TRIGONOMETRY

THE DOUBLE ANGLE IDENTITIES

Question 1

Prove the validity of each of the following trigonometric identities.

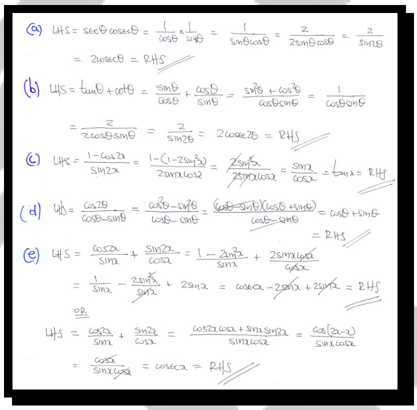
a) $\sec\theta \operatorname{cosec}\theta \equiv 2\operatorname{cosec} 2\theta$

b) $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$

c) $\dfrac{1-\cos 2x}{\sin 2x} \equiv \tan x$

d) $\dfrac{\cos 2\theta}{\cos\theta - \sin\theta} \equiv \cos\theta + \sin\theta$

e) $\dfrac{\cos 2x}{\sin x} + \dfrac{\sin 2x}{\cos x} \equiv \operatorname{cosec} x$



(a) $\text{LHS} = \sec\theta\operatorname{cosec}\theta = \frac{1}{\cos\theta}\times\frac{1}{\sin\theta} = \frac{1}{\sin\theta\cos\theta} = \frac{2}{2\sin\theta\cos\theta} = \frac{2}{\sin 2\theta}$

$= 2\operatorname{cosec}\theta = \text{RHS}$

(b) $\text{LHS} = \tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\cos\theta\sin\theta}$

$= \frac{2}{2\cos\theta\sin\theta} = \frac{2}{\sin 2\theta} = 2\operatorname{cosec}2\theta = \text{RHS}$

(c) $\text{LHS} = \frac{1-\cos 2x}{\sin 2x} = \frac{1-(1-2\sin^2 x)}{2\sin x\cos x} = \frac{2\sin^2 x}{2\sin x\cos x} = \frac{\sin x}{\cos x} = \tan x = \text{RHS}$

(d) $\text{LHS} = \frac{\cos 2\theta}{\cos\theta - \sin\theta} = \frac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta} = \frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{\cos\theta - \sin\theta} = \cos\theta + \sin\theta$

$= \text{RHS}$

(e) $\text{LHS} = \frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \frac{1-2\sin^2 x}{\sin x} + \frac{2\sin x\cos x}{\cos x}$

$= \frac{1}{\sin x} - \frac{2\sin^2 x}{\sin x} + 2\sin x = \operatorname{cosec}x - 2\sin x + 2\sin x = \text{RHS}$

OR

$\text{LHS} = \frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \frac{\cos 2x\cos x + \sin x\sin 2x}{\sin x\cos x} = \frac{\cos(2x-x)}{\sin x\cos x}$

$= \frac{\cos x}{\sin x\cos x} = \operatorname{cosec}x = \text{RHS}$

Question 2

Prove the validity of each of the following trigonometric identities.

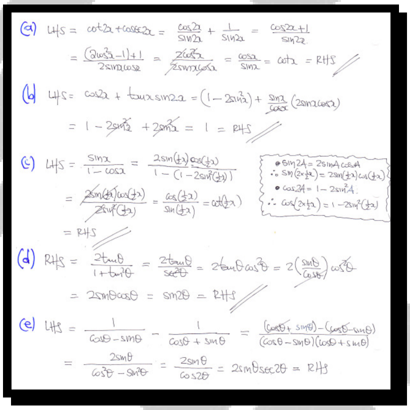
a) $\cot 2x + \operatorname{cosec} 2x \equiv \cot x$

b) $\cos 2x + \tan x \sin 2x \equiv 1$

c) $\dfrac{\sin x}{1-\cos x} \equiv \cot \dfrac{1}{2}x$

d) $\sin 2\theta \equiv \dfrac{2\tan\theta}{1+\tan^2\theta}$

e) $\dfrac{1}{\cos\theta - \sin\theta} - \dfrac{1}{\cos\theta + \sin\theta} \equiv 2\sin\theta \sec 2\theta$


(a) $LHS = \cot 2x + \operatorname{cosec} 2x = \frac{\cos 2x}{\sin 2x} + \frac{1}{\sin 2x} = \frac{\cos 2x + 1}{\sin 2x}$

$= \frac{(2\cos^2 x - 1) + 1}{2\sin x \cos x} = \frac{2\cos^2 x}{2\sin x \cos x} = \frac{\cos x}{\sin x} = \cot x = RHS$

(b) $LHS = \cos 2x + \tan x \sin 2x = (1 - 2\sin^2 x) + \frac{\sin x}{\cos x}(2\sin x \cos x)$

$= 1 - 2\sin^2 x + 2\sin^2 x = 1 = RHS$

(c) $LHS = \frac{\sin x}{1-\cos x} = \frac{2\sin(\frac{1}{2}x)\cos(\frac{1}{2}x)}{1 - (1 - 2\sin^2(\frac{1}{2}x))}$

$= \frac{2\sin(\frac{1}{2}x)\cos(\frac{1}{2}x)}{2\sin^2(\frac{1}{2}x)} = \frac{\cos(\frac{1}{2}x)}{\sin(\frac{1}{2}x)} = \cot(\frac{1}{2}x)$

$= RHS$

- $\sin 2A = 2\sin A \cos A$
- $\therefore \sin(2\times\frac{1}{2}x) = 2\sin(\frac{1}{2}x)\cos(\frac{1}{2}x)$
- $\cos 2A = 1 - 2\sin^2 A$
- $\therefore \cos(2\times\frac{1}{2}x) = 1 - 2\sin^2(\frac{1}{2}x)$

(d) $RHS = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\tan\theta}{\sec^2\theta} = 2\tan\theta\cos^2\theta = 2\left(\frac{\sin\theta}{\cos\theta}\right)\cos^2\theta$

$= 2\sin\theta\cos\theta = \sin 2\theta = RHS$

(e) $LHS = \frac{1}{\cos\theta - \sin\theta} - \frac{1}{\cos\theta + \sin\theta} = \frac{(\cos\theta + \sin\theta) - (\cos\theta - \sin\theta)}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$

$= \frac{2\sin\theta}{\cos^2\theta - \sin^2\theta} = \frac{2\sin\theta}{\cos 2\theta} = 2\sin\theta\sec 2\theta = RHS$

Question 3

Prove the validity of each of the following trigonometric identities.

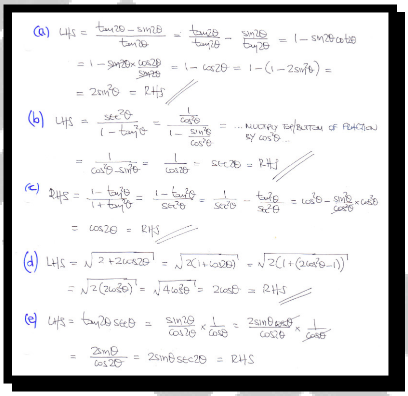
a) $\dfrac{\tan 2\theta - \sin 2\theta}{\tan 2\theta} \equiv 2\sin^2\theta$

b) $\dfrac{\sec^2\theta}{1-\tan^2\theta} \equiv \sec 2\theta$

c) $\cos 2\theta \equiv \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

d) $\sqrt{2+2\cos 2\theta} \equiv 2\cos\theta$

e) $\tan 2\theta \sec\theta \equiv 2\sin\theta \sec 2\theta$



(a) $\text{LHS} = \dfrac{\tan 2\theta - \sin 2\theta}{\tan 2\theta} = \dfrac{\tan 2\theta}{\tan 2\theta} - \dfrac{\sin 2\theta}{\tan 2\theta} = 1 - \sin 2\theta \cot 2\theta$

$= 1 - \sin 2\theta \times \dfrac{\cos 2\theta}{\sin 2\theta} = 1 - \cos 2\theta = 1 - (1 - 2\sin^2\theta) =$

$= 2\sin^2\theta = \text{RHS}$

(b) $\text{LHS} = \dfrac{\sec^2\theta}{1-\tan^2\theta} = \dfrac{\frac{1}{\cos^2\theta}}{1 - \frac{\sin^2\theta}{\cos^2\theta}} = \ldots$ MULTIPLY TOP/BOTTOM OF FRACTION BY $\cos^2\theta$ ...

$= \dfrac{1}{\cos^2\theta - \sin^2\theta} = \dfrac{1}{\cos 2\theta} = \sec 2\theta = \text{RHS}$

(c) $\text{RHS} = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-\tan^2\theta}{\sec^2\theta} = \dfrac{1}{\sec^2\theta} - \dfrac{\tan^2\theta}{\sec^2\theta} = \cos^2\theta - \dfrac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta$

$= \cos 2\theta = \text{RHS}$

(d) $\text{LHS} = \sqrt{2+2\cos 2\theta} = \sqrt{2(1+\cos 2\theta)} = \sqrt{2(1+(2\cos^2\theta - 1))}$

$= \sqrt{2(2\cos^2\theta)} = \sqrt{4\cos^2\theta} = 2\cos\theta = \text{RHS}$

(e) $\text{LHS} = \tan 2\theta \sec\theta = \dfrac{\sin 2\theta}{\cos 2\theta} \times \dfrac{1}{\cos\theta} = \dfrac{2\sin\theta\cos\theta}{\cos 2\theta} \times \dfrac{1}{\cos\theta}$

$= \dfrac{2\sin\theta}{\cos 2\theta} = 2\sin\theta \sec 2\theta = \text{RHS}$

Question 4

Prove the validity of each of the following trigonometric identities.

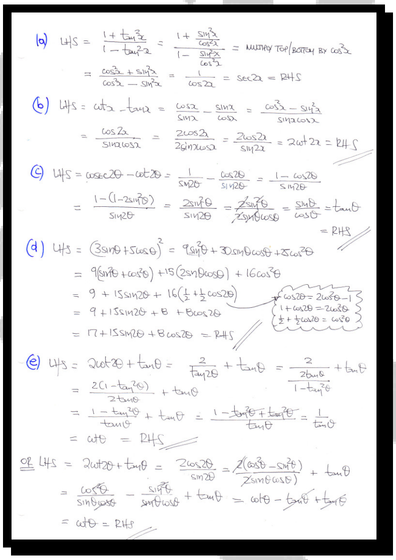
a) $\dfrac{1+\tan^2 x}{1-\tan^2 x} \equiv \sec 2x$

b) $\cot x - \tan x \equiv 2\cot 2x$

c) $\operatorname{cosec} 2\theta - \cot 2\theta \equiv \tan\theta$

d) $(3\sin\theta + 5\cos\theta)^2 \equiv 17 + 8\cos 2\theta + 15\sin 2\theta$

e) $2\cot 2\theta + \tan\theta \equiv \cot\theta$



Question 5

Prove the validity of each of the following trigonometric identities.

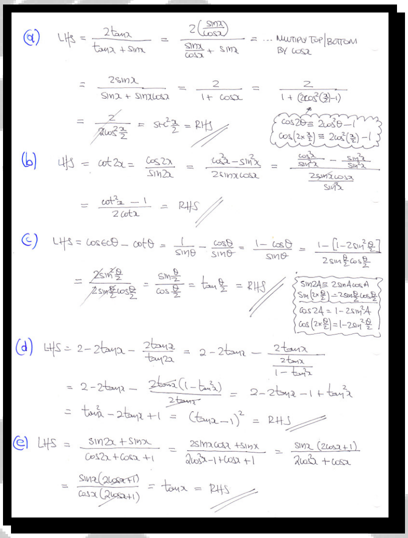
a) $\dfrac{2\tan x}{\tan x+\sin x}\equiv \sec^2\left(\dfrac{x}{2}\right)$

b) $\cot 2x \equiv \dfrac{\cot^2 x-1}{2\cot x}$

c) $\operatorname{cosec}\theta-\cot\theta\equiv \tan\dfrac{1}{2}\theta$

d) $2-2\tan x-\dfrac{2\tan x}{\tan 2x}\equiv(1-\tan x)^2$

e) $\dfrac{\sin 2x+\sin x}{\cos 2x+\cos x+1}\equiv \tan x$



Question 6

Prove the validity of each of the following trigonometric identities.

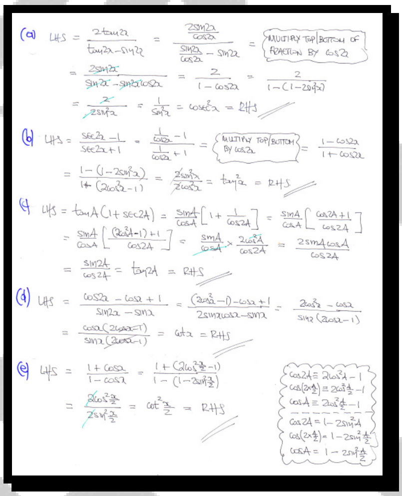
a) $\dfrac{2\tan 2x}{\tan 2x - \sin 2x} \equiv \text{cosec}^2 x$

b) $\dfrac{\sec 2x - 1}{\sec 2x + 1} \equiv \tan^2 x$

c) $\tan A\left(1+\sec 2A\right) \equiv \tan 2A$

d) $\dfrac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} \equiv \cot x$

e) $\dfrac{1+\cos x}{1-\cos x} \equiv \cot^2 \dfrac{1}{2}x$



Question 7

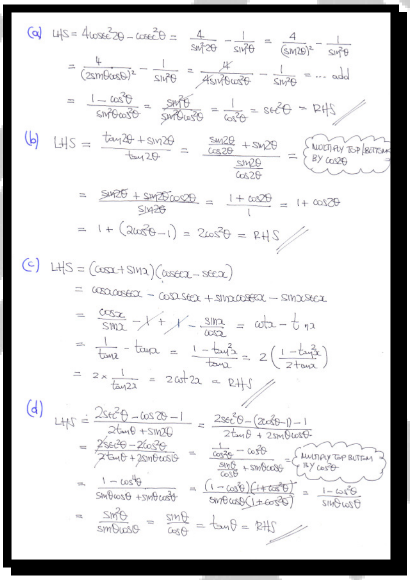
Prove the validity of each of the following trigonometric identities.

a) $4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta \equiv \sec^2\theta$

b) $\dfrac{\tan 2\theta + \sin 2\theta}{\tan 2\theta} \equiv 2\cos^2\theta$

c) $(\cos x + \sin x)(\operatorname{cosec} x - \sec x) \equiv 2\cot 2x$

d) $\dfrac{2\sec^2\theta - \cos 2\theta - 1}{2\tan\theta + \sin 2\theta} \equiv \tan\theta$



Question 8

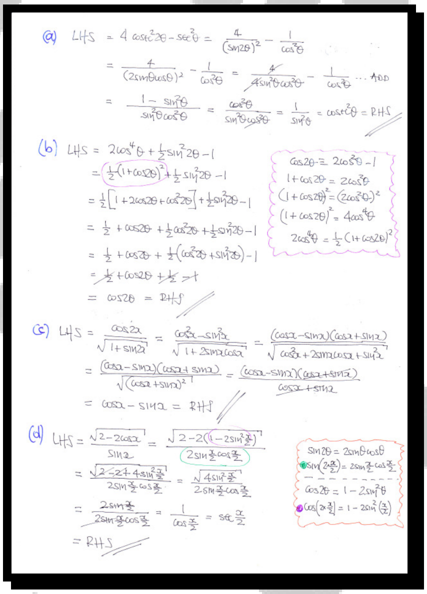
Prove the validity of each of the following trigonometric identities.

a) $4\operatorname{cosec}^2 2\theta - \sec^2\theta \equiv \operatorname{cosec}^2\theta$

b) $2\cos^4\theta + \frac{1}{2}\sin^2 2\theta - 1 \equiv \cos 2\theta$

c) $\frac{\cos 2x}{\sqrt{1+\sin 2x}} \equiv \cos x - \sin x$

d) $\frac{\sqrt{2-2\cos x}}{\sin x} \equiv \sec\frac{x}{2}$



(a) $\text{LHS} = 4\operatorname{cosec}^2 2\theta - \sec^2\theta = \frac{4}{(\sin 2\theta)^2} - \frac{1}{\cos^2\theta}$

$= \frac{4}{(2\sin\theta\cos\theta)^2} - \frac{1}{\cos^2\theta} = \frac{4}{4\sin^2\theta\cos^2\theta} - \frac{1}{\cos^2\theta}$ ... ADD

$= \frac{1-\sin^2\theta}{\sin^2\theta\cos^2\theta} = \frac{\cos^2\theta}{\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta} = \operatorname{cosec}^2\theta = \text{RHS}$

(b) $\text{LHS} = 2\cos^4\theta + \frac{1}{2}\sin^2 2\theta - 1$

$= \frac{1}{2}(1+\cos 2\theta)^2 + \frac{1}{2}\sin^2 2\theta - 1$

$= \frac{1}{2}\left[1 + 2\cos 2\theta + \cos^2 2\theta\right] + \frac{1}{2}\sin^2 2\theta - 1$

$= \frac{1}{2} + \cos 2\theta + \frac{1}{2}\cos^2 2\theta + \frac{1}{2}\sin^2 2\theta - 1$

$= \frac{1}{2} + \cos 2\theta + \frac{1}{2}(\cos^2 2\theta + \sin^2 2\theta) - 1$

$= \frac{1}{2} + \cos 2\theta + \frac{1}{2} - 1$

$= \cos 2\theta = \text{RHS}$

$\cos 2\theta \equiv 2\cos^2\theta - 1$
$1 + \cos 2\theta = 2\cos^2\theta$
$(1+\cos 2\theta)^2 = (2\cos^2\theta)^2$
$(1+\cos 2\theta)^2 = 4\cos^4\theta$
$2\cos^4\theta = \frac{1}{2}(1+\cos 2\theta)^2$

(c) $\text{LHS} = \frac{\cos 2x}{\sqrt{1+\sin 2x}} = \frac{\cos^2 x - \sin^2 x}{\sqrt{1+2\sin x\cos x}} = \frac{(\cos x - \sin x)(\cos x + \sin x)}{\sqrt{\cos^2 x + 2\sin x\cos x + \sin^2 x}}$

$= \frac{(\cos x - \sin x)(\cos x + \sin x)}{\sqrt{(\cos x + \sin x)^2}} = \frac{(\cos x - \sin x)(\cos x + \sin x)}{\cos x + \sin x}$

$= \cos x - \sin x = \text{RHS}$

(d) $\text{LHS} = \frac{\sqrt{2-2\cos x}}{\sin x} = \frac{\sqrt{2-2\left(1-2\sin^2\frac{x}{2}\right)}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$= \frac{\sqrt{2-2+4\sin^2\frac{x}{2}}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\sqrt{4\sin^2\frac{x}{2}}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$= \frac{2\sin\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1}{\cos\frac{x}{2}} = \sec\frac{x}{2}$

$= \text{RHS}$

$\sin 2\theta = 2\sin\theta\cos\theta$
$\sin\left(2\times\frac{x}{2}\right) = 2\sin\frac{x}{2}\cos\frac{x}{2}$
$\cos 2\theta = 1 - 2\sin^2\theta$
$\cos\left(2\times\frac{x}{2}\right) = 1 - 2\sin^2\left(\frac{x}{2}\right)$

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Question 9

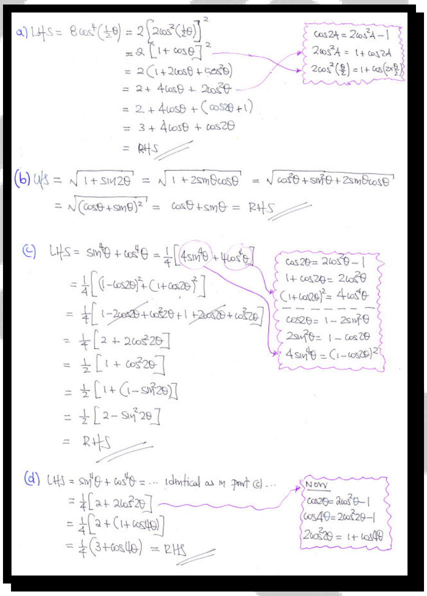
Prove the validity of each of the following trigonometric identities.

a) $8\cos^4\left(\frac{1}{2}\theta\right) \equiv \cos 2\theta + 4\cos\theta + 3$

b) $\sqrt{1+\sin 2\theta} \equiv \sin\theta + \cos\theta$

c) $\sin^4\theta + \cos^4\theta \equiv \frac{1}{2}\left(2 - \sin^2 2\theta\right)$

d) $\sin^4\theta + \cos^4\theta \equiv \frac{1}{4}\left(3 + \cos 4\theta\right)$



Question 10

Solve each of the following trigonometric equations.

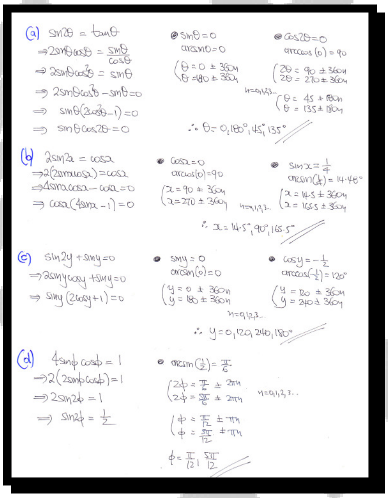
a) $\sin 2\theta = \tan\theta$, $\quad 0 \le \theta \le 180°$

b) $2\sin 2x = \cos x$, $\quad 0 \le x < 180°$

c) $\sin 2y + \sin y = 0$, $\quad 0 \le y < 360°$

d) $4\sin\varphi\cos\varphi = 1$, $\quad 0 \le \varphi < \pi$

$\boxed{\theta = 0°,\ 45°,\ 135°,\ 180°}$, $\boxed{x = 90°,\ x \approx 14.5°,\ 165.5°}$, $\boxed{y = 0°,\ 120°,\ 180°,\ 240°}$, $\boxed{\varphi = \frac{\pi}{12},\ \frac{5\pi}{12}}$



(a) $\sin 2\theta = \tan\theta$
$\Rightarrow 2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}$
$\Rightarrow 2\sin\theta\cos^2\theta = \sin\theta$
$\Rightarrow 2\sin\theta\cos^2\theta - \sin\theta = 0$
$\Rightarrow \sin\theta(2\cos^2\theta - 1) = 0$
$\Rightarrow \sin\theta\cos 2\theta = 0$

• $\sin\theta = 0$, $\arcsin 0 = 0$: $\theta = 0 \pm 360n$, $\theta = 180 \pm 360n$
• $\cos 2\theta = 0$, $\arccos(0) = 90$: $2\theta = 90 \pm 360n$, $2\theta = 270 \pm 360n$; $\theta = 45 \pm 180n$, $\theta = 135 \pm 180n$, $n = 0, 1, 2, 3 \ldots$

$\therefore \theta = 0, 180°, 45°, 135°$

(b) $2\sin 2x = \cos x$
$\Rightarrow 2(2\sin x\cos x) = \cos x$
$\Rightarrow 4\sin x\cos x - \cos x = 0$
$\Rightarrow \cos x(4\sin x - 1) = 0$

• $\cos x = 0$, $\arccos(0) = 90$: $x = 90 \pm 360n$, $x = 270 \pm 360n$, $n = 0, 1, 2, 3 \ldots$
• $\sin x = \frac{1}{4}$, $\arcsin(\frac{1}{4}) = 14.48°$: $x = 14.5 \pm 360n$, $x = 165.5 \pm 360n$

$\therefore x = 14.5°, 90°, 165.5°$

(c) $\sin 2y + \sin y = 0$
$\Rightarrow 2\sin y\cos y + \sin y = 0$
$\Rightarrow \sin y(2\cos y + 1) = 0$

• $\sin y = 0$, $\arcsin(0) = 0$: $y = 0 \pm 360n$, $y = 180 \pm 360n$
• $\cos y = -\frac{1}{2}$, $\arccos(-\frac{1}{2}) = 120°$: $y = 120 \pm 360n$, $y = 240 \pm 360n$, $n = 0, 1, 2, 3 \ldots$

$\therefore y = 0, 120, 240, 180°$

(d) $4\sin\phi\cos\phi = 1$
$\Rightarrow 2(2\sin\phi\cos\phi) = 1$
$\Rightarrow 2\sin 2\phi = 1$
$\Rightarrow \sin 2\phi = \frac{1}{2}$

• $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$: $2\phi = \frac{\pi}{6} \pm 2\pi n$, $2\phi = \frac{5\pi}{6} \pm 2\pi n$, $n = 0, 1, 2, 3 \ldots$
$\phi = \frac{\pi}{12} \pm \pi n$, $\phi = \frac{5\pi}{12} \pm \pi n$

$\phi = \frac{\pi}{12}, \frac{5\pi}{12}$

Question 11

Solve each of the following trigonometric equations.

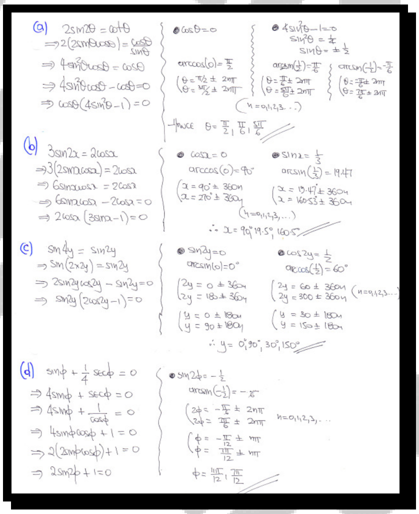
a) $2\sin 2\theta = \cot\theta$, $\quad 0 \le \theta \le \pi$

b) $3\sin 2x = 2\cos x$, $\quad 0 \le x < 180°$

c) $\sin 4y = \sin 2y$, $\quad 0 \le y < 180°$

d) $\sin\varphi + \frac{1}{4}\sec\varphi = 0$, $\quad 0 \le \varphi < \pi$

$$\boxed{\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}}, \ \boxed{x = 90°, \ x \approx 19.5°, \ 160.5°}, \ \boxed{y = 0°, \ 30°, \ 90°, \ 150°}, \ \boxed{\varphi = \frac{7\pi}{12}, \ \frac{11\pi}{12}}$$



Question 12

Solve each of the following trigonometric equations.

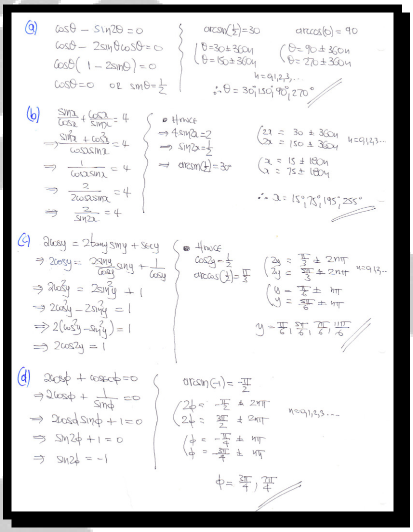
a) $\cos\theta - \sin 2\theta = 0$, $\quad 0 \le \theta \le 360°$

b) $\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = 4$, $\quad 0 \le x < 360°$

c) $2\cos y = 2\tan y \sin y + \sec y$, $\quad 0 \le y < 2\pi$

d) $2\cos\varphi + \operatorname{cosec}\varphi = 0$, $\quad 0 \le \varphi < 2\pi$

$\boxed{\theta = 30°,\ 90°,\ 150°,\ 270°}$, $\boxed{x = 15°,\ 75°,\ 195°,\ 255°}$, $\boxed{y = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$, $\boxed{\varphi = \frac{3\pi}{4}, \frac{7\pi}{4}}$



Question 13

Solve each of the following trigonometric equations.

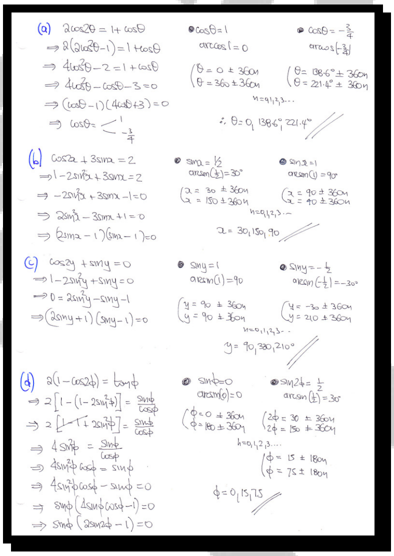
a) $2\cos 2\theta = 1 + \cos\theta$, $\quad 0 \le \theta < 360°$

b) $\cos 2x + 3\sin x = 2$, $\quad 0 \le x < 360°$

c) $\cos 2y + \sin y = 0$, $\quad 0 \le y < 360°$

d) $2(1 - \cos 2\varphi) = \tan\varphi$, $\quad 0 \le \varphi < 180°$

$\boxed{\theta = 0°,\ \theta \approx 138.6°,\ 221.4°}$, $\boxed{x = 30°,\ 90°,\ 150°}$, $\boxed{y = 90°,\ 210°,\ 330°}$, $\boxed{\varphi = 0°,\ 15°,\ 75°}$



Question 14

Solve each of the following trigonometric equations.

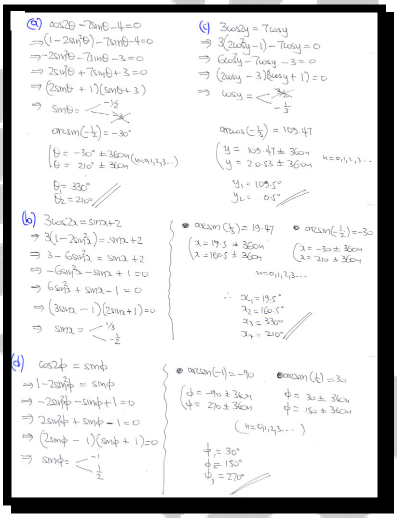
a) $\cos 2\theta - 7\sin\theta - 4 = 0$, $\quad 0 \le \theta < 360°$

b) $3\cos 2x = \sin x + 2$, $\quad 0 \le x < 360°$

c) $3\cos 2y = 7\cos y$, $\quad 0 \le y < 360°$

d) $\cos 2\varphi = \sin\varphi$, $\quad 0 \le \varphi < 360°$

$\boxed{\theta = 210°,\ 330°}$, $\boxed{x \approx 19.5°,\ 160.5° \quad x = 210°,\ 330°}$, $\boxed{y \approx 109.5°,\ 250.5°}$, $\boxed{\varphi = 30°,\ 150°,\ 270°}$



Question 15

Solve each of the following trigonometric equations.

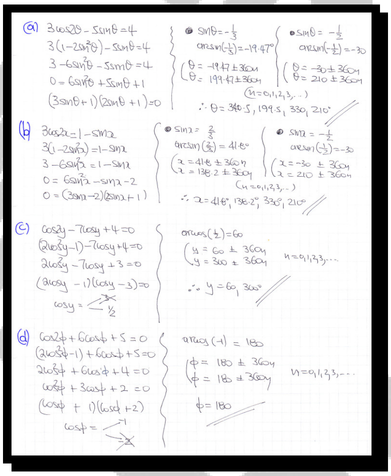
a) $3\cos 2\theta - 5\sin\theta = 4\,, \qquad 0 \le \theta < 360°$

b) $3\cos 2x = 1 - \sin x\,, \qquad 0 \le x < 360°$

c) $\cos 2y - 7\cos y + 4 = 0\,, \qquad 0 \le y < 360°$

d) $\cos 2\varphi + 6\cos\varphi + 5 = 0\,, \qquad 0 \le \varphi < 360°$

$\boxed{\theta = 210°,\ 330° \quad \theta \approx 199.5°,\ 340.5°}\,,\ \boxed{x \approx 41.8°,\ 138.2° \quad x = 210°,\ 330°}\,,$
$\boxed{y = 60°,\ 300°}\,,\ \boxed{\varphi = 180°}$



Question 16

Solve each of the following trigonometric equations.

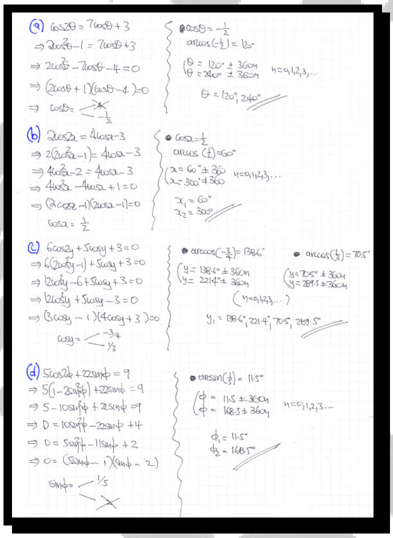
a) $\cos 2\theta = 7\cos\theta + 3$, $\quad 0 \le \theta < 360°$

b) $2\cos 2x = 4\cos x - 3$, $\quad 0 \le x < 360°$

c) $6\cos 2y + 5\cos y + 3 = 0$, $\quad 0 \le y < 360°$

d) $5\cos 2\varphi + 22\sin\varphi = 9$, $\quad 0 \le \varphi < 360°$

$\boxed{\theta = 120°,\ 240°}$, $\boxed{x = 60°,\ 300°}$, $\boxed{y \approx 70.5°,\ 138.6°,\ 221.4°,\ 289.5°}$, $\boxed{\varphi \approx 11.5°,\ 168.5°}$


(a) $\cos 2\theta = 7\cos\theta + 3$
$\Rightarrow 2\cos^2\theta - 1 = 7\cos\theta + 3$
$\Rightarrow 2\cos^2\theta - 7\cos\theta - 4 = 0$
$\Rightarrow (2\cos\theta + 1)(\cos\theta - 4) = 0$
$\Rightarrow \cos\theta = -\frac{1}{2}$ (4 rejected)

$\cos\theta = -\frac{1}{2}$, $\arccos(-\frac{1}{2}) = 120°$
$\theta = 120° \pm 360n$, $\theta = 240° \pm 360n$, $n = 0, 1, 2, 3, \ldots$
$\theta = 120°, 240°$

(b) $2\cos 2x = 4\cos x - 3$
$\Rightarrow 2(2\cos^2 x - 1) = 4\cos x - 3$
$\Rightarrow 4\cos^2 x - 2 = 4\cos x - 3$
$\Rightarrow 4\cos^2 x - 4\cos x + 1 = 0$
$\Rightarrow (2\cos x - 1)(2\cos x - 1) = 0$
$\cos x = \frac{1}{2}$

$\cos x = \frac{1}{2}$, $\arccos(\frac{1}{2}) = 60°$
$x = 60° \pm 360n$, $x = 300° \pm 360n$, $n = 0, 1, 2, 3, \ldots$
$x_1 = 60°$, $x_2 = 300°$

(c) $6\cos 2y + 5\cos y + 3 = 0$
$\Rightarrow 6(2\cos^2 y - 1) + 5\cos y + 3 = 0$
$\Rightarrow 12\cos^2 y - 6 + 5\cos y + 3 = 0$
$\Rightarrow 12\cos^2 y + 5\cos y - 3 = 0$
$\Rightarrow (3\cos y - 1)(4\cos y + 3) = 0$
$\cos y = -3/4$ or $1/3$

$\arccos(-\frac{3}{4}) = 138.6°$: $y = 138.6° \pm 360n$, $y = 221.4° \pm 360n$
$\arccos(\frac{1}{3}) = 70.5°$: $y = 70.5° \pm 360n$, $y = 289.5 \pm 360n$
$(n = 0, 1, 2, 3, \ldots)$
$y = 138.6°, 221.4°, 70.5°, 289.5°$

(d) $5\cos 2\phi + 22\sin\phi = 9$
$\Rightarrow 5(1 - 2\sin^2\phi) + 22\sin\phi = 9$
$\Rightarrow 5 - 10\sin^2\phi + 22\sin\phi = 9$
$\Rightarrow 0 = 10\sin^2\phi - 22\sin\phi + 4$
$\Rightarrow 0 = 5\sin^2\phi - 11\sin\phi + 2$
$\Rightarrow 0 = (5\sin\phi - 1)(\sin\phi - 2)$
$\sin\phi = 1/5$ (2 rejected)

$\arcsin(\frac{1}{5}) = 11.5°$
$\phi = 11.5 \pm 360n$, $\phi = 168.5 \pm 360n$, $n = 0, 1, 2, 3, \ldots$
$\phi_1 = 11.5°$, $\phi_2 = 168.5°$

Question 17

Solve each of the following trigonometric equations.

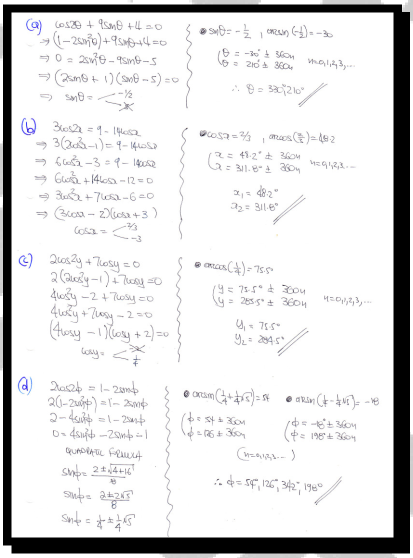
a) $\cos 2\theta + 9\sin\theta + 4 = 0$, $\quad 0 \le \theta < 360°$

b) $3\cos 2x = 9 - 14\cos x$, $\quad 0 \le x < 360°$

c) $2\cos 2y + 7\cos y = 0$, $\quad 0 \le y < 360°$

d) $2\cos 2\varphi = 1 - 2\sin\varphi$, $\quad 0 \le \varphi < 360°$

$\boxed{\theta = 210°,\ 330°}$, $\boxed{x \approx 48.2°,\ 311.8°}$, $\boxed{y \approx 75.5°,\ 284.5°}$, $\boxed{\varphi = 54°,\ 126°,\ 198°,\ 342°}$



Question 18

Solve each of the following trigonometric equations.

a) $\cos 2\theta = 1 + \sin\theta$, $\quad 0 \le \theta < 360°$

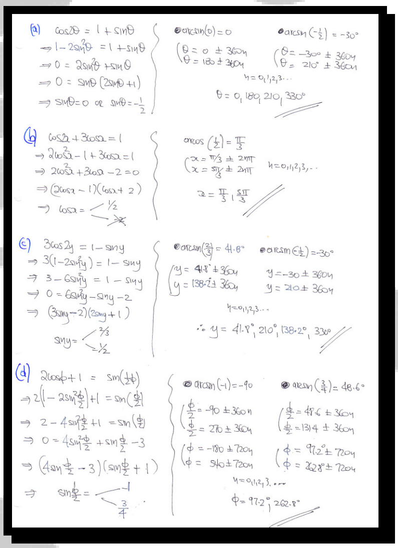
b) $\cos 2x + 3\cos x = 1$, $\quad 0 \le x < 2\pi$

c) $3\cos 2y = 1 - \sin y$, $\quad 0 \le y < 360°$

d) $2\cos\varphi + 1 = \sin\left(\frac{1}{2}\varphi\right)$, $\quad 0 \le \varphi < 360°$

$\boxed{\theta = 0°,\ 180°,\ 210°,\ 330°}$, $\boxed{x = \frac{\pi}{3},\ \frac{5\pi}{3}}$, $\boxed{y \approx 41.8°,\ 138.2° \quad y = 210°,\ 330°}$,

$\boxed{\varphi = 97.2°,\ 262.8°}$



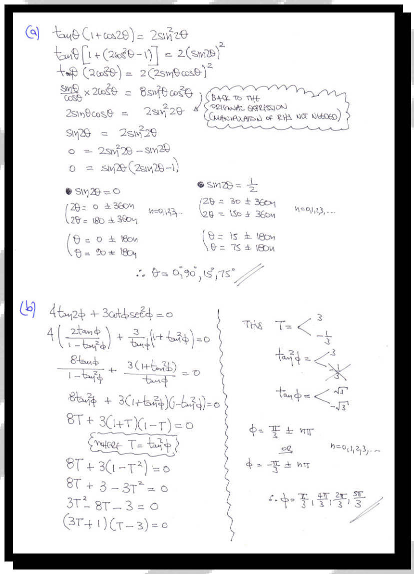
Question 19

Solve each of the following trigonometric equations.

a) $\tan\theta(1+\cos 2\theta) = 2\sin^2 2\theta$, $\quad 0 \le \theta \le 90°$

b) $4\tan 2\varphi + 3\cot\varphi\sec^2\varphi = 0$, $\quad 0 \le \varphi < 2\pi$ **(hard)**

$\boxed{\theta = 0°,\ 15°,\ 75°,\ 90°}$, $\boxed{\varphi = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$



(a) $\tan\theta(1+\cos 2\theta) = 2\sin^2 2\theta$

$\tan\theta[1+(2\cos^2\theta-1)] = 2(\sin 2\theta)^2$

$\tan\theta(2\cos^2\theta) = 2(2\sin\theta\cos\theta)^2$

$\frac{\sin\theta}{\cos\theta}\times 2\cos^2\theta = 8\sin^2\theta\cos^2\theta$

$2\sin\theta\cos\theta = 2\sin^2 2\theta$ (BACK TO THE ORIGINAL EXPRESSION (MANIPULATION OF RHS NOT NEEDED))

$\sin 2\theta = 2\sin^2 2\theta$

$0 = 2\sin^2 2\theta - \sin 2\theta$

$0 = \sin 2\theta(2\sin 2\theta - 1)$

- $\sin 2\theta = 0$: $2\theta = 0 \pm 360n$, $2\theta = 180 \pm 360n$, $n = 0,1,2,3,\ldots$; $\theta = 0 \pm 180n$, $\theta = 90 \pm 180n$
- $\sin 2\theta = \frac{1}{2}$: $2\theta = 30 \pm 360n$, $2\theta = 150 \pm 360n$, $n = 0,1,2,3,\ldots$; $\theta = 15 \pm 180n$, $\theta = 75 \pm 180n$

$\therefore \theta = 0°, 90°, 15°, 75°$

(b) $4\tan 2\phi + 3\cot\phi\sec^2\phi = 0$

$4\left(\frac{2\tan\phi}{1-\tan^2\phi}\right) + \frac{3}{\tan\phi}(1+\tan^2\phi) = 0$

$\frac{8\tan\phi}{1-\tan^2\phi} + \frac{3(1+\tan^2\phi)}{\tan\phi} = 0$

$8\tan^2\phi + 3(1+\tan^2\phi)(1-\tan^2\phi) = 0$

$8T + 3(1+T)(1-T) = 0$ {where $T = \tan^2\phi$}

$8T + 3(1-T^2) = 0$

$8T + 3 - 3T^2 = 0$

$3T^2 - 8T - 3 = 0$

$(3T+1)(T-3) = 0$

Thus $T = 3$ or $-\frac{1}{3}$

$\tan^2\phi = 3$ (~~$-\frac{1}{3}$~~)

$\tan\phi = \sqrt{3}$ or $-\sqrt{3}$

$\phi = \frac{\pi}{3} \pm n\pi$ or $\phi = -\frac{\pi}{3} \pm n\pi$, $n = 0,1,2,3,\ldots$

$\therefore \phi = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$

Question 20

Show clearly that each of the following trigonometric equations has no real roots, regardless of the solution interval.

a) $\cos 2\theta = 23 + 14\cos\theta$

b) $\cos 2x + \cos x + 2 = 0$

proof