TRIGONOMETRY

THE COMPOUND ANGLE IDENTITIES

Question 1

Prove the validity of each of the following trigonometric identities.

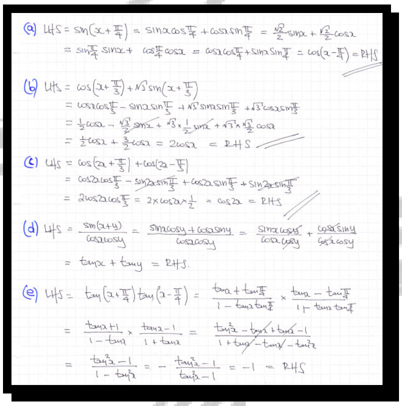
a) $\sin\left(x+\frac{\pi}{4}\right) \equiv \cos\left(x-\frac{\pi}{4}\right)$

b) $\cos\left(x+\frac{\pi}{3}\right)+\sqrt{3}\sin\left(x+\frac{\pi}{3}\right) \equiv 2\cos x$

c) $\cos\left(2x+\frac{\pi}{3}\right)+\cos\left(2x-\frac{\pi}{3}\right) \equiv \cos 2x$

d) $\frac{\sin(x+y)}{\cos x \cos y} \equiv \tan x + \tan y$

e) $\tan\left(x+\frac{\pi}{4}\right)\tan\left(x-\frac{\pi}{4}\right) \equiv -1$



(a) $\text{LHS} = \sin\left(x+\frac{\pi}{4}\right) = \sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x$
$= \sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}\cos x = \cos x\cos\frac{\pi}{4} + \sin x\sin\frac{\pi}{4} = \cos\left(x-\frac{\pi}{4}\right) = \text{RHS}$

(b) $\text{LHS} = \cos\left(x+\frac{\pi}{3}\right) + \sqrt{3}\sin\left(x+\frac{\pi}{3}\right)$
$= \cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3} + \sqrt{3}\sin x\sin\frac{\pi}{3} + \sqrt{3}\cos x\sin\frac{\pi}{3}$
$= \frac{1}{2}\cos x - \cancel{\frac{\sqrt{3}}{2}\sin x} + \cancel{\sqrt{3}\times\frac{1}{2}\sin x} + \sqrt{3}\times\frac{\sqrt{3}}{2}\cos x$
$= \frac{1}{2}\cos x + \frac{3}{2}\cos x = 2\cos x = \text{RHS}$

(c) $\text{LHS} = \cos\left(2x+\frac{\pi}{3}\right) + \cos\left(2x-\frac{\pi}{3}\right)$
$= \cos 2x\cos\frac{\pi}{3} - \cancel{\sin 2x\sin\frac{\pi}{3}} + \cos 2x\sin\frac{\pi}{3} + \cancel{\sin 2x\sin\frac{\pi}{3}}$
$= 2\cos 2x\cos\frac{\pi}{3} = 2\times\cos 2x\times\frac{1}{2} = \cos 2x = \text{RHS}$

(d) $\text{LHS} = \frac{\sin(x+y)}{\cos x\cos y} = \frac{\sin x\cos y + \cos x\sin y}{\cos x\cos y} = \frac{\sin x\cancel{\cos y}}{\cos x\cancel{\cos y}} + \frac{\cancel{\cos x}\sin y}{\cancel{\cos x}\cos y}$
$= \tan x + \tan y = \text{RHS}.$

(e) $\text{LHS} = \tan\left(x+\frac{\pi}{4}\right)\tan\left(x-\frac{\pi}{4}\right) = \frac{\tan x + \tan\frac{\pi}{4}}{1-\tan x\tan\frac{\pi}{4}} \times \frac{\tan x - \tan\frac{\pi}{4}}{1+\tan x\tan\frac{\pi}{4}}$
$= \frac{\tan x+1}{1-\tan x} \times \frac{\tan x-1}{1+\tan x} = \frac{\tan^2 x - \cancel{\tan x} + \cancel{\tan x} - 1}{1+\cancel{\tan x}-\cancel{\tan x}-\tan^2 x}$
$= \frac{\tan^2 x - 1}{1-\tan^2 x} = -\frac{\tan^2 x-1}{\tan^2 x - 1} = -1 = \text{RHS}$

Question 2

Prove the validity of each of the following trigonometric identities.

a) $\sin\left(x+\frac{\pi}{3}\right)-\sqrt{3}\cos\left(x+\frac{\pi}{3}\right)\equiv 2\sin x$

b) $\frac{\cos x}{\sin y}-\frac{\sin x}{\cos y}\equiv\frac{\cos(x+y)}{\sin y\cos y}$

c) $\tan(x+60°)\tan(x-60°)\equiv\frac{\tan^2 x-3}{1-3\tan^2 x}$

d) $\sin(x+y)\sin(x-y)\equiv\cos^2 y-\cos^2 x$

e) $\cot(x+y)\equiv\frac{\cot x\cot y-1}{\cot x+\cot y}$

(a) $\sin\left(x+\frac{\pi}{3}\right)-\sqrt{3}\cos\left(x+\frac{\pi}{3}\right)=\sin x\cos\frac{\pi}{3}+\cos x\sin\frac{\pi}{3}-\left[\sqrt{3}\cos x\cos\frac{\pi}{3}-\sqrt{3}\sin x\sin\frac{\pi}{3}\right]$
$=\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x-\left[\frac{\sqrt{3}}{2}\cos x-\frac{3}{2}\sin x\right]$
$=\frac{1}{2}\sin x+\cancel{\frac{\sqrt{3}}{2}\cos x}-\cancel{\frac{\sqrt{3}}{2}\cos x}+\frac{3}{2}\sin x$
$=2\sin x=\text{RHS}$

(b) $\text{LHS}=\frac{\cos x}{\sin y}-\frac{\sin x}{\cos y}=\frac{\cos x\cos y-\sin x\sin y}{\sin y\cos y}=\frac{\cos(x+y)}{\sin y\cos y}=\text{RHS}$

(c) $\text{LHS}=\tan(x+60)\tan(x-60)=\frac{\tan x+\tan 60}{1-\tan x\tan 60}\times\frac{\tan x-\tan 60}{1+\tan x\tan 60}$
$=\frac{\tan x+\sqrt{3}}{1-\sqrt{3}\tan x}\times\frac{\tan x-\sqrt{3}}{1+\sqrt{3}\tan x}=\frac{\tan^2 x-3}{1-3\tan^2 x}=\text{RHS}$

(d) $\text{LHS}=\sin(x+y)\sin(x-y)=(\sin x\cos y+\cos x\sin y)(\sin x\cos y-\cos x\sin y)$
$=\sin^2 x\cos^2 y-\cancel{\sin x\cos y\cos x\sin y}+\cancel{\cos x\sin y\sin x\cos y}-\cos^2 x\sin^2 y$
$=(1-\cos^2 x)\cos^2 y-\cos^2 x(1-\cos^2 y)$
$=\cos^2 y-\cancel{\cos^2 x\cos^2 y}-\cos^2 x+\cancel{\cos^2 x\cos^2 y}$
$=\cos^2 y-\cos^2 x$
$=\text{RHS}$

(e) $\text{LHS}=\cot(x+y)=\frac{\cos(x+y)}{\sin(x+y)}=\frac{\cos x\cos y-\sin x\sin y}{\sin x\cos y+\cos x\sin y}$
$=\frac{\frac{\cos x\cos y}{\sin x\sin y}-\frac{\sin x\sin y}{\sin x\sin y}}{\frac{\sin x\cos y}{\sin x\sin y}+\frac{\cos x\sin y}{\sin x\sin y}}=\frac{\cot x\cot y-1}{\cot y+\cot x}=\text{RHS}$

Question 3

Prove the validity of each of the following trigonometric identities.

a) $\cos(x+y)\cos(x-y) \equiv \cos^2 x - \sin^2 y$

b) $\sin P - \sin Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

c) $\sin^2\left(\theta+\frac{\pi}{4}\right)+\sin^2\left(\theta-\frac{\pi}{4}\right) \equiv 1$

d) $\cos x + \sin x \tan 2x \equiv \frac{\cos x}{\cos 2x}$

e) $\cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

(a) LHS $= \cos(x+y)\cos(x-y)$
$= [\cos x\cos y - \sin x\sin y][\cos x\cos y + \sin x\sin y]$ = difference of squares
$= \cos^2 x\cos^2 y - \sin^2 x\sin^2 y$
$= \cos^2 x(1-\sin^2 y) - (1-\cos^2 x)\sin^2 y$
$= \cos^2 x - \cancel{\cos^2 x\sin^2 y} - \sin^2 y + \cancel{\cos^2 x\sin^2 y}$
$= \cos^2 x - \sin^2 y$
$=$ RHS

(b)
$\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin(A-B) = \sin A\cos B - \cos A\sin B$ subtract equations

$\boxed{\sin(A+B) - \sin(A-B) = 2\cos A\sin B}$ $(*)$

LET $P = A+B$, $Q = A-B$

Add equations: $P+Q = 2A$, $\boxed{\frac{P+Q}{2} = A}$

subtract equations: $P-Q = 2B$, $\boxed{\frac{P-Q}{2} = B}$

THUS $(*)$ BECOMES
$\sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

(c) LHS $= \sin^2\left(\theta+\frac{\pi}{4}\right) + \cos^2\left(\theta+\frac{\pi}{4}\right) = \left[\sin\left(\theta+\frac{\pi}{4}\right)\right]^2 + \left[\cos\left(\theta+\frac{\pi}{4}\right)\right]^2$
$= \left(\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right)^2 + \left(\cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4}\right)^2$
$= \left(\frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta\right)^2 + \left(\frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta\right)^2$
$= \frac{1}{2}\sin^2\theta + \cancel{\sin\theta\cos\theta} + \frac{1}{2}\cos^2\theta + \frac{1}{2}\cos^2\theta - \cancel{\sin\theta\cos\theta} + \frac{1}{2}\sin^2\theta$
$= \sin^2\theta + \cos^2\theta = 1$
$=$ RHS

(d) LHS $= \cos x + \sin x\tan 2x = \cos x + \sin x\frac{\sin 2x}{\cos 2x}$
$= \frac{\cos x\cos 2x + \sin x\sin 2x}{\cos 2x} = \frac{\cos(2x-x)}{\cos 2x}$
$= \frac{\cos x}{\cos 2x} =$ RHS

(e)
$\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\cos(A-B) = \cos A\cos B + \sin A\sin B$ Add equations

$\boxed{\cos(A+B) + \cos(A-B) = 2\cos A\cos B}$ $(*)$

LET $P = A+B$, $Q = A-B$

add equations: $P+Q = 2A$, $\boxed{\frac{P+Q}{2} = A}$; subtract equations: $P-Q = 2B$, $\boxed{\frac{P-Q}{2} = B}$

HENCE $(*)$ BECOMES
$\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

Question 4

If $\sin(\theta+\alpha)=2\sin\theta$, show clearly that

$$\tan\theta=\frac{\sin\alpha}{2-\cos\alpha}.$$

$\boxed{\text{proof}}$

$\sin(\theta+\alpha)=2\sin\theta$

$\sin\theta\cos\alpha+\cos\theta\sin\alpha=2\sin\theta$

$\cos\theta\sin\alpha=2\sin\theta-\sin\theta\cos\alpha$

$\cos\theta\sin\alpha=\sin\theta(2-\cos\alpha)$

$\frac{\cos\theta\sin\alpha}{\cos\theta}=\frac{\sin\theta(2-\cos\alpha)}{\cos\theta}$

$\sin\alpha=\tan\theta(2-\cos\alpha)$

$\therefore\ \tan\theta=\frac{\sin\alpha}{2-\cos\alpha}$ // AS REQUIRED

Question 5

By expanding $\tan(\theta+45°)$ with a suitable value for θ, show clearly that

$$\tan 75°=2+\sqrt{3}.$$

$\boxed{\text{proof}}$

$\tan(75)=\tan(45+30)=\frac{\tan45°+\tan30°}{1-\tan45°\tan30°}=\frac{1+\frac{\sqrt{3}}{3}}{1-1\times\frac{\sqrt{3}}{3}}$

$=\frac{3+\sqrt{3}}{3-\sqrt{3}}=\frac{(3+\sqrt{3})(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})}=\frac{9+6\sqrt{3}+3}{9-3}=\frac{12+6\sqrt{3}}{6}$

$=2+\sqrt{3}$ //

Question 6

By expanding $\sin(45°-x)$ with a suitable value for x, show clearly that

$$\operatorname{cosec}15°=\sqrt{2}+\sqrt{6}.$$

$\boxed{\text{proof}}$

LET $x=30°$

- $\sin(15)=\sin(45-30)=\sin45\cos30-\cos45\sin30=\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}$

$=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}=\frac{\sqrt{6}-\sqrt{2}}{4}$

$\therefore\ \operatorname{cosec}15=\frac{4}{\sqrt{6}-\sqrt{2}}=\frac{4(\sqrt{6}+\sqrt{2})}{(\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2})}=\frac{4(\sqrt{6}+\sqrt{2})}{6-2}=\sqrt{6}+\sqrt{2}$ // AS REQUIRED

Question 7

By expanding $\tan(\theta + 45°)$ with a suitable value for θ, show clearly that

$$\tan 105° = -2 - \sqrt{3}.$$

$\boxed{\text{proof}}$

LET $\theta = 60$

$$\tan 105 = \tan(60 + 45°) = \frac{\tan 60 + \tan 45}{1 - \tan 60 \tan 45} = \frac{\sqrt{3} + 1}{1 - \sqrt{3} \times 1} = \frac{\sqrt{3} + 1}{1 - \sqrt{3}}$$

$$= \frac{(\sqrt{3}+1)(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{\sqrt{3} + 3 + 1 + \sqrt{3}}{1 - 3} = -\frac{1}{2}(4 + 2\sqrt{3}) = -2 - \sqrt{3}$$

AS REQUIRED

Question 8

By expanding $\cos(y + 45°)$ with a suitable value for y, show clearly that

$$\sec 75° = \sqrt{2} + \sqrt{6}.$$

$\boxed{\text{proof}}$

$$\sec 75 = \frac{1}{\cos 75} = \frac{1}{\cos(45 + 30)} = \frac{1}{\cos 45 \cos 30 - \sin 45 \sin 30}$$

$$= \frac{1}{\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}} = \frac{1}{\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}} = \frac{4}{\sqrt{6} - \sqrt{2}} = \frac{4(\sqrt{6} + \sqrt{2})}{(\sqrt{6} - \sqrt{2})(\sqrt{6} + \sqrt{2})}$$

$$= \frac{4(\sqrt{6} + \sqrt{2})}{6 - 2} = \frac{\cancel{4}(\sqrt{6} + \sqrt{2})}{\cancel{4}} = \sqrt{6} + \sqrt{2}$$

Question 9

By considering the expansion of $\tan(A + B)$ with suitable values for A and B, show clearly that

$$\cot 75° = 2 - \sqrt{3}.$$

$\boxed{\text{proof}}$

$$\cot 75 = \frac{1}{\tan 75} = \frac{1}{\tan(45 + 30)} = \frac{1}{\frac{\tan 45 + \tan 30}{1 - \tan 45 \tan 30}} = \frac{1}{\frac{1 + \sqrt{3}/3}{1 - 1 \times \sqrt{3}/3}}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}} = \frac{(3 - \sqrt{3})(3 - \sqrt{3})}{(3 + \sqrt{3})(3 - \sqrt{3})} = \frac{9 - 3\sqrt{3} - 3\sqrt{3} + 3}{9 - 3} = \frac{12 - 6\sqrt{3}}{6}$$

$$= 2 - \sqrt{3}$$

Question 10

Show clearly, by using the compound angle identities, that

$$\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

$\boxed{\text{proof}}$

$$\sin 15° = \sin(45°-30°) = \sin 45°\cos 30° - \cos 45°\sin 30°$$
$$= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\times\frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

AS REQUIRED

Question 11

Show clearly, by using the compound angle identities, that

$$\cos 105° = \frac{\sqrt{2}-\sqrt{6}}{4}.$$

$\boxed{\text{proof}}$

$$\cos 105° = \cos(60°+45°) = \cos 60\cos 45 - \sin 60\sin 45$$
$$= \frac{1}{2}\times\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

AS REQUIRED

Question 12

Show clearly, by using the compound angle identities, that

$$\tan 15° = 2-\sqrt{3}.$$

$\boxed{\text{proof}}$

$$\tan 15° = \tan(60°-45°) = \frac{\tan 60 - \tan 45}{1+\tan 60\tan 45} = \frac{\sqrt{3}-1}{1+\sqrt{3}\times 1}$$
$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{3-2\sqrt{3}+1}{3-1} = \frac{4-2\sqrt{3}}{2}$$
$$= 2-\sqrt{3}$$

AS REQUIRED

Question 13

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B .$$

a) Use the above trigonometric identity with suitable values for A and B, to show that

$$\sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

b) Hence by using the trigonometric expansion of $\cos(75° + \alpha)$ with a suitable value for α, show clearly that

$$\cos 165° = -\sin 75° .$$

proof

(a) $\sin 75 = \sin(45+30) = \sin 45 \cos 30 + \cos 45 \sin 30$
$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$

(b) use $\alpha = 90$
$\cos 165 = \cos(90+75) = \cos 90 \cos 75 - \sin 90 \sin 75$
$= 0 \times \cos 75 - 1 \times \frac{\sqrt{6}+\sqrt{2}}{4}$
$= -\frac{\sqrt{6}+\sqrt{2}}{4} = -\sin 75$

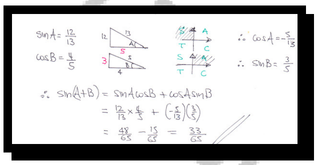
Question 14

$$\sin A = \frac{12}{13} \quad \text{and} \quad \cos B = \frac{4}{5} .$$

If A is obtuse and B is acute, show clearly that

$$\sin(A+B) = \frac{33}{65} .$$

proof



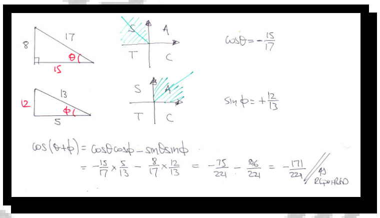
Question 15

$$\sin\theta = \frac{8}{17} \quad \text{and} \quad \cos\varphi = \frac{5}{13}.$$

If θ is obtuse and φ is acute, show clearly that

$$\cos(\theta+\varphi) = -\frac{171}{221}.$$

$\boxed{\text{proof}}$



Question 16

The constants a and b are such so that

$$\tan a = \frac{1}{3} \quad \text{and} \quad \tan b = \frac{1}{7}.$$

Determine the exact value of $\cot(a-b)$, showing all the steps in the workings.

$\boxed{\cot(a-b) = \frac{11}{2}}$

$$\cot(a-b) = \frac{1}{\tan(a-b)} = \frac{1}{\frac{\tan a - \tan b}{1+\tan a\tan b}} = \frac{1+\tan a\tan b}{\tan a - \tan b}$$

$$= \frac{1+\frac{1}{3}\times\frac{1}{7}}{\frac{1}{3}-\frac{1}{7}} = \frac{\frac{22}{\cancel{21}}}{\frac{4}{\cancel{21}}} = \frac{11}{2}$$

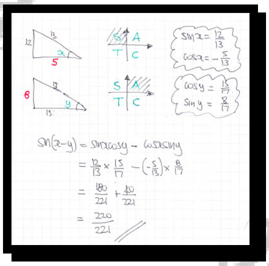
Question 17

$$\sin x = \frac{12}{13} \quad \text{and} \quad \cos y = \frac{15}{17}.$$

If x is obtuse and y is acute, show clearly that

$$\sin(x-y) = \frac{220}{221}.$$

proof



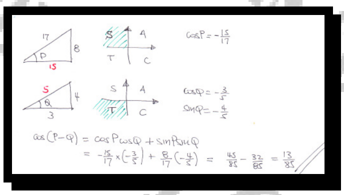
Question 18

$$\sin P = \frac{8}{17} \quad \text{and} \quad \tan Q = \frac{4}{3}.$$

If P is obtuse and Q is reflex, show clearly that

$$\cos(P-Q) = \frac{13}{85}.$$

proof



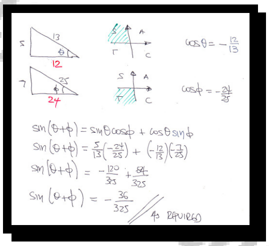
Question 19

$$\sin\theta = \frac{5}{13} \quad \text{and} \quad \sin\varphi = -\frac{7}{25}.$$

If θ is obtuse and φ is such so that $180° < \varphi < 270°$, show clearly that

$$\sin(\theta + \varphi) = -\frac{36}{325}.$$

$\boxed{\text{proof}}$

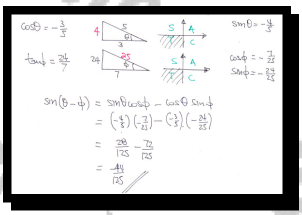


Question 20

$$\cos\theta = -\frac{3}{5} \quad \text{and} \quad \tan\varphi = \frac{24}{7}.$$

If θ is reflex, and φ is also reflex, show clearly that

$$\sin(\theta - \varphi) = -\frac{44}{125}.$$



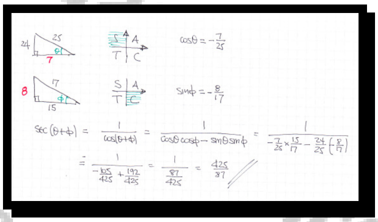
Question 21

$$\sin\theta = \frac{24}{25} \quad \text{and} \quad \cos\varphi = \frac{15}{17}.$$

If θ is obtuse and φ is reflex, show clearly that

$$\sec(\theta+\varphi) = \frac{425}{87}.$$

$\boxed{\text{proof}}$



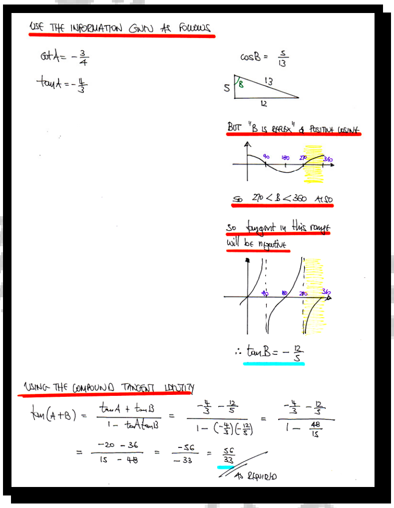
Question 22

$$\cot A = -\frac{3}{4} \quad \text{and} \quad \cos B = \frac{5}{13}.$$

If A is reflex and B is also reflex, show clearly that

$$\tan(A+B) = \frac{56}{33}.$$

$\boxed{\text{proof}}$


USE THE INFORMATION GIVEN AS FOLLOWS

$\cot A = -\frac{3}{4}$

$\tan A = -\frac{4}{3}$

$\cos B = \frac{5}{13}$

BUT "B IS REFLEX" & POSITIVE COSINE

SO $270 < B < 360$ ALSO

SO tangent in this range will be negative

$\therefore \tan B = -\frac{12}{5}$

USING THE COMPOUND TANGENT IDENTITY

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{-\frac{4}{3} - \frac{12}{5}}{1 - \left(-\frac{4}{3}\right)\left(-\frac{12}{5}\right)} = \frac{-\frac{4}{3} - \frac{12}{5}}{1 - \frac{48}{15}}$$

$$= \frac{-20 - 36}{15 - 48} = \frac{-56}{-33} = \frac{56}{33}$$

AS REQUIRED

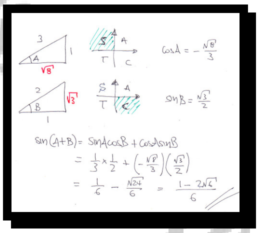
Question 23

$$\sin A = \frac{1}{3} \quad \text{and} \quad \cos B = \frac{1}{2}.$$

If A is obtuse and B is reflex, show clearly that

$$\sin(A+B) = \frac{1-2\sqrt{6}}{6}.$$

proof



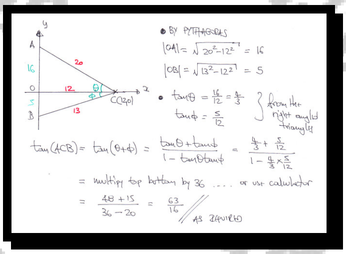
Question 24

The point A lies on the y axis above the origin O and the point B lies on the y axis below the origin O.

The point $C(12,0)$ is at a distance of 20 units from A and at a distance of 13 units from B.

By considering the tangent ratios of $\measuredangle OCA$ and $\measuredangle OCB$, show that the tangent of the angle ACB is exactly $\frac{63}{16}$.

proof



Question 25

Solve each of the following trigonometric equations.

a) $\cos(\theta + 30°) = \sin\theta$, $\quad 0 \le \theta < 360°$

b) $3\cos(x + 30°) = \sin(x - 60°)$, $\quad 0 \le x < 360°$

c) $\sin(y - 30°) = \sin(y + 45°)$, $\quad 0 \le y < 360°$

d) $\sin(\varphi + 30°) = \cos(\varphi - 45°)$, $\quad 0 \le \varphi < 360°$

e) $\cos(\alpha - 60°) = \cos(\alpha - 45°)$, $\quad 0 \le \alpha < 360°$

$\boxed{\theta = 30°,\ 210°}$, $\boxed{x = 60°,\ 240°}$, $\boxed{y = 82.5°,\ 262.5°}$, $\boxed{\varphi = 52.5°,\ 232.5°}$, $\boxed{\alpha = 52.5°,\ 232.5°}$

Question 26

Solve each of the following trigonometric equations.

a) $\sin(\theta - 45°) = \sin\theta, \qquad 0 \le \theta < 360°$

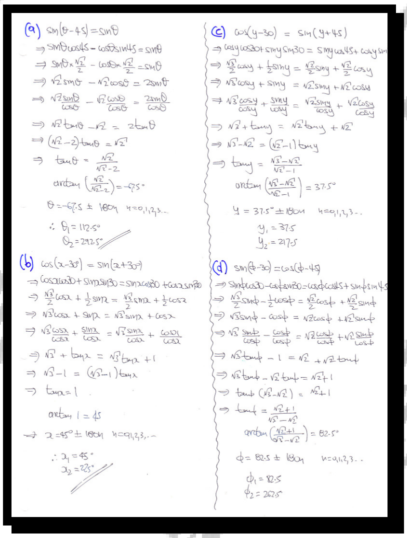
b) $\cos(x - 30°) = \sin(x + 30°), \qquad 0 \le x < 360°$

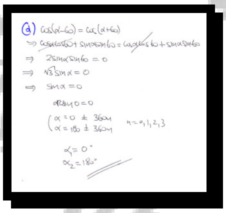
c) $\cos(y - 30°) = \sin(y + 45°), \qquad 0 \le y < 360°$

d) $\sin(\varphi - 30°) = \cos(\varphi - 45°), \qquad 0 \le \varphi < 360°$

e) $\cos(\alpha - 60°) = \cos(\alpha + 60°), \qquad 0 \le \alpha < 360°$

$\boxed{\theta = 112.5°,\ 292.5°}$, $\boxed{x = 45°,\ 225°}$, $\boxed{y = 37.5°,\ 217.5°}$, $\boxed{\varphi = 82.5°,\ 262.5°}$, $\boxed{\alpha = 0°,\ 180°}$





Question 27

Solve each of the following trigonometric equations.

a) $\sin\left(\theta+\frac{\pi}{4}\right)=\sin\theta\,, \qquad 0 \le \theta < 2\pi$

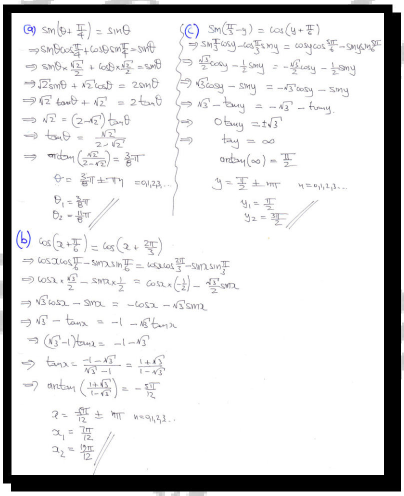
b) $\cos\left(x+\frac{\pi}{6}\right)=\cos\left(x+\frac{2\pi}{3}\right), \qquad 0 \le \theta < 2\pi$

c) $\sin\left(\frac{\pi}{3}-y\right)=\cos\left(y+\frac{5\pi}{6}\right), \qquad 0 \le y < 2\pi$ (very hard)

d) $2\cos\left(\varphi+\frac{\pi}{2}\right)+\sin\left(\varphi+\frac{\pi}{3}\right)=0\,, \qquad 0 \le \varphi < 2\pi$

e) $\sqrt{2}\cos\left(\alpha+\frac{\pi}{4}\right)=\sin\left(\alpha+\frac{\pi}{6}\right), \qquad 0 \le \alpha < 2\pi$

$\boxed{\theta=\frac{3\pi}{8},\ \frac{11\pi}{8}}$, $\boxed{x=\frac{7\pi}{12},\ \frac{19\pi}{12}}$, $\boxed{y=\frac{\pi}{2},\ \frac{3\pi}{2}}$, $\boxed{\varphi=\frac{\pi}{6},\ \frac{7\pi}{6}}$, $\boxed{\alpha=\frac{\pi}{12},\ \frac{13\pi}{12}}$



(a) $\sin\left(\theta+\frac{\pi}{4}\right)=\sin\theta$

$\Rightarrow \sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}=\sin\theta$

$\Rightarrow \sin\theta\times\frac{\sqrt{2}}{2}+\cos\theta\times\frac{\sqrt{2}}{2}=\sin\theta$

$\Rightarrow \sqrt{2}\sin\theta+\sqrt{2}\cos\theta=2\sin\theta$

$\Rightarrow \sqrt{2}\tan\theta+\sqrt{2}=2\tan\theta$

$\Rightarrow \sqrt{2}=(2-\sqrt{2})\tan\theta$

$\Rightarrow \tan\theta=\frac{\sqrt{2}}{2-\sqrt{2}}$

$\Rightarrow \arctan\left(\frac{\sqrt{2}}{2-\sqrt{2}}\right)=\frac{3}{8}\pi$

$\theta=\frac{3}{8}\pi\pm\pi n \quad =0,1,2,3,\ldots$

$\theta_1=\frac{3}{8}\pi$

$\theta_2=\frac{11}{8}\pi$

(c) $\sin\left(\frac{\pi}{3}-y\right)=\cos\left(y+\frac{\pi}{6}\right)$

$\Rightarrow \sin\frac{\pi}{3}\cos y-\cos\frac{\pi}{3}\sin y=\cos y\cos\frac{5\pi}{6}-\sin y\sin\frac{5\pi}{6}$

$\Rightarrow \frac{\sqrt{3}}{2}\cos y-\frac{1}{2}\sin y=-\frac{\sqrt{3}}{2}\cos y-\frac{1}{2}\sin y$

$\Rightarrow \sqrt{3}\cos y-\sin y=-\sqrt{3}\cos y-\sin y$

$\Rightarrow \sqrt{3}-\tan y=-\sqrt{3}-\tan y$

$\Rightarrow 0\tan y=\pm\sqrt{3}$

$\Rightarrow \tan y=\infty$

$\arctan(\infty)=\frac{\pi}{2}$

$y=\frac{\pi}{2}\pm n\pi \quad n=0,1,2,3\ldots$

$y_1=\frac{\pi}{2}$

$y_2=\frac{3\pi}{2}$

(b) $\cos\left(x+\frac{\pi}{6}\right)=\cos\left(x+\frac{2\pi}{3}\right)$

$\Rightarrow \cos x\cos\frac{\pi}{6}-\sin x\sin\frac{\pi}{6}=\cos x\cos\frac{2\pi}{3}-\sin x\sin\frac{\pi}{3}$

$\Rightarrow \cos x\times\frac{\sqrt{3}}{2}-\sin x\times\frac{1}{2}=\cos x\times\left(-\frac{1}{2}\right)-\frac{\sqrt{3}}{2}\sin x$

$\Rightarrow \sqrt{3}\cos x-\sin x=-\cos x-\sqrt{3}\sin x$

$\Rightarrow \sqrt{3}-\tan x=-1-\sqrt{3}\tan x$

$\Rightarrow (\sqrt{3}-1)\tan x=-1-\sqrt{3}$

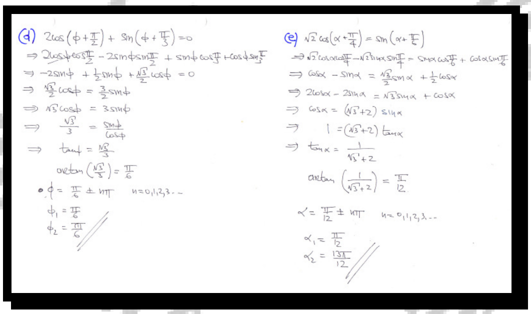
$\Rightarrow \tan x=\frac{-1-\sqrt{3}}{\sqrt{3}-1}=\frac{1+\sqrt{3}}{1-\sqrt{3}}$

$\Rightarrow \arctan\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right)=-\frac{5\pi}{12}$

$x=\frac{5\pi}{12}\pm n\pi \quad n=0,1,2,3\ldots$

$x_1=\frac{7\pi}{12}$

$x_2=\frac{19\pi}{12}$



(d) $2\cos\left(\phi+\frac{\pi}{2}\right)+\sin\left(\phi+\frac{\pi}{3}\right)=0$

$\Rightarrow 2\cos\phi\cos\frac{\pi}{2}-2\sin\phi\sin\frac{\pi}{2}+\sin\phi\cos\frac{\pi}{3}+\cos\phi\sin\frac{\pi}{3}$

$\Rightarrow -2\sin\phi+\frac{1}{2}\sin\phi+\frac{\sqrt{3}}{2}\cos\phi=0$

$\Rightarrow \frac{\sqrt{3}}{2}\cos\phi=\frac{3}{2}\sin\phi$

$\Rightarrow \sqrt{3}\cos\phi=3\sin\phi$

$\Rightarrow \frac{\sqrt{3}}{3}=\frac{\sin\phi}{\cos\phi}$

$\Rightarrow \tan\phi=\frac{\sqrt{3}}{3}$

$\arctan\left(\frac{\sqrt{3}}{3}\right)=\frac{\pi}{6}$

$\phi=\frac{\pi}{6}\pm n\pi \quad n=0,1,2,3\ldots$

$\phi_1=\frac{\pi}{6}$

$\phi_2=\frac{7\pi}{6}$

(e) $\sqrt{2}\cos\left(\alpha+\frac{\pi}{4}\right)=\sin\left(\alpha+\frac{\pi}{6}\right)$

$\Rightarrow \sqrt{2}\cos\alpha\cos\frac{\pi}{4}-\sqrt{2}\sin\alpha\sin\frac{\pi}{4}=\sin\alpha\cos\frac{\pi}{6}+\cos\alpha\sin\frac{\pi}{6}$

$\Rightarrow \cos\alpha-\sin\alpha=\frac{\sqrt{3}}{2}\sin\alpha+\frac{1}{2}\cos\alpha$

$\Rightarrow 2\cos\alpha-2\sin\alpha=\sqrt{3}\sin\alpha+\cos\alpha$

$\Rightarrow \cos\alpha=(\sqrt{3}+2)\sin\alpha$

$\Rightarrow 1=(\sqrt{3}+2)\tan\alpha$

$\Rightarrow \tan\alpha=\frac{1}{\sqrt{3}+2}$

$\arctan\left(\frac{1}{\sqrt{3}+2}\right)=\frac{\pi}{12}$

$\alpha=\frac{\pi}{12}\pm n\pi \quad n=0,1,2,3,\ldots$

$\alpha_1=\frac{\pi}{12}$

$\alpha_2=\frac{13\pi}{12}$

Question 28

Solve each of the following trigonometric equations.

a) $\sin(\theta - 20°) = \sin(\theta + 60°)$, $\quad 0 \le \theta < 360°$

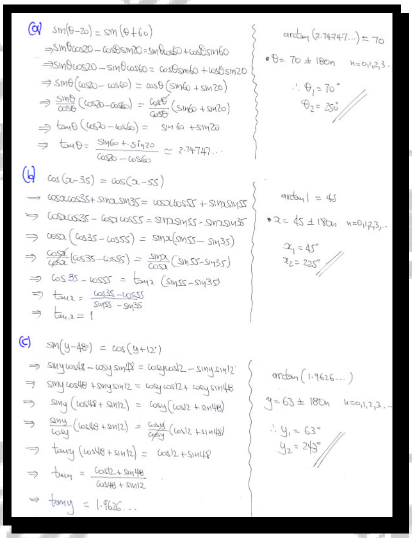
b) $\cos(x - 35°) = \cos(x - 55°)$, $\quad 0 \le x < 360°$

c) $\sin(y - 48°) = \cos(y + 12°)$, $\quad 0 \le y < 360°$

d) $\sin(\varphi + 72°) = \cos(\varphi - 38°)$, $\quad 0 \le \varphi < 360°$

e) $\cos(\alpha - 36°) = \cos(\alpha - 72°)$, $\quad 0 \le \alpha < 360°$

$\boxed{\theta = 70°,\ 250°}$, $\boxed{x = 45°,\ 225°}$, $\boxed{y = 63°,\ 243°}$, $\boxed{\varphi = 28°,\ 208°}$, $\boxed{\alpha = 54°,\ 234°}$



(a) $\sin(\theta-20) = \sin(\theta+60)$

$\Rightarrow \sin\theta\cos20 - \cos\theta\sin20 = \sin\theta\cos60 + \cos\theta\sin60$

$\Rightarrow \sin\theta\cos20 - \sin\theta\cos60 = \cos\theta\sin60 + \cos\theta\sin20$

$\Rightarrow \sin\theta(\cos20 - \cos60) = \cos\theta(\sin60 + \sin20)$

$\Rightarrow \frac{\sin\theta}{\cos\theta}(\cos20 - \cos60) = \frac{\cos\theta}{\cos\theta}(\sin60 + \sin20)$

$\Rightarrow \tan\theta(\cos20 - \cos60) = \sin60 + \sin20$

$\Rightarrow \tan\theta = \frac{\sin60 + \sin20}{\cos20 - \cos60} \simeq 2.74747\ldots$

$\arctan(2.74747\ldots) = 70$

• $\theta = 70 \pm 180n \quad n = 0,1,2,3\ldots$

$\therefore \theta_1 = 70°$

$\theta_2 = 250°$

(b) $\cos(x-35) = \cos(x-55)$

$\Rightarrow \cos x\cos35 + \sin x\sin35 = \cos x\cos55 + \sin x\sin55$

$\Rightarrow \cos x\cos35 - \cos x\cos55 = \sin x\sin55 - \sin x\sin35$

$\Rightarrow \cos x(\cos35 - \cos55) = \sin x(\sin55 - \sin35)$

$\Rightarrow \frac{\cos x}{\cos x}(\cos35 - \cos55) = \frac{\sin x}{\cos x}(\sin55 - \sin35)$

$\Rightarrow \cos35 - \cos55 = \tan x(\sin55 - \sin35)$

$\Rightarrow \tan x = \frac{\cos35 - \cos55}{\sin55 - \sin35}$

$\Rightarrow \tan x = 1$

$\arctan 1 = 45$

• $x = 45 \pm 180n \quad n = 0,1,2,3,\ldots$

$x_1 = 45°$

$x_2 = 225°$

(c) $\sin(y-48°) = \cos(y+12°)$

$\Rightarrow \sin y\cos48 - \cos y\sin48 = \cos y\cos12 - \sin y\sin12$

$\Rightarrow \sin y\cos48 + \sin y\sin12 = \cos y\cos12 + \cos y\sin48$

$\Rightarrow \sin y(\cos48 + \sin12) = \cos y(\cos12 + \sin48)$

$\Rightarrow \frac{\sin y}{\cos y}(\cos48 + \sin12) = \frac{\cos y}{\cos y}(\cos12 + \sin48)$

$\Rightarrow \tan y(\cos48 + \sin12) = \cos12 + \sin48$

$\Rightarrow \tan y = \frac{\cos12 + \sin48}{\cos48 + \sin12}$

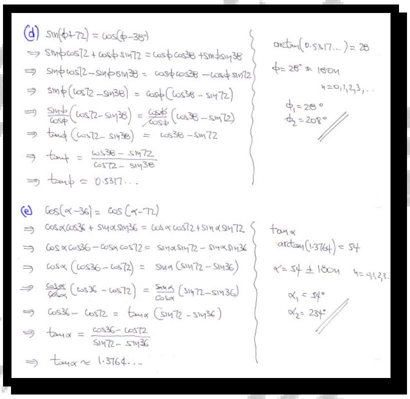
$\Rightarrow \tan y = 1.9626\ldots$

$\arctan(1.9626\ldots)$

$y = 63 \pm 180n \quad n = 0,1,2,3\ldots$

$\therefore y_1 = 63°$

$y_2 = 243°$



(d) $\sin(\phi+72) = \cos(\phi-38°)$

$\Rightarrow \sin\phi\cos72 + \cos\phi\sin72 = \cos\phi\cos38 + \sin\phi\sin38$

$\Rightarrow \sin\phi\cos72 - \sin\phi\sin38 = \cos\phi\cos38 - \cos\phi\sin72$

$\Rightarrow \sin\phi(\cos72 - \sin38) = \cos\phi(\cos38 - \sin72)$

$\Rightarrow \frac{\sin\phi}{\cos\phi}(\cos72 - \sin38) = \frac{\cos\phi}{\cos\phi}(\cos38 - \sin72)$

$\Rightarrow \tan\phi(\cos72 - \sin38) = \cos38 - \sin72$

$\Rightarrow \tan\phi = \frac{\cos38 - \sin72}{\cos72 - \sin38}$

$\Rightarrow \tan\phi = 0.5317\ldots$

$\arctan(0.5317\ldots) = 28$

$\phi = 28° \pm 180n \quad n = 0,1,2,3,\ldots$

$\phi_1 = 28°$

$\phi_2 = 208°$

(e) $\cos(\alpha-36) = \cos(\alpha-72)$

$\Rightarrow \cos\alpha\cos36 + \sin\alpha\sin36 = \cos\alpha\cos72 + \sin\alpha\sin72$

$\Rightarrow \cos\alpha\cos36 - \cos\alpha\cos72 = \sin\alpha\sin72 - \sin\alpha\sin36$

$\Rightarrow \cos\alpha(\cos36 - \cos72) = \sin\alpha(\sin72 - \sin36)$

$\Rightarrow \frac{\cos\alpha}{\cos\alpha}(\cos36 - \cos72) = \frac{\sin\alpha}{\cos\alpha}(\sin72 - \sin36)$

$\Rightarrow \cos36 - \cos72 = \tan\alpha(\sin72 - \sin36)$

$\Rightarrow \tan\alpha = \frac{\cos36 - \cos72}{\sin72 - \sin36}$

$\Rightarrow \tan\alpha \approx 1.3764\ldots$

$\tan\alpha$

$\arctan(1.3764) = 54$

$\alpha = 54 \pm 180n \quad n = 0,1,2,3\ldots$

$\alpha_1 = 54°$

$\alpha_2 = 234°$