lambda+\mu-2 \\ 2\lambda+2\mu+3 \\ \nu+3\mu-1 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 5 \\ 1 \end{pmatrix} = 51$$

$$\Rightarrow \begin{matrix} 5\lambda-5\mu+10 \\ 10\lambda+10\mu+15 \\ \nu+3\mu-1 \end{matrix} = 51$$

$$\Rightarrow 15\lambda+8\mu+\nu = 27$$

FINALLY SOLVING THE 3 EQUATIONS

$$\lambda-16\mu-5\nu = -7$$
$$2\lambda-13\mu-5\nu = -8$$
$$15\lambda+8\mu+\nu = 27$$

$$\boxed{\lambda = 16\mu+5\nu-7}$$

$$\Rightarrow \begin{matrix} 2(16\mu+5\nu-7)-13\mu-5\nu = -8 \\ 15(16\mu+5\nu-7)+8\mu+\nu = 27 \end{matrix}$$

$$19\mu+5\nu = 6$$
$$248\mu+76\nu = 132$$

$$\Rightarrow \begin{matrix} 19\mu+5\nu = 6 & \times(-19) \\ 62\mu+19\nu = 33 & \times 5 \end{matrix}$$

$$\Rightarrow \begin{matrix} -361\mu-95\nu = -114 \\ 310\mu+95\nu = 165 \end{matrix}$$

$$\Rightarrow -51\mu = 51$$

$$\therefore \mu = -1$$

$$\therefore 19\mu+5\nu = 6$$
$$-19+5\nu = 6$$
$$5\nu = 25$$
$$\nu = 5$$

$$\lambda = 16\mu+5\nu-7$$
$$\lambda = -16+25-7$$
$$\lambda = 2$$