470

TRIGONOMETRY EXAM QUESTIONS

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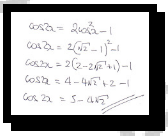
BASIC QUESTIONS

Question 1 **(**)**

Given that $\cos x = \sqrt{2} - 1$, show clearly that

$$\cos 2x = 5 - 4\sqrt{2} \; .$$

proof

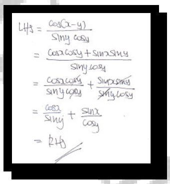


Question 2 **(**)**

Show clearly that

$$\frac{\cos(x-y)}{\sin y \cos y} \equiv \frac{\cos x}{\sin y} + \frac{\sin x}{\cos y} \; .$$

proof

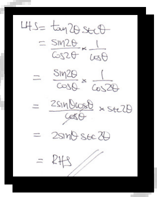


Question 3 **(**+)**

Prove the validity of the trigonometric identity

$$\tan 2\theta \sec\theta \equiv 2\sin\theta \sec 2\theta \; .$$

proof



Question 4 (**+)

If $\sin x = \frac{3}{5}$, show clearly that

$$\sec 2x = \frac{25}{7}.$$

[illegible], proof

$$\sec 2x = \frac{1}{\cos 2x} = \frac{1}{1-2\sin^2 x} = \frac{1}{1-2\left(\frac{3}{5}\right)^2} = \frac{1}{1-\frac{18}{25}} = \frac{25}{7}$$

Question 5 (**+)

Solve the following trigonometric equation

$$\cos x \cot x + \sin x + 2\cot x = 0, \quad 0 \le x < 360°, \quad x \ne 0°, 180°.$$

$x = 120°, 240°$

$$\cos x \cot x + \sin x + 2\cot x = 0$$
$$\Rightarrow \cos x \times \frac{\cos x}{\sin x} + \sin x + 2\left(\frac{\cos x}{\sin x}\right) = 0$$
$$\Rightarrow \frac{\cos^2 x}{\sin x} + \sin x + \frac{2\cos x}{\sin x} = 0$$
$$\Rightarrow \cos^2 x + \sin^2 x + 2\cos x = 0$$
$$\Rightarrow 1 + 2\cos x = 0$$
$$\Rightarrow 2\cos x = -1$$
$$\Rightarrow \cos x = -\frac{1}{2}$$

$$\arccos\left(-\tfrac{1}{2}\right) = 120°$$

$$\begin{cases} x = 120 \pm 360n \\ x = 240 \pm 360n \end{cases} \quad n = 0, 1, 2, 3, \ldots$$

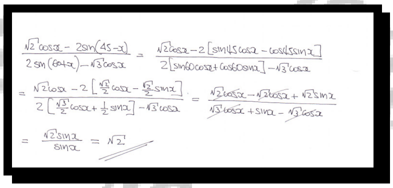
$$x_1 = 120°$$
$$x_2 = 240°$$

Question 6 **(**+)**

Simplify fully the following trigonometric expression

$$\frac{\sqrt{2}\cos x^\circ - 2\sin(45-x)^\circ}{2\sin(60+x)^\circ - \sqrt{3}\cos x^\circ}.$$

$\boxed{\sqrt{2}}$



$$\frac{\sqrt{2}\cos x - 2\sin(45-x)}{2\sin(60+x) - \sqrt{3}\cos x} = \frac{\sqrt{2}\cos x - 2[\sin 45\cos x - \cos 45 \sin x]}{2[\sin 60\cos x + \cos 60 \sin x] - \sqrt{3}\cos x}$$

$$= \frac{\sqrt{2}\cos x - 2\left[\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x\right]}{2\left[\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right] - \sqrt{3}\cos x} = \frac{\sqrt{2}\cos x - \sqrt{2}\cos x + \sqrt{2}\sin x}{\sqrt{3}\cos x + \sin x - \sqrt{3}\cos x}$$

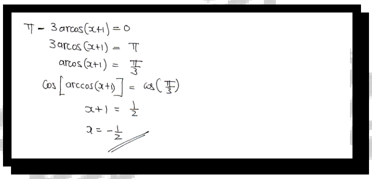
$$= \frac{\sqrt{2}\sin x}{\sin x} = \sqrt{2}$$

Question 7 **(**+)**

Solve the following trigonometric equation

$$\pi - 3\arccos(x+1) = 0.$$

$\boxed{}$, $\boxed{x = -\frac{1}{2}}$



$$\pi - 3\arccos(x+1) = 0$$
$$3\arccos(x+1) = \pi$$
$$\arccos(x+1) = \frac{\pi}{3}$$
$$\cos[\arccos(x+1)] = \cos\left(\frac{\pi}{3}\right)$$
$$x+1 = \frac{1}{2}$$
$$x = -\frac{1}{2}$$

Question 8 (**+)

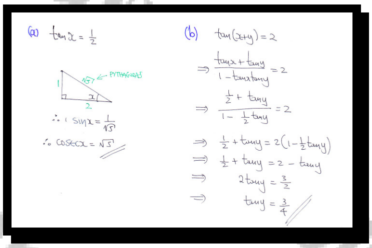
The angle x is acute so that $\tan x = \frac{1}{2}$.

a) Find the exact value of $\text{cosec}\, x$.

It is further given that $\tan(x+y) = 2$, where y is another angle.

b) Determine the value of $\tan y$.

$\boxed{\text{cosec}\, x = \sqrt{5}}$, $\boxed{\tan y = \frac{3}{4}}$

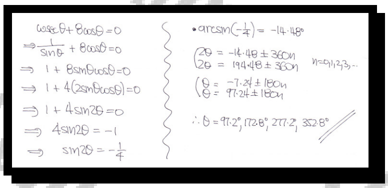


Question 9 (***)

$$\text{cosec}\,\theta + 8\cos\theta = 0, \ 0° \le \theta < 360°.$$

Find the solutions of the above trigonometric equation, giving the answers in degrees correct to one decimal place.

$\boxed{\theta = 97.2°, \ 172.8°, \ 277.2°, \ 352.8°}$

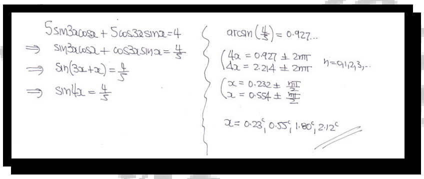


Question 10 (***)

$$5\sin 3x\cos x + 5\cos 3x\sin x = 4\,,\ \ 0 \le x < \pi\,.$$

Use a compound angle trigonometric identity to find the solutions of the above trigonometric equation, giving the answers in radians correct to two decimal places.

[ ] , $\boxed{x = 0.23^{\text{c}}, 0.55^{\text{c}}, 1.80^{\text{c}}, 2.12^{\text{c}}}$

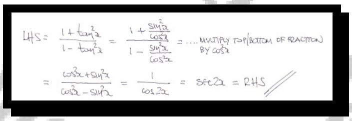


Question 11 (***)

Prove the validity of the trigonometric identity

$$\frac{1+\tan^2 x}{1-\tan^2 x} \equiv \sec 2x\,.$$

proof

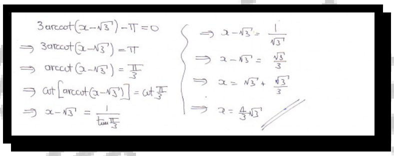


Question 12 (***)

Solve the following trigonometric equation

$$3\operatorname{arccot}\left(x-\sqrt{3}\right)-\pi=0.$$

$$\boxed{x=\frac{4}{3}\sqrt{3}}$$


$3\operatorname{arccot}(x-\sqrt{3})-\pi=0$
$\Rightarrow 3\operatorname{arccot}(x-\sqrt{3})=\pi$
$\Rightarrow \operatorname{arccot}(x-\sqrt{3})=\frac{\pi}{3}$
$\Rightarrow \cot\left[\operatorname{arccot}(x-\sqrt{3})\right]=\cot\frac{\pi}{3}$
$\Rightarrow x-\sqrt{3}=\frac{1}{\tan\frac{\pi}{3}}$
$\Rightarrow x-\sqrt{3}=\frac{1}{\sqrt{3}}$
$\Rightarrow x-\sqrt{3}=\frac{\sqrt{3}}{3}$
$\Rightarrow x=\sqrt{3}+\frac{\sqrt{3}}{3}$
$\Rightarrow x=\frac{4}{3}\sqrt{3}$

Question 13 (***)

$$f(x)\equiv\sqrt{3}\sin x+\cos x,\quad 0\le x<2\pi.$$

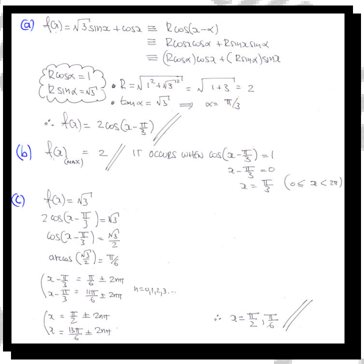
a) Express $f(x)$ in the form $R\cos(x-\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

b) State the maximum value of $f(x)$ and find the value of x for which this maximum value occurs.

c) Solve the equation

$$f(x)=\sqrt{3}.$$

$\boxed{f(x)\equiv 2\cos\left(x-\frac{\pi}{3}\right)}$, $\boxed{f(x)_{\max}=2}$, $\boxed{x=\frac{\pi}{3}}$, $\boxed{x=\frac{\pi}{6},\frac{\pi}{2}}$


(a) $f(x)=\sqrt{3}\sin x+\cos x\equiv R\cos(x-\alpha)$
$\equiv R\cos x\cos\alpha+R\sin x\sin\alpha$
$\equiv(R\cos\alpha)\cos x+(R\sin\alpha)\sin x$
$R\cos\alpha=1$, $R\sin\alpha=\sqrt{3}$
• $R=\sqrt{1^2+\sqrt{3}^2}=\sqrt{1+3}=2$
• $\tan\alpha=\sqrt{3}\Rightarrow\alpha=\pi/3$
$\therefore f(x)=2\cos\left(x-\frac{\pi}{3}\right)$
(b) $f(x)_{(\text{MAX})}=2$ IT OCCURS WHEN $\cos\left(x-\frac{\pi}{3}\right)=1$
$x-\frac{\pi}{3}=0$
$x=\frac{\pi}{3}$ $(0\le x<2\pi)$
(c) $f(x)=\sqrt{3}$
$2\cos\left(x-\frac{\pi}{3}\right)=\sqrt{3}$
$\cos\left(x-\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$
$\arccos\left(\frac{\sqrt{3}}{2}\right)=\pi/6$
$x-\frac{\pi}{3}=\frac{\pi}{6}\pm2n\pi$
$x-\frac{\pi}{3}=\frac{11\pi}{6}\pm2n\pi$ $n=0,1,2,3\ldots$
$x=\frac{\pi}{2}\pm2n\pi$
$x=\frac{13\pi}{6}\pm2n\pi$
$\therefore x=\frac{\pi}{2},\frac{\pi}{6}$

Question 14 (***)

It is given that

$$\frac{1+\cot^2 x}{\cot x \operatorname{cosec} x} \equiv \sec x .$$

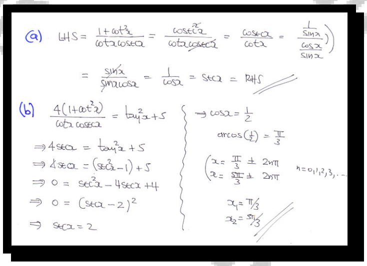
a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$\frac{4\left(1+\cot^2 x\right)}{\cot x \operatorname{cosec} x} = \tan^2 x + 5 , \quad 0 \le x < 2\pi ,$$

giving the answers in terms of π.

$\square$, $x = \frac{\pi}{3}, \frac{5\pi}{3}$


(a) $\text{LHS} = \frac{1+\cot^2 x}{\cot x \operatorname{cosec} x} = \frac{\operatorname{cosec}^2 x}{\cot x \operatorname{cosec} x} = \frac{\operatorname{cosec} x}{\cot x} = \frac{\frac{1}{\sin x}}{\frac{\cos x}{\sin x}}$

$= \frac{\sin x}{\sin x \cos x} = \frac{1}{\cos x} = \sec x = \text{RHS}$

(b) $\frac{4(1+\cot^2 x)}{\cot x \operatorname{cosec} x} = \tan^2 x + 5$

$\Rightarrow 4\sec x = \tan^2 x + 5$

$\Rightarrow 4\sec x = (\sec^2 x - 1) + 5$

$\Rightarrow 0 = \sec^2 x - 4\sec x + 4$

$\Rightarrow 0 = (\sec x - 2)^2$

$\Rightarrow \sec x = 2$

$\Rightarrow \cos x = \frac{1}{2}$

$\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$x = \frac{\pi}{3} \pm 2n\pi$, $x = \frac{5\pi}{3} \pm 2n\pi$, $n = 0, 1, 2, 3, \ldots$

$x_1 = \pi/3$

$x_2 = 5\pi/3$

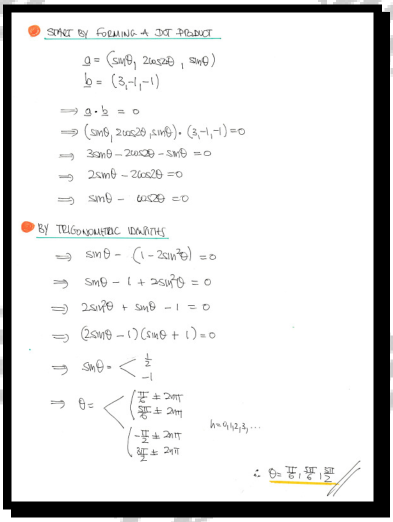
Question 15 (***)

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are given below.

$$\mathbf{a} = (\sin\theta)\mathbf{i} + (2\cos 2\theta)\mathbf{j} + (\sin\theta)\mathbf{k} \qquad \text{and} \qquad \mathbf{a} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}\,.$$

Find the values of θ, $0 \le \theta < 2\pi$, for which $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

$\boxed{}$, $\boxed{\theta = \frac{1}{6}\pi,\ \frac{5}{6}\pi,\ \frac{3}{2}\pi}$



START BY FORMING A DOT PRODUCT

$\underline{a} = (\sin\theta, 2\cos2\theta, \sin\theta)$
$\underline{b} = (3,-1,-1)$

$\Rightarrow \underline{a}\cdot\underline{b} = 0$
$\Rightarrow (\sin\theta, 2\cos2\theta, \sin\theta)\cdot(3,-1,-1) = 0$
$\Rightarrow 3\sin\theta - 2\cos2\theta - \sin\theta = 0$
$\Rightarrow 2\sin\theta - 2\cos2\theta = 0$
$\Rightarrow \sin\theta - \cos2\theta = 0$

BY TRIGONOMETRIC IDENTITIES

$\Rightarrow \sin\theta - (1 - 2\sin^2\theta) = 0$
$\Rightarrow \sin\theta - 1 + 2\sin^2\theta = 0$
$\Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$
$\Rightarrow (2\sin\theta - 1)(\sin\theta + 1) = 0$
$\Rightarrow \sin\theta = \begin{cases} \frac{1}{2} \\ -1 \end{cases}$
$\Rightarrow \theta = \begin{cases} \frac{\pi}{6} \pm 2n\pi \\ \frac{5\pi}{6} \pm 2n\pi \\ -\frac{\pi}{2} \pm 2n\pi \\ \frac{3\pi}{2} \pm 2n\pi \end{cases}$ $\quad n = 0,1,2,3,\ldots$

$\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

Question 16 (***)

It is given that

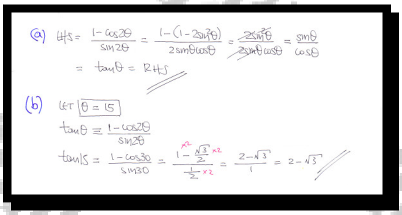
$$\frac{1-\cos 2\theta}{\sin 2\theta} \equiv \tan\theta\,, \quad \theta \neq 90k°,\ k \in \mathbb{Z}\,.$$

a) Prove the validity of the above trigonometric identity.

b) Hence show that

$$\tan 15° = 2-\sqrt{3}\,.$$

[ ] , [proof]

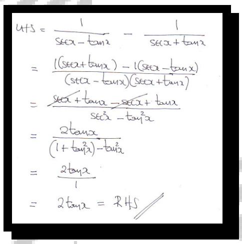


Question 17 (***)

Prove the validity of the trigonometric identity

$$\frac{1}{\sec x - \tan x} - \frac{1}{\sec x + \tan x} \equiv 2\tan x\,.$$

[proof]



Question 18 (***)

Solve the trigonometric equation

$$\cos\theta + \sec\theta = \frac{5}{2}, \quad 0° \le \theta < 360° .$$

$\boxed{\theta = 60°, 300°}$

$\cos\theta + \sec\theta = \frac{5}{2}$

$\Rightarrow \cos\theta + \frac{1}{\cos\theta} = \frac{5}{2}$

$\Rightarrow 2\cos\theta + \frac{2}{\cos\theta} = 5$

$\Rightarrow 2\cos^2\theta + 2 = 5\cos\theta$

$\Rightarrow 2\cos^2\theta - 5\cos\theta + 2 = 0$

$\Rightarrow (2\cos\theta - 1)(\cos\theta - 2) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$ or 2 (rejected)

$\Rightarrow \cos\theta = \frac{1}{2}$

$\arccos(\frac{1}{2}) = 60°$

$\theta = 60° \pm 360n$, $\theta = 300 \pm 360n$, $n = 0, 1, 2, 3, \ldots$

$\theta_1 = 60°$

$\theta_2 = 300°$

Question 19 (***)

Prove the validity of the trigonometric identity

$$\sqrt{2 + 2\cos 2\theta} \equiv 2\cos\theta .$$

$\boxed{\text{proof}}$

$\text{LHS} = \sqrt{2 + 2\cos 2\theta} = \sqrt{2 + 2(2\cos^2\theta - 1)} = \sqrt{2 + 4\cos^2\theta - 2}$

$= \sqrt{4\cos^2\theta} = 2\cos\theta = \text{RHS}$

Question 20 (***)

$$y \equiv 2\sqrt{2}\cos x + 2\sqrt{2}\sin x\,,\ \ x \in \mathbb{R}\,.$$

a) Express y in the form $R\sin(x+\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

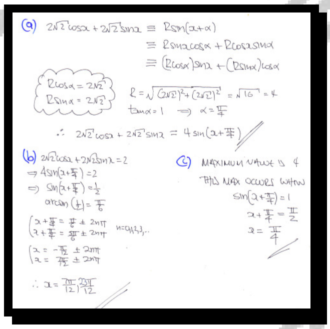
b) Solve the equation

$$y = 2 \ \text{ for } \ 0 < x < 2\pi\,.$$

c) Write down the maximum value of y.

d) Find the smallest positive value of x for which this maximum value occurs.

$\boxed{y \equiv 4\sin\left(x+\frac{\pi}{4}\right)}$, $\boxed{x = \frac{7\pi}{12}, \frac{23\pi}{12}}$, $\boxed{y_{max} = 4}$, $\boxed{x = \frac{\pi}{4}}$


(a) $2\sqrt{2}\cos x + 2\sqrt{2}\sin x \equiv R\sin(x+\alpha)$
$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$
$\equiv (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$
$R\cos\alpha = 2\sqrt{2}$
$R\sin\alpha = 2\sqrt{2}$
$R = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{16} = 4$
$\tan\alpha = 1 \implies \alpha = \frac{\pi}{4}$
$\therefore 2\sqrt{2}\cos x + 2\sqrt{2}\sin x = 4\sin\left(x+\frac{\pi}{4}\right)$

(b) $2\sqrt{2}\cos x + 2\sqrt{2}\sin x = 2$
$\implies 4\sin\left(x+\frac{\pi}{4}\right) = 2$
$\implies \sin\left(x+\frac{\pi}{4}\right) = \frac{1}{2}$
$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$
$x+\frac{\pi}{4} = \frac{\pi}{6} \pm 2n\pi$
$x+\frac{\pi}{4} = \frac{5\pi}{6} \pm 2n\pi$ $\quad n = 0,1,2,3,\ldots$
$x = -\frac{\pi}{12} \pm 2n\pi$
$x = \frac{7\pi}{12} \pm 2n\pi$
$\therefore x = \frac{7\pi}{12}, \frac{23\pi}{12}$

(c) MAXIMUM VALUE IS 4
THIS MAX OCCURS WHEN
$\sin\left(x+\frac{\pi}{4}\right) = 1$
$x+\frac{\pi}{4} = \frac{\pi}{2}$
$x = \frac{\pi}{4}$

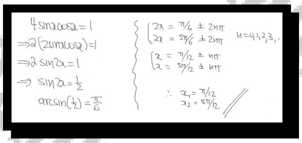
Question 21 (***)

Solve the following trigonometric equation

$$4\sin x\cos x = 1, \quad 0 \le x < \pi,$$

giving the answers in terms of π.

$$x = \frac{\pi}{12}, \frac{5\pi}{12}$$

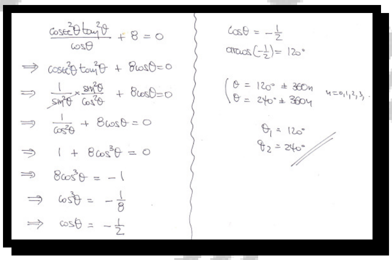


Question 22 (***)

Solve the following trigonometric equation

$$\frac{\text{cosec}^2\theta\tan^2\theta}{\cos\theta} + 8 = 0, \quad 0° \le \theta < 360°.$$

$$\theta = 120°, 240°$$



Question 23 (***)

Solve the following trigonometric equation

$$8\tan x = \cot^2 x, \quad 0 \le x < 2\pi .$$

$x = 0.464^c, 3.61^c$

$8\tan x = \cot^2 x$

$\Rightarrow 8\tan x = \frac{1}{\tan^2 x}$

$\Rightarrow 8\tan^3 x = 1$

$\Rightarrow \tan^3 x = \frac{1}{8}$

$\Rightarrow \tan x = \frac{1}{2}$

Hence $\arctan\left(\frac{1}{2}\right) = 0.464^c$

• $x = 0.464 \pm n\pi \quad n = 0,1,2,3\ldots$

$x_1 = 0.464^c$

$x_2 = 3.61^c$

Question 24 (***)

Prove the validity of the following trigonometric identity

$$\frac{1}{\cos\theta - \sin\theta} - \frac{1}{\cos\theta + \sin\theta} \equiv 2\sin\theta\sec 2\theta .$$

proof

$\text{LHS} = \frac{1}{\cos\theta - \sin\theta} - \frac{1}{\cos\theta + \sin\theta} = \frac{(\cos\theta + \sin\theta) - (\cos\theta - \sin\theta)}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$

$= \frac{2\sin\theta}{\cos^2\theta - \sin^2\theta} = \frac{2\sin\theta}{\cos 2\theta} = 2\sin\theta\sec 2\theta = \text{RHS}$

Question 25 (***)

It is given that

$$\frac{1+\tan^2 x}{1-\tan^2 x} \equiv \sec 2x .$$

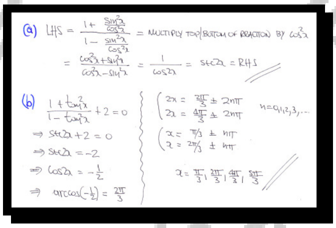
a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$\frac{1+\tan^2 x}{1-\tan^2 x} + 2 = 0, \quad 0 \le x < 2\pi ,$$

giving the answers in terms of π.

$$\boxed{x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$$


(a) $\text{LHS} = \dfrac{1+\frac{\sin^2 x}{\cos^2 x}}{1-\frac{\sin^2 x}{\cos^2 x}}$ = MULTIPLY TOP/BOTTOM OF FRACTION BY $\cos^2 x$

$= \dfrac{\cos^2 x+\sin^2 x}{\cos^2 x-\sin^2 x} = \dfrac{1}{\cos 2x} = \sec 2x = \text{RHS}$

(b) $\dfrac{1+\tan^2 x}{1-\tan^2 x}+2=0$

$\Rightarrow \sec 2x + 2 = 0$

$\Rightarrow \sec 2x = -2$

$\Rightarrow \cos 2x = -\frac{1}{2}$

$\Rightarrow \arccos(-\frac{1}{2}) = \frac{2\pi}{3}$

$2x = \frac{2\pi}{3} \pm 2n\pi$, $2x = \frac{4\pi}{3} \pm 2n\pi$, $n = 0,1,2,3,\ldots$

$x = \pi/3 \pm n\pi$, $x = 2\pi/3 \pm n\pi$

$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Question 26 (***)

$$f(x) \equiv \sin x + \sqrt{3}\cos x, \ 0 \le x < 2\pi .$$

a) Express $f(x)$ in the form $R\cos(x-\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

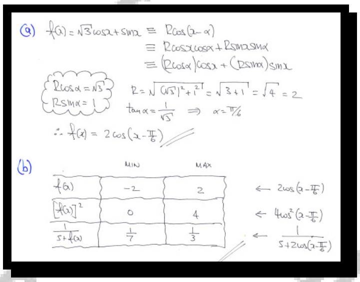
b) State the minimum and maximum value of …

i. … $f(x)$.

ii. … $\left[f(x)\right]^2$.

iii. … $\dfrac{1}{5+f(x)}$.

$\boxed{}$, $\boxed{f(x) \equiv 2\cos\left(x-\frac{\pi}{6}\right)}$, $\boxed{[-2,2]}$, $\boxed{[0,4]}$, $\boxed{\left[\frac{1}{7},\frac{1}{3}\right]}$



Question 27 (***)

$$\cos^2 x + \sin^2 x \equiv 1 .$$

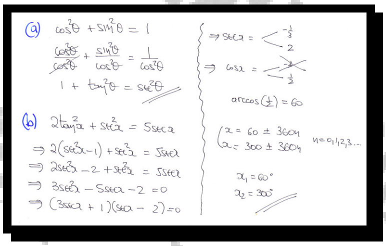
a) Starting with the above identity prove that

$$1 + \tan^2 x \equiv \sec^2 x .$$

b) Hence, or otherwise, solve the following trigonometric equation

$$2\tan^2 x + \sec^2 x = 5\sec x , \quad 0 \le x < 360° .$$

$\boxed{}$, $\boxed{x = 60°, 300°}$



(a) $\cos^2\theta + \sin^2\theta = 1$

$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$

$1 + \tan^2\theta = \sec^2\theta$

(b) $2\tan^2 x + \sec^2 x = 5\sec x$

$\Rightarrow 2(\sec^2 x - 1) + \sec^2 x = 5\sec x$

$\Rightarrow 2\sec^2 x - 2 + \sec^2 x = 5\sec x$

$\Rightarrow 3\sec^2 x - 5\sec x - 2 = 0$

$\Rightarrow (3\sec x + 1)(\sec x - 2) = 0$

$\Rightarrow \sec x = -\frac{1}{3}, \; 2$

$\Rightarrow \cos x = -3 \text{ (rejected)}, \; \frac{1}{2}$

$\arccos\left(\frac{1}{2}\right) = 60$

$x = 60 \pm 360n$, $x = 300 \pm 360n$, $n = 0, 1, 2, 3 \ldots$

$x_1 = 60°$

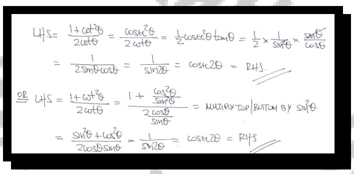
$x_2 = 300°$

Question 28 (***)

Prove the validity of the following trigonometric identity

$$\frac{1 + \cot^2\theta}{2\cot\theta} \equiv \operatorname{cosec} 2\theta .$$

$\boxed{}$, $\boxed{\text{proof}}$



$\text{LHS} = \frac{1+\cot^2\theta}{2\cot\theta} = \frac{\operatorname{cosec}^2\theta}{2\cot\theta} = \frac{1}{2}\operatorname{cosec}^2\theta\tan\theta = \frac{1}{2} \times \frac{1}{\sin^2\theta} \times \frac{\sin\theta}{\cos\theta}$

$= \frac{1}{2\sin\theta\cos\theta} = \frac{1}{\sin 2\theta} = \operatorname{cosec} 2\theta = \text{RHS}$

OR $\text{LHS} = \frac{1+\cot^2\theta}{2\cot\theta} = \frac{1 + \frac{\cos^2\theta}{\sin^2\theta}}{\frac{2\cos\theta}{\sin\theta}}$ = MULTIPLY TOP/BOTTOM BY $\sin^2\theta$

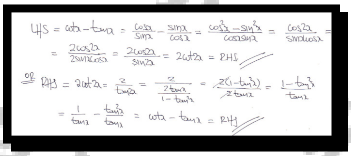
$= \frac{\sin^2\theta + \cos^2\theta}{2\cos\theta\sin\theta} = \frac{1}{\sin 2\theta} = \operatorname{cosec} 2\theta = \text{RHS}$

Question 29 (***)

Prove the validity of the following trigonometric identity

$$\cot x - \tan x \equiv 2\cot 2x .$$

proof



$$\text{LHS} = \cot x - \tan x = \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\cos x \sin x} = \frac{\cos 2x}{\sin x \cos x} = \frac{2\cos 2x}{2\sin x \cos x} = \frac{2\cos 2x}{\sin 2x} = 2\cot 2x = \text{RHS}$$

or

$$\text{RHS} = 2\cot 2x = \frac{2}{\tan 2x} = \frac{2}{\frac{2\tan x}{1-\tan^2 x}} = \frac{2(1-\tan^2 x)}{2\tan x} = \frac{1-\tan^2 x}{\tan x} = \frac{1}{\tan x} - \frac{\tan^2 x}{\tan x} = \cot x - \tan x = \text{RHS}$$

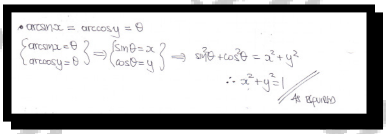
Question 30 (***)

It is given that $\arcsin x = \arccos y$.

Show, by a clear method, that

$$x^2 + y^2 = 1 .$$

proof



- $\arcsin x = \arccos y = \theta$

$$\left\{\begin{array}{l}\arcsin x = \theta \\ \arccos y = \theta\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\sin\theta = x \\ \cos\theta = y\end{array}\right\} \Rightarrow \sin^2\theta + \cos^2\theta = x^2 + y^2$$

$$\therefore x^2 + y^2 = 1$$

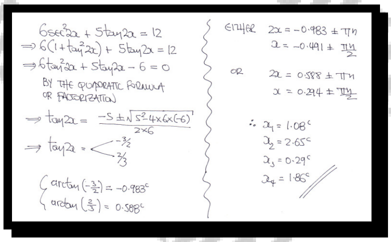
As required

Question 31 (***)

$$6\sec^2 2x + 5\tan 2x = 12, \ 0 \le \theta < \pi.$$

Find the solutions of the above trigonometric equation, giving the answers in radians correct to two decimal places.

$x = 0.29^c, 1.08^c, 1.86^c, 2.65^c$



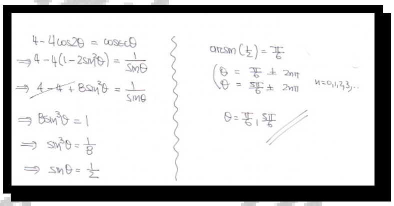
Question 32 (***)

Solve the trigonometric equation

$$4 - 4\cos 2\theta = \operatorname{cosec}\theta, \ 0 \le \theta < 2\pi,$$

giving the answers in terms of π.

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$



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STANDARD QUESTIONS

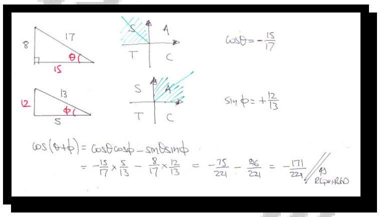
Question 1 (***+)

$$\sin\theta = \frac{8}{17} \quad \text{and} \quad \cos\varphi = \frac{5}{13}.$$

If θ is obtuse and φ is acute, show that

$$\cos(\theta+\varphi) = -\frac{171}{221}.$$

[ ], [proof]



Question 2 (***+)

A curve C is defined by the equation

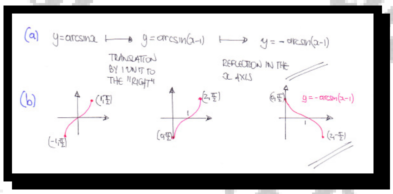
$$y = -\arcsin(x-1), \; 0 \le x \le 2.$$

a) Describe the 2 geometric transformations that map the graph of $\arcsin x$ onto the graph of C.

b) Sketch the graph of C.

The sketch must include the coordinates of any points where the graph of C meets the coordinate axes and the coordinates of the endpoints of C.

[ ], [translation by 1 unit to the right, followed by reflection in the x axis]



Question 3 (***+)

It is given that

$$\frac{\sec\theta}{\sec\theta-\cos\theta}\equiv \operatorname{cosec}^2\theta,\quad \theta\neq\frac{n\pi}{2},\ n\in\mathbb{Z}.$$

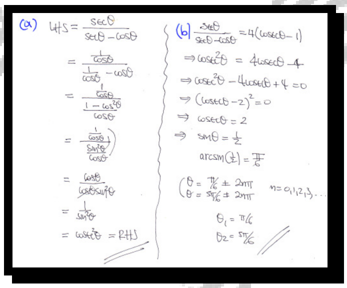
a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$\frac{\sec\theta}{\sec\theta-\cos\theta}=4(\operatorname{cosec}\theta-1),\quad 0\leq\theta<2\pi,$$

giving the answers in terms of π.

$$\boxed{\theta=\frac{\pi}{6},\frac{5\pi}{6}}$$



(a) $\text{LHS}=\dfrac{\sec\theta}{\sec\theta-\cos\theta}=\dfrac{\frac{1}{\cos\theta}}{\frac{1}{\cos\theta}-\cos\theta}=\dfrac{\frac{1}{\cos\theta}}{\frac{1-\cos^2\theta}{\cos\theta}}=\dfrac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos\theta}}=\dfrac{\cos\theta}{\cos\theta\sin^2\theta}=\dfrac{1}{\sin^2\theta}=\operatorname{cosec}^2\theta=\text{RHS}$

(b) $\dfrac{\sec\theta}{\sec\theta-\cos\theta}=4(\operatorname{cosec}\theta-1)$

$\Rightarrow \operatorname{cosec}^2\theta=4\operatorname{cosec}\theta-4$

$\Rightarrow \operatorname{cosec}^2\theta-4\operatorname{cosec}\theta+4=0$

$\Rightarrow (\operatorname{cosec}\theta-2)^2=0$

$\Rightarrow \operatorname{cosec}\theta=2$

$\Rightarrow \sin\theta=\frac{1}{2}$

$\arcsin(\frac{1}{2})=\frac{\pi}{6}$

$\theta=\pi/6\pm2n\pi$, $\theta=5\pi/6\pm2n\pi$, $n=0,1,2,3,\ldots$

$\theta_1=\pi/6$

$\theta_2=5\pi/6$

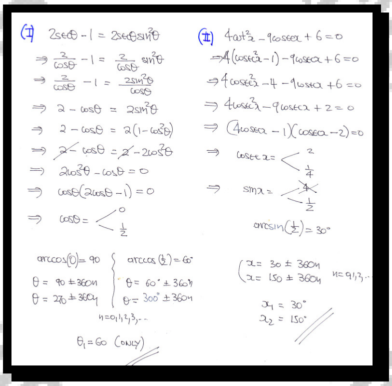
Question 4 (***+)

Solve each of the following trigonometric equations.

i. $2\sec\theta - 1 = 2\sec\theta\sin^2\theta$, $0° \le \theta < 180°$, $\theta \ne 90°$

ii. $4\cot^2 x - 9\operatorname{cosec} x + 6 = 0$, $0° \le x < 360°$, $x \ne 0°, 180°$

$\boxed{}$, $\boxed{\theta = 60°}$, $\boxed{x = 30°, 150°}$



(I) $2\sec\theta - 1 = 2\sec\theta\sin^2\theta$

$\Rightarrow \frac{2}{\cos\theta} - 1 = \frac{2}{\cos\theta}\sin^2\theta$

$\Rightarrow \frac{2}{\cos\theta} - 1 = \frac{2\sin^2\theta}{\cos\theta}$

$\Rightarrow 2 - \cos\theta = 2\sin^2\theta$

$\Rightarrow 2 - \cos\theta = 2(1-\cos^2\theta)$

$\Rightarrow 2 - \cos\theta = 2 - 2\cos^2\theta$

$\Rightarrow 2\cos^2\theta - \cos\theta = 0$

$\Rightarrow \cos\theta(2\cos\theta - 1) = 0$

$\Rightarrow \cos\theta = 0$ or $\frac{1}{2}$

$\arccos(0) = 90$: $\theta = 90 \pm 360n$, $\theta = 270 \pm 360n$

$\arccos(\frac{1}{2}) = 60°$: $\theta = 60° \pm 360n$, $\theta = 300° \pm 360n$

$n = 0, 1, 2, 3, \ldots$

$\theta_1 = 60$ (ONLY)

(II) $4\cot^2 x - 9\operatorname{cosec} x + 6 = 0$

$\Rightarrow 4(\operatorname{cosec}^2 x - 1) - 9\operatorname{cosec} x + 6 = 0$

$\Rightarrow 4\operatorname{cosec}^2 x - 4 - 9\operatorname{cosec} x + 6 = 0$

$\Rightarrow 4\operatorname{cosec}^2 x - 9\operatorname{cosec} x + 2 = 0$

$\Rightarrow (4\operatorname{cosec} x - 1)(\operatorname{cosec} x - 2) = 0$

$\Rightarrow \operatorname{cosec} x = 2$ or $\frac{1}{4}$

$\Rightarrow \sin x = 4$ (rejected) or $\frac{1}{2}$

$\arcsin(\frac{1}{2}) = 30°$

$x = 30 \pm 360n$, $x = 150 \pm 360n$, $n = 0, 1, 2, 3, \ldots$

$x_1 = 30°$, $x_2 = 150°$

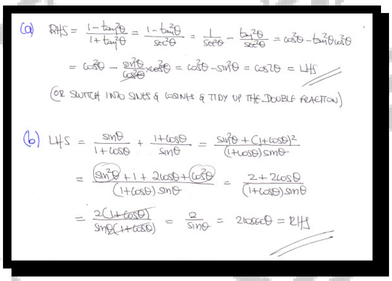
Question 5 (***+)

Prove the validity of each of the following trigonometric identities.

a) $\dfrac{1-\tan^2\theta}{1+\tan^2\theta} \equiv \cos 2\theta$.

b) $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} \equiv 2\operatorname{cosec}\theta$.

$\boxed{\text{proof}}$



(a) RHS $= \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\theta}{\sec^2\theta} = \frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta} = \cos^2\theta - \tan^2\theta\cos^2\theta$

$= \cos^2\theta - \frac{\sin^2\theta}{\cos^2\theta}\times\cos^2\theta = \cos^2\theta - \sin^2\theta = \cos 2\theta =$ LHS

(OR SWITCH INTO SINES & COSINES & TIDY UP THE DOUBLE FRACTION)

(b) LHS $= \frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{\sin^2\theta + (1+\cos\theta)^2}{(1+\cos\theta)\sin\theta}$

$= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{(1+\cos\theta)\sin\theta} = \frac{2 + 2\cos\theta}{(1+\cos\theta)\sin\theta}$

$= \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = \frac{2}{\sin\theta} = 2\operatorname{cosec}\theta =$ RHS

Question 6 (***+)

It is given that

$$\frac{1+\cos 2\theta}{\sin 2\theta} \equiv \cot\theta, \quad \theta \neq \frac{k\pi}{2}, \; k \in \mathbb{Z}.$$

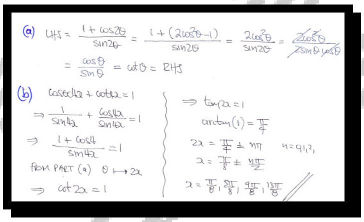
a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$\operatorname{cosec} 4x + \cot 4x = 1, \quad 0 \le x < 2\pi,$$

giving the answers in terms of π.

$\boxed{}$, $\boxed{x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}}$



Question 7 (***+)

$$y \equiv \sqrt{2}\cos\theta - \sqrt{6}\sin\theta, \; 0 < \theta < 360° .$$

a) Express y in the form $R\cos(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

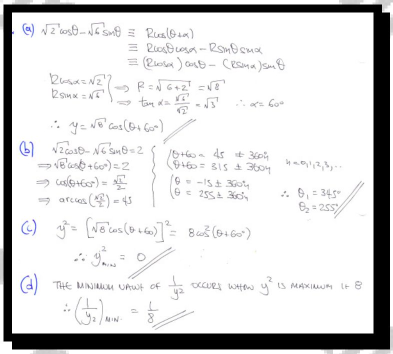
b) Solve the equation $y = 2$.

c) Write down the minimum value of …

i. … y^2.

ii. … $\dfrac{1}{y^2}$.

$\boxed{y \equiv \sqrt{8}\cos(\theta + 60°)}$, $\boxed{\theta = 255°, 345°}$, $\boxed{\min = 0}$, $\boxed{\min = \frac{1}{8}}$


(a) $\sqrt{2}\cos\theta - \sqrt{6}\sin\theta \equiv R\cos(\theta+\alpha)$
$\equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$
$\equiv (R\cos\alpha)\cos\theta - (R\sin\alpha)\sin\theta$
$R\cos\alpha = \sqrt{2}$, $R\sin\alpha = \sqrt{6}$ $\Rightarrow R = \sqrt{6+2} = \sqrt{8}$ $\Rightarrow \tan\alpha = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$ $\therefore \alpha = 60°$
$\therefore y = \sqrt{8}\cos(\theta + 60°)$

(b) $\sqrt{2}\cos\theta - \sqrt{6}\sin\theta = 2$
$\Rightarrow \sqrt{8}\cos(\theta+60°) = 2$
$\Rightarrow \cos(\theta+60°) = \frac{\sqrt{2}}{2}$
$\Rightarrow \arccos\left(\frac{\sqrt{2}}{2}\right) = 45$
$\theta + 60 = 45 \pm 360n$, $\theta + 60 = 315 \pm 360n$, $n = 0, 1, 2, 3, \ldots$
$\theta = -15 \pm 360n$, $\theta = 255 \pm 360n$
$\therefore \theta_1 = 345°$, $\theta_2 = 255°$

(c) $y^2 = \left[\sqrt{8}\cos(\theta+60)\right]^2 = 8\cos^2(\theta+60°)$
$\therefore y^2_{MIN} = 0$

(d) THE MINIMUM VALUE OF $\frac{1}{y^2}$ OCCURS WHEN y^2 IS MAXIMUM i.e. 8
$\therefore \left(\frac{1}{y^2}\right)_{MIN} = \frac{1}{8}$

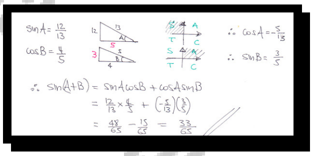
Question 8 (***+)

$$\sin A = \frac{12}{13} \quad \text{and} \quad \cos B = \frac{4}{5}.$$

If A is obtuse and B is acute, show clearly that

$$\sin(A+B) = \frac{33}{65}.$$

[ ], [proof]


$\sin A = \frac{12}{13}$, $\cos B = \frac{4}{5}$

$\therefore \cos A = -\frac{5}{13}$, $\therefore \sin B = \frac{3}{5}$

$\therefore \sin(A+B) = \sin A\cos B + \cos A \sin B$

$= \frac{12}{13}\times\frac{4}{5} + \left(-\frac{5}{13}\right)\left(\frac{3}{5}\right)$

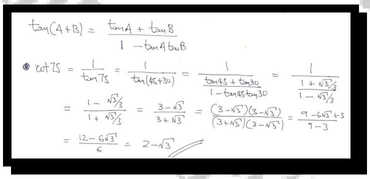
$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$

Question 9 (***+)

By considering the compound angle identity for $\tan(A+B)$, with suitable values for A and B, show that

$$\cot 75° = 2 - \sqrt{3}.$$

[ ], [proof]


$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\cot 75 = \frac{1}{\tan 75} = \frac{1}{\tan(45+30)} = \frac{1}{\frac{\tan 45 + \tan 30}{1 - \tan 45 \tan 30}} = \frac{1}{\frac{1+\sqrt{3}/3}{1-\sqrt{3}/3}}$

$= \frac{1-\sqrt{3}/3}{1+\sqrt{3}/3} = \frac{3-\sqrt{3}}{3+\sqrt{3}} = \frac{(3-\sqrt{3})(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})} = \frac{9-6\sqrt{3}+3}{9-3}$

$= \frac{12-6\sqrt{3}}{6} = 2-\sqrt{3}$

Question 10 (***+)

It is given that

$$\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta, \quad \theta \neq \frac{k\pi}{2}, \ k \in \mathbb{Z}.$$

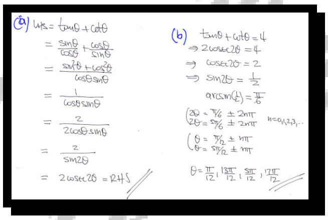
a) Prove the validity of the above trigonometric identity.

b) Hence find, in terms of π, the solutions of the equation

$$\tan\theta + \cot\theta = 4, \quad 0 \le \theta < 2\pi,$$

giving the answers in terms of π.

$$\boxed{\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}}$$

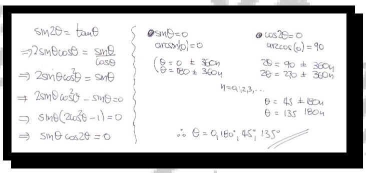


Question 11 (***+)

Solve the following trigonometric equation

$$\sin 2\theta = \tan\theta, \qquad 0 \le \theta \le 180°.$$

$$\boxed{\theta = 0°, 45°, 135°, 180°}$$



Question 12 **(***+)**

It is given that

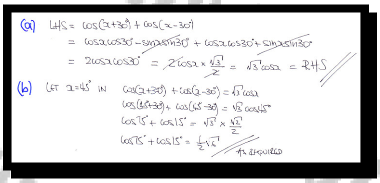
$$\cos(x+30°)+\cos(x-30°)\equiv\sqrt{3}\cos x\,.$$

a) Prove the validity of the above trigonometric identity.

b) Hence show that

$$\cos 75°+\cos 15°=\tfrac{1}{2}\sqrt{6}\,.$$

proof


(a) $LHS = \cos(x+30°)+\cos(x-30°)$
$= \cos x\cos 30° - \sin x\sin 30° + \cos x\cos 30° + \sin x\sin 30°$
$= 2\cos x\cos 30° = 2\cos x \times \frac{\sqrt{3}}{2} = \sqrt{3}\cos x = RHS$

(b) LET $x=45°$ IN $\cos(x+30°)+\cos(x-30°)=\sqrt{3}\cos x$
$\cos(45°+30°)+\cos(45°-30°)=\sqrt{3}\cos 45°$
$\cos 75°+\cos 15° = \sqrt{3}\times\frac{\sqrt{2}}{2}$
$\cos 75°+\cos 15° = \frac{1}{2}\sqrt{6}$ AS REQUIRED

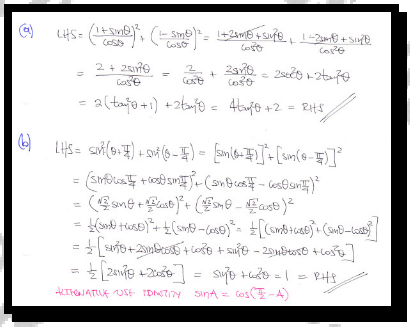
Question 13 **(***+)**

Prove the validity of each of the following trigonometric identities.

a) $\left(\dfrac{1+\sin\theta}{\cos\theta}\right)^2+\left(\dfrac{1-\sin\theta}{\cos\theta}\right)^2\equiv 4\tan^2\theta+2\,.$

b) $\sin^2\left(\theta+\dfrac{\pi}{4}\right)+\sin^2\left(\theta-\dfrac{\pi}{4}\right)\equiv 1\,.$

proof


(a) $LHS = \left(\frac{1+\sin\theta}{\cos\theta}\right)^2+\left(\frac{1-\sin\theta}{\cos\theta}\right)^2 = \frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta}+\frac{1-2\sin\theta+\sin^2\theta}{\cos^2\theta}$
$= \frac{2+2\sin^2\theta}{\cos^2\theta} = \frac{2}{\cos^2\theta}+\frac{2\sin^2\theta}{\cos^2\theta} = 2\sec^2\theta+2\tan^2\theta$
$= 2(\tan^2\theta+1)+2\tan^2\theta = 4\tan^2\theta+2 = RHS$

(b) $LHS = \sin^2(\theta+\frac{\pi}{4})+\sin^2(\theta-\frac{\pi}{4}) = [\sin(\theta+\frac{\pi}{4})]^2+[\sin(\theta-\frac{\pi}{4})]^2$
$= (\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4})^2+(\sin\theta\cos\frac{\pi}{4}-\cos\theta\sin\frac{\pi}{4})^2$
$= (\frac{\sqrt{2}}{2}\sin\theta+\frac{\sqrt{2}}{2}\cos\theta)^2+(\frac{\sqrt{2}}{2}\sin\theta-\frac{\sqrt{2}}{2}\cos\theta)^2$
$= \frac{1}{2}(\sin\theta+\cos\theta)^2+\frac{1}{2}(\sin\theta-\cos\theta)^2 = \frac{1}{2}[(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2]$
$= \frac{1}{2}[\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta+\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta]$
$= \frac{1}{2}[2\sin^2\theta+2\cos^2\theta] = \sin^2\theta+\cos^2\theta = 1 = RHS$
ALTERNATIVE USE IDENTITY $\sin A = \cos(\frac{\pi}{2}-A)$

Question 14 (***+)

It is given that

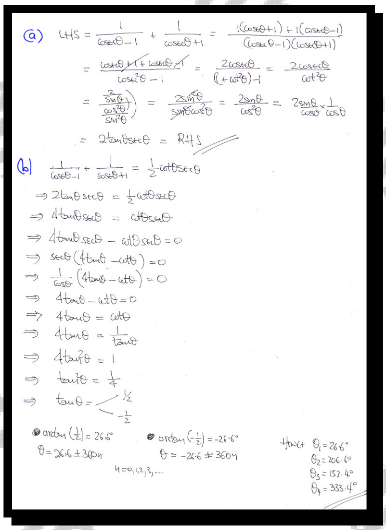
$$\frac{1}{\operatorname{cosec}\theta - 1} + \frac{1}{\operatorname{cosec}\theta + 1} \equiv 2\tan\theta\sec\theta .$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$\frac{1}{\operatorname{cosec}\theta - 1} + \frac{1}{\operatorname{cosec}\theta + 1} = \frac{1}{2}\cot\theta\sec\theta, \quad 0 \le \theta < 360° .$$

$\boxed{\theta = 26.6°, 153.4°, 206.6°, 333.4°}$



(a) $\text{LHS} = \frac{1}{\operatorname{cosec}\theta - 1} + \frac{1}{\operatorname{cosec}\theta + 1} = \frac{1(\operatorname{cosec}\theta + 1) + 1(\operatorname{cosec}\theta - 1)}{(\operatorname{cosec}\theta - 1)(\operatorname{cosec}\theta + 1)}$

$= \frac{\operatorname{cosec}\theta + 1 + \operatorname{cosec}\theta - 1}{\operatorname{cosec}^2\theta - 1} = \frac{2\operatorname{cosec}\theta}{(1+\cot^2\theta) - 1} = \frac{2\operatorname{cosec}\theta}{\cot^2\theta}$

$= \frac{\frac{2}{\sin\theta}}{\frac{\cos^2\theta}{\sin^2\theta}} = \frac{2\sin^2\theta}{\sin\theta\cos^2\theta} = \frac{2\sin\theta}{\cos^2\theta} = \frac{2\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta}$

$= 2\tan\theta\sec\theta = \text{RHS}$

(b) $\frac{1}{\operatorname{cosec}\theta - 1} + \frac{1}{\operatorname{cosec}\theta + 1} = \frac{1}{2}\cot\theta\sec\theta$

$\Rightarrow 2\tan\theta\sec\theta = \frac{1}{2}\cot\theta\sec\theta$

$\Rightarrow 4\tan\theta\sec\theta = \cot\theta\sec\theta$

$\Rightarrow 4\tan\theta\sec\theta - \cot\theta\sec\theta = 0$

$\Rightarrow \sec\theta(4\tan\theta - \cot\theta) = 0$

$\Rightarrow \frac{1}{\cos\theta}(4\tan\theta - \cot\theta) = 0$

$\Rightarrow 4\tan\theta - \cot\theta = 0$

$\Rightarrow 4\tan\theta = \cot\theta$

$\Rightarrow 4\tan\theta = \frac{1}{\tan\theta}$

$\Rightarrow 4\tan^2\theta = 1$

$\Rightarrow \tan^2\theta = \frac{1}{4}$

$\Rightarrow \tan\theta = \frac{1}{2}$ or $-\frac{1}{2}$

- $\arctan\left(\frac{1}{2}\right) = 26.6°$, $\theta = 26.6 \pm 360n$
- $\arctan\left(-\frac{1}{2}\right) = -26.6°$, $\theta = -26.6 \pm 360n$

$n = 0, 1, 2, 3, \ldots$

Hence $\theta_1 = 26.6°$, $\theta_2 = 206.6°$, $\theta_3 = 153.4°$, $\theta_4 = 333.4°$

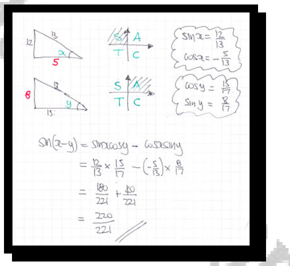
Question 15 (***+)

$$\sin x = \frac{12}{13} \quad \text{and} \quad \cos y = \frac{15}{17}.$$

If x is obtuse and y is acute, show clearly that

$$\sin(x-y) = \frac{220}{221}.$$

[ ] , [proof]

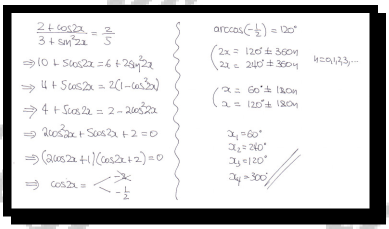


Question 16 (***+)

Solve the following trigonometric equation

$$\frac{2+\cos 2x}{3+\sin^2 2x} = \frac{2}{5}, \quad \text{for} \quad 0° \le x < 360°,$$

$x = 60°, 120°, 240°, 300°$



Question 17 **(***+)**

It is given that

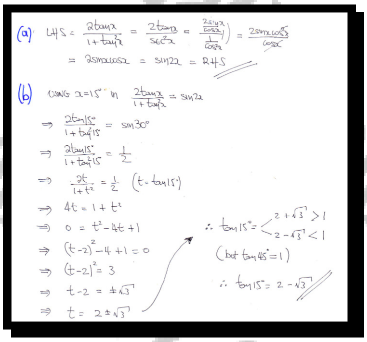
$$\frac{2\tan x}{1+\tan^2 x} \equiv \sin 2x \, .$$

a) Prove the validity of the above trigonometric identity.

b) Use part **(a)** to show that

$$\tan 15° = 2-\sqrt{3} \, .$$

[ ] , proof

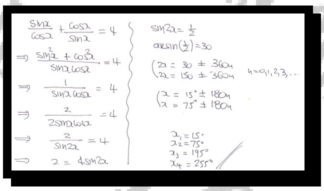


Question 18 **(***+)**

Solve the trigonometric equation

$$\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=4 \, , \qquad 0 \le x < 360° \, .$$

$x = 15°, 75°, 195°, 255°$

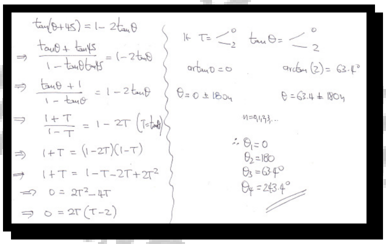


Question 19 (***+)

Solve the following trigonometric equation

$$\tan(\theta + 45°) = 1 - 2\tan\theta, \quad 0 \le \theta < 360°.$$

$\boxed{x = 0, 180°, 63.4°, 243.4°}$



Question 20 (***+)

It is given that

$$\tan x \sec x + \operatorname{cosec} x \equiv \operatorname{cosec} x \sec^2 x, \quad x \ne \frac{n\pi}{2}, n \in \mathbb{Z}.$$

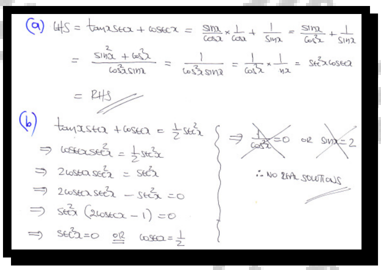
a) Prove the validity of the above trigonometric identity.

b) Hence show that the equation

$$\tan x \sec x + \operatorname{cosec} x = \tfrac{1}{2}\sec^2 x$$

has no real solutions.

$\boxed{\text{proof}}$



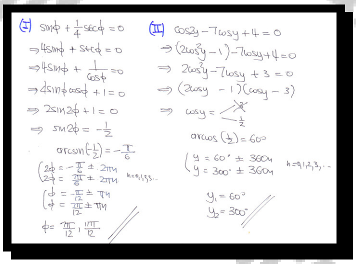
Question 21 (***+)

Solve each of the following trigonometric equations.

i. $\sin\varphi + \frac{1}{4}\sec\varphi = 0$, $\quad 0 \le \varphi < \pi$.

ii. $\cos 2y - 7\cos y + 4 = 0$, $\quad 0 \le y < 360°$.

$\boxed{\varphi = \frac{7\pi}{12}, \frac{11\pi}{12}}$, $\boxed{y = 60°, 300°}$


(I) $\sin\phi + \frac{1}{4}\sec\phi = 0$
$\Rightarrow 4\sin\phi + \sec\phi = 0$
$\Rightarrow 4\sin\phi + \frac{1}{\cos\phi} = 0$
$\Rightarrow 4\sin\phi\cos\phi + 1 = 0$
$\Rightarrow 2\sin 2\phi + 1 = 0$
$\Rightarrow \sin 2\phi = -\frac{1}{2}$
$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$
$2\phi = -\frac{\pi}{6} \pm 2\pi n$, $2\phi = \frac{7\pi}{6} \pm 2\pi n$, $n = 0, 1, 2, 3, \dots$
$\phi = -\frac{\pi}{12} \pm \pi n$, $\phi = \frac{7\pi}{12} \pm \pi n$
$\phi = \frac{7\pi}{12}, \frac{11\pi}{12}$

(II) $\cos 2y - 7\cos y + 4 = 0$
$\Rightarrow (2\cos^2 y - 1) - 7\cos y + 4 = 0$
$\Rightarrow 2\cos^2 y - 7\cos y + 3 = 0$
$\Rightarrow (2\cos y - 1)(\cos y - 3)$
$\Rightarrow \cos y = 3$ (rejected), $\frac{1}{2}$
$\arccos\left(\frac{1}{2}\right) = 60°$
$y = 60° \pm 360n$, $y = 300° \pm 360n$, $n = 0, 1, 2, 3, \dots$
$y_1 = 60°$
$y_2 = 300°$

Question 22 (***+)

The acute angles α and β satisfy the relationships

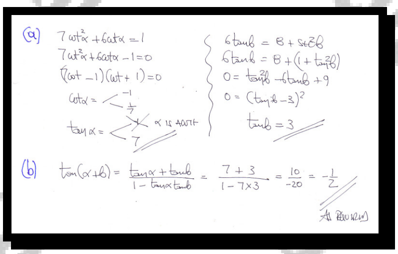
$$7\cot^2\alpha + 6\cot\alpha = 1 \quad \text{and} \quad 6\tan\beta = 8 + \sec^2\beta.$$

a) Determine the value of $\tan\alpha$ and the value of $\tan\beta$.

b) Show clearly that

$$\tan(\alpha + \beta) = -\frac{1}{2}.$$

$\boxed{\quad}$, $\boxed{\tan\alpha = 7}$, $\boxed{\tan\beta = 3}$


(a) $7\cot^2\alpha + 6\cot\alpha = 1$
$7\cot^2\alpha + 6\cot\alpha - 1 = 0$
$(7\cot - 1)(\cot + 1) = 0$
$\cot\alpha = -1, \frac{1}{7}$
$\tan\alpha = -1$ (rejected, α is acute), 7

$6\tan\beta = 8 + \sec^2\beta$
$6\tan\beta = 8 + (1 + \tan^2\beta)$
$0 = \tan^2\beta - 6\tan\beta + 9$
$0 = (\tan\beta - 3)^2$
$\tan\beta = 3$

(b) $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{7 + 3}{1 - 7 \times 3} = \frac{10}{-20} = -\frac{1}{2}$

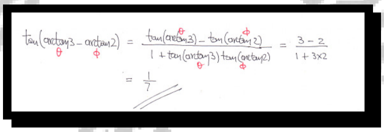
Question 23 (***+)

Simplify, showing all steps in the calculation, the following expression

$$\tan\left(\arctan 3 - \arctan 2\right),$$

giving the final answer as an exact fraction.

$$\boxed{\frac{1}{7}}$$

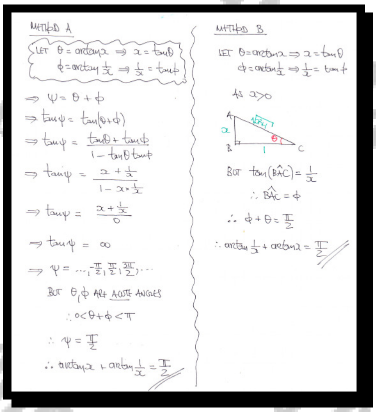


Question 24 (***+)

Show clearly that if $x > 0$

$$\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}.$$

proof



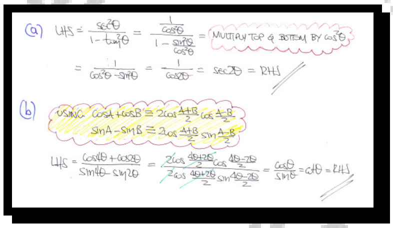
Question 25 (***+)

Prove the validity of each of the following trigonometric identities.

a) $\dfrac{\sec^2\theta}{1-\tan^2\theta} \equiv \sec 2\theta$.

b) $\dfrac{\cos 4\theta + \cos 2\theta}{\sin 4\theta - \sin 2\theta} \equiv \cot\theta$.

proof



Question 26 (***+)

It is given that

$$\frac{\sec^2\theta}{1-\tan^2\theta} \equiv \sec 2\theta \, .$$

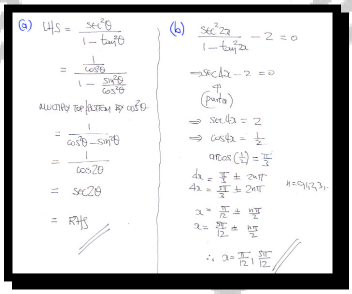
a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, solve the equation

$$\frac{\sec^2 2x}{1-\tan^2 2x} - 2 = 0 \, , \quad 0 \le x < \frac{\pi}{2} \, ,$$

giving the answers in terms of π.

$$\boxed{x = \frac{\pi}{12}, \frac{5\pi}{12}}$$



(a) $\text{LHS} = \frac{\sec^2\theta}{1-\tan^2\theta}$

$= \frac{\frac{1}{\cos^2\theta}}{1-\frac{\sin^2\theta}{\cos^2\theta}}$

MULTIPLY TOP/BOTTOM BY $\cos^2\theta$

$= \frac{1}{\cos^2\theta - \sin^2\theta}$

$= \frac{1}{\cos 2\theta}$

$= \sec 2\theta$

$= \text{RHS}$

(b) $\frac{\sec^2 2x}{1-\tan^2 2x} - 2 = 0$

$\Rightarrow \sec 4x - 2 = 0$ (part a)

$\Rightarrow \sec 4x = 2$

$\Rightarrow \cos 4x = \frac{1}{2}$

$\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$4x = \frac{\pi}{3} \pm 2n\pi$

$4x = \frac{5\pi}{3} \pm 2n\pi$ $\quad n = 0, 1, 2, 3, \ldots$

$x = \frac{\pi}{12} \pm \frac{n\pi}{2}$

$x = \frac{5\pi}{12} \pm \frac{n\pi}{2}$

$\therefore x = \frac{\pi}{12}, \frac{5\pi}{12}$

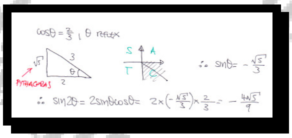
Question 27 (***+)

It is given that θ is a reflex angle such that

$$\cos\theta = \frac{2}{3}.$$

Find the exact value of $\sin 2\theta$.

$$\boxed{\sin 2\theta = -\frac{4\sqrt{5}}{9}}$$



Question 28 (***+)

It is given that

$$\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sin\theta} \equiv \sec^2\theta, \quad \theta \neq \frac{n\pi}{2}, \ n \in \mathbb{Z}.$$

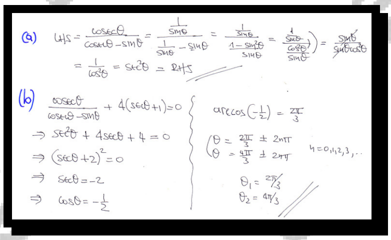
a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$\frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sin\theta} + 4(\sec\theta + 1) = 0, \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of π.

$$\boxed{\theta = \frac{2\pi}{3}, \frac{4\pi}{3}}$$

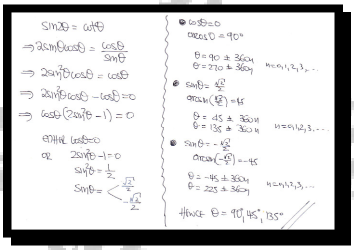


Question 29 (***+)

Solve the following trigonometric equation

$$\sin 2\theta = \cot\theta, \qquad 0 \le \theta \le 180°.$$

$\boxed{}$, $\boxed{\theta = 45°, 90°, 135°}$



$\sin 2\theta = \cot\theta$

$\Rightarrow 2\sin\theta\cos\theta = \frac{\cos\theta}{\sin\theta}$

$\Rightarrow 2\sin^2\theta\cos\theta = \cos\theta$

$\Rightarrow 2\sin^2\theta\cos\theta - \cos\theta = 0$

$\Rightarrow \cos\theta(2\sin^2\theta - 1) = 0$

either $\cos\theta = 0$

or $2\sin^2\theta - 1 = 0$, $\sin^2\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$

- $\cos\theta = 0$: $\arccos 0 = 90°$; $\theta = 90 \pm 360n$, $\theta = 270 \pm 360n$, $n = 0, 1, 2, 3, \ldots$
- $\sin\theta = \frac{\sqrt{2}}{2}$: $\arcsin\left(\frac{\sqrt{2}}{2}\right) = 45$; $\theta = 45 \pm 360n$, $\theta = 135 \pm 360n$, $n = 0, 1, 2, 3, \ldots$
- $\sin\theta = -\frac{\sqrt{2}}{2}$: $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -45$; $\theta = -45 \pm 360n$, $\theta = 225 \pm 360n$, $n = 0, 1, 2, 3, \ldots$

Hence $\theta = 90°, 45°, 135°$

Question 30 (***+)

It is given that

$$(\operatorname{cosec}\theta - \sin\theta)\sec^2\theta \equiv \operatorname{cosec}\theta, \quad \theta \ne \frac{n\pi}{2}, n \in \mathbb{Z}.$$

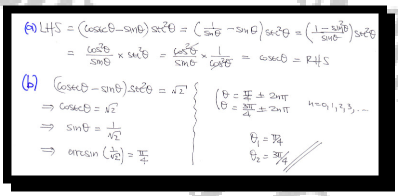
a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$(\operatorname{cosec}\theta - \sin\theta)\sec^2\theta = \sqrt{2}, \quad 0 \le \theta < 2\pi,$$

giving the answers in terms of π.

$\boxed{\theta = \frac{\pi}{4}, \frac{3\pi}{4}}$



(a) LHS $= (\operatorname{cosec}\theta - \sin\theta)\sec^2\theta = \left(\frac{1}{\sin\theta} - \sin\theta\right)\sec^2\theta = \left(\frac{1 - \sin^2\theta}{\sin\theta}\right)\sec^2\theta$

$= \frac{\cos^2\theta}{\sin\theta} \times \sec^2\theta = \frac{\cos^2\theta}{\sin\theta} \times \frac{1}{\cos^2\theta} = \operatorname{cosec}\theta =$ RHS

(b) $(\operatorname{cosec}\theta - \sin\theta)\sec^2\theta = \sqrt{2}$

$\Rightarrow \operatorname{cosec}\theta = \sqrt{2}$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$\theta = \frac{\pi}{4} \pm 2n\pi$, $\theta = \frac{3\pi}{4} \pm 2n\pi$, $n = 0, 1, 2, 3, \ldots$

$\theta_1 = \pi/4$, $\theta_2 = 3\pi/4$

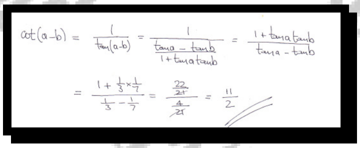
Question 31 (***+)

The constants a and b are such that

$$\tan a = \frac{1}{3} \quad \text{and} \quad \tan b = \frac{1}{7}.$$

Determine the exact value of $\cot(a-b)$, showing all the steps in the workings.

$\boxed{\cot(a-b) = \frac{11}{2}}$


$\cot(a-b) = \frac{1}{\tan(a-b)} = \frac{1}{\frac{\tan a - \tan b}{1+\tan a \tan b}} = \frac{1+\tan a \tan b}{\tan a - \tan b}$

$= \frac{1+\frac{1}{3}\times\frac{1}{7}}{\frac{1}{3}-\frac{1}{7}} = \frac{\frac{22}{21}}{\frac{4}{21}} = \frac{11}{2}$

Question 32 (***+)

$$f(y) = 6 + 3\cos y + 4\sin y, \quad 0 < y < 2\pi.$$

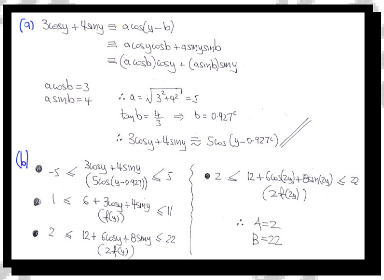
a) Express $3\cos y + 4\sin y$ in the form $a\cos(y-b)$, $a>0$, $0<b<\frac{\pi}{2}$.

It is further given that for $0 < y < 2\pi$

$$A \le 2f(2y) \le B.$$

b) Determine the value of each of the constants A and B.

C3P, $\boxed{3\cos y + 4\sin y \equiv 5\cos\left(y - 0.927^{c}\right)}$, $\boxed{A=2}$, $\boxed{B=22}$


(a) $3\cos y + 4\sin y \equiv a\cos(y-b)$

$\equiv a\cos y\cos b + a\sin y\sin b$

$\equiv (a\cos b)\cos y + (a\sin b)\sin y$

$a\cos b = 3$, $a\sin b = 4$

$\therefore a = \sqrt{3^2+4^2} = 5$

$\tan b = \frac{4}{3} \Rightarrow b = 0.927^c$

$\therefore 3\cos y + 4\sin y \equiv 5\cos(y - 0.927^c)$

(b)

- $-5 \le 3\cos y + 4\sin y \le 5$ $(5\cos(y-0.927))$
- $1 \le 6 + 3\cos y + 4\sin y \le 11$ $(f(y))$
- $2 \le 12 + 6\cos y + 8\sin y \le 22$ $(2f(y))$
- $2 \le 12 + 6\cos(2y) + 8\sin(2y) \le 22$ $(2f(2y))$

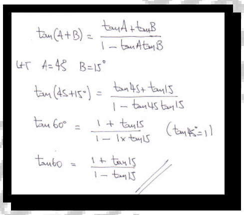
$\therefore A = 2$, $B = 22$

Question 33 (***+)

Use the compound angle identity for $\tan(A+B)$, with suitable values for A and B, to show that

$$\frac{1+\tan 15°}{1-\tan 15°} = \tan 60° .$$

proof

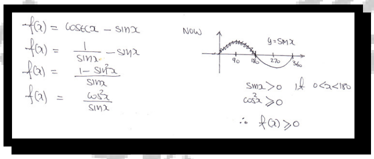


Question 34 (***+)

$$f(x) = \operatorname{cosec} x - \sin x , \; 0 < x < 180° .$$

Show that $f(x) \geq 0$ for the entire domain of the function.

proof

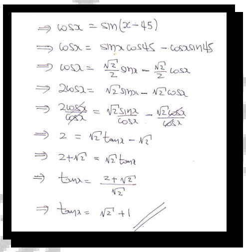


Question 35 (***+)

Given that $\cos x° = \sin\left(x-45\right)°$, show that

$$\tan x° = 1+\sqrt{2} .$$

	proof



Question 36 (***+)

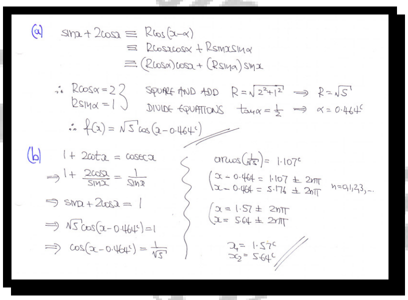
$$f\left(x\right) \equiv \sin x + 2\cos x .$$

a) Express $f\left(x\right)$ in the form $R\cos\left(x-\alpha\right)$, where $R>0$, $0<\alpha<\frac{\pi}{2}$.

b) Hence solve the equation

$$1+2\cot x = \operatorname{cosec} x , \quad 0<x<2\pi .$$

$f\left(x\right) = \sqrt{5}\cos\left(x-0.464^{c}\right)$, $x = 1.57^{c} \cup x = 5.64^{c}$

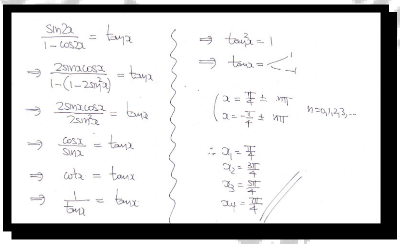


Question 37 **(***+)**

$$\frac{\sin 2x}{1-\cos 2x} = \tan x, \quad 0 < x < 2\pi .$$

Find the solutions of the above trigonometric equation, giving the answers in radians in terms of π.

$$\boxed{x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$$



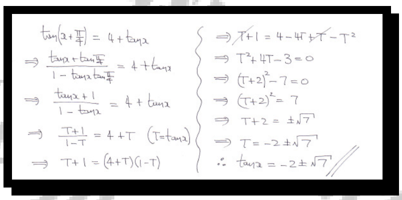
Question 38 **(***+)**

Given that

$$\tan\left(x + \frac{\pi}{4}\right) = 4 + \tan x,$$

find as an exact surd the exact value of $\tan x$.

$$\boxed{\tan x = -2 \pm \sqrt{7}}$$

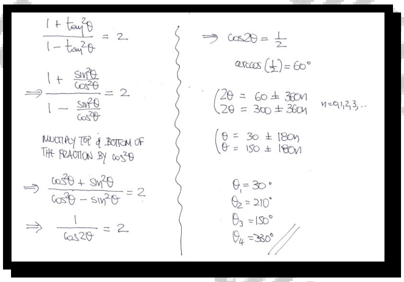


Question 39 (***+)

$$\frac{1+\tan^2\theta}{1-\tan^2\theta}=2,\ 0\le\theta<360.$$

Find the solutions of the above trigonometric equation, giving the answers in degrees.

$$\boxed{\theta=30°,150°,210°,330°}$$

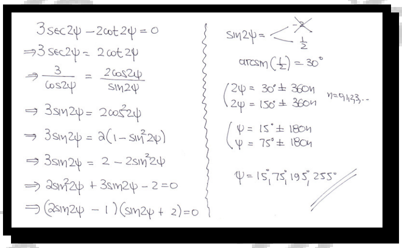


Question 40 (***+)

$$3\sec 2\psi-2\cot 2\psi=0,\ 0\le\psi<360.$$

Find the solutions of the above trigonometric equation, giving the answers in degrees.

$$\boxed{\psi=15°,75°,195°,255°}$$



Question 41 (***+)

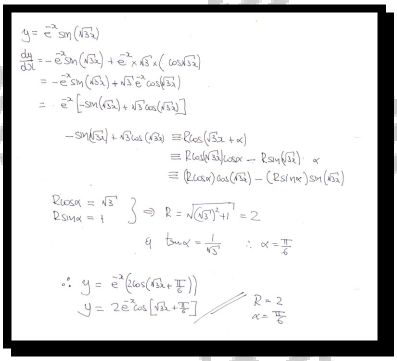
$$y = e^{-x}\sin\left(\sqrt{3}x\right),\ x \in \mathbb{R}.$$

Find the exact values of the constants R and α so that

$$\frac{dy}{dx} = R e^{-x}\cos\left(\sqrt{3}x + \alpha\right),$$

where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

$\boxed{R = 2}$, $\boxed{\alpha = \frac{\pi}{6}}$


$y = e^{-x}\sin(\sqrt{3}x)$

$\frac{dy}{dx} = -e^{-x}\sin(\sqrt{3}x) + e^{-x} \times \sqrt{3} \times (\cos\sqrt{3}x)$

$= -e^{-x}\sin(\sqrt{3}x) + \sqrt{3}e^{-x}\cos(\sqrt{3}x)$

$= e^{-x}\left[-\sin(\sqrt{3}x) + \sqrt{3}\cos(\sqrt{3}x)\right]$

$-\sin(\sqrt{3}x) + \sqrt{3}\cos(\sqrt{3}x) \equiv R\cos(\sqrt{3}x + \alpha)$

$\equiv R\cos(\sqrt{3}x)\cos\alpha - R\sin(\sqrt{3}x)\sin\alpha$

$\equiv (R\cos\alpha)\cos(\sqrt{3}x) - (R\sin\alpha)\sin(\sqrt{3}x)$

$R\cos\alpha = \sqrt{3}$, $R\sin\alpha = 1$ $\Rightarrow R = \sqrt{(\sqrt{3})^2 + 1} = 2$

& $\tan\alpha = \frac{1}{\sqrt{3}}$ $\therefore \alpha = \frac{\pi}{6}$

$\therefore y = e^{-x}\left(2\cos\left(\sqrt{3}x + \frac{\pi}{6}\right)\right)$

$y = 2e^{-x}\cos\left[\sqrt{3}x + \frac{\pi}{6}\right]$ // $R = 2$, $\alpha = \frac{\pi}{6}$

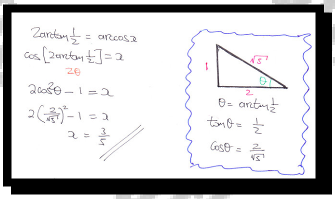
Question 42 (***+)

Solve the following trigonometric equation

$$2\arctan\left(\frac{1}{2}\right) = \arccos x \, ,$$

showing clearly all the workings.

$$\boxed{x = \frac{3}{5}}$$



Question 43 (***+)

The functions f and g are defined as

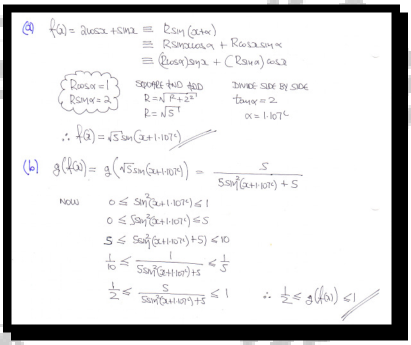
$$f(x) = 2\cos x + \sin x,\ x \in \mathbb{R}.$$

$$g(x) = \frac{5}{x^2+5},\ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\sin(x+\alpha)$, where $R>0$ and $0<\alpha<\frac{\pi}{2}$.

b) Determine the range of $gf(x)$, showing clearly all the relevant workings.

$\boxed{}$, $\boxed{f(x) = \sqrt{5}\sin\left(x+1.107^{c}\right)}$, $\boxed{\frac{1}{2} \le gf(x) \le 1}$


(a) $f(x) = 2\cos x + \sin x \equiv R\sin(x+\alpha)$

$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

$\equiv (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$

$R\cos\alpha = 1$, $R\sin\alpha = 2$

SQUARE AND ADD: $R = \sqrt{1^2+2^2}$, $R = \sqrt{5}$

DIVIDE SIDE BY SIDE: $\tan\alpha = 2$, $\alpha = 1.107^{c}$

$\therefore f(x) = \sqrt{5}\sin(x+1.107^{c})$

(b) $g(f(x)) = g\left(\sqrt{5}\sin(x+1.107^{c})\right) = \frac{5}{5\sin^2(x+1.107^{c})+5}$

NOW

$0 \le \sin^2(x+1.107^{c}) \le 1$

$0 \le 5\sin^2(x+1.107^{c}) \le 5$

$5 \le 5\sin^2(x+1.107^{c})+5 \le 10$

$\frac{1}{10} \le \frac{1}{5\sin^2(x+1.107^{c})+5} \le \frac{1}{5}$

$\frac{1}{2} \le \frac{5}{5\sin^2(x+1.107^{c})+5} \le 1$

$\therefore \frac{1}{2} \le g(f(x)) \le 1$

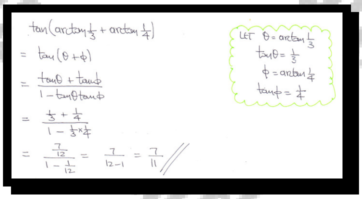
Question 44 (***+)

Simplify, showing clearly all the workings, the expression

$$\tan\left[\arctan\frac{1}{3}+\arctan\frac{1}{4}\right],$$

giving the final answer as an exact fraction.

$\boxed{\frac{7}{11}}$

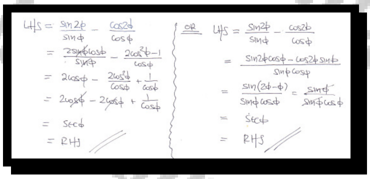


Question 45 (***+)

Prove the validity of the following trigonometric identity

$$\frac{\sin 2\varphi}{\sin\varphi}-\frac{\cos 2\varphi}{\cos\varphi}\equiv\sec\varphi .$$

proof



Question 46 (***+)

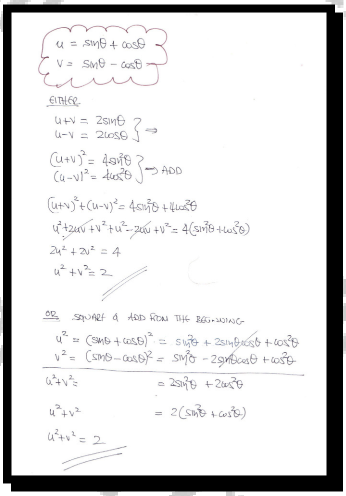
It is given that

$$u = \sin\theta + \cos\theta \ , \quad v = \sin\theta - \cos\theta \ .$$

Use trigonometric identities to show that

$$u^2 + v^2 = 2 \ .$$

proof



$u = \sin\theta + \cos\theta$
$v = \sin\theta - \cos\theta$

EITHER

$u+v = 2\sin\theta$
$u-v = 2\cos\theta$ $\Rightarrow$

$(u+v)^2 = 4\sin^2\theta$
$(u-v)^2 = 4\cos^2\theta$ $\Rightarrow$ ADD

$(u+v)^2 + (u-v)^2 = 4\sin^2\theta + 4\cos^2\theta$

$u^2 + 2uv + v^2 + u^2 - 2uv + v^2 = 4(\sin^2\theta + \cos^2\theta)$

$2u^2 + 2v^2 = 4$

$u^2 + v^2 = 2$

OR SQUARE & ADD FROM THE BEGINNING

$u^2 = (\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$
$v^2 = (\sin\theta - \cos\theta)^2 = \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$

$u^2 + v^2 = 2\sin^2\theta + 2\cos^2\theta$

$u^2 + v^2 = 2(\sin^2\theta + \cos^2\theta)$

$u^2 + v^2 = 2$

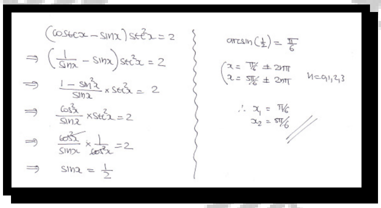
Question 47 **(***+)**

Solve the trigonometric equation

$$(\operatorname{cosec} x - \sin x)\sec^2 x = 2, \quad 0 \le x < \pi, \quad x \ne \frac{\pi}{2},$$

giving the answers in terms of π.

$$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}}$$



$(\operatorname{cosec}x - \sin x)\sec^2 x = 2$

$\Rightarrow \left(\frac{1}{\sin x} - \sin x\right)\sec^2 x = 2$

$\Rightarrow \frac{1-\sin^2 x}{\sin x} \times \sec^2 x = 2$

$\Rightarrow \frac{\cos^2 x}{\sin x} \times \sec^2 x = 2$

$\Rightarrow \frac{\cos^2 x}{\sin x} \times \frac{1}{\cos^2 x} = 2$

$\Rightarrow \sin x = \frac{1}{2}$

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\begin{cases} x = \pi/6 \pm 2n\pi \\ x = 5\pi/6 \pm 2n\pi \end{cases} \quad n = 0, 1, 2, 3$

$\therefore x_1 = \pi/6$

$x_2 = 5\pi/6$

Question 48 **(***+)**

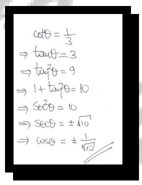
The angle θ is such so that

$$\cot\theta = \frac{1}{3}.$$

Show clearly that

$$\cos\theta = \pm\frac{\sqrt{10}}{10}.$$

$$\boxed{\text{proof}}$$



$\cot\theta = \frac{1}{3}$

$\Rightarrow \tan\theta = 3$

$\Rightarrow \tan^2\theta = 9$

$\Rightarrow 1 + \tan^2\theta = 10$

$\Rightarrow \sec^2\theta = 10$

$\Rightarrow \sec\theta = \pm\sqrt{10}$

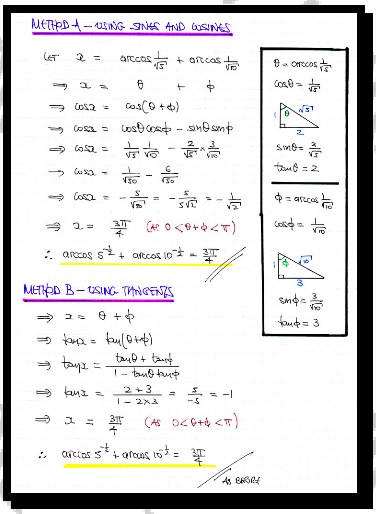
$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{10}}$

Question 49 (***+)

Use a detailed method to show that

$$\arccos\left(5^{-\frac{1}{2}}\right)+\arccos\left(10^{-\frac{1}{2}}\right)=\frac{3\pi}{4}.$$

proof



Question 50 (***+)

A relationship is defined as

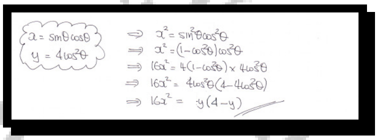
$$x=\sin\theta\cos\theta,\quad 0\le\theta<2\pi$$

$$y=4\cos^2\theta,\quad 0\le\theta<2\pi.$$

Use trigonometric identities to show that

$$16x^2=y(4-y).$$

proof

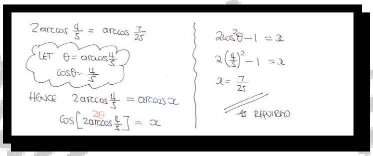


Question 51 (*+)**

Show clearly that

$$2\arccos\left(\frac{4}{5}\right) = \arccos\left(\frac{7}{25}\right).$$

proof



Question 52 (*+)**

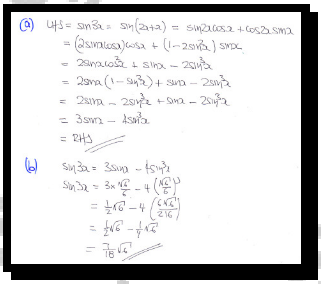
It is given that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x.$$

a) Prove the validity of the above trigonometric identity, by writing $\sin 3x$ as $\sin(2x + x)$.

b) Given that $\sin x = \dfrac{\sqrt{6}}{6}$, find the exact value of $\sin 3x$.

$$\sin 3x = \frac{7\sqrt{6}}{18}$$



Question 53 (***+)

It is given that

$$4\operatorname{cosec}^2 2\theta - \sec^2\theta \equiv \operatorname{cosec}^2\theta, \quad \theta \neq \frac{n\pi}{2}, \; n \in \mathbb{Z}.$$

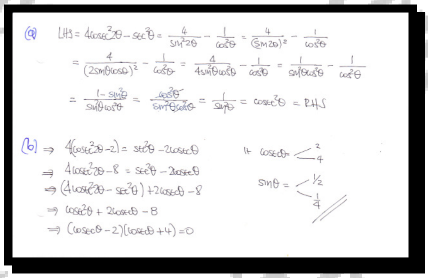
a) Prove the validity of the above trigonometric identity.

b) Hence show that if

$$4\left(\operatorname{cosec}^2 2\theta - 2\right) = \sec^2\theta - 2\operatorname{cosec}\theta,$$

then either $\sin\theta = \frac{1}{2}$ or $\sin\theta = -\frac{1}{4}$.

C3U , proof



(a) $\text{LHS} = 4\operatorname{cosec}^2 2\theta - \sec^2\theta = \frac{4}{\sin^2 2\theta} - \frac{1}{\cos^2\theta} = \frac{4}{(\sin 2\theta)^2} - \frac{1}{\cos^2\theta}$

$= \frac{4}{(2\sin\theta\cos\theta)^2} - \frac{1}{\cos^2\theta} = \frac{4}{4\sin^2\theta\cos^2\theta} - \frac{1}{\cos^2\theta} = \frac{1}{\sin^2\theta\cos^2\theta} - \frac{1}{\cos^2\theta}$

$= \frac{1-\sin^2\theta}{\sin^2\theta\cos^2\theta} = \frac{\cos^2\theta}{\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta} = \operatorname{cosec}^2\theta = \text{RHS}$

(b) $\Rightarrow 4(\operatorname{cosec}^2 2\theta - 2) = \sec^2\theta - 2\operatorname{cosec}\theta$

$\Rightarrow 4\operatorname{cosec}^2 2\theta - 8 = \sec^2\theta - 2\operatorname{cosec}\theta$

$\Rightarrow (4\operatorname{cosec}^2 2\theta - \sec^2\theta) + 2\operatorname{cosec}\theta - 8$

$\Rightarrow \operatorname{cosec}^2\theta + 2\operatorname{cosec}\theta - 8$

$\Rightarrow (\operatorname{cosec}\theta - 2)(\operatorname{cosec}\theta + 4) = 0$

$\therefore \operatorname{cosec}\theta = 2$ or -4

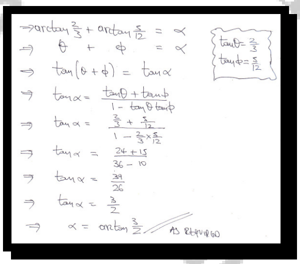
$\sin\theta = \frac{1}{2}$ or $-\frac{1}{4}$

Question 54 (***+)

Show clearly that

$$\arctan\frac{2}{3} + \arctan\frac{5}{12} = \arctan\frac{3}{2}.$$

proof



$\Rightarrow \arctan\frac{2}{3} + \arctan\frac{5}{12} = \alpha$ (with $\tan\theta = \frac{2}{3}$, $\tan\phi = \frac{5}{12}$)

$\Rightarrow \theta + \phi = \alpha$

$\Rightarrow \tan(\theta+\phi) = \tan\alpha$

$\Rightarrow \tan\alpha = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

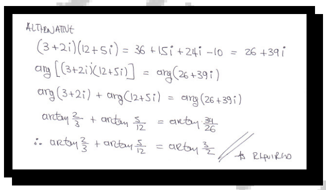
$\Rightarrow \tan\alpha = \frac{\frac{2}{3} + \frac{5}{12}}{1 - \frac{2}{3}\times\frac{5}{12}}$

$\Rightarrow \tan\alpha = \frac{24+15}{36-10}$

$\Rightarrow \tan\alpha = \frac{39}{26}$

$\Rightarrow \tan\alpha = \frac{3}{2}$

$\Rightarrow \alpha = \arctan\frac{3}{2}$ AS REQUIRED



ALTERNATIVE

$(3+2i)(12+5i) = 36 + 15i + 24i - 10 = 26 + 39i$

$\arg[(3+2i)(12+5i)] = \arg(26+39i)$

$\arg(3+2i) + \arg(12+5i) = \arg(26+39i)$

$\arctan\frac{2}{3} + \arctan\frac{5}{12} = \arctan\frac{39}{26}$

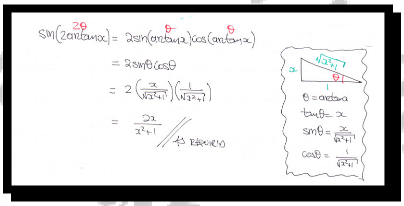
$\therefore \arctan\frac{2}{3} + \arctan\frac{5}{12} = \arctan\frac{3}{2}$ AS REQUIRED

Question 55 **(***+)**

Show clearly that

$$\sin\left(2\arctan x\right)=\frac{2x}{x^2+1}.$$

proof

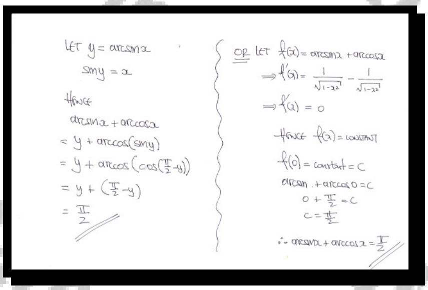


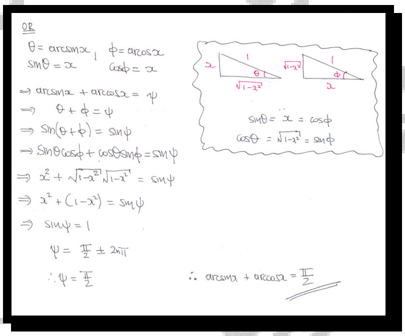
Question 56 **(***+)**

Prove the validity of the following trigonometric identity

$$\arcsin x+\arccos x=\frac{\pi}{2}.$$

proof



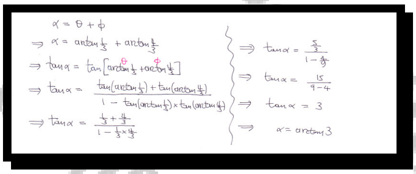


Question 57 (***+)

Show clearly that

$$\arctan\frac{1}{3} + \arctan\frac{4}{3} = \arctan 3 .$$

proof

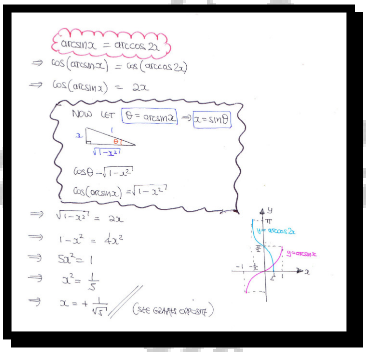


Question 58 (***+)

Solve the trigonometric equation

$$\arcsin x = \arccos 2x .$$

$$x = \frac{1}{\sqrt{5}}$$

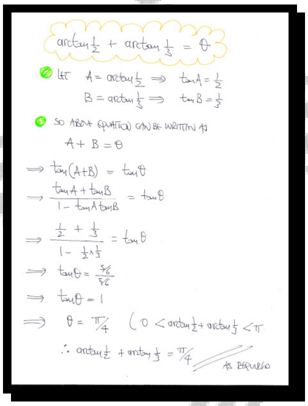


Question 59 **(***+)**

Using a detailed method, show that

$$\arctan\frac{1}{2}+\arctan\frac{1}{3}=\frac{1}{4}\pi .$$

proof

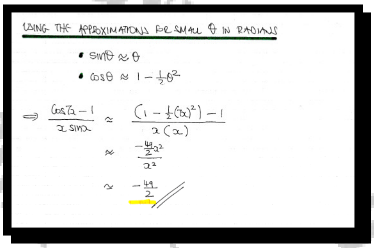


Question 60 **(***+)**

Given that x is measured in radians, use small angle approximations to simplify the following expression.

$$\frac{\cos 7x \; - \; 1}{x\sin x}$$

[ ] , $-\frac{49}{2}$



Question 61 (***+)

Given that x is measured in radians, use small angle approximations to simplify the following expression.

$$\frac{\cos^2(3x) - 1}{2x\sin\left(\frac{3}{4}x\right)}$$

$\boxed{}$, $\boxed{-6}$

USING THE APPROXIMATIONS FOR SMALL θ, IN RADIANS

- $\sin\theta \simeq \theta$
- $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$

$$\frac{\cos^2 3x - 1}{2x\sin\left(\frac{3}{4}x\right)} \simeq \frac{\left[1 - \frac{1}{2}(3x)^2\right]^2 - 1}{2x\left(\frac{3}{4}x\right)} \simeq \frac{\left(1 - \frac{9}{2}x^2\right)^2 - 1}{\frac{3}{2}x^2} \simeq \frac{\left(1 - 9x^2 + \frac{81}{4}x^4\right) - 1}{\frac{3}{2}x^2} \simeq \frac{-9x^2}{\frac{3}{2}x^2} \simeq -6$$

ALTERNATIVE

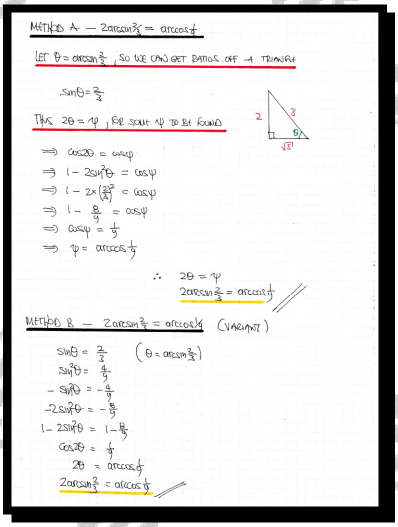
$$\frac{\cos^2 3x - 1}{2x\sin\left(\frac{3}{4}x\right)} = -\frac{1 - \cos^2 3x}{2x\sin\left(\frac{3}{4}x\right)} = -\frac{\sin^2 3x}{2x\sin\left(\frac{3}{4}x\right)} \approx -\frac{(3x)^2}{2x\left(\frac{3}{4}x\right)} \approx -\frac{9x^2}{\frac{3}{2}x^2} \approx -6$$

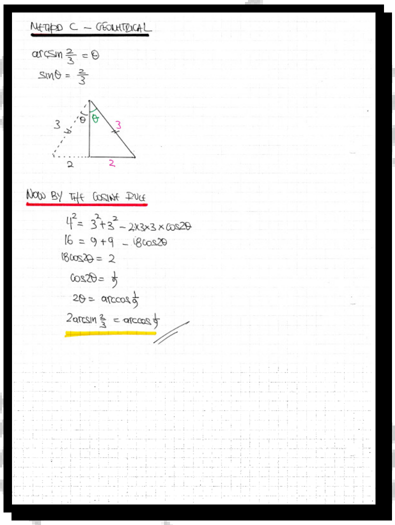
Question 62 **(***+)**

Prove that

$$2\arcsin\left(\frac{2}{3}\right) = \arccos\left(\frac{1}{9}\right).$$

V , , proof





Question 63 (***+)

Solve the following trigonometric equation

$$2\arctan\left(\frac{3}{x}\right) = \arctan\left(\frac{6x}{25}\right).$$

$x = \pm 6$

LET $\theta = \arctan\left(\frac{3}{x}\right)$ & $\phi = \arctan\left(\frac{2x}{9}\right)$

$$\Rightarrow 2\arctan\left(\frac{3}{x}\right) = \arctan\left(\frac{2x}{9}\right)$$

$$\Rightarrow 2\theta = \phi$$

$$\Rightarrow \tan 2\theta = \tan\phi$$

$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \tan\phi$$

BUT IF $\theta = \arctan\left(\frac{3}{x}\right) \Rightarrow \tan\theta = \frac{3}{x}$

$\phi = \arctan\left(\frac{2x}{9}\right) \Rightarrow \tan\phi = \frac{2x}{9}$

$$\Rightarrow \frac{2\left(\frac{3}{x}\right)}{1-\left(\frac{3}{x}\right)^2} = \frac{2x}{9}$$

$$\Rightarrow \frac{\frac{6}{x}}{1-\frac{9}{x^2}} = \frac{2x}{9}$$

MULTIPLY "TOP" & "BOTTOM" OF THE DOUBLE FRACTION BY x^2

$$\Rightarrow \frac{6x}{x^2-9} = \frac{2x}{9}$$

AS $x \neq 0$, WE MAY DIVIDE BOTH SIDES BY x

$$\Rightarrow \frac{6}{x^2-9} = \frac{2}{9}$$

$$\Rightarrow 2(x^2-9) = 54$$

$$\Rightarrow x^2-9 = 27$$

$$\Rightarrow x^2 = 36$$

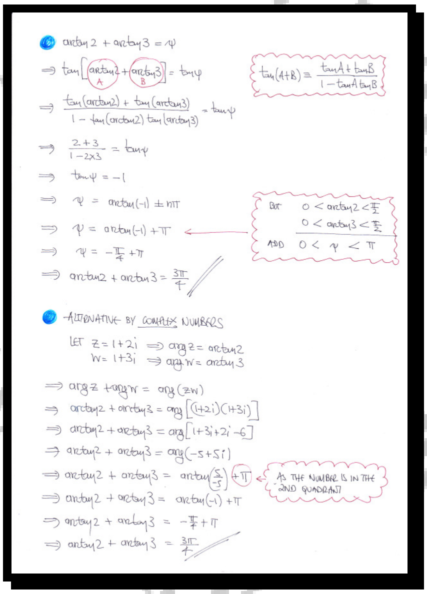
$$\Rightarrow x = \pm 6$$

Question 64 (***+)

Show, by detailed workings, that

$$\arctan 2 + \arctan 3 = \frac{3\pi}{4}.$$

proof

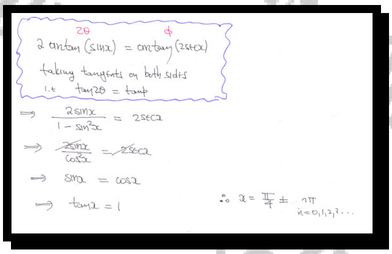


Question 65 (***+)

Find the general solution of the following trigonometric equation

$$2\arctan(\sin x) = \arctan(\sec x).$$

$$x = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$$



Question 66 (****)

$$f(x) = \left(x^2+1\right)(x-1), \ x \in \mathbb{R}.$$

a) Simplify $f(x)$.

b) Prove the validity of the trigonometric identity

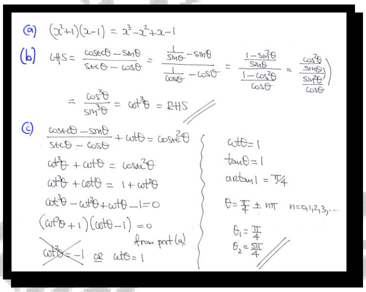
$$\frac{\operatorname{cosec}\theta - \sin\theta}{\sec\theta - \cos\theta} \equiv \cot^3\theta.$$

c) Hence, or otherwise, solve the equation

$$\frac{\operatorname{cosec}\theta - \sin\theta}{\sec\theta - \cos\theta} + \cot\theta = \operatorname{cosec}^2\theta, \quad 0 \le x < 2\pi,$$

giving the answers in terms of π.

$\boxed{x^3 - x^2 + x - 1}$, $\boxed{x = \frac{\pi}{4}, \frac{5\pi}{4}}$


(a) $(x^2+1)(x-1) = x^3-x^2+x-1$

(b) $\text{LHS} = \frac{\operatorname{cosec}\theta - \sin\theta}{\sec\theta - \cos\theta} = \frac{\frac{1}{\sin\theta} - \sin\theta}{\frac{1}{\cos\theta} - \cos\theta} = \frac{\frac{1-\sin^2\theta}{\sin\theta}}{\frac{1-\cos^2\theta}{\cos\theta}} = \frac{\frac{\cos^2\theta}{\sin\theta}}{\frac{\sin^2\theta}{\cos\theta}}$

$= \frac{\cos^3\theta}{\sin^3\theta} = \cot^3\theta = \text{RHS}$

(c) $\frac{\operatorname{cosec}\theta - \sin\theta}{\sec\theta - \cos\theta} + \cot\theta = \operatorname{cosec}^2\theta$

$\cot^3\theta + \cot\theta = \operatorname{cosec}^2\theta$

$\cot^3\theta + \cot\theta = 1 + \cot^2\theta$

$\cot^3\theta - \cot^2\theta + \cot\theta - 1 = 0$

$(\cot^2\theta + 1)(\cot\theta - 1) = 0$ from part (a).

~~$\cot^2\theta = -1$~~ or $\cot\theta = 1$

$\cot\theta = 1$

$\tan\theta = 1$

$\arctan 1 = \pi/4$

$\theta = \frac{\pi}{4} \pm n\pi \quad n = 0, 1, 2, 3, \ldots$

$\theta_1 = \frac{\pi}{4}$

$\theta_2 = \frac{5\pi}{4}$

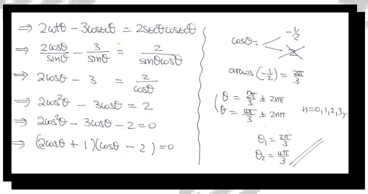
Question 67 (****)

Solve the following trigonometric equation

$$2\cot\theta - 3\operatorname{cosec}\theta = 2\sec\theta\operatorname{cosec}\theta, \qquad 0 < \theta < 2\pi, \quad \theta \neq \frac{k\pi}{2}, k \in \mathbb{Z},$$

giving the answers in terms of π.

$$\boxed{\theta = \frac{2\pi}{3}, \frac{4\pi}{3}}$$



Question 68 (****)

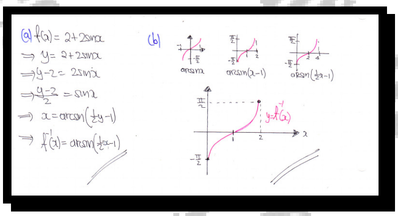
$$f(x) = 2 + 2\sin x, \quad -\pi \leq x \leq \pi.$$

a) Find an expression for $f^{-1}(x)$.

b) Sketch the graph of $f^{-1}(x)$.

The sketch must include the coordinates of any points where the graph of $f^{-1}(x)$ meet the coordinate axes as well as the coordinates of its endpoints.

$$\boxed{f^{-1}(x) = \arcsin\left(\tfrac{1}{2}x - 1\right)}$$



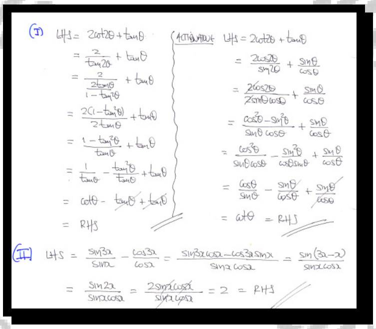
Question 69 (****)

Prove the validity of each of the following trigonometric identities.

a) $2\cot 2\theta + \tan\theta \equiv \cot\theta$.

b) $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} \equiv 2$.

proof



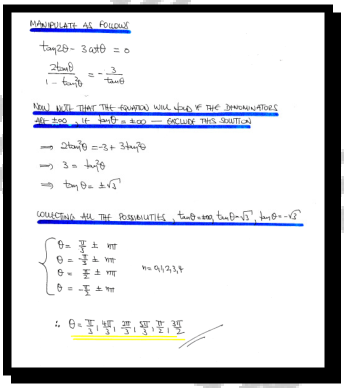
Question 70 (****)

Show that the following trigonometric equation

$$\tan 2\theta - 3\cot\theta = 0, \quad 0 < \theta < 2\pi, ,$$

has six solutions in the interval $0 < \theta < 2\pi$, giving the answers in terms of π.

$\theta = \frac{1}{3}\pi, \ \frac{1}{2}\pi, \ \frac{2}{3}\pi, \ \frac{4}{3}\pi, \ \frac{3}{2}\pi, \ \frac{5}{3}\pi$



Question 71 (****)

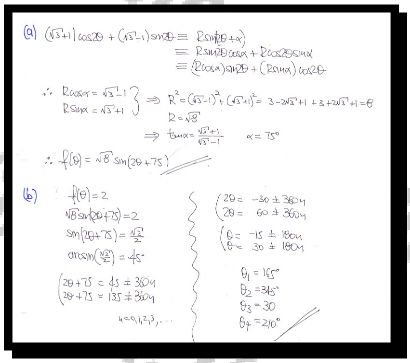
$$f(\theta°) \equiv \left(\sqrt{3}+1\right)\cos 2\theta° + \left(\sqrt{3}-1\right)\sin 2\theta° .$$

a) Express $f(\theta)$ in the form $R\sin(2\theta+\alpha)$, $R>0$, $0 \le \theta° < 90$.

b) Solve the equation

$$f(\theta°) = 2 ,\ 0 \le \theta° < 360 .$$

$\boxed{}$, $\boxed{f(\theta°) \equiv \sqrt{8}\sin(2\theta+75)°}$, $\boxed{\theta = 30°, 165°, 210°, 345°}$



Question 72 (****)

It is given that

$$\frac{\sec x}{1+\sec x} - \frac{\sec x}{1-\sec x} \equiv 2\operatorname{cosec}^2 x \, .$$

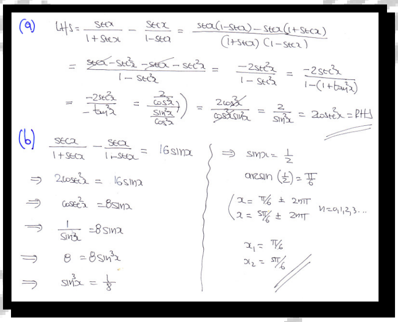
a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, solve the equation

$$\frac{\sec x}{1+\sec x} - \frac{\sec x}{1-\sec x} = 16\sin x\,, \quad 0 \le x < 2\pi\,,$$

giving the answers in terms of π.

$$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}}$$



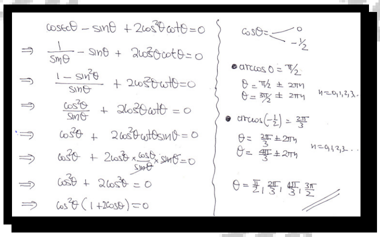
Question 73 (****)

Solve the trigonometric equation

$$\operatorname{cosec}\theta - \sin\theta + 2\cos^2\theta\cot\theta = 0, \quad 0 < \theta < 2\pi, \ \theta \neq \pi,$$

giving the answers in terms of π.

$$\boxed{\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}}$$



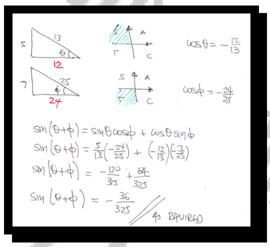
Question 74 (****)

$$\sin\theta = \frac{5}{13} \quad \text{and} \quad \sin\varphi = -\frac{7}{25}.$$

If θ is obtuse and φ is such so that $180° < \varphi < 270°$, show that

$$\sin(\theta + \varphi) = -\frac{36}{325}.$$

$$\boxed{\text{proof}}$$



Question 75 (****)

It is given that

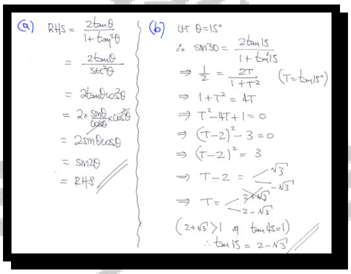
$$\sin 2\theta \equiv \frac{2\tan\theta}{1+\tan^2\theta}.$$

a) Prove the validity of the above trigonometric identity.

b) By using $\theta = 15°$ in the above identity, show that

$$\tan 15° = 2-\sqrt{3}.$$

proof



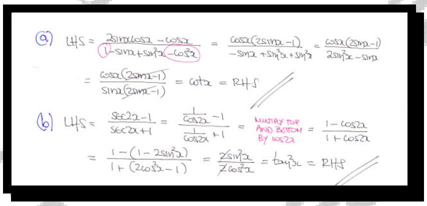
Question 76 (****)

Prove the validity of each of the following trigonometric identities.

a) $\dfrac{2\sin x\cos x-\cos x}{1-\sin x+\sin^2 x-\cos^2 x} \equiv \cot x$.

b) $\dfrac{\sec 2x-1}{\sec 2x+1} \equiv \tan^2 x$.

proof



Question 77 (****)

It is given that

$$\cos 3x \equiv 4\cos^3 x - 3\cos x .$$

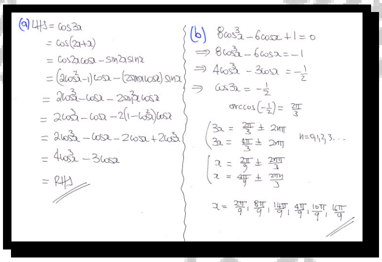
a) Prove the validity of the above trigonometric identity by writing $\cos 3x$ as $\cos(2x+x)$.

b) Hence, or otherwise, solve the trigonometric equation

$$8\cos^3 x - 6\cos x + 1 = 0, \ 0 \le x < 2\pi ,$$

giving the answers in terms of π.

$$x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$$



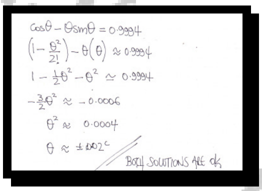
Question 78 (****)

It is given that

$$\cos\theta - \theta\sin\theta = 0.9994 .$$

Given that the above equation has a solution that is numerically small, show by using a quadratic approximation that $\theta = \pm 0.02^{c}$.

proof



Question 79 (****)

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B\,.$$

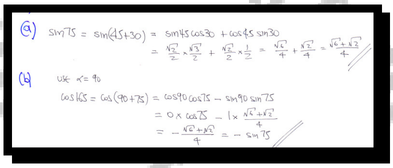
a) Use the above trigonometric identity with suitable values for A and B, to show

$$\sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}\,.$$

b) Hence by using the trigonometric expansion of $\cos(75°+\alpha)$ with a suitable value for α, show that

$$\cos 165° = -\sin 75°\,.$$

proof



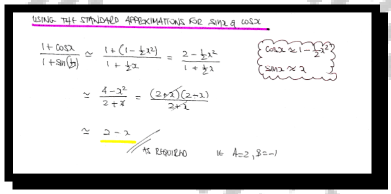
Question 80 (****)

Use small angle approximations to show that if x is measured in radians then

$$\frac{1+\cos x}{1+\sin\left(\frac{1}{2}x\right)} \approx A + Bx\,,$$

where A and B are constants to be found.

, $A=2$, $B=-1$



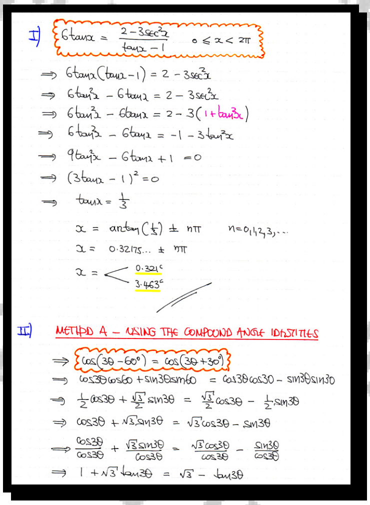
Question 81 (****)

Solve each of the following trigonometric equations.

i. $6\tan x = \dfrac{2 - 3\sec^2 x}{\tan x - 1}$, $\quad 0 \le x < 2\pi$, $\quad x \ne \dfrac{\pi}{4}, \dfrac{5\pi}{4}$.

ii. $\cos(3\theta - 60°) = \cos(3\theta + 30°)$, $\quad 0 \le \theta \le 180°$.

$\boxed{}$, $\boxed{x \approx 0.322^c, \ x \approx 3.46^c}$, $\boxed{\theta = 5°, \ 65°, \ 125°}$



I) $6\tan x = \dfrac{2-3\sec^2 x}{\tan x - 1}$ $\quad 0 \le x < 2\pi$

$\Rightarrow 6\tan x(\tan x - 1) = 2 - 3\sec^2 x$

$\Rightarrow 6\tan^2 x - 6\tan x = 2 - 3\sec^2 x$

$\Rightarrow 6\tan^2 x - 6\tan x = 2 - 3(1+\tan^2 x)$

$\Rightarrow 6\tan^2 x - 6\tan x = -1 - 3\tan^2 x$

$\Rightarrow 9\tan^2 x - 6\tan x + 1 = 0$

$\Rightarrow (3\tan x - 1)^2 = 0$

$\Rightarrow \tan x = \frac{1}{3}$

$x = \arctan\left(\frac{1}{3}\right) \pm n\pi \quad n = 0,1,2,3,\ldots$

$x = 0.32175\ldots \pm n\pi$

$x = 0.321^c, \ 3.463^c$

II) METHOD A – USING THE COMPOUND ANGLE IDENTITIES

$\Rightarrow \cos(3\theta - 60°) = \cos(3\theta + 30°)$

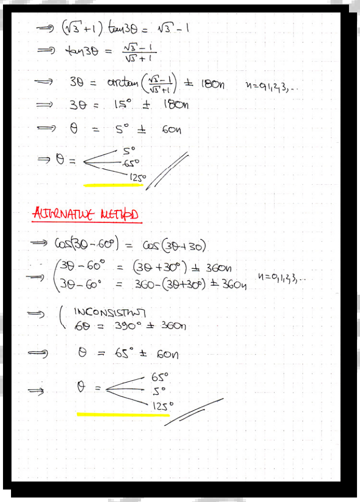
$\Rightarrow \cos 3\theta \cos 60 + \sin 3\theta \sin 60 = \cos 3\theta \cos 30 - \sin 3\theta \sin 30$

$\Rightarrow \frac{1}{2}\cos 3\theta + \frac{\sqrt{3}}{2}\sin 3\theta = \frac{\sqrt{3}}{2}\cos 3\theta - \frac{1}{2}\sin 3\theta$

$\Rightarrow \cos 3\theta + \sqrt{3}\sin 3\theta = \sqrt{3}\cos 3\theta - \sin 3\theta$

$\Rightarrow \frac{\cos 3\theta}{\cos 3\theta} + \frac{\sqrt{3}\sin 3\theta}{\cos 3\theta} = \frac{\sqrt{3}\cos 3\theta}{\cos 3\theta} - \frac{\sin 3\theta}{\cos 3\theta}$

$\Rightarrow 1 + \sqrt{3}\tan 3\theta = \sqrt{3} - \tan 3\theta$



$\Rightarrow (\sqrt{3}+1)\tan 3\theta = \sqrt{3} - 1$

$\Rightarrow \tan 3\theta = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$\Rightarrow 3\theta = \arctan\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) \pm 180n \quad n = 0,1,2,3,\ldots$

$\Rightarrow 3\theta = 15° \pm 180n$

$\Rightarrow \theta = 5° \pm 60n$

$\Rightarrow \theta = 5°, \ 65°, \ 125°$

ALTERNATIVE METHOD

$\Rightarrow \cos(3\theta - 60°) = \cos(3\theta + 30)$

$\Rightarrow \begin{cases} 3\theta - 60° = (3\theta + 30°) \pm 360n \\ 3\theta - 60° = 360 - (3\theta + 30°) \pm 360n \end{cases} \quad n = 0,1,2,3,\ldots$

$\Rightarrow \begin{cases} \text{INCONSISTENT} \\ 6\theta = 390° \pm 360n \end{cases}$

$\Rightarrow \theta = 65° \pm 60n$

$\Rightarrow \theta = 65°, \ 5°, \ 125°$

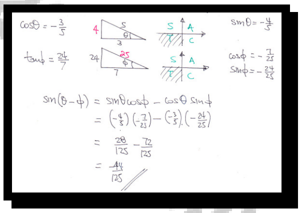
Question 82 (****)

$$\cos\theta = -\frac{3}{5} \quad \text{and} \quad \tan\varphi = \frac{24}{7}.$$

If θ is reflex, and φ is also reflex, show that

$$\sin(\theta - \varphi) = -\frac{44}{125}.$$

proof

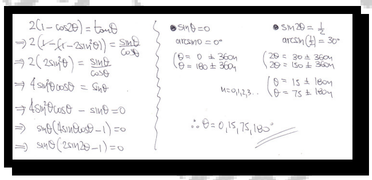


Question 83 (****)

Solve the trigonometric equation

$$2(1 - \cos 2\theta) = \tan\theta, \qquad 0 \le \theta \le 180°.$$

$\theta = 0°, 15°, 75°, 180°$



Question 84 **(****)**

A curve has equation

$$y = \pi - \arccos(x+1),\ -2 \le x \le 0.$$

a) Describe geometrically the 3 transformations that map the graph of

$$y = \arccos x,\ -1 \le x \le 1,$$

onto the graph of

$$y = \pi - \arccos(x+1),\ -2 \le x \le 0.$$

b) Sketch the graph of

$$y = \pi - \arccos(x+1),\ -2 \le x \le 0.$$

The sketch must include the coordinates of any points where the graph meets the coordinate axes.

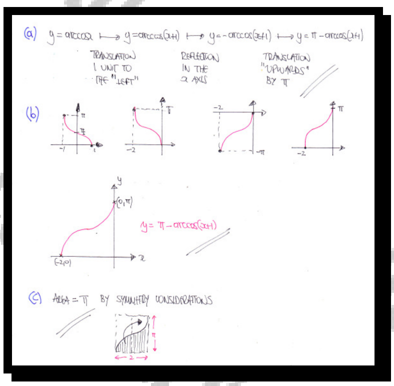
c) Use symmetry arguments to find the area of the finite region bounded by

$$y = \pi - \arccos(x+1),\ -2 \le x \le 0,$$

and the coordinate axes.

[ ], [translation by 1 unit to the right, followed by reflection in the x axis],

[area $= \pi$]



Question 85 (****)

It is given that

$$(\cos x + \sec x)^2 \equiv \cos^2 x + \tan^2 x + 3 .$$

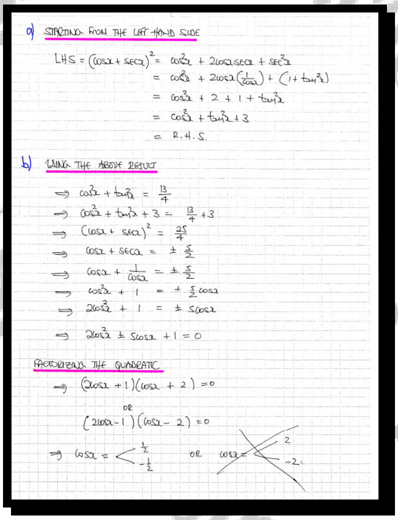
a) Prove the validity of the above trigonometric identity.

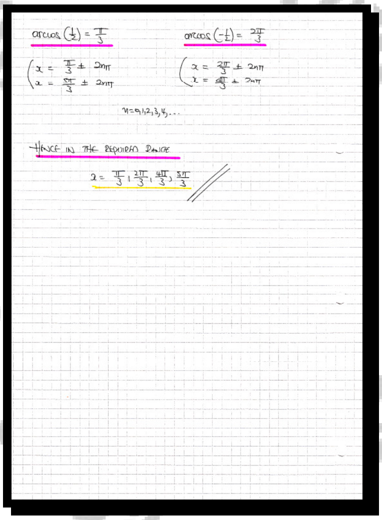
b) Hence solve the trigonometric equation

$$\cos^2 x + \tan^2 x = \frac{13}{4}, \quad 0 \le x < 2\pi ,$$

giving the answers in terms of π.

$\boxed{}$, $\boxed{x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$





Question 86 (****)

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$$

a) By using the above identities show that

$$\cos(A+B) + \cos(A-B) \equiv 2\cos A \cos B .$$

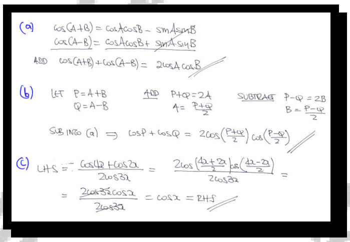
b) Hence show that

$$\cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right).$$

c) Deduce that

$$\frac{\cos 4x + \cos 2x}{2\cos 3x} \equiv \cos x .$$

proof



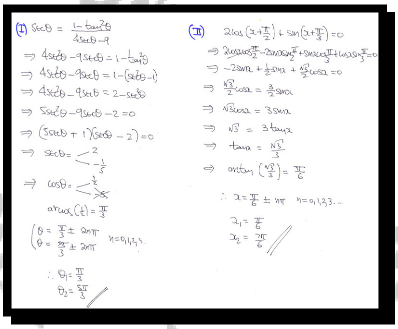
Question 87 (****)

Solve each of the following trigonometric equations.

i. $\sec\theta = \dfrac{1-\tan^2\theta}{4\sec\theta - 9}, \qquad 0 \le \theta < 2\pi$.

ii. $2\cos\left(x+\dfrac{\pi}{2}\right) + \sin\left(x+\dfrac{\pi}{3}\right) = 0, \qquad 0 \le x < 2\pi$.

$\boxed{\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}}, \quad \boxed{x = \dfrac{\pi}{6}, \dfrac{7\pi}{6}}$



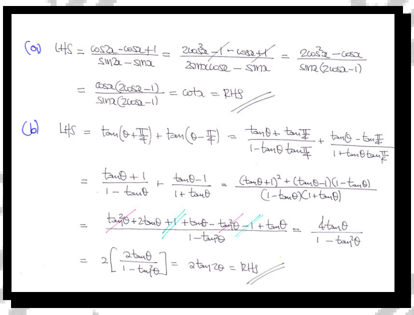
Question 88 (****)

Prove the validity of each of the following trigonometric identities.

a) $\dfrac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} \equiv \cot x$.

b) $\tan\left(\theta+\dfrac{\pi}{4}\right) + \tan\left(\theta-\dfrac{\pi}{4}\right) \equiv 2\tan 2\theta$.

proof



Question 89 (****)

It is given that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

a) Use the above trigonometric identity to express $\tan 2\theta$ in terms of $\tan\theta$.

b) Hence determine the exact value of $\tan 22.5°$, showing clearly all the relevant workings.

$$\boxed{\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}}, \quad \boxed{\tan 22.5° = -1+\sqrt{2}}$$

(a) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

• LET $A = B = \theta$

$\Rightarrow \tan(\theta+\theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$

$\Rightarrow \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

(b) LET $\theta = 22.5°$

$\Rightarrow \tan 45 = \frac{2\tan 22.5°}{1-\tan^2 22.5°}$

$\Rightarrow 1 = \frac{2T}{1-T^2}$ WHERE $T = \tan 22.5$

$\Rightarrow 1 - T^2 = 2T$

$\Rightarrow 0 = T^2 + 2T - 1$

$\Rightarrow 0 = (T+1)^2 - 2$

$\Rightarrow 2 = (T+1)^2$

$\Rightarrow \pm\sqrt{2} = T+1$

$\Rightarrow T = -1 \pm \sqrt{2}$

$\Rightarrow \tan 22.5° = -1+\sqrt{2}$ or ~~$-1-\sqrt{2}$~~ < 0 ($\tan 22.5° > 0$)

$\therefore \tan 22.5° = -1+\sqrt{2}$

Question 90 (****)

Prove the validity of the trigonometric identity

$$\frac{1-\cos 2x + \sin 2x}{1+\cos 2x + \sin 2x} \equiv \tan x.$$

proof

$\text{LHS} = \frac{1-\cos 2x + \sin 2x}{1+\cos 2x + \sin 2x} = \frac{1-(1-2\sin^2 x) + \sin 2x}{1 + (2\cos^2 x - 1) + \sin 2x} = \frac{2\sin^2 x + \sin 2x}{2\cos^2 x + \sin 2x}$

$= \frac{2\sin^2 x + 2\sin x\cos x}{2\cos^2 x + 2\sin x\cos x} = \frac{2\sin x(\sin x + \cos x)}{2\cos x(\cos x + \sin x)} = \tan x = \text{RHS}$

Question 91 (****)

In this question it is given that the exact value of $\tan 20° = t$.

a) Express $\tan 25°$ in terms of t.

b) By using the result of part **(a)** show that

$$\tan 25° \tan 65° = 1.$$

c) Show further that if

$$2\cos(\theta° + 20°) = 5\sin(\theta° - 20°),$$

then

$$\tan\theta = \frac{2+5t}{5+2t}.$$

$\boxed{}$, $\boxed{\tan 25° = \dfrac{1-t}{1+t}}$

(a) $\tan 25 = \tan(45-20) = \dfrac{\tan 45 - \tan 20}{1+\tan 45 \tan 20} = \dfrac{1-t}{1+1\times t} = \dfrac{1-t}{1+t}$

• SIMILARLY

$\tan 65 = \tan(45+20) = \dfrac{\tan 45 + \tan 20}{1-\tan 45 \tan 20} = \dfrac{1+t}{1-1\times t} = \dfrac{1+t}{1-t}$

$\therefore \tan 25 \tan 65 = \dfrac{1-t}{1+t} \times \dfrac{1+t}{1-t} = 1$

(a) $2\cos(\theta+20) = 5\sin(\theta-20)$

$\Rightarrow 2\cos\theta\cos 20 - 2\sin\theta\sin 20 = 5\sin\theta\cos 20 - 5\cos\theta\sin 20$

$\Rightarrow \dfrac{2\cos\theta\cos 20°}{\cos\theta\cos 20°} - \dfrac{2\sin\theta\sin 20°}{\cos\theta\cos 20°} = \dfrac{5\sin\theta\cos 20°}{\cos\theta\cos 20°} - \dfrac{5\cos\theta\sin 20°}{\cos\theta\cos 20°}$

$\Rightarrow 2 - 2\tan\theta\tan 20° = 5\tan\theta - 5\tan 20$

$\Rightarrow 2 - 2t\tan\theta = 5\tan\theta - 5t$

$\Rightarrow 2 + 5t = 5\tan\theta + 2t\tan\theta$

$\Rightarrow (2+5t) = (5+2t)\tan\theta$

$\Rightarrow \tan\theta = \dfrac{2+5t}{5+2t}$ AS REQUIRED

Question 92 (****)

$$f(x) = -2 + 2\tan\left(\tfrac{1}{2}x\right), \quad -\pi \le x \le \pi .$$

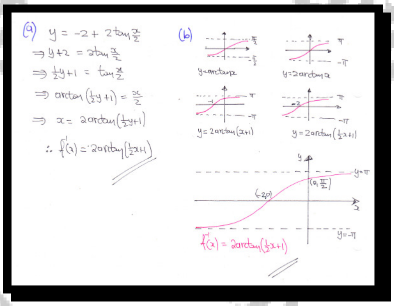
a) Find an expression for $f^{-1}(x)$.

b) Sketch the graph of $f^{-1}(x)$.

The sketch must include …

- …the equations of the asymptotes of $f^{-1}(x)$
- …the coordinates of any points where the graph of $f^{-1}(x)$ meets the coordinate axes.

$$\boxed{f^{-1}(x) = 2\arctan\left(\tfrac{1}{2}x + 1\right)}$$



Question 93 (****)

The function f is defined as

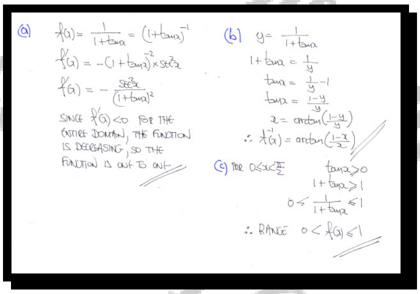
$$f(x) = \frac{1}{1+\tan x}, \quad 0 \le x < \frac{\pi}{2}.$$

a) Use differentiation to show that f is a one to one function.

b) Find a simplified expression for the inverse of f.

c) Determine the range of f.

$\boxed{}$, $\boxed{f^{-1}(x) = \arctan\left(\frac{1-x}{x}\right)}$, $\boxed{0 < f(x) \le 1}$



(a) $f(x) = \frac{1}{1+\tan x} = (1+\tan x)^{-1}$

$f'(x) = -(1+\tan x)^{-2} \times \sec^2 x$

$f'(x) = -\frac{\sec^2 x}{(1+\tan x)^2}$

SINCE $f'(x) < 0$ FOR THE ENTIRE DOMAIN, THE FUNCTION IS DECREASING, SO THE FUNCTION IS ONE TO ONE

(b) $y = \frac{1}{1+\tan x}$

$1+\tan x = \frac{1}{y}$

$\tan x = \frac{1}{y} - 1$

$\tan x = \frac{1-y}{y}$

$x = \arctan\left(\frac{1-y}{y}\right)$

$\therefore f^{-1}(x) = \arctan\left(\frac{1-x}{x}\right)$

(c) FOR $0 \le x < \frac{\pi}{2}$

$\tan x \ge 0$

$1+\tan x \ge 1$

$0 < \frac{1}{1+\tan x} \le 1$

$\therefore$ RANGE $0 < f(x) \le 1$

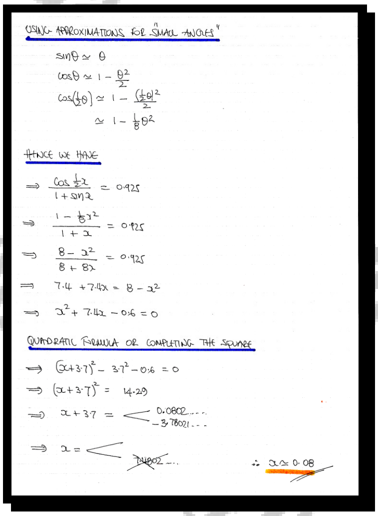
Question 94 (****)

$$\frac{\cos\left(\frac{1}{2}x\right)}{1+\sin x} = 0.925 .$$

Given that the above equation has a solution that is numerically small, find this solution by using a quadratic approximation.

No credit will be given for solving a trigonometric equation.

[   ] , $x \approx 0.08$



USING APPROXIMATIONS FOR "SMALL ANGLES"

$\sin\theta \simeq \theta$

$\cos\theta \simeq 1 - \frac{\theta^2}{2}$

$\cos\left(\frac{1}{2}\theta\right) \simeq 1 - \frac{\left(\frac{1}{2}\theta\right)^2}{2} \simeq 1 - \frac{1}{8}\theta^2$

HENCE WE HAVE

$\Rightarrow \frac{\cos\frac{1}{2}x}{1+\sin x} = 0.925$

$\Rightarrow \frac{1-\frac{1}{8}x^2}{1+x} = 0.925$

$\Rightarrow \frac{8-x^2}{8+8x} = 0.925$

$\Rightarrow 7.4 + 7.4x = 8 - x^2$

$\Rightarrow x^2 + 7.4x - 0.6 = 0$

QUADRATIC FORMULA OR COMPLETING THE SQUARE

$\Rightarrow (x+3.7)^2 - 3.7^2 - 0.6 = 0$

$\Rightarrow (x+3.7)^2 = 14.29$

$\Rightarrow x + 3.7 = 0.0802\ldots$ or $-3.78021\ldots$

$\Rightarrow x =$ ~~7.4802...~~

$\therefore x \simeq 0.08$

Question 95 (****)

It is given that

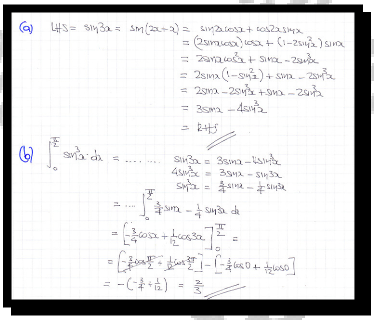
$$\sin 3x \equiv 3\sin x - 4\sin^3 x .$$

a) Prove the validity of the above trigonometric identity, by writing $\sin 3x$ as $\sin(2x + x)$.

b) Hence, or otherwise, find the exact value of

$$\int_0^{\frac{\pi}{2}} \sin^3 x \ dx .$$

$\boxed{\frac{2}{3}}$



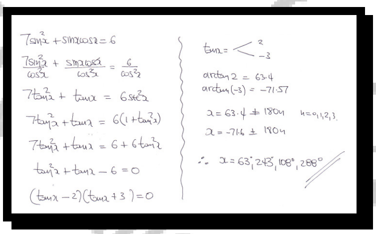
Question 96 (****)

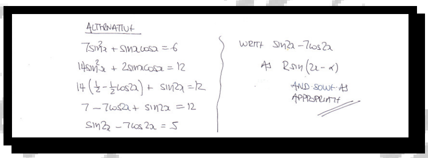
Solve the trigonometric equation

$$7\sin^2 x + \sin x \cos x = 6 ,\ 0° \le x \le 360° ,$$

giving the answers to the nearest degree.

$\boxed{}$, $\boxed{x \approx 63°,\ 108°,\ 243°,\ 288°}$





Question 97 (****)

$$f(x) = \sqrt{3}\cos x - \sin x, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\cos(x+\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

b) State the maximum value of $f(x)$ and find the smallest positive value of x for which this maximum occurs.

The depth of the water, D metres, in a harbour is modelled by the equation

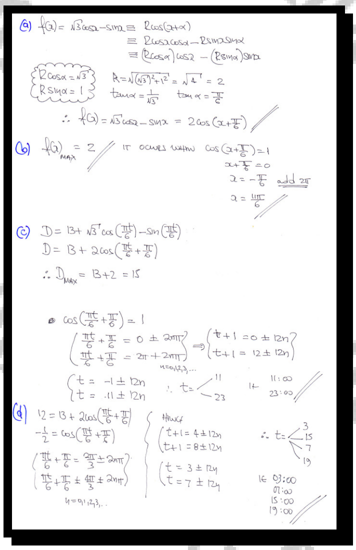
$$D = 13 + \sqrt{3}\cos\left(\frac{\pi t}{6}\right) - \sin\left(\frac{\pi t}{6}\right), \quad 0 \le t < 24$$

where t is the time in hours measured since midnight.

c) State the maximum depth of the water in the harbour and a time when this maximum depth occurs.

d) Find the times when the depth of the water in the harbour is 12 metres.

$\boxed{}$, $\boxed{\sqrt{3}\cos x - \sin x \equiv 2\cos\left(x+\frac{\pi}{6}\right)}$, $\boxed{\max = 2}$, $\boxed{x = \frac{11\pi}{6}}$, $\boxed{D_{max} = 15}$,
$\boxed{11:00/23:00}$, $\boxed{03:00/07:00/15:00/19:00}$



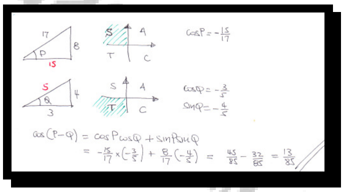
Question 98 (****)

$$\sin P = \frac{8}{17} \quad \text{and} \quad \tan Q = \frac{4}{3}.$$

If P is obtuse and Q is reflex, show that

$$\cos(P-Q) = \frac{13}{85}.$$

proof



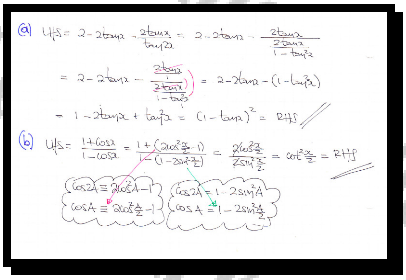
Question 99 (****)

Prove the validity of each of the following trigonometric identities.

a) $2 - 2\tan x - \dfrac{2\tan x}{\tan 2x} \equiv (1-\tan x)^2$.

b) $\dfrac{1-\cos x}{1+\cos x} \equiv \cot^2\left(\dfrac{x}{2}\right)$.

proof



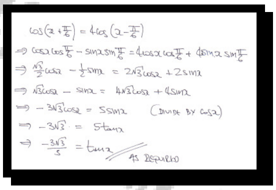
Question 100 (****)

$$\cos\left(x+\frac{\pi}{6}\right)=4\cos\left(x-\frac{\pi}{6}\right).$$

Show by using an appropriate compound angle identity that

$$\tan x=-\frac{3}{5}\sqrt{3}.$$

proof



$\cos\left(x+\frac{\pi}{6}\right)=4\cos\left(x-\frac{\pi}{6}\right)$

$\Rightarrow \cos x\cos\frac{\pi}{6}-\sin x\sin\frac{\pi}{6}=4\cos x\cos\frac{\pi}{6}+4\sin x\sin\frac{\pi}{6}$

$\Rightarrow \frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x=2\sqrt{3}\cos x+2\sin x$

$\Rightarrow \sqrt{3}\cos x-\sin x=4\sqrt{3}\cos x+4\sin x$

$\Rightarrow -3\sqrt{3}\cos x=5\sin x$ (DIVIDE BY $\cos x$)

$\Rightarrow -3\sqrt{3}=5\tan x$

$\Rightarrow -\frac{3\sqrt{3}}{5}=\tan x$ AS REQUIRED

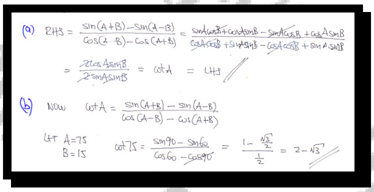
Question 101 (****)

Show clearly that …

a) … $\cot A\equiv\dfrac{\sin(A+B)-\sin(A-B)}{\cos(A-B)-\cos(A+B)}$.

b) … $\cot 75°=2-\sqrt{3}$,
(use part (a) with suitable values of A and B)

proof



(a) RHS $=\frac{\sin(A+B)-\sin(A-B)}{\cos(A-B)-\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B-\sin A\cos B+\cos A\sin B}{\cos A\cos B+\sin A\sin B-\cos A\cos B+\sin A\sin B}$

$=\frac{2\cos A\sin B}{2\sin A\sin B}=\cot A=$ LHS

(b) NOW $\cot A=\frac{\sin(A+B)-\sin(A-B)}{\cos(A-B)-\cos(A+B)}$

LET $A=75$, $B=15$: $\cot 75=\frac{\sin 90-\sin 60}{\cos 60-\cos 90}=\frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}=2-\sqrt{3}$

Question 102 (****)

The curves C_1 and C_2 have respective equations

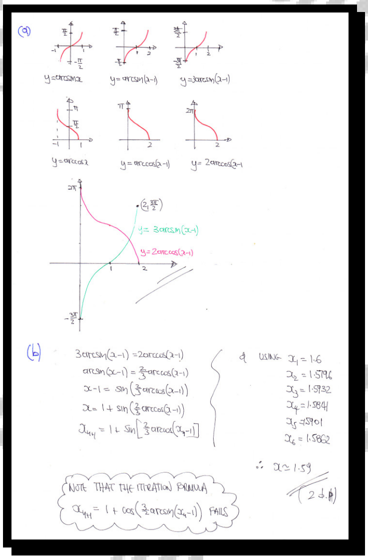
$$y_1 = 3\arcsin(x-1) \quad \text{and} \quad y_1 = 2\arccos(x-1).$$

a) Sketch in the same diagram the graph of C_1 and the graph of C_2.

The sketch must include the coordinates of any points where the graphs of C_1 and C_2 meet the coordinate axes as well as the coordinates of the endpoints of the curves.

b) Use a suitable iteration formula of the form $x_{n+1} = f(x_n)$ with $x_1 = 1.6$ to find the x coordinate of the point of intersection between the graph of C_1 and the graph of C_2.

$\boxed{x \approx 1.59}$



Question 103 (****)

It is given that

$$\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right).$$

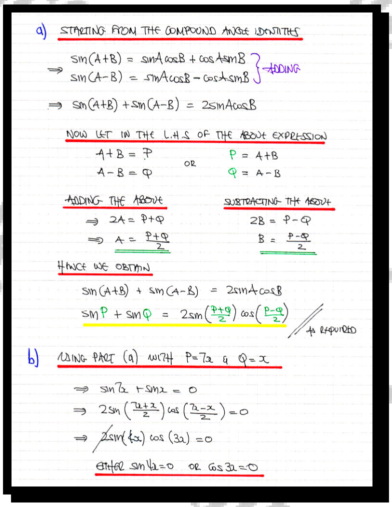
a) Prove the validity of the above trigonometric identity by using the compound angle identities for $\sin(A+B)$ and $\sin(A-B)$.

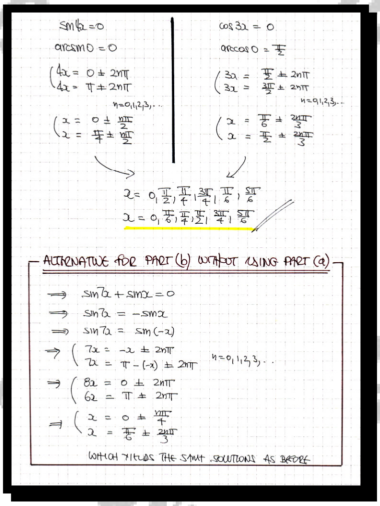
b) Hence, or otherwise, solve the trigonometric equation

$$\sin 7x + \sin x = 0,\quad 0 \le x < \pi,$$

giving the answers in terms of π.

$\boxed{}$, $\boxed{x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}}$





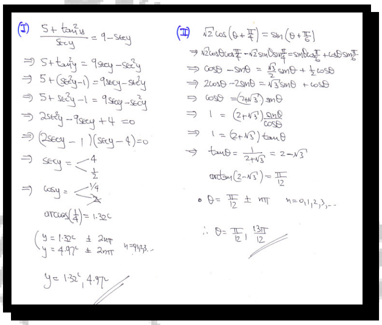
Question 104 (****)

Solve each of the following trigonometric equations.

i. $\dfrac{5+\tan^2 y}{\sec y} = 9 - \sec y, \quad 0 \le y < 2\pi, \quad y \ne \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

ii. $\sqrt{2}\cos\left(\theta + \dfrac{\pi}{4}\right) = \sin\left(\theta + \dfrac{\pi}{6}\right), \qquad 0 \le \theta < 2\pi$.

$\boxed{y = 1.32^{c}, 4.97^{c}}$, $\boxed{\theta = \dfrac{\pi}{12}, \dfrac{13\pi}{12}}$



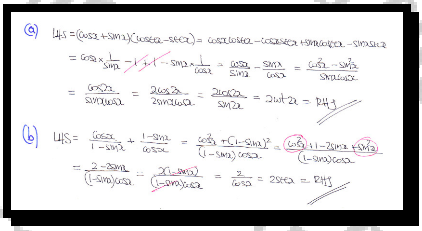
Question 105 (****)

Prove the validity of each of the following trigonometric identities.

a) $(\cos x + \sin x)(\operatorname{cosec} x - \sec x) \equiv 2\cot 2x$.

b) $\dfrac{\cos x}{1-\sin x} + \dfrac{1-\sin x}{\cos x} \equiv \sec x$.

$\boxed{\text{proof}}$



Question 106 (****)

It is given that

$$\cos(A+B)+\cos(A-B)\equiv 2\cos A\cos B\,.$$

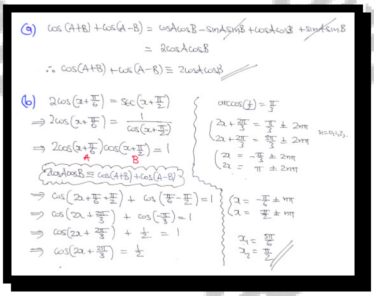
a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, solve the trigonometric equation

$$2\cos\left(x+\frac{\pi}{6}\right)=\sec\left(x+\frac{\pi}{2}\right),\quad 0\leq x\leq\pi\,,$$

giving the answers in terms of π.

$\boxed{}$, $\boxed{x=\frac{\pi}{2},\ \frac{5\pi}{6}}$



Question 107 (****)

It is given that

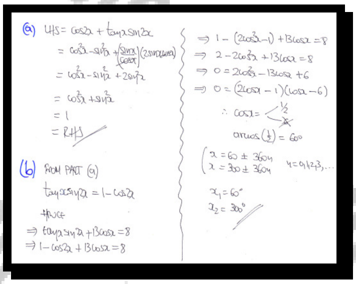
$$\cos 2x + \tan x \sin 2x \equiv 1, \quad x \neq 90°(n+1),\ n \in \mathbb{Z}.$$

a) Prove the validity of the above trigonometric identity.

b) Use the above result to solve the trigonometric equation

$$\tan x \sin 2x + 13\cos x = 8, \quad 0 \leq x < 360°.$$

$$\boxed{x = 60°,\ 300°}$$


(a) $LHS = \cos 2x + \tan x \sin 2x$
$= \cos^2 x - \sin^2 x + \left(\frac{\sin x}{\cos x}\right)(2\sin x \cos x)$
$= \cos^2 x - \sin^2 x + 2\sin^2 x$
$= \cos^2 x + \sin^2 x$
$= 1$
$= RHS$

(b) FROM PART (a)
$\tan x \sin 2x = 1 - \cos 2x$
HENCE
$\Rightarrow \tan x \sin 2x + 13\cos x = 8$
$\Rightarrow 1 - \cos 2x + 13\cos x = 8$
$\Rightarrow 1 - (2\cos^2 x - 1) + 13\cos x = 8$
$\Rightarrow 2 - 2\cos^2 x + 13\cos x = 8$
$\Rightarrow 0 = 2\cos^2 x - 13\cos x + 6$
$\Rightarrow 0 = (2\cos x - 1)(\cos x - 6)$
$\therefore \cos x = \frac{1}{2}$ (or 6, rejected)
$\arccos\left(\frac{1}{2}\right) = 60°$
$x = 60 \pm 360n$
$x = 300 \pm 360n$ $\quad n = 0, 1, 2, 3, \ldots$
$x_1 = 60°$
$x_2 = 300°$

Question 108 (****)

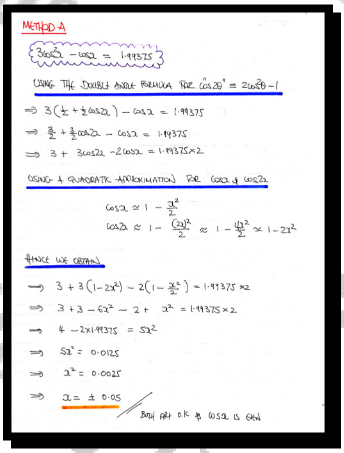
$$3\cos^2 x - \cos x = 1.99375 .$$

It is given that the above trigonometric equation has a solution that is numerically small.

Use small angle approximations to find this solution.

No credit will be given for standard solution methods.

[ ] , $x \approx \pm 0.05$



METHOD A

$3\cos^2 x - \cos x = 1.99375$

USING THE DOUBLE ANGLE FORMULA FOR "$\cos 2\theta$" $= 2\cos^2\theta - 1$

$\Rightarrow 3\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) - \cos x = 1.99375$

$\Rightarrow \frac{3}{2} + \frac{3}{2}\cos 2x - \cos x = 1.99375$

$\Rightarrow 3 + 3\cos 2x - 2\cos x = 1.99375 \times 2$

USING A QUADRATIC APPROXIMATION FOR $\cos x$ & $\cos 2x$

$\cos x \simeq 1 - \frac{x^2}{2}$

$\cos 2x \approx 1 - \frac{(2x)^2}{2} \approx 1 - \frac{4x^2}{2} \simeq 1 - 2x^2$

HENCE WE OBTAIN

$\Rightarrow 3 + 3(1 - 2x^2) - 2\left(1 - \frac{x^2}{2}\right) = 1.99375 \times 2$

$\Rightarrow 3 + 3 - 6x^2 - 2 + x^2 = 1.99375 \times 2$

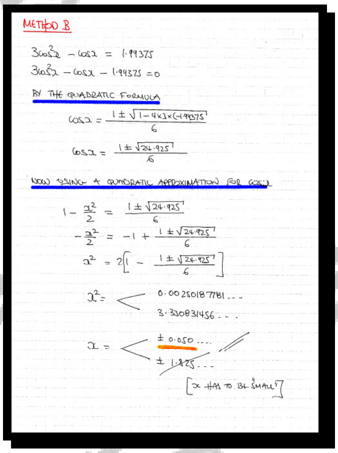
$\Rightarrow 4 - 2 \times 1.99375 = 5x^2$

$\Rightarrow 5x^2 = 0.0125$

$\Rightarrow x^2 = 0.0025$

$\Rightarrow x = \pm 0.05$

BOTH ARE O.K AS $\cos x$ IS EVEN



METHOD B

$3\cos^2 x - \cos x = 1.99375$

$3\cos^2 x - \cos x - 1.99375 = 0$

BY THE QUADRATIC FORMULA

$\cos x = \frac{1 \pm \sqrt{1 - 4 \times 3 \times (-1.99375)}}{6}$

$\cos x = \frac{1 \pm \sqrt{24.925}}{6}$

NOW USING A QUADRATIC APPROXIMATION FOR $\cos x$

$1 - \frac{x^2}{2} = \frac{1 \pm \sqrt{24.925}}{6}$

$-\frac{x^2}{2} = -1 + \frac{1 \pm \sqrt{24.925}}{6}$

$x^2 = 2\left[1 - \frac{1 \pm \sqrt{24.925}}{6}\right]$

$x^2 = 0.0025018778\ldots$ or $3.330831456\ldots$

$x = \pm 0.050\ldots$ or ~~$\pm 1.825\ldots$~~

[x HAS TO BE "SMALL"]

Question 109 (****)

It is given that

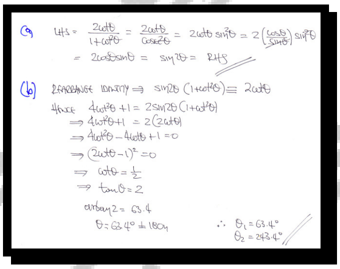
$$\frac{2\cot\theta}{1+\cot^2\theta} \equiv \sin 2\theta .$$

a) Prove the validity of the above trigonometric identity.

b) Use the above result to solve the trigonometric equation

$$4\cot^2\theta + 1 = 2\sin 2\theta\left(1+\cot^2\theta\right), \quad 0 \le \theta < 360° .$$

[ ] , $x \approx 63.4°,\ 243.4°$


(a) $\text{LHS} = \frac{2\cot\theta}{1+\cot^2\theta} = \frac{2\cot\theta}{\operatorname{cosec}^2\theta} = 2\cot\theta\sin^2\theta = 2\left(\frac{\cos\theta}{\sin\theta}\right)\sin^2\theta$

$= 2\cos\theta\sin\theta = \sin 2\theta = \text{RHS}$

(b) REARRANGE IDENTITY $\Rightarrow \sin 2\theta(1+\cot^2\theta) \equiv 2\cot\theta$

HENCE $4\cot^2\theta + 1 = 2\sin 2\theta(1+\cot^2\theta)$

$\Rightarrow 4\cot^2\theta + 1 = 2(2\cot\theta)$

$\Rightarrow 4\cot^2\theta - 4\cot\theta + 1 = 0$

$\Rightarrow (2\cot\theta - 1)^2 = 0$

$\Rightarrow \cot\theta = \frac{1}{2}$

$\Rightarrow \tan\theta = 2$

$\arctan 2 = 63.4$

$\theta = 63.4° \pm 180n$

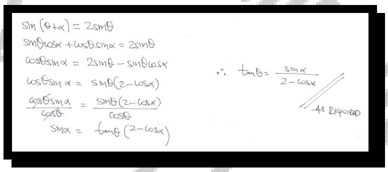
$\therefore \theta_1 = 63.4°$, $\theta_2 = 243.4°$

Question 110 (****)

If $\sin(\theta+\alpha) = 2\sin\theta$, show clearly that

$$\tan\theta = \frac{\sin\alpha}{2-\cos\alpha} .$$

proof


$\sin(\theta+\alpha) = 2\sin\theta$

$\sin\theta\cos\alpha + \cos\theta\sin\alpha = 2\sin\theta$

$\cos\theta\sin\alpha = 2\sin\theta - \sin\theta\cos\alpha$

$\cos\theta\sin\alpha = \sin\theta(2-\cos\alpha)$

$\frac{\cos\theta\sin\alpha}{\cos\theta} = \frac{\sin\theta(2-\cos\alpha)}{\cos\theta}$

$\sin\alpha = \tan\theta(2-\cos\alpha)$

$\therefore \tan\theta = \frac{\sin\alpha}{2-\cos\alpha}$

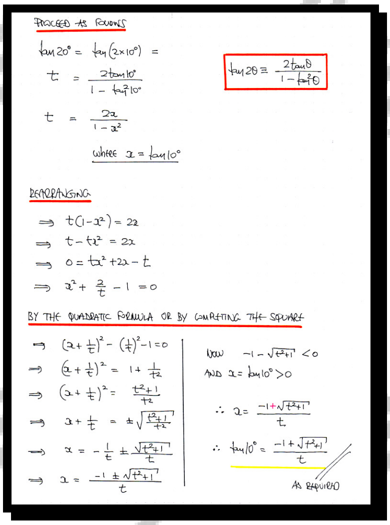
AS REQUIRED

Question 111 (****)

Given that the exact value of $\tan 20° = t$, show that

$$\tan 10° = \frac{-1+\sqrt{t^2+1}}{t}.$$

SYGB, proof


PROCEED AS FOLLOWS

$\tan 20° = \tan(2\times 10°) =$

$t = \frac{2\tan 10°}{1-\tan^2 10°}$ $\qquad$ $\tan 2\theta \equiv \frac{2\tan\theta}{1-\tan^2\theta}$

$t = \frac{2x}{1-x^2}$

where $x = \tan 10°$

REARRANGING

$\Rightarrow t(1-x^2) = 2x$

$\Rightarrow t - tx^2 = 2x$

$\Rightarrow 0 = tx^2 + 2x - t$

$\Rightarrow x^2 + \frac{2}{t}x - 1 = 0$

BY THE QUADRATIC FORMULA OR BY COMPLETING THE SQUARE

$\Rightarrow \left(x+\frac{1}{t}\right)^2 - \left(\frac{1}{t}\right)^2 - 1 = 0$

$\Rightarrow \left(x+\frac{1}{t}\right)^2 = 1 + \frac{1}{t^2}$

$\Rightarrow \left(x+\frac{1}{t}\right)^2 = \frac{t^2+1}{t^2}$

$\Rightarrow x + \frac{1}{t} = \pm\sqrt{\frac{t^2+1}{t^2}}$

$\Rightarrow x = -\frac{1}{t} \pm \frac{\sqrt{t^2+1}}{t}$

$\Rightarrow x = \frac{-1 \pm \sqrt{t^2+1}}{t}$

NOW $-1-\sqrt{t^2+1} < 0$

AND $x = \tan 10° > 0$

$\therefore x = \frac{-1+\sqrt{t^2+1}}{t}$

$\therefore \tan 10° = \frac{-1+\sqrt{t^2+1}}{t}$

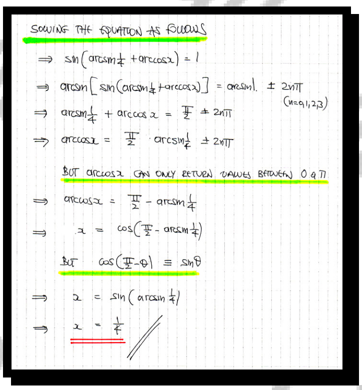
AS REQUIRED

Question 112 (****)

Use trigonometric algebra to solve the equation

$$\sin\left[\arcsin\frac{1}{4} + \arccos x\right] = 1.$$

$x = \frac{1}{4}$


SOLVING THE EQUATION AS FOLLOWS

$\Rightarrow \sin\left(\arcsin\frac{1}{4} + \arccos x\right) = 1$

$\Rightarrow \arcsin\left[\sin\left(\arcsin\frac{1}{4} + \arccos x\right)\right] = \arcsin 1 \pm 2n\pi$ $\quad (n=0,1,2,3)$

$\Rightarrow \arcsin\frac{1}{4} + \arccos x = \frac{\pi}{2} \pm 2n\pi$

$\Rightarrow \arccos x = \frac{\pi}{2} - \arcsin\frac{1}{4} \pm 2n\pi$

BUT $\arccos x$ CAN ONLY RETURN VALUES BETWEEN 0 & π

$\Rightarrow \arccos x = \frac{\pi}{2} - \arcsin\frac{1}{4}$

$\Rightarrow x = \cos\left(\frac{\pi}{2} - \arcsin\frac{1}{4}\right)$

BUT $\cos\left(\frac{\pi}{2}-\theta\right) \equiv \sin\theta$

$\Rightarrow x = \sin\left(\arcsin\frac{1}{4}\right)$

$\Rightarrow x = \frac{1}{4}$

Question 113 (****)

It is given that

$$\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right).$$

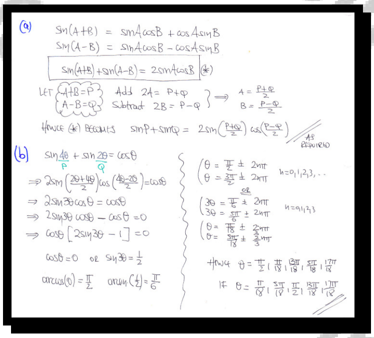
a) Prove the validity of the above trigonometric identity, by using the compound angle identities for $\sin(A+B)$ and $\sin(A-B)$.

b) Hence, or otherwise, solve the trigonometric equation

$$\sin 4\theta + \sin 2\theta = \cos\theta, \quad 0 \le \theta < \pi,$$

giving the answers in terms of π.

$$\boxed{\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{\pi}{2}, \frac{13\pi}{18}, \frac{17\pi}{18}}$$


(a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\boxed{\sin(A+B) + \sin(A-B) = 2\sin A\cos B}$ (*)

LET $A+B=P$, $A-B=Q$. Add $2A = P+Q$, Subtract $2B = P-Q$ $\Rightarrow A = \frac{P+Q}{2}$, $B = \frac{P-Q}{2}$

HENCE (*) BECOMES $\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$ AS REQUIRED

(b) $\sin 4\theta + \sin 2\theta = \cos\theta$

$\Rightarrow 2\sin\left(\frac{2\theta+4\theta}{2}\right)\cos\left(\frac{4\theta-2\theta}{2}\right) = \cos\theta$

$\Rightarrow 2\sin 3\theta\cos\theta = \cos\theta$

$\Rightarrow 2\sin 3\theta\cos\theta - \cos\theta = 0$

$\Rightarrow \cos\theta\left[2\sin 3\theta - 1\right] = 0$

$\cos\theta = 0$ OR $\sin 3\theta = \frac{1}{2}$

$\arccos(0) = \frac{\pi}{2}$ $\quad \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\theta = \frac{\pi}{2} \pm 2n\pi$, $\theta = \frac{3\pi}{2} \pm 2n\pi$, $n = 0,1,2,3,\dots$

OR

$3\theta = \frac{\pi}{6} \pm 2n\pi$, $3\theta = \frac{5\pi}{6} \pm 2n\pi$, $n = 0,1,2,3$

$\theta = \frac{\pi}{18} \pm \frac{2}{3}n\pi$, $\theta = \frac{5\pi}{18} \pm \frac{2}{3}n\pi$

THUS $\theta = \frac{\pi}{2}, \frac{\pi}{18}, \frac{13\pi}{18}, \frac{5\pi}{18}, \frac{17\pi}{18}$

IE $\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{\pi}{2}, \frac{13\pi}{18}, \frac{17\pi}{18}$

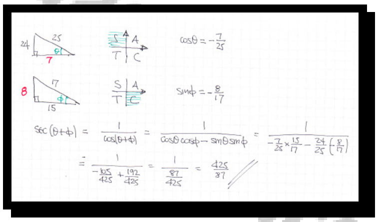
Question 114 (****)

$$\sin\theta = \frac{24}{25} \quad \text{and} \quad \cos\varphi = \frac{15}{17}.$$

If θ is obtuse and φ is reflex, show clearly that

$$\sec(\theta+\varphi) = \frac{425}{87}.$$

proof



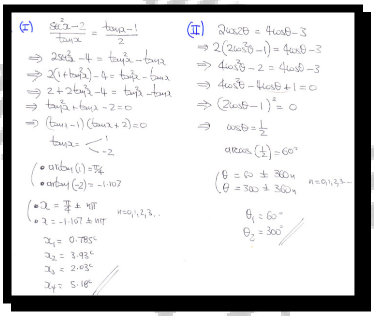
Question 115 (****)

Solve each of the following trigonometric equations.

i. $\dfrac{\sec^2 x - 2}{\tan x} = \dfrac{\tan x - 1}{2}$, $\quad 0 \le x < 2\pi$, $\quad x \ne \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

ii. $2\cos 2\theta = 4\cos\theta - 3$, $\quad 0 \le \theta < 360°$.

, $x = 0.785^c,\ 2.03^c,\ 3.93^c,\ 5.18^c$, $\theta = 60°,\ 300°$

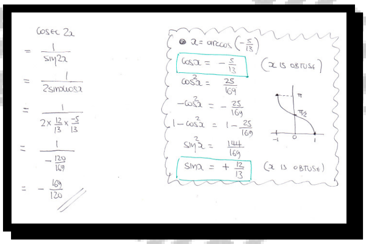


Question 116 (****)

$$x = \arccos\left(-\frac{5}{13}\right).$$

Determine the exact value of $\operatorname{cosec} 2x$.

$$\boxed{-\frac{169}{120}}$$



Question 117 (****)

$$f(x) = \sec x,\ x \in \mathbb{R},\ 0 \le x \le 4\pi.$$

a) Sketch the graph of $f(x)$, showing clearly the coordinates of any stationary points and equations of asymptotes.

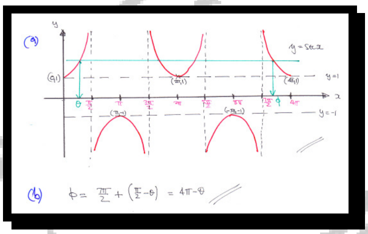
It is now given that

$$\sec\theta = \sec\varphi,$$

where $0 < \theta < \frac{\pi}{2}$ and $\frac{7\pi}{2} < \varphi < 4\pi$.

b) Express φ in terms of θ.

$\boxed{}$, $\boxed{\varphi = 4\pi - \theta}$

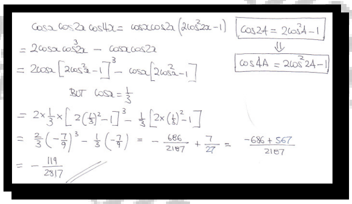


Question 118 (****)

If $\cos x = \dfrac{1}{3}$, show by detailed workings that

$$\cos x \cos 2x \cos 4x = -\frac{119}{2187}.$$

proof



Question 119 (****)

$$4\sin\theta + \cos\theta = 2, \ 0 \le \theta < 360°.$$

a) Show that the above trigonometric equation can be written as

$$16\sin^2\theta = 4 - 4\cos\theta + \cos^2\theta.$$

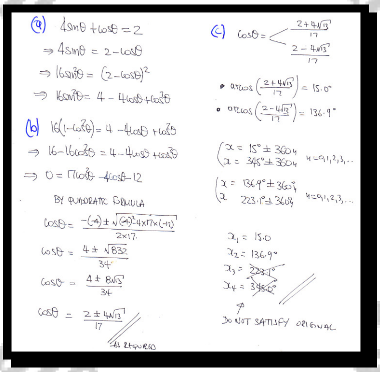
b) Show further that

$$\cos\theta = \frac{2 \pm 4\sqrt{13}}{17}.$$

c) Hence, or otherwise, find the **two** values of θ that satisfy the equation

$$4\sin\theta + \cos\theta = 2, \ 0 \le \theta < 360°.$$

$\boxed{\theta \approx 15.0°, \ 136.9°}$


(a) $4\sin\theta + \cos\theta = 2$

$\Rightarrow 4\sin\theta = 2 - \cos\theta$

$\Rightarrow 16\sin^2\theta = (2-\cos\theta)^2$

$\Rightarrow 16\sin^2\theta = 4 - 4\cos\theta + \cos^2\theta$

(b) $16(1-\cos^2\theta) = 4 - 4\cos\theta + \cos^2\theta$

$\Rightarrow 16 - 16\cos^2\theta = 4 - 4\cos\theta + \cos^2\theta$

$\Rightarrow 0 = 17\cos^2\theta - 4\cos\theta - 12$

BY QUADRATIC FORMULA

$\cos\theta = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\times17\times(-12)}}{2\times17}$

$\cos\theta = \frac{4 \pm \sqrt{832}}{34}$

$\cos\theta = \frac{4 \pm 8\sqrt{13}}{34}$

$\cos\theta = \frac{2 \pm 4\sqrt{13}}{17}$ // AS REQUIRED

(c) $\cos\theta = \frac{2+4\sqrt{13}}{17}$ or $\frac{2-4\sqrt{13}}{17}$

• $\arccos\left(\frac{2+4\sqrt{13}}{17}\right) = 15.0°$

• $\arccos\left(\frac{2-4\sqrt{13}}{17}\right) = 136.9°$

$x = 15° \pm 360n$, $x = 345° \pm 360n$, $n = 0, 1, 2, 3, \dots$

$x = 136.9° \pm 360n$, $x = 223.1° \pm 360n$, $n = 0, 1, 2, 3, \dots$

$x_1 = 15.0$

$x_2 = 136.9°$

$x_3 = 223.1°$ (crossed out)

$x_4 = 345.0°$ (crossed out) //

DO NOT SATISFY ORIGINAL

Question 120 (****)

It is given that

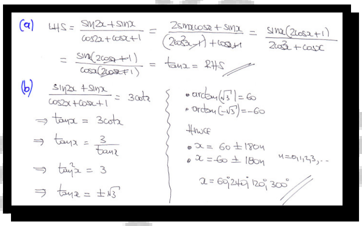
$$\frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} \equiv \tan x .$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, solve for $0 \le x < 360°$

$$\frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} = 3\cot x .$$

$$\boxed{x = 60°,\ 120°,\ 180°,\ 300°}$$



(a) $\text{LHS} = \frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} = \frac{2\sin x\cos x + \sin x}{(2\cos^2 x - 1) + \cos x + 1} = \frac{\sin x(2\cos x + 1)}{2\cos^2 x + \cos x}$

$= \frac{\sin x(2\cos x + 1)}{\cos x(2\cos x + 1)} = \tan x = \text{RHS}$

(b) $\frac{\sin 2x + \sin x}{\cos 2x + \cos x + 1} = 3\cot x$

$\Rightarrow \tan x = 3\cot x$

$\Rightarrow \tan x = \frac{3}{\tan x}$

$\Rightarrow \tan^2 x = 3$

$\Rightarrow \tan x = \pm\sqrt{3}$

- $\arctan(\sqrt{3}) = 60$
- $\arctan(-\sqrt{3}) = -60$

Hence

- $x = 60 \pm 180n$
- $x = -60 \pm 180n$ $\quad n = 0, 1, 2, 3, \ldots$

$x = 60°, 240°, 120°, 300°$

Question 121 (****)

It is given that

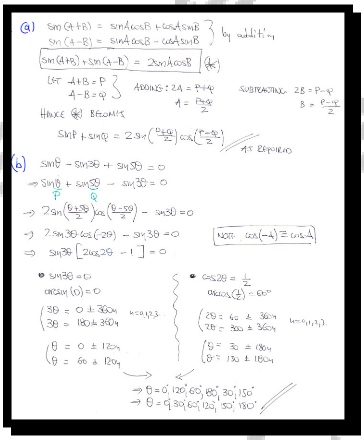
$$\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right).$$

a) Prove the validity of the above trigonometric identity, by using the compound angle identities for $\sin(A+B)$ and $\sin(A-B)$.

b) Hence, or otherwise, solve the trigonometric equation

$$\sin\theta - \sin 3\theta + \sin 5\theta = 0, \quad 0 \le \theta \le 180°.$$

$\boxed{}$, $\boxed{\theta = 0°,\ 30°,\ 60°,\ 120°,\ 150°,\ 180°}$


(a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin(A-B) = \sin A\cos B - \cos A\sin B$ } by addition

$\boxed{\sin(A+B) + \sin(A-B) = 2\sin A\cos B}$ (*)

LET $A+B = P$, $A-B = Q$ } ADDING: $2A = P+Q$, $A = \frac{P+Q}{2}$; SUBTRACTING: $2B = P-Q$, $B = \frac{P-Q}{2}$

HENCE (*) BECOMES

$\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$ // AS REQUIRED

(b) $\sin\theta - \sin 3\theta + \sin 5\theta = 0$

$\Rightarrow \sin\theta + \sin 5\theta - \sin 3\theta = 0$ (P = θ, Q = 5θ)

$\Rightarrow 2\sin\left(\frac{\theta+5\theta}{2}\right)\cos\left(\frac{\theta-5\theta}{2}\right) - \sin 3\theta = 0$

$\Rightarrow 2\sin 3\theta\cos(-2\theta) - \sin 3\theta = 0$ NOTE $\cos(-A) \equiv \cos A$

$\Rightarrow \sin 3\theta\,[2\cos 2\theta - 1] = 0$

- $\sin 3\theta = 0$, $\arcsin(0) = 0$
  $3\theta = 0 \pm 360n$, $3\theta = 180 \pm 360n$, $n = 0,1,2,3\ldots$
  $\theta = 0 \pm 120n$, $\theta = 60 \pm 120n$
- $\cos 2\theta = \frac{1}{2}$, $\arccos\left(\frac{1}{2}\right) = 60°$
  $2\theta = 60 \pm 360n$, $2\theta = 300 \pm 360n$, $n = 0,1,2,3\ldots$
  $\theta = 30 \pm 180n$, $\theta = 150 \pm 180n$

$\Rightarrow \theta = 0°, 120°, 60°, 180°, 30°, 150°$
$\Rightarrow \theta = 0°, 30°, 60°, 120°, 150°, 180°$ //

Question 122 (****)

It is given that

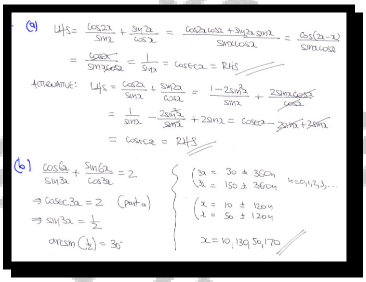
$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} \equiv \operatorname{cosec} x .$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, solve for $0 \le x < 180°$

$$\frac{\cos 6x}{\sin 3x} + \frac{\sin 6x}{\cos 3x} = 2 .$$

$x = 10°,\ 50°,\ 130°, 170°$

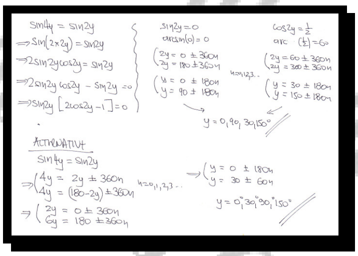


Question 123 (****)

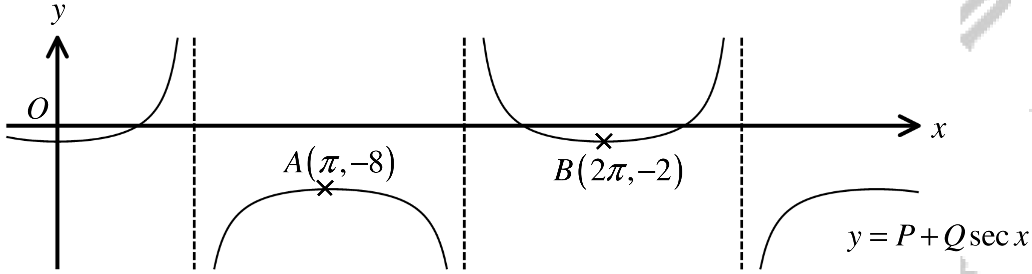
Solve the following trigonometric equation

$$\sin 4y = \sin 2y , \quad 0 \le y < 180° .$$

$y = 0°,\ 30°, 90°, 150°$



Question 124 (****)



The figure above shows part of the curve with equation

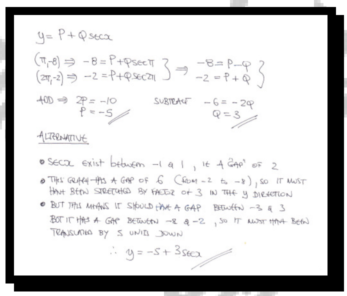
$$y = P + Q\sec x,$$

where P and Q are non zero constants.

The curve has turning points at $A(\pi,-8)$ and $B(2\pi,-2)$.

Determine the value of P and the value of Q.

$\boxed{}$, $\boxed{P=-5}$, $\boxed{Q=3}$



$y = P + Q\sec x$

$(\pi,-8) \Rightarrow -8 = P + Q\sec\pi$
$(2\pi,-2) \Rightarrow -2 = P + Q\sec 2\pi$
$\Rightarrow -8 = P - Q$, $-2 = P + Q$

ADD $\Rightarrow 2P = -10$, $P = -5$

SUBTRACT $-6 = -2Q$, $Q = 3$

ALTERNATIVE

- $\sec x$ exist between -1 & 1, ie A "GAP" OF 2
- THIS GRAPH HAS A GAP OF 6 (FROM -2 to -8), SO IT MUST HAVE BEEN STRETCHED BY FACTOR OF 3 IN THE y DIRECTION
- BUT THIS MEANS IT SHOULD HAVE A GAP BETWEEN -3 & 3 BUT IT HAS A GAP BETWEEN -8 & -2, SO IT MUST HAVE BEEN TRANSLATED BY 5 UNITS DOWN

$\therefore y = -5 + 3\sec x$

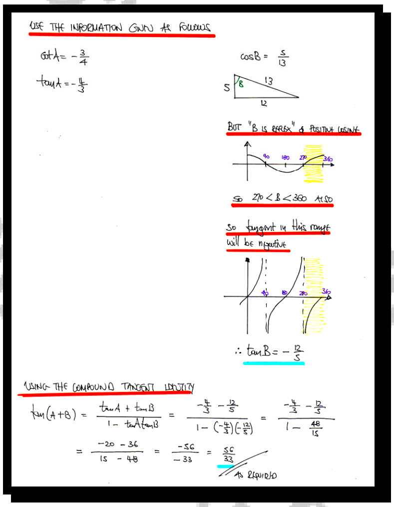
Question 125 (****)

$$\cot A = -\frac{3}{4} \quad \text{and} \quad \cos B = \frac{5}{13}.$$

If A is reflex and B is also reflex, show that

$$\tan(A+B) = \frac{56}{33}.$$

[ ] , [proof]



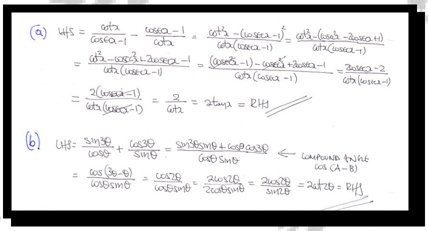
Question 126 (****)

Prove the validity of each of the following trigonometric identities.

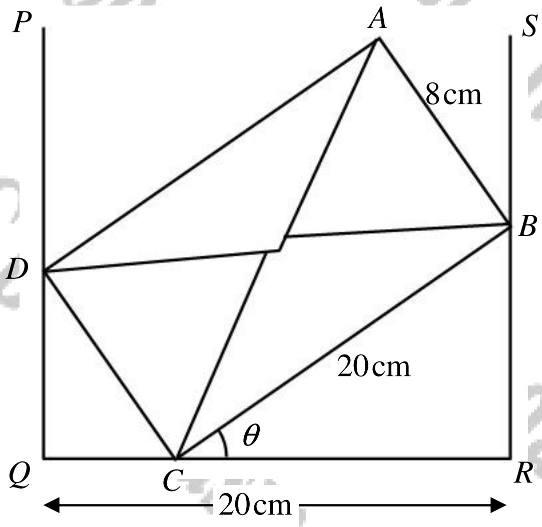
a) $\dfrac{\cot x}{\operatorname{cosec} x - 1} - \dfrac{\operatorname{cosec} x - 1}{\cot x} \equiv 2\tan x$.

b) $\dfrac{\sin 3\theta}{\cos\theta} + \dfrac{\cos 3\theta}{\sin\theta} \equiv 2\cot 2\theta$.

proof



Question 127 (****)



The figure above shows the cross section of a letter inside a filling slot.

The letter $ABCD$ is modelled as a rectangle with $|AB| = 8\text{cm}$ and $|BC| = 20\text{cm}$.

The width of the filling slot QR is also 20 cm and the angle BCR is θ.

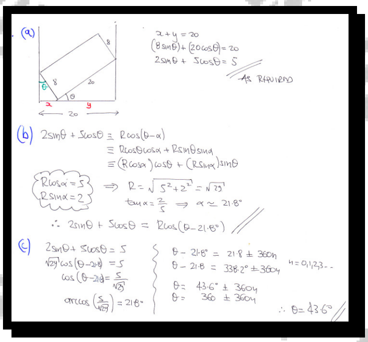
a) Show clearly that

$$5\cos\theta + 2\sin\theta = 5.$$

b) Express $5\cos\theta + 2\sin\theta$ in the form $R\cos(\theta - \alpha)$, $R > 0$, $0 < a < 90°$.

c) Hence, determine the value of θ.

[ ], $5\cos\theta + 2\sin\theta = \sqrt{29}\cos(\theta - 21.8°)$, $\theta \approx 43.6°$

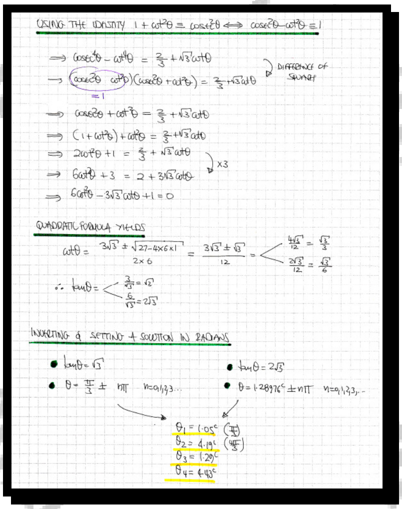


Question 128 (****)

Solve the following trigonometric equation

$$\operatorname{cosec}^4\theta - \cot^4\theta = \frac{2}{3} + \sqrt{3}\cot\theta, \ 0 \le \theta < 2\pi .$$

$\boxed{\theta \approx 1.05^c,\ 1.29^c,\ 4.19^c,\ 4.43^c}$



USING THE IDENTITY $1+\cot^2\theta \equiv \operatorname{cosec}^2\theta \Leftrightarrow \operatorname{cosec}^2\theta - \cot^2\theta \equiv 1$

$\Rightarrow \operatorname{cosec}^4\theta - \cot^4\theta = \frac{2}{3} + \sqrt{3}\cot\theta$

$\Rightarrow (\operatorname{cosec}^2\theta - \cot^2\theta)(\operatorname{cosec}^2\theta + \cot^2\theta) = \frac{2}{3} + \sqrt{3}\cot\theta$ (DIFFERENCE OF SQUARES; first bracket $=1$)

$\Rightarrow \operatorname{cosec}^2\theta + \cot^2\theta = \frac{2}{3} + \sqrt{3}\cot\theta$

$\Rightarrow (1+\cot^2\theta) + \cot^2\theta = \frac{2}{3} + \sqrt{3}\cot\theta$

$\Rightarrow 2\cot^2\theta + 1 = \frac{2}{3} + \sqrt{3}\cot\theta$ ($\times 3$)

$\Rightarrow 6\cot^2\theta + 3 = 2 + 3\sqrt{3}\cot\theta$

$\Rightarrow 6\cot^2\theta - 3\sqrt{3}\cot\theta + 1 = 0$

QUADRATIC FORMULA YIELDS

$\cot\theta = \frac{3\sqrt{3} \pm \sqrt{27-4\times6\times1}}{2\times6} = \frac{3\sqrt{3}\pm\sqrt{3}}{12} = \begin{cases} \frac{4\sqrt{3}}{12} = \frac{\sqrt{3}}{3} \\ \frac{2\sqrt{3}}{12} = \frac{\sqrt{3}}{6} \end{cases}$

$\therefore \tan\theta = \begin{cases} \frac{3}{\sqrt{3}} = \sqrt{2} \\ \frac{6}{\sqrt{3}} = 2\sqrt{3} \end{cases}$

INVERTING & SETTING A SOLUTION IN RADIANS

- $\tan\theta = \sqrt{3}$
- $\theta = \frac{\pi}{3} \pm n\pi \quad n=0,1,2,3\dots$
- $\tan\theta = 2\sqrt{3}$
- $\theta = 1.28976^c \pm n\pi \quad n=0,1,2,3,\dots$

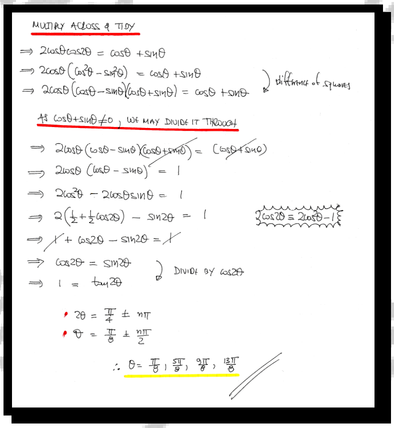
$\theta_1 = 1.05^c \left(\frac{\pi}{3}\right)$
$\theta_2 = 4.19^c \left(\frac{4\pi}{3}\right)$
$\theta_3 = 1.29^c$
$\theta_4 = 4.43^c$

Question 129 (****)

$$\frac{\cos\theta\cos 2\theta}{\cos\theta+\sin\theta}=\frac{1}{2}, \quad 0 \le x < 2\pi.$$

Given that $\cos\theta+\sin\theta \neq 0$, find the solutions of the above trigonometric equation, giving the answers in radians in terms of π.

$$x=\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

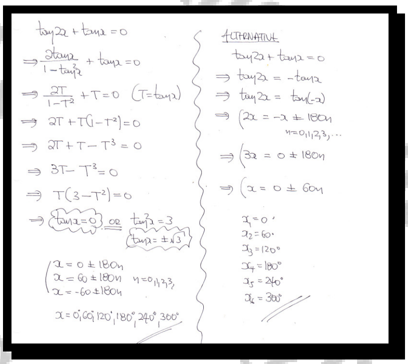


Question 130 (****)

Solve in degrees the following trigonometric equation

$$\tan 2x+\tan x=0, \quad 0 \le x < 360.$$

$$x=0°, 60°, 120°, 180°, 240°, 300°$$



Question 131 (****)

It is given that

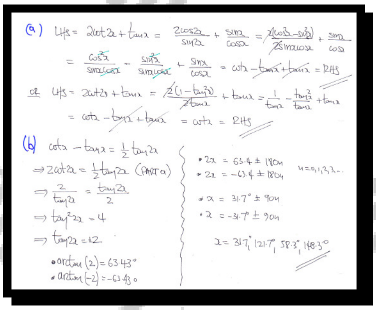
$$2\cot 2x + \tan x \equiv \cot x \,.$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the following trigonometric equation

$$\cot x - \tan x = \frac{1}{2}\tan 2x\,, \quad 0 \le x < 180^\circ\,.$$

$x \approx 31.7^\circ,\ 58.3^\circ,\ 121.7^\circ,\ 148.3^\circ$



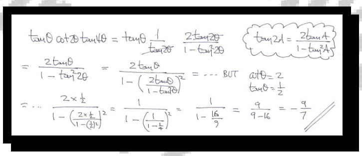
Question 132 (****)

If $\cot\theta = 2$, use the tangent double angle identity to show

$$\tan\theta \cot 2\theta \tan 4\theta = -\frac{9}{7}\,.$$

You must show detailed workings in this question

, proof



Question 133 (****)

It is given that

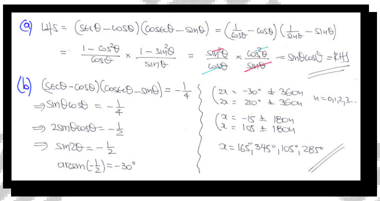
$$(\sec\theta - \cos\theta)(\operatorname{cosec}\theta - \sin\theta) \equiv \sin\theta\cos\theta\,.$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the trigonometric equation

$$(\sec\theta - \cos\theta)(\operatorname{cosec}\theta - \sin\theta) = -\frac{1}{4}, \quad 0 \le \theta < 360°\,.$$

$\theta = 105°,\ 165°,\ 285°,\ 345°$



Question 134 (****)

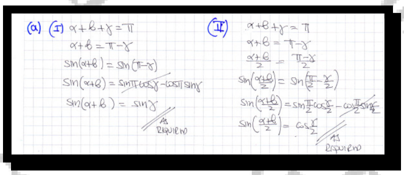
The three angles of a triangle are denoted by α, β and γ.

Show clearly that …

a) … $\sin(\alpha+\beta) = \sin\gamma$.

b) … $\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\gamma}{2}\right)$.

proof



Question 135 (****)

$$f(x) = \sqrt{3}\sin x + \cos x, \quad x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\cos(x-\alpha)$, $R>0$, $0<\alpha<90°$.

b) State the maximum value of $f(x)$ and find the smallest positive value of x for which this maximum occurs.

The temperature of the water T °C in a tropical fish tank is modelled by the equation

$$T = 32 + \sqrt{3}\sin(15t)° + \cos(15t)°, \quad 0 \le t < 24,$$

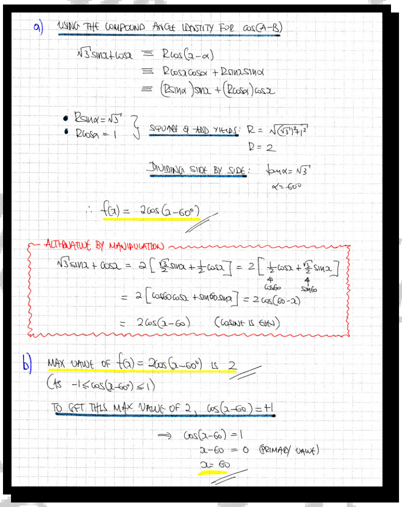
where t is the time in hours measured since midnight.

c) State the maximum temperature of the water in the tank and the time when this maximum temperature occurs.

d) Show that the temperature of the water in the tank reaches 30.5 °C at 13:14 and at 18:46.

[*You may not verify the answers in this part*]

$\boxed{}$, $\boxed{\sqrt{3}\sin x + \cos x \equiv 2\cos(x-60°)}$, $\boxed{\max = 2}$, $\boxed{x = 60°}$, $\boxed{T_{max} = 34}$, $\boxed{04:00}$


a) USING THE COMPOUND ANGLE IDENTITY FOR cos(A−B)

$\sqrt{3}\sin x + \cos x \equiv R\cos(x-\alpha)$

$\equiv R\cos x\cos\alpha + R\sin x\sin\alpha$

$\equiv (R\sin\alpha)\sin x + (R\cos\alpha)\cos x$

- $R\sin\alpha = \sqrt{3}$
- $R\cos\alpha = 1$

SQUARE & ADD YIELDS: $R = \sqrt{(\sqrt{3})^2 + 1^2}$, $R = 2$

DIVIDING SIDE BY SIDE: $\tan\alpha = \sqrt{3}$, $\alpha = 60°$

$\therefore f(x) = 2\cos(x-60°)$

ALTERNATIVE BY MANIPULATION

$\sqrt{3}\sin x + \cos x = 2\left[\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right] = 2\left[\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right]$ (cos60, sin60)

$= 2\left[\cos 60\cos x + \sin 60 \sin x\right] = 2\cos(60-x)$

$= 2\cos(x-60)$ (COSINE IS EVEN)

b) MAX VALUE OF $f(x) = 2\cos(x-60°)$ IS 2

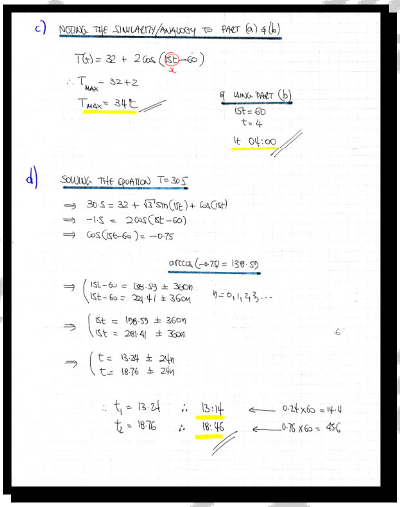
(AS $-1 \le \cos(x-60°) \le 1$)

TO GET THIS MAX VALUE OF 2, $\cos(x-60) = +1$

$\Rightarrow \cos(x-60) = 1$

$x - 60 = 0$ (PRIMARY VALUE)

$x = 60$


c) NOTING THE SIMILARITY/ANALOGY TO PART (a) & (b)

$T(t) = 32 + 2\cos(15t - 60)$

$\therefore T_{MAX} = 32 + 2$

$T_{MAX} = 34°C$

& USING PART (b)

$15t = 60$

$t = 4$

IE 04:00

d) SOLVING THE EQUATION $T = 30.5$

$\Rightarrow 30.5 = 32 + \sqrt{3}\sin(15t) + \cos(15t)$

$\Rightarrow -1.5 = 2\cos(15t - 60)$

$\Rightarrow \cos(15t-60) = -0.75$

$\arccos(-0.75) = 138.59$

$\Rightarrow 15t - 60 = 138.59 \pm 360n$, $15t - 60 = 221.41 \pm 360n$, $n = 0, 1, 2, 3, \ldots$

$\Rightarrow 15t = 198.59 \pm 360n$, $15t = 281.41 \pm 360n$

$\Rightarrow t = 13.24 \pm 24n$, $t = 18.76 \pm 24n$

$\therefore t_1 = 13.24 \therefore 13:14 \leftarrow 0.24 \times 60 = 14.4$

$t_2 = 18.76 \therefore 18:46 \leftarrow 0.76 \times 60 = 45.6$

Question 136 (****)

It is given that

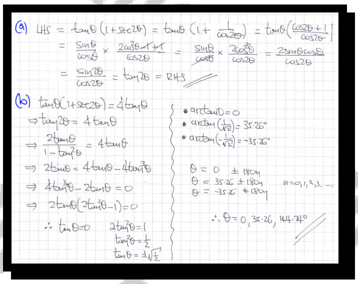
$$\tan\theta(1+\sec 2\theta) \equiv \tan 2\theta .$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, solve for $0 \le \theta < 180°$

$$\tan\theta(1+\sec 2\theta) = 4\tan\theta .$$

$\boxed{\theta = 0° \quad \theta \approx 35.3°,\ 144.7°}$



Question 137 (****)

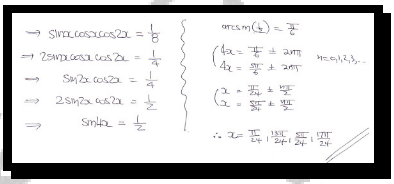
By twice applying the identity

$$\sin 2\theta = 2\sin\theta\cos\theta ,$$

solve the trigonometric equation

$$\sin x \cos x \cos 2x = \frac{1}{8}, \quad 0 \le x < \pi .$$

$\boxed{x = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}}$



Question 138 (****)

$$f(x) = 2.5\sin 2x + 6\cos 2x, \quad 0 < x < 2\pi.$$

a) Express $f(x)$ in the form $R\sin(2x+\alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

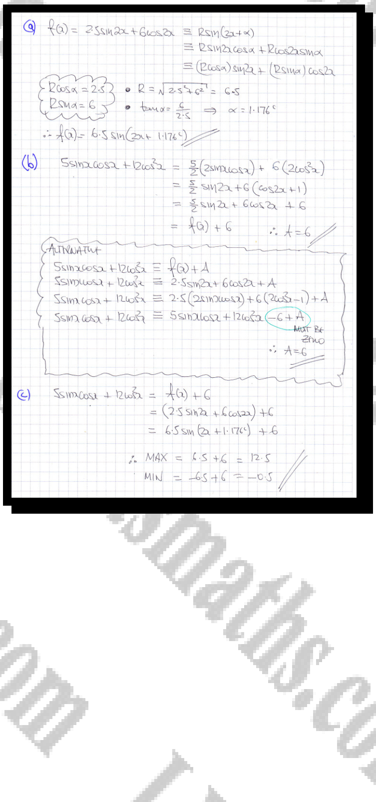
b) Determine the value of the constant A so that

$$5\sin x\cos x + 12\cos^2 x \equiv f(x) + A.$$

c) Hence, or otherwise, find the minimum and the maximum value of

$$5\sin x\cos x + 12\cos^2 x.$$

$\boxed{f(x) = 6.5\sin\left(2x + 1.176^c\right)}$, $\boxed{A = 6}$, $\boxed{\max = 12.5}$, $\boxed{\min = -0.5}$



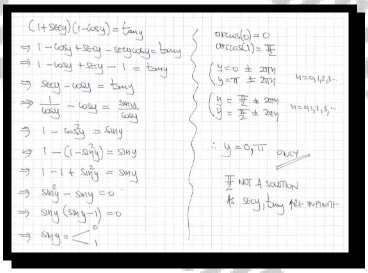
Question 139 (****)

Find the **two** solutions of the trigonometric equation

$$(1+\sec y)(1-\cos y)=\tan y, \qquad 0 \le y < 2\pi,$$

giving the answers in terms of π.

$\boxed{y=0,\ \pi}$



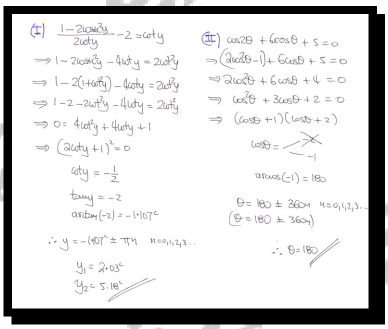
Question 140 (****)

Solve each of the following trigonometric equations.

i. $\dfrac{1-2\operatorname{cosec}^2 y}{2\cot y}-2=\cot y, \qquad 0<y<2\pi,\ \ y\neq\pi$.

ii. $\cos 2\theta+6\cos\theta+5=0, \qquad 0\le\theta<360°$.

$\boxed{y\approx 2.03^{c},\ 5.18^{c}}$, $\boxed{\theta=180°}$



Question 141 (****)

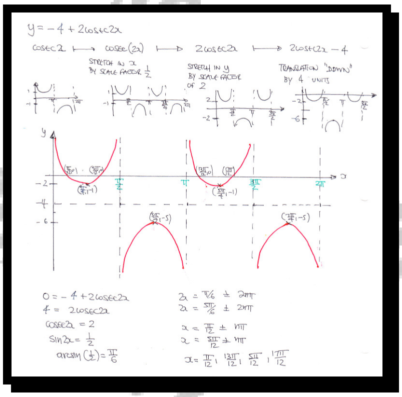
Sketch the graph of

$$y = -4 + 2\operatorname{cosec} 2x, \quad 0 \le x \le 2\pi .$$

The sketch must include

- the equations of any asymptotes to the curve
- the exact coordinates of any stationary points.
- the exact coordinates of any points where the curve crosses the coordinate axes.

graph



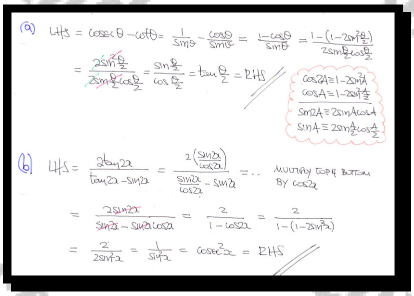
Question 142 (****)

Prove the validity of each of the trigonometric identities.

a) $\operatorname{cosec}\theta - \cot\theta \equiv \tan\left(\frac{\theta}{2}\right)$.

b) $\dfrac{2\tan 2x}{\tan 2x - \sin 2x} \equiv \operatorname{cosec}^2 x$.

proof



Question 143 (****)

$$f(x) \equiv 27x^3 - 9x - 2, \; x \in \mathbb{R}.$$

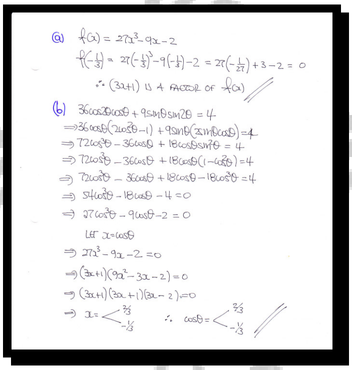
a) Show that $(3x+1)$ is a factor of $f(x)$.

It is further given that

$$36\cos 2\theta \cos\theta + 9\sin 2\theta \sin\theta = 4.$$

b) Find the possible values of $\cos\theta$.

[illegible], $\boxed{\cos\theta = -\frac{1}{3}, \; \frac{2}{3}}$


(a) $f(x) = 27x^3 - 9x - 2$

$f\left(-\frac{1}{3}\right) = 27\left(-\frac{1}{3}\right)^3 - 9\left(-\frac{1}{3}\right) - 2 = 27\left(-\frac{1}{27}\right) + 3 - 2 = 0$

$\therefore (3x+1)$ IS A FACTOR OF $f(x)$ //

(b) $36\cos 2\theta\cos\theta + 9\sin\theta\sin 2\theta = 4$

$\Rightarrow 36\cos\theta(2\cos^2\theta - 1) + 9\sin\theta(2\sin\theta\cos\theta) = 4$

$\Rightarrow 72\cos^3\theta - 36\cos\theta + 18\cos\theta\sin^2\theta = 4$

$\Rightarrow 72\cos^3\theta - 36\cos\theta + 18\cos\theta(1 - \cos^2\theta) = 4$

$\Rightarrow 72\cos^3\theta - 36\cos\theta + 18\cos\theta - 18\cos^3\theta = 4$

$\Rightarrow 54\cos^3\theta - 18\cos\theta - 4 = 0$

$\Rightarrow 27\cos^3\theta - 9\cos\theta - 2 = 0$

LET $x = \cos\theta$

$\Rightarrow 27x^3 - 9x - 2 = 0$

$\Rightarrow (3x+1)(9x^2 - 3x - 2) = 0$

$\Rightarrow (3x+1)(3x+1)(3x-2) = 0$

$\Rightarrow x = \frac{2}{3}, \; -\frac{1}{3}$ $\quad \therefore \cos\theta = \frac{2}{3}, \; -\frac{1}{3}$ //

Question 144 (****)

It is given that

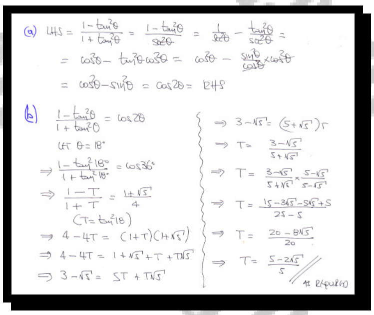
$$\frac{1-\tan^2\theta}{1+\tan^2\theta} \equiv \cos 2\theta .$$

a) Prove the validity of the above trigonometric identity.

b) Given that $\cos 36° = \frac{1+\sqrt{5}}{4}$, show clearly that

$$\tan^2 18° = \frac{5-2\sqrt{5}}{5} .$$

proof



(a) $\text{LHS} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\theta}{\sec^2\theta} = \frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta} =$

$= \cos^2\theta - \tan^2\theta\cos^2\theta = \cos^2\theta - \frac{\sin^2\theta}{\cos^2\theta}\times\cos^2\theta$

$= \cos^2\theta - \sin^2\theta = \cos 2\theta = \text{RHS}$

(b) $\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

LET $\theta = 18°$

$\Rightarrow \frac{1-\tan^2 18°}{1+\tan^2 18°} = \cos 36°$

$\Rightarrow \frac{1-T}{1+T} = \frac{1+\sqrt{5}}{4}$

$(T = \tan^2 18)$

$\Rightarrow 4-4T = (1+T)(1+\sqrt{5})$

$\Rightarrow 4-4T = 1+\sqrt{5}+T+T\sqrt{5}$

$\Rightarrow 3-\sqrt{5} = 5T + T\sqrt{5}$

$\Rightarrow 3-\sqrt{5} = (5+\sqrt{5})T$

$\Rightarrow T = \frac{3-\sqrt{5}}{5+\sqrt{5}}$

$\Rightarrow T = \frac{3-\sqrt{5}}{5+\sqrt{5}} \times \frac{5-\sqrt{5}}{5-\sqrt{5}}$

$\Rightarrow T = \frac{15-3\sqrt{5}-5\sqrt{5}+5}{25-5}$

$\Rightarrow T = \frac{20-8\sqrt{5}}{20}$

$\Rightarrow T = \frac{5-2\sqrt{5}}{5}$

AS REQUIRED

Question 145 (****)

$$f(\theta) \equiv 5\cos\theta - 12\sin\theta\,,\ \theta \in \mathbb{R}\,.$$

a) Express $f(\theta)$ in the form $R\cos(\theta+\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

Give the value of α correct to 3 decimal places.

b) State the maximum value of $f(\theta)$ and find the smallest positive value of θ for which this maximum occurs.

The pressure P, in suitable units, in a nuclear plant is modelled by the equation

$$P = 20 + 5\cos\left(\frac{4\pi t}{25}\right) - 12\sin\left(\frac{4\pi t}{25}\right),\quad 0 \le t < 12\,,$$

where t is the time in hours measured from midnight.

c) State the maximum pressure in the plant and the value of t when this maximum pressure occurs.

d) Find the times, to the nearest minute, when $P = 15$.

$\boxed{}$, $\boxed{5\cos\theta - 12\sin\theta \equiv 13\cos\left(\theta + 1.176^c\right)}$, $\boxed{\max = 13}$, $\boxed{\theta = 5.107^c}$, $\boxed{P_{\max} = 33}$, $\boxed{t_{\max} = 10.16}$, $\boxed{01:34\,/\,06:15}$

a) USING THE STANDARD METHOD

$5\cos\theta - 12\sin\theta \equiv R\cos(\theta+\alpha)$

$5\cos\theta - 12\sin\theta \equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$5\cos\theta - 12\sin\theta \equiv (R\cos\alpha)\cos\theta - (R\sin\alpha)\sin\theta$

COMPARING & SOLVING

$R\cos\alpha = 5$, $R\sin\alpha = 12$ $\Rightarrow$ SQUARING & ADDING YIELDS $R = +\sqrt{5^2+12^2} = 13$

$\Rightarrow$ DIVIDING THE EQUATIONS SIDE BY SIDE $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \frac{12}{5}$, $\alpha \approx 1.176^c$

$\therefore f(\theta) \equiv 13\cos(\theta + 1.176^c)$

b) FIRSTLY $f(\theta)_{MAX} = 13$

NEXT, TO GET A MAX OF 13

$\Rightarrow \cos(\theta + 1.176^c) = 1$

$\Rightarrow \theta + 1.176^c = 0$

$\Rightarrow \theta = -1.176^c$ (+2π)

$\Rightarrow \theta = 5.107^c$

c) USING PARTS (a) & (b)

$\Rightarrow P = 20 + 5\cos\left(\frac{4\pi t}{25}\right) - 12\sin\left(\frac{4\pi t}{25}\right)$

$\Rightarrow P = 20 + 13\cos\left(\frac{4\pi t}{25} + 1.176^c\right)$

$\therefore P_{MAX} = 20 + 13 = 33$

AND FROM PART (b) $\Rightarrow \theta = 5.107^c$

$\Rightarrow \frac{4\pi t}{25} = 5.107^c$

$\Rightarrow t \approx 10.16$

d) SOLVING THE EQUATION $P = 15$

$\Rightarrow 15 = 20 + 13\cos\left(\frac{4\pi t}{25} + 1.176^c\right)$

$\Rightarrow -5 = 13\cos\left(\frac{4\pi t}{25} + 1.176^c\right)$

$\Rightarrow \cos\left(\frac{4\pi t}{25} + 1.176^c\right) = -\frac{5}{13}$ $\left[\arccos\left(-\frac{5}{13}\right) = 1.965587\ldots\right]$

$\Rightarrow \begin{cases} \frac{4\pi t}{25} + 1.176^c = 1.9656\ldots \pm 2n\pi \\ \frac{4\pi t}{25} + 1.176^c = 4.3176\ldots \pm 2n\pi \end{cases}$ $n = 0,1,2,3,\ldots$

$\Rightarrow \begin{cases} \frac{4\pi t}{25} = 0.78958\ldots \pm 2n\pi \\ \frac{4\pi t}{25} = \pi \pm 2n\pi \end{cases}$

$\Rightarrow \begin{cases} t = 1.5708 \pm \frac{25}{2}n \\ t = 6.25 \pm \frac{25}{2}n \end{cases}$

$t = 1.5707$ or 6.25; TIME $= 01:34$ or $06:15$

Question 146 (****)

It is given that

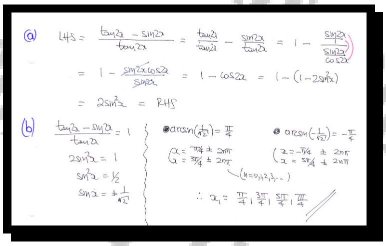
$$\frac{\tan 2x - \sin 2x}{\tan 2x} \equiv 2\sin^2 x\,, \ \tan 2x \neq 0\,.$$

a) Prove the validity of the above trigonometric identity.

b) Hence find, in terms of π, the solutions of the trigonometric equation

$$\frac{\tan 2x - \sin 2x}{\tan 2x} = 1\,, \quad 0 \le x < 2\pi\,.$$

$$\boxed{x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$$

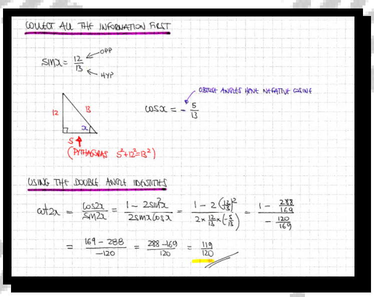


Question 147 (****)

If $\sin x = \frac{12}{13}$ and x is obtuse, show clearly that

$$\cot 2x = \frac{119}{120}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 148 (****)

It is given that

$$\cos x + \sin x \tan 2x \equiv \frac{\cos x}{\cos 2x}.$$

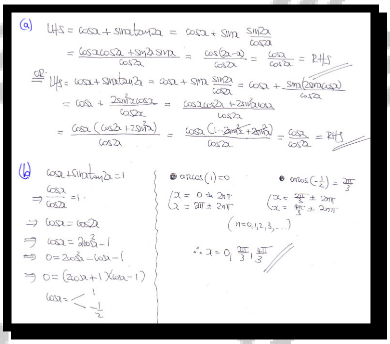
a) Prove the validity of the above trigonometric identity.

b) Hence find, in terms of π, the solutions of the trigonometric equation

$$\cos x + \sin x \tan 2x = 1, \quad 0 \le x < 2\pi,$$

giving the answers in terms of π.

$$\boxed{x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}}$$



(a) $LHS = \cos x + \sin x \tan 2x = \cos x + \sin x \frac{\sin 2x}{\cos 2x}$

$= \frac{\cos x \cos 2x + \sin 2x \sin x}{\cos 2x} = \frac{\cos(2x - x)}{\cos 2x} = \frac{\cos x}{\cos 2x} = RHS$

OR: $LHS = \cos x + \sin x \tan 2x = \cos x + \sin x \frac{\sin 2x}{\cos 2x} = \cos x + \frac{\sin x(2\sin x \cos x)}{\cos 2x}$

$= \cos x + \frac{2\sin^2 x \cos x}{\cos 2x} = \frac{\cos x \cos 2x + 2\sin^2 x \cos x}{\cos 2x}$

$= \frac{\cos x(\cos 2x + 2\sin^2 x)}{\cos 2x} = \frac{\cos x(1 - 2\sin^2 x + 2\sin^2 x)}{\cos 2x} = \frac{\cos x}{\cos 2x} = RHS$

(b) $\cos x + \sin x \tan 2x = 1$

$\Rightarrow \frac{\cos x}{\cos 2x} = 1$

$\Rightarrow \cos x = \cos 2x$

$\Rightarrow \cos x = 2\cos^2 x - 1$

$\Rightarrow 0 = 2\cos^2 x - \cos x - 1$

$\Rightarrow 0 = (2\cos x + 1)(\cos x - 1)$

$\cos x = 1, -\frac{1}{2}$

- $\arccos(1) = 0$: $x = 0 \pm 2n\pi$, $x = 2\pi \pm 2n\pi$
- $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$: $x = \frac{2\pi}{3} \pm 2n\pi$, $x = \frac{4\pi}{3} \pm 2n\pi$

$(n = 0, 1, 2, 3, \dots)$

$\therefore x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

Question 149 (****)

It is given that

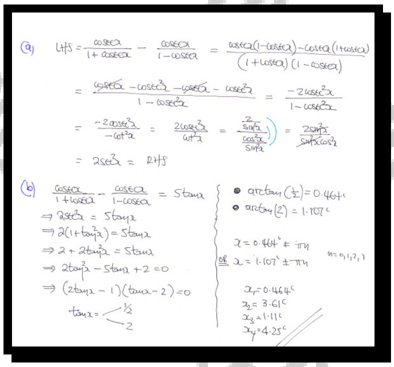
$$\frac{\operatorname{cosec} x}{1+\operatorname{cosec} x}-\frac{\operatorname{cosec} x}{1-\operatorname{cosec} x}\equiv 2\sec^2 x, \qquad \operatorname{cosec} x \neq \pm 1.$$

a) Prove the validity of the above trigonometric identity.

b) Hence find the solutions of the trigonometric equation

$$\frac{\operatorname{cosec} x}{1+\operatorname{cosec} x}-\frac{\operatorname{cosec} x}{1-\operatorname{cosec} x}=5\tan x, \quad 0 \le x < 2\pi.$$

$x \approx 0.464^{c},\ 1.11^{c},\ 3.61^{c},\ 4.25^{c}$


(a) $LHS = \frac{\operatorname{cosec}x}{1+\operatorname{cosec}x} - \frac{\operatorname{cosec}x}{1-\operatorname{cosec}x} = \frac{\operatorname{cosec}x(1-\operatorname{cosec}x)-\operatorname{cosec}x(1+\operatorname{cosec}x)}{(1+\operatorname{cosec}x)(1-\operatorname{cosec}x)}$

$= \frac{\operatorname{cosec}x - \operatorname{cosec}^2x - \operatorname{cosec}x - \operatorname{cosec}^2x}{1-\operatorname{cosec}^2x} = \frac{-2\operatorname{cosec}^2x}{1-\operatorname{cosec}^2x}$

$= \frac{-2\operatorname{cosec}^2x}{-\cot^2x} = \frac{2\operatorname{cosec}^2x}{\cot^2x} = \frac{\frac{2}{\sin^2x}}{\frac{\cos^2x}{\sin^2x}} = \frac{2\sin^2x}{\sin^2x\cos^2x}$

$= 2\sec^2x = RHS$

(b) $\frac{\operatorname{cosec}x}{1+\operatorname{cosec}x} - \frac{\operatorname{cosec}x}{1-\operatorname{cosec}x} = 5\tan x$

$\Rightarrow 2\sec^2x = 5\tan x$

$\Rightarrow 2(1+\tan^2x) = 5\tan x$

$\Rightarrow 2 + 2\tan^2x = 5\tan x$

$\Rightarrow 2\tan^2x - 5\tan x + 2 = 0$

$\Rightarrow (2\tan x - 1)(\tan x - 2) = 0$

$\tan x = \frac{1}{2}$ or 2

- $\arctan\left(\frac{1}{2}\right) = 0.464^c$
- $\arctan(2) = 1.107^c$

$x = 0.464^c \pm \pi n$

OR $x = 1.107^c \pm \pi n$ $\quad n = 0, 1, 2, 3$

$x_1 = 0.464^c$
$x_2 = 3.61^c$
$x_3 = 1.11^c$
$x_4 = 4.25^c$

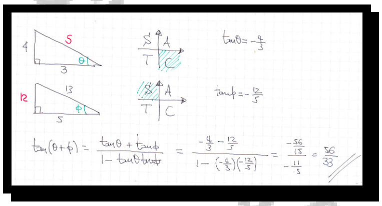
Question 150 (****)

$$\cot A = -\frac{3}{4} \quad \text{and} \quad \cos B = \frac{5}{13}.$$

If A is reflex and B is also reflex, show that

$$\tan(A+B) = \frac{56}{33}.$$

[ ] , [proof]



Question 151 (****)

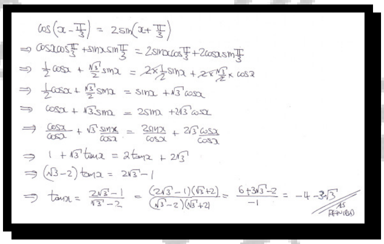
Given that

$$\cos\left(x - \frac{\pi}{3}\right) = 2\sin\left(x + \frac{\pi}{3}\right),$$

show clearly that

$$\tan x = -4 - 3\sqrt{3}.$$

[proof]

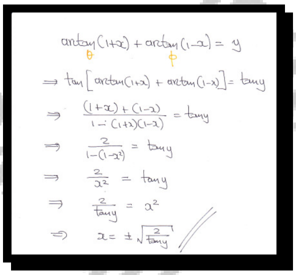


Question 152 (****)

Make x the subject of the equation

$$\arctan(1+x)+\arctan(1-x)=y.$$

$$\boxed{x=\pm\sqrt{\frac{2}{\tan y}}}$$



Question 153 (****)

It is given that

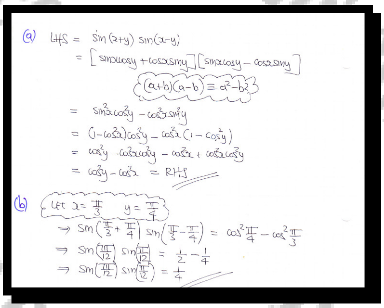
$$\sin(x+y)\sin(x-y)\equiv\cos^2 y-\cos^2 x.$$

a) Prove the validity of the above trigonometric identity.

b) Hence, show that

$$\sin\left(\frac{7\pi}{12}\right)\sin\left(\frac{\pi}{12}\right)=\frac{1}{4}.$$

proof



Question 154 (****)

It is given that

$$\frac{1}{\cos x - \sin x} + \frac{1}{\cos x + \sin x} \equiv \frac{2\sec x}{1 - \tan^2 x}.$$

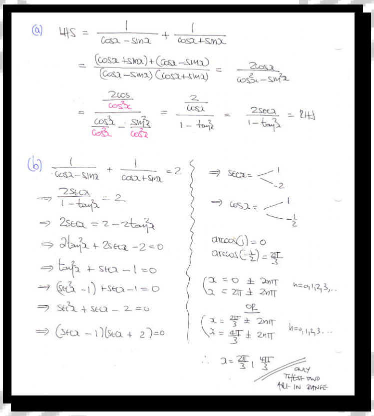
a) Prove the validity of the above trigonometric identity.

b) Hence solve the trigonometric equation

$$\frac{1}{\cos x - \sin x} + \frac{1}{\cos x + \sin x} = 2, \quad 0 < x < 2\pi,$$

giving the answers in terms of π.

$$\boxed{x = \frac{2\pi}{3}, \frac{4\pi}{3}}$$



Question 155 (****)

$$f(x)=\frac{6}{2\cos x+2\sin x} \quad \text{for} \quad 0<x<\pi,\ x\neq\beta.$$

a) Express $2\cos x+2\sin x$ in the form $R\cos(x-\alpha)$, $R>0$, $0<a<\frac{\pi}{2}$.

The curve with equation $y=f(x)$ has a vertical asymptote at $x=\beta$.

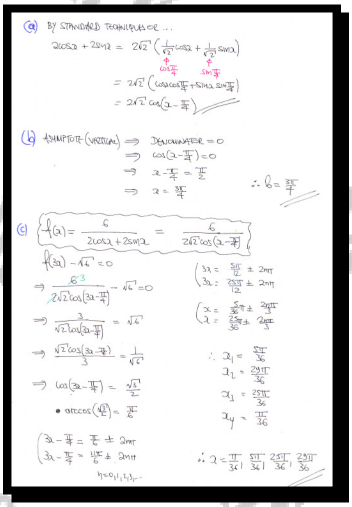
b) Determine the value of β.

c) Solve the equation

$$f(3x)-\sqrt{6}=0,$$

giving the answers in terms of π.

$\boxed{}$, $\boxed{2\cos x+2\sin x\equiv 2\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)}$, $\boxed{\beta=\frac{3\pi}{4}}$, $\boxed{x=\frac{\pi}{36},\ \frac{5\pi}{36},\ \frac{25\pi}{36},\ \frac{29\pi}{36}}$



Question 156 (****)

It is given that

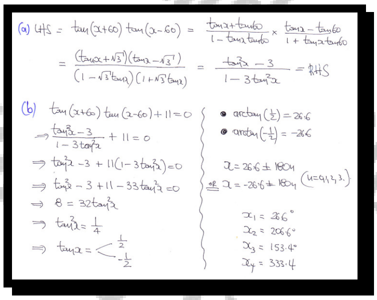
$$\tan(x+60°)\tan(x-60°) \equiv \frac{\tan^2 x - 3}{1 - 3\tan^2 x}.$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, solve the trigonometric equation

$$\tan(x+60°)\tan(x-60°) + 11 = 0, \quad 0 \le x < 360°.$$

$$\boxed{x \approx 26.6°,\ 153.4°,\ 186.6°,\ 333.4°}$$


(a) LHS $= \tan(x+60)\tan(x-60) = \frac{\tan x + \tan 60}{1 - \tan x \tan 60} \times \frac{\tan x - \tan 60}{1 + \tan x \tan 60}$

$= \frac{(\tan x + \sqrt{3})(\tan x - \sqrt{3})}{(1 - \sqrt{3}\tan x)(1 + \sqrt{3}\tan x)} = \frac{\tan^2 x - 3}{1 - 3\tan^2 x} =$ RHS

(b) $\tan(x+60)\tan(x-60) + 11 = 0$

$\Rightarrow \frac{\tan^2 x - 3}{1 - 3\tan^2 x} + 11 = 0$

$\Rightarrow \tan^2 x - 3 + 11(1 - 3\tan^2 x) = 0$

$\Rightarrow \tan^2 x - 3 + 11 - 33\tan^2 x = 0$

$\Rightarrow 8 = 32\tan^2 x$

$\Rightarrow \tan^2 x = \frac{1}{4}$

$\Rightarrow \tan x = \frac{1}{2}$ or $-\frac{1}{2}$

- $\arctan\left(\frac{1}{2}\right) = 26.6$
- $\arctan\left(-\frac{1}{2}\right) = -26.6$

$x = 26.6 \pm 180n$

or $x = -26.6 \pm 180n$ $(n = 0, 1, 2, 3.)$

$x_1 = 26.6°$

$x_2 = 206.6°$

$x_3 = 153.4°$

$x_4 = 333.4$

Question 157 (****)

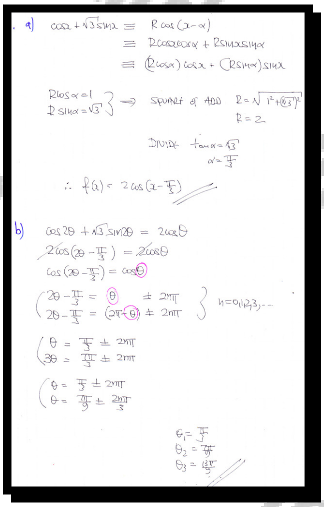
$$f(x) \equiv \cos x + \sqrt{3}\sin x, \quad x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\cos(x-\alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$.

b) Hence solve the equation

$$\cos 2\theta + \sqrt{3}\sin 2\theta = 2\cos\theta, \quad 0 \le \theta < 2\pi.$$

$\boxed{}$, $\boxed{f(x) = 2\cos\left(x - \dfrac{\pi}{3}\right)}$, $\boxed{\theta = \dfrac{\pi}{3}, \dfrac{7\pi}{9}, \dfrac{13\pi}{9}}$



Question 158 (****)

It is given that

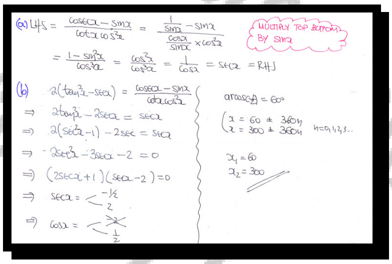
$$\frac{\operatorname{cosec} x - \sin x}{\cot x \cos^2 x} \equiv \sec x .$$

a) Prove the validity of the above trigonometric identity.

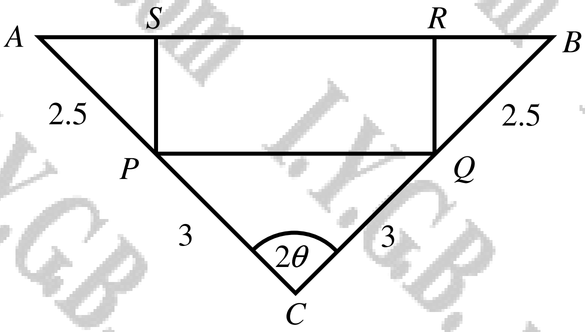
b) Hence, or otherwise, solve the trigonometric equation

$$\tan^2 x - \sec x = \frac{\operatorname{cosec} x - \sin x}{2\cot x \cos^2 x}, \quad 0 \le x < 360° .$$

$\boxed{x = 60°,\ 300°}$



Question 159 (****)



The figure above shows an isosceles triangle ABC where the angle $ACB = 2\theta$.

A rectangle $PQRS$ is drawn inside ABC, so that S and R lie on AB, P lies on AC and Q lies on BC.

It is further given that $|AP| = |BQ| = 2.5$ and $|PC| = |QC| = 3$.

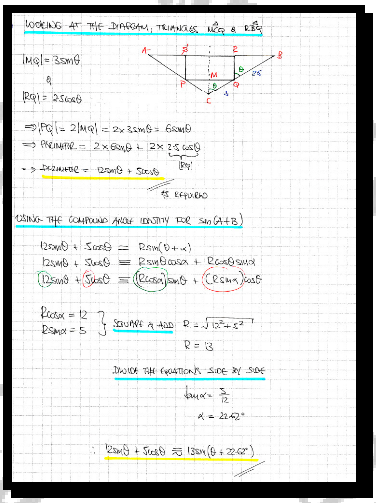
a) Show clearly that the perimeter of $PQRS$ is

$$5\cos\theta + 12\sin\theta .$$

b) Express $5\cos\theta + 12\sin\theta$ in the form $R\sin(\theta+\alpha)$, $R>0$, $0<\alpha<90°$.

c) Find the value of θ, given that the perimeter of $PQRS$ is 10.

$\boxed{}$, $\boxed{5\cos\theta + 12\sin\theta \cong 13\sin(\theta + 22.62)°}$, $\boxed{\theta \approx 27.7°}$



LOOKING AT THE DIAGRAM, TRIANGLES $\hat{MCQ}$ & $\hat{RBQ}$

$|MQ| = 3\sin\theta$
&
$|RQ| = 2.5\cos\theta$

$\Rightarrow |PQ| = 2|MQ| = 2 \times 3\sin\theta = 6\sin\theta$
$\Rightarrow$ PERIMETER $= 2 \times 6\sin\theta + 2 \times \underbrace{2.5\cos\theta}_{|RQ|}$
$\rightarrow$ PERIMETER $= 12\sin\theta + 5\cos\theta$
AS REQUIRED

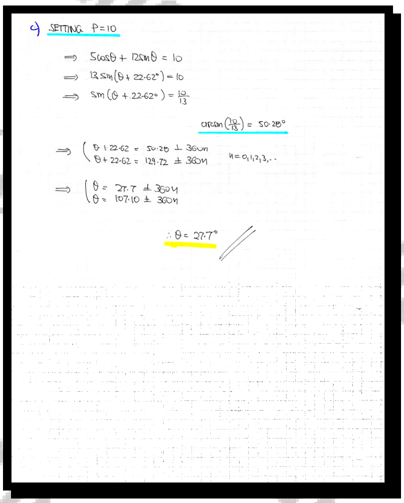
USING THE COMPOUND ANGLE IDENTITY FOR $\sin(A+B)$

$12\sin\theta + 5\cos\theta \equiv R\sin(\theta+\alpha)$
$12\sin\theta + 5\cos\theta \equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$
$12\sin\theta + 5\cos\theta \equiv (R\cos\alpha)\sin\theta + (R\sin\alpha)\cos\theta$

$R\cos\alpha = 12$, $R\sin\alpha = 5$ } SQUARE & ADD $R = \sqrt{12^2+5^2}$
$R = 13$

DIVIDE THE EQUATIONS SIDE BY SIDE
$\tan\alpha = \frac{5}{12}$
$\alpha = 22.62°$

$\therefore 12\sin\theta + 5\cos\theta \equiv 13\sin(\theta + 22.62°)$



c) SETTING $P=10$

$\Rightarrow 5\cos\theta + 12\sin\theta = 10$
$\Rightarrow 13\sin(\theta + 22.62°) = 10$
$\Rightarrow \sin(\theta + 22.62°) = \frac{10}{13}$

$\arcsin\left(\frac{10}{13}\right) = 50.28°$

$\Rightarrow \begin{cases} \theta + 22.62 = 50.28 \pm 360n \\ \theta + 22.62 = 129.72 \pm 360n \end{cases}$ $n = 0,1,2,3,\ldots$

$\Rightarrow \begin{cases} \theta = 27.7 \pm 360n \\ \theta = 107.10 \pm 360n \end{cases}$

$\therefore \theta = 27.7°$

Question 160 (****)

$$f(\theta) \equiv \sqrt{3}\sin\theta + \sin\theta, \; 0 \le \theta < 2\pi .$$

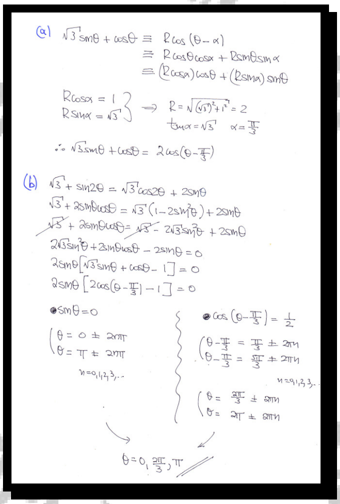
a) Express $f(\theta)$ in the form $R\cos(\theta - \alpha)$, $R > 0$, $0 < \alpha < \dfrac{\pi}{2}$.

b) Find the solutions of the trigonometric equation

$$\sqrt{3} + \sin 2\theta = \sqrt{3}\cos 2\theta + 2\sin\theta, \; 0 \le \theta < 2\pi ,$$

giving the answers in radians in terms of π.

$$\boxed{\theta = 0, \; \frac{2\pi}{3}, \; \pi}$$



(a) $\sqrt{3}\sin\theta + \cos\theta \equiv R\cos(\theta - \alpha)$
$\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$
$\equiv (R\cos\alpha)\cos\theta + (R\sin\alpha)\sin\theta$

$R\cos\alpha = 1$, $R\sin\alpha = \sqrt{3}$ $\Rightarrow$ $R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$, $\tan\alpha = \sqrt{3}$, $\alpha = \frac{\pi}{3}$

$\therefore \sqrt{3}\sin\theta + \cos\theta = 2\cos\left(\theta - \frac{\pi}{3}\right)$

(b) $\sqrt{3} + \sin 2\theta = \sqrt{3}\cos 2\theta + 2\sin\theta$
$\sqrt{3} + 2\sin\theta\cos\theta = \sqrt{3}(1 - 2\sin^2\theta) + 2\sin\theta$
$\sqrt{3} + 2\sin\theta\cos\theta = \sqrt{3} - 2\sqrt{3}\sin^2\theta + 2\sin\theta$
$2\sqrt{3}\sin^2\theta + 2\sin\theta\cos\theta - 2\sin\theta = 0$
$2\sin\theta\left[\sqrt{3}\sin\theta + \cos\theta - 1\right] = 0$
$2\sin\theta\left[2\cos\left(\theta - \frac{\pi}{3}\right) - 1\right] = 0$

• $\sin\theta = 0$
$\theta = 0 \pm 2n\pi$
$\theta = \pi \pm 2n\pi$
$n = 0, 1, 2, 3, \ldots$

• $\cos\left(\theta - \frac{\pi}{3}\right) = \frac{1}{2}$
$\theta - \frac{\pi}{3} = \frac{\pi}{3} \pm 2\pi n$
$\theta - \frac{\pi}{3} = \frac{5\pi}{3} \pm 2\pi n$
$n = 0, 1, 2, 3, \ldots$
$\theta = \frac{2\pi}{3} \pm 2\pi n$
$\theta = 2\pi \pm 2\pi n$

$\theta = 0, \frac{2\pi}{3}, \pi$

Question 161 (****)

It is given that

$$\cos P - \cos Q \equiv -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right).$$

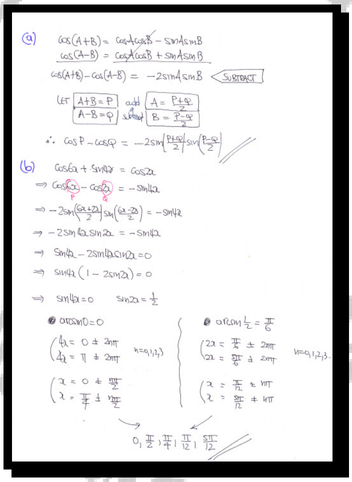
a) Prove the validity of the above trigonometric identity by using the compound angle identities for $\cos(A+B)$ and $\cos(A-B)$.

b) Hence, or otherwise, solve the trigonometric equation

$$\cos 6x + \sin 4x = \cos 2x, \quad 0 \le x \le \frac{\pi}{2},$$

giving the answers in terms of π.

$$\boxed{x = 0, \ \frac{\pi}{12}, \ \frac{\pi}{4}, \ \frac{5\pi}{12}, \ \frac{\pi}{2}}$$


(a) $\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\cos(A-B) = \cos A\cos B + \sin A\sin B$
$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$ ⟸ SUBTRACT
LET $A+B=P$, $A-B=Q$; add / subtract: $A = \frac{P+Q}{2}$, $B = \frac{P-Q}{2}$
$\therefore \cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$
(b) $\cos 6x + \sin 4x = \cos 2x$
$\Rightarrow \cos 6x - \cos 2x = -\sin 4x$ (P = 6x, Q = 2x)
$\Rightarrow -2\sin\left(\frac{6x+2x}{2}\right)\sin\left(\frac{6x-2x}{2}\right) = -\sin 4x$
$\Rightarrow -2\sin 4x\sin 2x = -\sin 4x$
$\Rightarrow \sin 4x - 2\sin 4x\sin 2x = 0$
$\Rightarrow \sin 4x(1 - 2\sin 2x) = 0$
$\Rightarrow \sin 4x = 0 \quad \sin 2x = \frac{1}{2}$
• $\arcsin 0 = 0$: $4x = 0 \pm 2n\pi$, $4x = \pi \pm 2n\pi$, $n = 0,1,2,3$; $x = 0 \pm \frac{n\pi}{2}$, $x = \frac{\pi}{4} \pm \frac{n\pi}{2}$
• $\arcsin\frac{1}{2} = \frac{\pi}{6}$: $2x = \frac{\pi}{6} \pm 2n\pi$, $2x = \frac{5\pi}{6} \pm 2n\pi$, $n = 0,1,2,3.$; $x = \frac{\pi}{12} \pm n\pi$, $x = \frac{5\pi}{12} \pm n\pi$
$0, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$

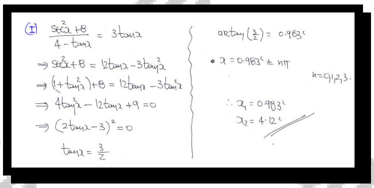
Question 162 (****)

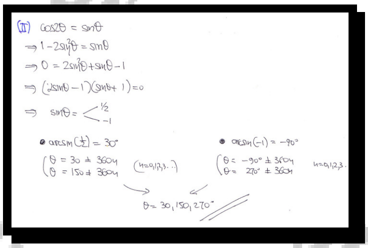
Solve each of the following trigonometric equations.

i. $\dfrac{\sec^2 x+8}{4-\tan x}=3\tan x, \qquad 0\le x<2\pi, \quad \tan x\ne 4.$

ii. $\cos 2\theta=\sin\theta, \qquad 0\le\theta<360°.$

$\boxed{}$, $\boxed{x\approx 0.983^{c},\ 4.12^{c}}$, $\boxed{\theta=30°,\ 150°,\ 270°}$



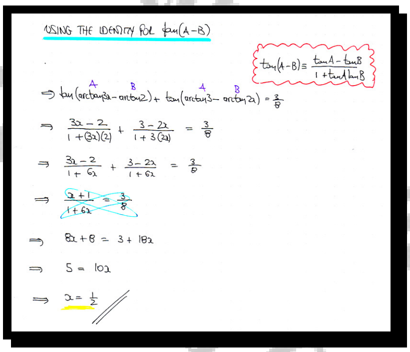


Question 163 (****)

Solve the following trigonometric equation

$$\tan(\arctan 3x-\arctan 2)+\tan(\arctan 3-\arctan 2x)=\frac{3}{8}.$$

$\boxed{}$, $\boxed{x=\frac{1}{2}}$



Question 164 (****)

It is given that

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$$

a) Use the above trigonometric identities show that

$$\cos(A+B) + \cos(A-B) \equiv 2\cos A \cos B .$$

b) Hence show further that

$$\cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right).$$

It is further given that

$$\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right).$$

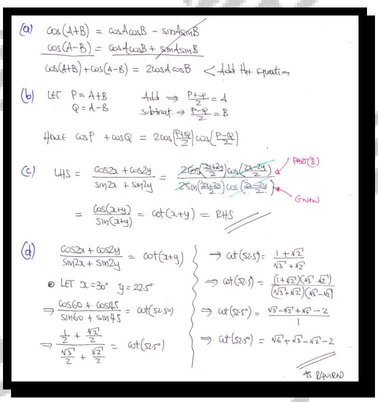
c) Show clearly that

$$\frac{\cos 2x + \cos 2y}{\sin 2x + \sin 2y} \equiv \cot(x+y) .$$

d) Use the above results to show that

$$\cot(52.5°) = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2$$

proof



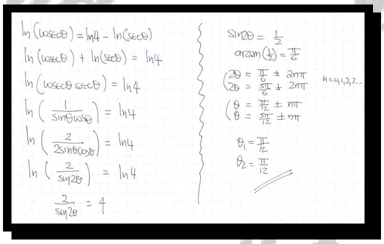
Question 165 (****)

Solve the trigonometric equation

$$\ln(\operatorname{cosec}\theta) = \ln 4 - \ln(\sec\theta), \quad 0 < \theta < \frac{\pi}{2},$$

giving the answers in terms of π.

$$\boxed{\theta = \frac{\pi}{12}, \frac{5\pi}{12}}$$



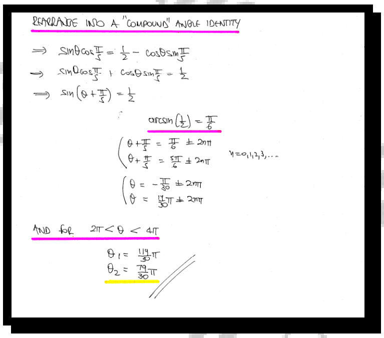
Question 166 (****)

Solve the trigonometric equation

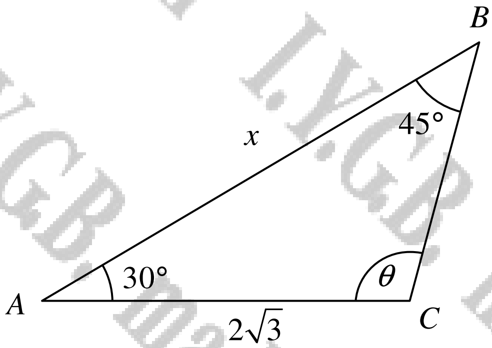
$$\sin\theta\cos\frac{\pi}{5} = \frac{1}{2} - \cos\theta\sin\frac{\pi}{5}, \quad 2\pi < \theta < 4\pi,$$

giving the answers in terms of π.

$$\boxed{\theta = \frac{79\pi}{30}, \frac{119\pi}{30}}$$



Question 167 **(****)** **non calculator**



The figure above shows a triangle ABC where $|AC| = 2\sqrt{3}$ and $|AB| = x$.

The angles ABC, CAB and BCA are $45°$, $30°$ and $\theta°$, respectively.

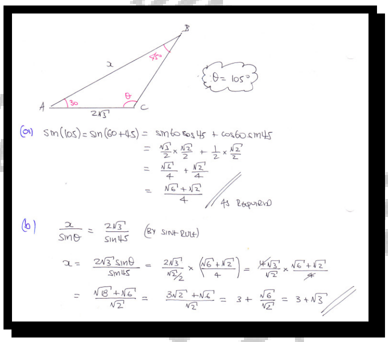
a) By using a suitable compound angle identity show clearly that

$$\sin 105° = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

b) Show without the use of a calculating aid that the exact length of AB, is

$$3+\sqrt{3}\ .$$

proof



Question 168 (****)

$$f(\theta) = 2\cos\theta + 3\sin\theta,\ \theta \in \mathbb{R}.$$

a) Express $f(\theta)$ in the form $R\cos(\theta - \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

Give the value of α correct to 3 decimal places.

b) State the maximum value of $f(\theta)$ and find the smallest positive value of θ for which this maximum occurs.

The temperature T °C in a warehouse is modelled by the equation

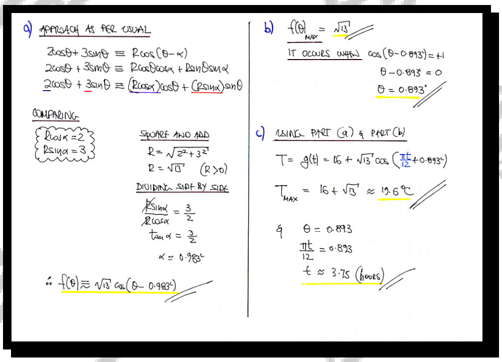
$$T = 16 + 2\cos\left(\frac{\pi t}{12}\right) + 3\sin\left(\frac{\pi t}{12}\right),\ \ 0 \le t < 24,$$

where t is the time in hours measured since midnight.

c) State the maximum temperature in the warehouse and a value of t when this maximum temperature occurs.

d) Find the times, to the nearest minute using 24 hour clock notation, when the temperature in the warehouse is 17 °C.

$\boxed{}$, $\boxed{2\cos\theta + 3\sin\theta \equiv \sqrt{13}\cos(\theta - 0.983^c)}$, $\boxed{\max = \sqrt{13}}$, $\boxed{\theta = 0.983^c}$, $\boxed{T_{max} \approx 19.6}$, $\boxed{t_{max} = 3.75}$, $\boxed{08:41/22:50}$



a) APPROACH AS PER USUAL

$2\cos\theta + 3\sin\theta \equiv R\cos(\theta - \alpha)$

$2\cos\theta + 3\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$2\cos\theta + 3\sin\theta \equiv (R\cos\alpha)\cos\theta + (R\sin\alpha)\sin\theta$

COMPARING

$R\cos\alpha = 2$, $R\sin\alpha = 3$

SQUARE AND ADD

$R = \sqrt{2^2 + 3^2}$

$R = \sqrt{13}$ $(R > 0)$

DIVIDING SIDE BY SIDE

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{2}$

$\tan\alpha = \frac{3}{2}$

$\alpha \simeq 0.983^c$

$\therefore f(\theta) \equiv \sqrt{13}\cos(\theta - 0.983^c)$

b) $f(\theta)_{MAX} = \sqrt{13}$

IT OCCURS WHEN $\cos(\theta - 0.893^c) = +1$

$\theta - 0.893^c = 0$

$\theta = 0.893^c$

c) USING PART (a) & PART (b)

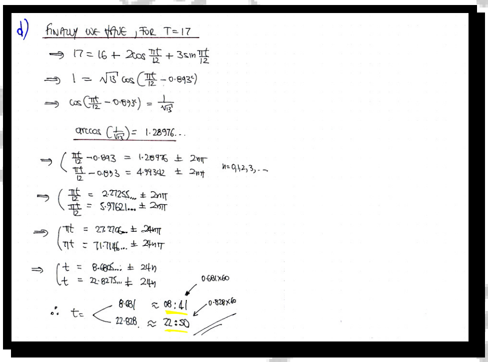
$T = g(t) = 16 + \sqrt{13}\cos\left(\frac{\pi t}{12} + 0.893^c\right)$

$T_{MAX} = 16 + \sqrt{13} \approx 19.6\,°C$

& $\theta = 0.893$

$\frac{\pi t}{12} = 0.893$

$t \approx 3.75$ (hours)



d) FINALLY WE HAVE, FOR $T = 17$

$\Rightarrow 17 = 16 + 2\cos\frac{\pi t}{12} + 3\sin\frac{\pi t}{12}$

$\Rightarrow 1 = \sqrt{13}\cos\left(\frac{\pi t}{12} - 0.893^c\right)$

$\Rightarrow \cos\left(\frac{\pi t}{12} - 0.893^c\right) = \frac{1}{\sqrt{13}}$

$\arccos\left(\frac{1}{\sqrt{13}}\right) = 1.28976\ldots$

$\Rightarrow \frac{\pi t}{12} - 0.893 = 1.28976 \pm 2n\pi$, $\frac{\pi t}{12} - 0.893 = 4.99342 \pm 2n\pi$ $n = 0,1,2,3,\ldots$

$\Rightarrow \frac{\pi t}{12} = 2.27255\ldots \pm 2n\pi$, $\frac{\pi t}{12} = 5.97621\ldots \pm 2n\pi$

$\Rightarrow \pi t = 27.2706\ldots \pm 24n\pi$, $\pi t = 71.7146\ldots \pm 24n\pi$

$\Rightarrow t = 8.6805\ldots \pm 24n$, $t = 22.8275\ldots \pm 24n$

$\therefore t = 8.681 \approx 08:41$ (0.681×60), $22.828 \approx 22:50$ (0.828×60)

Question 169 (****)

It is given that

$$\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right).$$

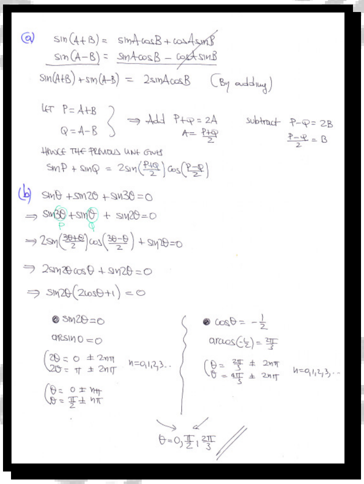
a) Prove the validity of the above trigonometric identity by using the compound angle identities for $\sin(A+B)$ and $\sin(A-B)$.

b) Hence, or otherwise, solve the equation

$$\sin\theta + \sin 2\theta + \sin 3\theta = 0, \quad 0 \le \theta < \pi,$$

giving the answers in terms of π.

$$\boxed{\theta = 0, \ \frac{\pi}{2}, \ \frac{2\pi}{3}}$$


(a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin(A-B) = \sin A\cos B - \cos A\sin B$
$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ (By adding)

LET $P = A+B$, $Q = A-B$ $\Rightarrow$ Add $P+Q = 2A$, $A = \frac{P+Q}{2}$; subtract $P-Q = 2B$, $\frac{P-Q}{2} = B$

HENCE THE PREVIOUS LINE GIVES
$\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

(b) $\sin\theta + \sin 2\theta + \sin 3\theta = 0$
$\Rightarrow \sin 3\theta + \sin\theta + \sin 2\theta = 0$ (P, Q)
$\Rightarrow 2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right) + \sin 2\theta = 0$
$\Rightarrow 2\sin 2\theta\cos\theta + \sin 2\theta = 0$
$\Rightarrow \sin 2\theta(2\cos\theta + 1) = 0$

• $\sin 2\theta = 0$
$\arcsin 0 = 0$
$2\theta = 0 \pm 2n\pi$, $2\theta = \pi \pm 2n\pi$, $n = 0, 1, 2, 3, \ldots$
$\theta = 0 \pm n\pi$, $\theta = \frac{\pi}{2} \pm n\pi$

• $\cos\theta = -\frac{1}{2}$
$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$
$\theta = \frac{2\pi}{3} \pm 2n\pi$, $\theta = \frac{4\pi}{3} \pm 2n\pi$, $n = 0, 1, 2, 3, \ldots$

$\theta = 0, \frac{\pi}{2}, \frac{2\pi}{3}$

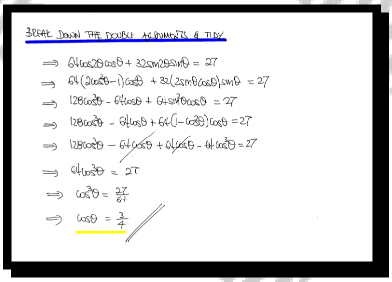
Question 170 (****)

Given that

$$64\cos 2\theta \cos\theta + 32\sin 2\theta \sin\theta = 27,$$

find the value of $\cos\theta$.

$\boxed{}$, $\boxed{\cos\theta = \frac{3}{4}}$


BREAK DOWN THE DOUBLE ARGUMENTS & TIDY

$\Rightarrow 64\cos 2\theta\cos\theta + 32\sin 2\theta\sin\theta = 27$

$\Rightarrow 64(2\cos^2\theta - 1)\cos\theta + 32(2\sin\theta\cos\theta)\sin\theta = 27$

$\Rightarrow 128\cos^3\theta - 64\cos\theta + 64\sin^2\theta\cos\theta = 27$

$\Rightarrow 128\cos^3\theta - 64\cos\theta + 64(1-\cos^2\theta)\cos\theta = 27$

$\Rightarrow 128\cos^3\theta - 64\cos\theta + 64\cos\theta - 64\cos^3\theta = 27$

$\Rightarrow 64\cos^3\theta = 27$

$\Rightarrow \cos^3\theta = \frac{27}{64}$

$\Rightarrow \cos\theta = \frac{3}{4}$

Question 171 (****)

$$f(x) = 2\sin x + 2\cos x, \quad x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\sin(x+\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

b) State the minimum and the maximum value of …

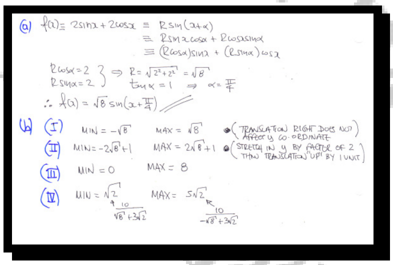
i. … $y = f\left(x - \frac{\pi}{2}\right)$.

ii. … $y = 2f(x)+1$.

iii. … $y = [f(x)]^2$.

iv. … $y = \dfrac{10}{f(x)+3\sqrt{2}}$.

$\boxed{}$, $\boxed{y=\sqrt{8}\sin\left(\theta+\frac{\pi}{4}\right)}$, $\boxed{\left[-\sqrt{8},\sqrt{8}\right]}$, $\boxed{\left[-2\sqrt{8}+1, 2\sqrt{8}+1\right]}$, $\boxed{[0,8]}$, $\boxed{\left[\sqrt{2}, 5\sqrt{2}\right]}$


(a) $f(x) \equiv 2\sin x + 2\cos x \equiv R\sin(x+\alpha)$
$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$
$\equiv (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$

$R\cos\alpha = 2$, $R\sin\alpha = 2$ $\Rightarrow$ $R=\sqrt{2^2+2^2}=\sqrt{8}$, $\tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$

$\therefore f(x) = \sqrt{8}\sin\left(x+\frac{\pi}{4}\right)$

(b) (I) MIN $= -\sqrt{8}$ MAX $= \sqrt{8}$ (TRANSLATION RIGHT DOES NOT AFFECT y CO-ORDINATE)

(II) MIN $= -2\sqrt{8}+1$ MAX $= 2\sqrt{8}+1$ (STRETCH IN y BY FACTOR OF 2 THEN TRANSLATION "UP" BY 1 UNIT)

(III) MIN $= 0$ MAX $= 8$

(IV) MIN $= \sqrt{2}$ ($\frac{10}{\sqrt{8}+3\sqrt{2}}$) MAX $= 5\sqrt{2}$ ($\frac{10}{-\sqrt{8}+3\sqrt{2}}$)

Question 172 (****)

It is given that θ and φ are such so that

$$\tan\theta = t \quad \text{and} \quad \tan\varphi = t-1,$$

where t is a constant.

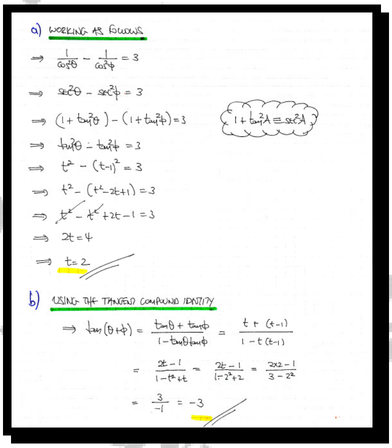
It is further given that

$$\frac{1}{\cos^2\theta} - \frac{1}{\cos^2\varphi} = 3.$$

a) Show clearly that $t = 2$.

b) Determine the exact value of $\tan(\theta+\varphi)$, showing clearly all the steps in the workings.

$\boxed{}$, $\boxed{\tan(\theta+\varphi) = -3}$



a) WORKING AS FOLLOWS

$$\Rightarrow \frac{1}{\cos^2\theta} - \frac{1}{\cos^2\varphi} = 3$$

$$\Rightarrow \sec^2\theta - \sec^2\varphi = 3$$

$$\Rightarrow (1+\tan^2\theta) - (1+\tan^2\varphi) = 3 \qquad 1+\tan^2 A \equiv \sec^2 A$$

$$\Rightarrow \tan^2\theta - \tan^2\varphi = 3$$

$$\Rightarrow t^2 - (t-1)^2 = 3$$

$$\Rightarrow t^2 - (t^2-2t+1) = 3$$

$$\Rightarrow t^2 - t^2 + 2t - 1 = 3$$

$$\Rightarrow 2t = 4$$

$$\Rightarrow t = 2$$

b) USING THE TANGENT COMPOUND IDENTITY

$$\Rightarrow \tan(\theta+\varphi) = \frac{\tan\theta + \tan\varphi}{1-\tan\theta\tan\varphi} = \frac{t+(t-1)}{1-t(t-1)}$$

$$= \frac{2t-1}{1-t^2+t} = \frac{2t-1}{1-2^2+2} = \frac{2\times2-1}{3-2^2}$$

$$= \frac{3}{-1} = -3$$

Question 173 (****)

$$\sin 2x + \cos 2x = 1 + \sin x, \quad 0 < x < \frac{\pi}{2}.$$

a) Show that the above trigonometric equation can be written as

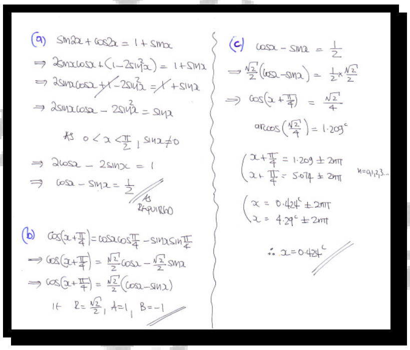
$$\cos x - \sin x = \frac{1}{2}.$$

b) Express $\cos\left(x + \frac{\pi}{4}\right)$ in the form $R\left(A\cos x + B\sin x\right)$, where R, A and B are constants to be found.

c) Use the results of part (a) and (b) to solve the trigonometric equation

$$\sin 2x + \cos 2x = 1 + \sin x, \quad 0 < x < \frac{\pi}{2}.$$

$\boxed{R = \frac{\sqrt{2}}{2}}$, $\boxed{A = 1}$, $\boxed{B = -1}$



Question 174 (****)

$$f(x) = \sin 2x\,, \quad x \in \mathbb{R}$$

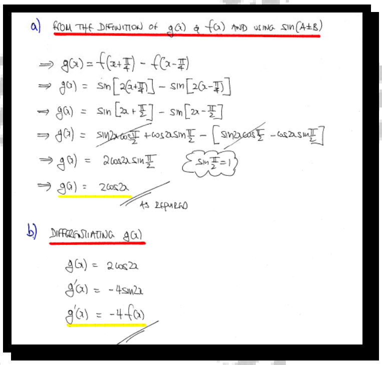
$$g(x) = f\left(x + \frac{\pi}{4}\right) - f\left(x - \frac{\pi}{4}\right), \; x \in \mathbb{R}\,.$$

a) Show clearly that

$$g(x) = 2\cos 2x\,.$$

b) Express $g'(x)$ in terms of $f(x)$.

$\boxed{}$, $\boxed{g'(x) = -4f(x)}$


a) FROM THE DEFINITION OF $g(x)$ & $f(x)$ AND USING $\sin(A \pm B)$

$\Rightarrow g(x) = f\left(x+\frac{\pi}{4}\right) - f\left(x-\frac{\pi}{4}\right)$

$\Rightarrow g(x) = \sin\left[2\left(x+\frac{\pi}{4}\right)\right] - \sin\left[2\left(x-\frac{\pi}{4}\right)\right]$

$\Rightarrow g(x) = \sin\left[2x+\frac{\pi}{2}\right] - \sin\left[2x-\frac{\pi}{2}\right]$

$\Rightarrow g(x) = \sin 2x\cos\frac{\pi}{2} + \cos 2x\sin\frac{\pi}{2} - \left[\sin 2x\cos\frac{\pi}{2} - \cos 2x\sin\frac{\pi}{2}\right]$

$\Rightarrow g(x) = 2\cos 2x\sin\frac{\pi}{2}$ $\quad \left\{\sin\frac{\pi}{2} = 1\right\}$

$\Rightarrow g(x) = 2\cos 2x$ AS REQUIRED

b) DIFFERENTIATING $g(x)$

$g(x) = 2\cos 2x$

$g'(x) = -4\sin 2x$

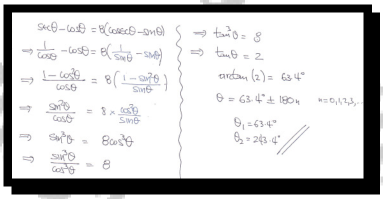
$g'(x) = -4f(x)$

Question 175 (****)

Solve the trigonometric equation

$$\sec\theta - \cos\theta = 8\left(\operatorname{cosec}\theta - \sin\theta\right), \quad 0 \le \theta < 360°.$$

$\boxed{\theta \approx 63.4°,\ 243.6°}$



Question 176 (****)

It is given that

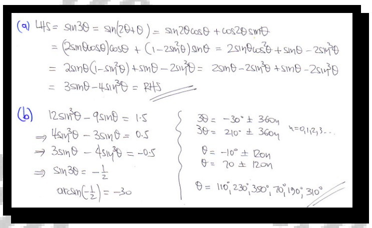
$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta.$$

a) Prove the validity of the above trigonometric identity by writing 3θ as $2\theta + \theta$.

b) Hence solve the trigonometric equation

$$12\sin^3\theta - 9\sin\theta = 1.5, \quad 0 \le \theta < 360°.$$

$\boxed{\theta = 70°,\ 110°,\ 190°,\ 230°,\ 310°,\ 350°}$



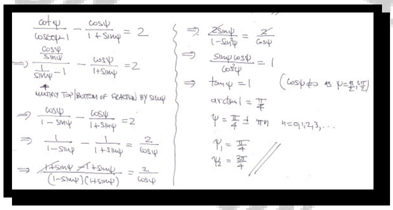
Question 177 (****)

Solve the trigonometric equation

$$\frac{\cot\psi}{\operatorname{cosec}\psi - 1} - \frac{\cos\psi}{1+\sin\psi} = 2, \qquad 0 < \psi < 2\pi,$$

giving the answers in terms of π.

$$\boxed{\psi = \frac{\pi}{4}, \frac{5\pi}{4}}$$



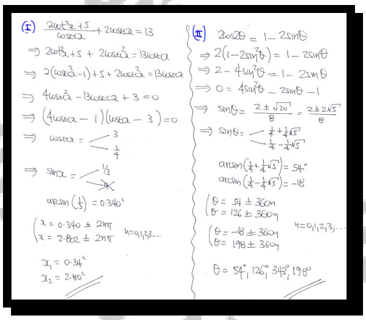
Question 178 (****)

Solve each of the following trigonometric equations.

i. $\dfrac{2\cot^2 x + 5}{\operatorname{cosec} x} + 2\operatorname{cosec} x = 13, \qquad 0 \le x < 2\pi.$

ii. $2\cos 2\theta = 1 - 2\sin\theta, \qquad 0 \le \theta < 360°.$

$\boxed{x \approx 0.340^c,\ 2.80^c}$, $\boxed{\theta = 54°,\ 126°,\ 198°,\ 342°}$

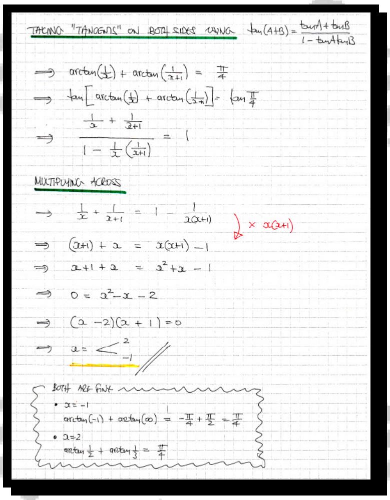


Question 179 (****)

Solve the following trigonometric equation

$$\arctan\left(\frac{1}{x}\right)+\arctan\left(\frac{1}{x+1}\right)=\frac{\pi}{4}.$$

$\boxed{}$, $\boxed{x=-1,\ 2}$



TAKING "TANGENTS" ON BOTH SIDES USING $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

$\Rightarrow \arctan\left(\frac{1}{x}\right)+\arctan\left(\frac{1}{x+1}\right)=\frac{\pi}{4}$

$\Rightarrow \tan\left[\arctan\left(\frac{1}{x}\right)+\arctan\left(\frac{1}{x+1}\right)\right]=\tan\frac{\pi}{4}$

$\Rightarrow \dfrac{\frac{1}{x}+\frac{1}{x+1}}{1-\frac{1}{x}\left(\frac{1}{x+1}\right)}=1$

MULTIPLYING ACROSS

$\Rightarrow \frac{1}{x}+\frac{1}{x+1}=1-\frac{1}{x(x+1)}$ $\quad \times\, x(x+1)$

$\Rightarrow (x+1)+x=x(x+1)-1$

$\Rightarrow x+1+x=x^2+x-1$

$\Rightarrow 0=x^2-x-2$

$\Rightarrow (x-2)(x+1)=0$

$\Rightarrow x=\begin{cases}2\\-1\end{cases}$

BOTH ARE FINE

- $x=-1$

$\arctan(-1)+\arctan(\infty)=-\frac{\pi}{4}+\frac{\pi}{2}=\frac{\pi}{4}$

- $x=2$

$\arctan\frac{1}{2}+\arctan\frac{1}{3}=\frac{\pi}{4}$

Question 180 (****)

Solve the following trigonometric equation

$$2\arctan\left(\frac{3}{x}\right) = \arcsin\left(\frac{6x}{25}\right).$$

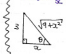
MP2-V, $\boxed{x = \pm 4}$

PROCEED AS FOLLOWS

$$\Rightarrow \underbrace{2\arctan\left(\frac{3}{x}\right)}_{\theta} = \underbrace{\arcsin\left(\frac{6x}{25}\right)}_{\phi}$$

$\theta = \arctan\frac{3}{x}$ $\qquad \phi = \arcsin\left(\frac{6x}{25}\right)$

$\tan\theta = \frac{3}{x}$ $\qquad \sin\phi = \frac{6x}{25}$



$\sin\theta = \frac{3}{\sqrt{9+x^2}}$

$\cos\theta = \frac{x}{\sqrt{9+x^2}}$ $\quad$ & $\quad \sin\phi = \frac{6x}{25}$

$\Rightarrow 2\theta = \phi$

$\Rightarrow \sin 2\theta = \sin\phi$

$\Rightarrow 2\sin\theta\cos\theta = \sin\phi$

USING THE VALUES FROM ABOVE

$\Rightarrow 2\left(\frac{3}{\sqrt{9+x^2}}\right)\left(\frac{x}{\sqrt{9+x^2}}\right) = \frac{6x}{25}$

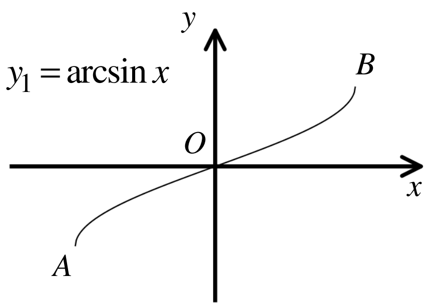
$\Rightarrow \frac{6x}{9+x^2} = \frac{6x}{25}$

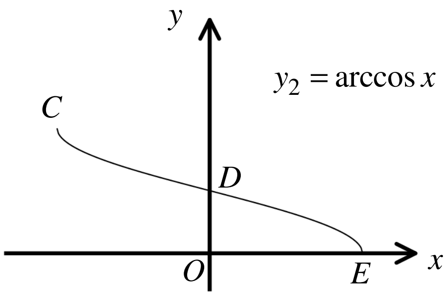
$\Rightarrow 9+x^2 = 25$

$\Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

Question 181 (****)





The figures above show the graph of $y_1 = \arcsin x$ and the graph of $y_2 = \arccos x$.

The graph of y_1 has endpoints at A and B.

The graph of y_2 has endpoints at C and E, and D is the point where the graph of y_2 crosses the y axis.

a) State the coordinates of A, B, C, D and E.

The graph of y_2 can be obtained from the graph of y_1 by a series of two geometric transformations which can be carried out in a specific order.

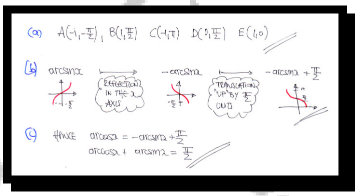
b) Describe these two geometric transformations.

c) Deduce using valid arguments that

$$\arcsin x + \arccos x = \text{constant},$$

stating the exact value of this constant.

$\boxed{A\left(-1,\frac{\pi}{2}\right)}$, $\boxed{B\left(1,\frac{\pi}{2}\right)}$, $\boxed{C(-1,\pi)}$, $\boxed{D\left(0,\frac{\pi}{2}\right)}$, $\boxed{E(1,0)}$, $\boxed{\text{constant} = \frac{\pi}{2}}$



Question 182 (****)

It is given that

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B.$$

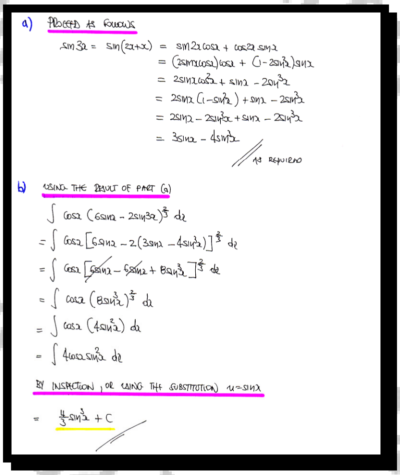
a) Use the above trigonometric identity to show that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x.$$

b) Hence find

$$\int \cos x (6\sin x - 2\sin 3x)^{\frac{2}{3}}\, dx.$$

$\boxed{}$, $\boxed{\frac{4}{3}\sin^3 x + C}$



a) PROCEED AS FOLLOWS

$\sin 3x = \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x$
$= (2\sin x\cos x)\cos x + (1-2\sin^2 x)\sin x$
$= 2\sin x\cos^2 x + \sin x - 2\sin^3 x$
$= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x$
$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$
$= 3\sin x - 4\sin^3 x$

AS REQUIRED

b) USING THE RESULT OF PART (a)

$\int \cos x(6\sin x - 2\sin 3x)^{\frac{2}{3}}\,dx$
$= \int \cos x\left[6\sin x - 2(3\sin x - 4\sin^3 x)\right]^{\frac{2}{3}}\,dx$
$= \int \cos x\left[6\sin x - 6\sin x + 8\sin^3 x\right]^{\frac{2}{3}}\,dx$
$= \int \cos x(8\sin^3 x)^{\frac{2}{3}}\,dx$
$= \int \cos x(4\sin^2 x)\,dx$
$= \int 4\cos x\sin^2 x\,dx$

BY INSPECTION, OR USING THE SUBSTITUTION $u = \sin x$

$= \frac{4}{3}\sin^3 x + C$

Question 183 (****)

$$f(x) = A\sec 2x + B\,,\ 0 \le x < 2\pi\,.$$

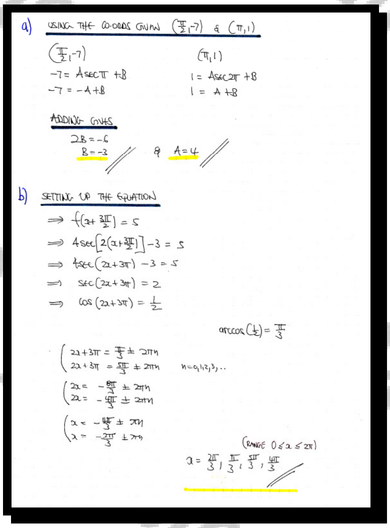
The graph of $f(x)$, where A and B are non zero constants, passes through the points $\left(\frac{\pi}{2}, -7\right)$ and $(\pi, 1)$.

a) Determine the value of A and the value of B.

b) Solve the equation

$$f\left(x + \frac{3\pi}{2}\right) = 5\,.$$

$\boxed{}$, $\boxed{A = 4}$, $\boxed{B = -3}$, $\boxed{x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$


a) USING THE CO-ORDS GIVEN $\left(\frac{\pi}{2}, -7\right)$ & $(\pi, 1)$

$\left(\frac{\pi}{2}, -7\right)$
$-7 = A\sec\pi + B$
$-7 = -A + B$

$(\pi, 1)$
$1 = A\sec 2\pi + B$
$1 = A + B$

ADDING GIVES
$2B = -6$
$B = -3$ // & $A = 4$ //

b) SETTING UP THE EQUATION
$\Rightarrow f\left(x + \frac{3\pi}{2}\right) = 5$
$\Rightarrow 4\sec\left[2\left(x + \frac{3\pi}{2}\right)\right] - 3 = 5$
$\Rightarrow 4\sec(2x + 3\pi) - 3 = 5$
$\Rightarrow \sec(2x + 3\pi) = 2$
$\Rightarrow \cos(2x + 3\pi) = \frac{1}{2}$

$\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$2x + 3\pi = \frac{\pi}{3} \pm 2\pi n$
$2x + 3\pi = \frac{5\pi}{3} \pm 2\pi n$ $\quad n = 0, 1, 2, 3, \ldots$

$2x = -\frac{8\pi}{3} \pm 2\pi n$
$2x = -\frac{4\pi}{3} \pm 2\pi n$

$x = -\frac{4\pi}{3} \pm \pi n$
$x = -\frac{2\pi}{3} \pm \pi n$

(RANGE $0 \le x \le 2\pi$)

$x = \frac{2\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{4\pi}{3}$ //

Question 184 (****)

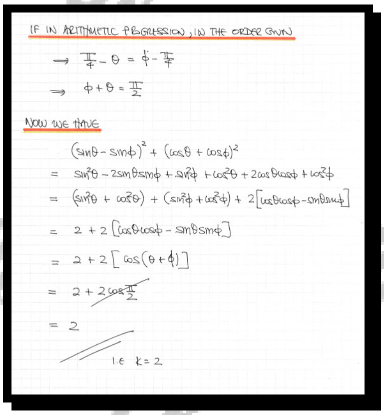
It is given that the angles θ, $\frac{\pi}{4}$ and φ are in arithmetic progression.

Show that

$$(\sin\theta - \sin\varphi)^2 + (\cos\theta + \cos\varphi)^2 = k \,,$$

where k is a constant to be found.

$\boxed{}$, $\boxed{k = 2}$


IF IN ARITHMETIC PROGRESSION, IN THE ORDER GIVEN

$\Rightarrow \frac{\pi}{4} - \theta = \phi - \frac{\pi}{4}$

$\Rightarrow \phi + \theta = \frac{\pi}{2}$

NOW WE HAVE

$(\sin\theta - \sin\phi)^2 + (\cos\theta + \cos\phi)^2$

$= \sin^2\theta - 2\sin\theta\sin\phi + \sin^2\phi + \cos^2\theta + 2\cos\theta\cos\phi + \cos^2\phi$

$= (\sin^2\theta + \cos^2\theta) + (\sin^2\phi + \cos^2\phi) + 2[\cos\theta\cos\phi - \sin\theta\sin\phi]$

$= 2 + 2[\cos\theta\cos\phi - \sin\theta\sin\phi]$

$= 2 + 2[\cos(\theta + \phi)]$

$= 2 + 2\cos\frac{\pi}{2}$

$= 2$

i.e. $k = 2$

Question 185 (****)

It is given that

$$\cos x \cos\left(x+\frac{\pi}{4}\right)-\cos\left(2x-\frac{\pi}{4}\right)=0 .$$

Given further that $x \neq k\pi$, $k \in \mathbb{Z}$, show clearly that $\tan x = 3$

proof

$$\cos x\cos\left(x+\tfrac{\pi}{4}\right)-\cos\left(2x-\tfrac{\pi}{4}\right)=0$$
$$\Rightarrow \cos x\left[\cos x\cos\tfrac{\pi}{4}-\sin x\sin\tfrac{\pi}{4}\right]-\left[\cos 2x\cos\tfrac{\pi}{4}+\sin 2x\sin\tfrac{\pi}{4}\right]=0$$
$$\Rightarrow \cos x\left[\tfrac{1}{\sqrt{2}}\cos x-\tfrac{1}{\sqrt{2}}\sin x\right]-\left[\tfrac{1}{\sqrt{2}}\cos 2x+\tfrac{1}{\sqrt{2}}\sin 2x\right]=0$$
$$\Rightarrow \tfrac{1}{\sqrt{2}}\cos^2 x-\tfrac{1}{\sqrt{2}}\cos x\sin x-\tfrac{1}{\sqrt{2}}\cos 2x-\tfrac{1}{\sqrt{2}}\sin 2x=0$$
$$\Rightarrow \cos^2 x-\cos x\sin x-\cos 2x-\sin 2x=0$$
$$\Rightarrow \cos^2 x-\cos x\sin x-(2\cos^2 x-1)-2\sin x\cos x=0$$
$$\Rightarrow 1-\cos^2 x-3\sin x\cos x=0$$
$$\Rightarrow \sin^2 x-3\sin x\cos x=0$$
$$\Rightarrow \sin x(\sin x-3\cos x)=0$$
$$\Rightarrow \sin x-3\cos x=0 \quad (\sin x\neq 0)$$
$$\Rightarrow \sin x=3\cos x$$
$$\Rightarrow \tan x=3$$

Question 186 (****)

$$y = \arcsin x\,,\ -1 \le x \le 1\,.$$

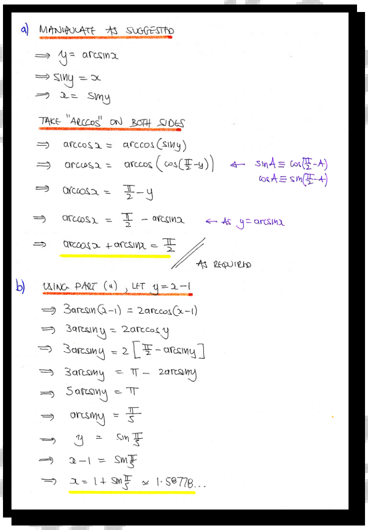
a) By expressing $\arccos x$ in terms of y, show that

$$\arcsin x + \arccos x = \frac{\pi}{2}\,.$$

b) Hence, or otherwise, solve the equation

$$3\arcsin(x-1) = 2\arccos(x-1)\,.$$

$\boxed{}$, $\boxed{x = 1 + \sin\left(\frac{\pi}{5}\right) \approx 1.5878}$


a) MANIPULATE AS SUGGESTED

$\Rightarrow y = \arcsin x$

$\Rightarrow \sin y = x$

$\Rightarrow x = \sin y$

TAKE "ARCCOS" ON BOTH SIDES

$\Rightarrow \arccos x = \arccos(\sin y)$

$\Rightarrow \arccos x = \arccos\left(\cos\left(\frac{\pi}{2} - y\right)\right)$ ← $\sin A \equiv \cos\left(\frac{\pi}{2} - A\right)$, $\cos A \equiv \sin\left(\frac{\pi}{2} - A\right)$

$\Rightarrow \arccos x = \frac{\pi}{2} - y$

$\Rightarrow \arccos x = \frac{\pi}{2} - \arcsin x$ ← AS $y = \arcsin x$

$\Rightarrow \arccos x + \arcsin x = \frac{\pi}{2}$ // AS REQUIRED

b) USING PART (a), LET $y = x - 1$

$\Rightarrow 3\arcsin(x-1) = 2\arccos(x-1)$

$\Rightarrow 3\arcsin y = 2\arccos y$

$\Rightarrow 3\arcsin y = 2\left[\frac{\pi}{2} - \arcsin y\right]$

$\Rightarrow 3\arcsin y = \pi - 2\arcsin y$

$\Rightarrow 5\arcsin y = \pi$

$\Rightarrow \arcsin y = \frac{\pi}{5}$

$\Rightarrow y = \sin\frac{\pi}{5}$

$\Rightarrow x - 1 = \sin\frac{\pi}{5}$

$\Rightarrow x = 1 + \sin\frac{\pi}{5} \approx 1.58778...$

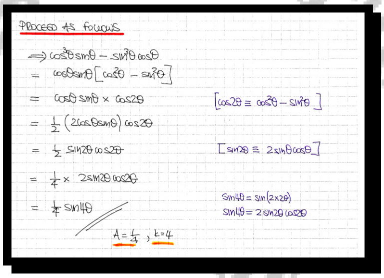
Question 187 (****)

Simplify, showing clearly all the workings, the trigonometric expression

$$\cos^3\theta\sin\theta - \sin^3\theta\cos\theta\,,$$

giving the final answer in the form $A\sin k\theta$, where A and k are constants.

$\boxed{}$, $\boxed{\frac{1}{4}\sin 4\theta}$


PROCEED AS FOLLOWS

$\Rightarrow \cos^3\theta\sin\theta - \sin^3\theta\cos\theta$

$= \cos\theta\sin\theta\left[\cos^2\theta - \sin^2\theta\right]$

$= \cos\theta\sin\theta \times \cos 2\theta$ $\quad\left[\cos 2\theta \equiv \cos^2\theta - \sin^2\theta\right]$

$= \frac{1}{2}\left(2\cos\theta\sin\theta\right)\cos 2\theta$

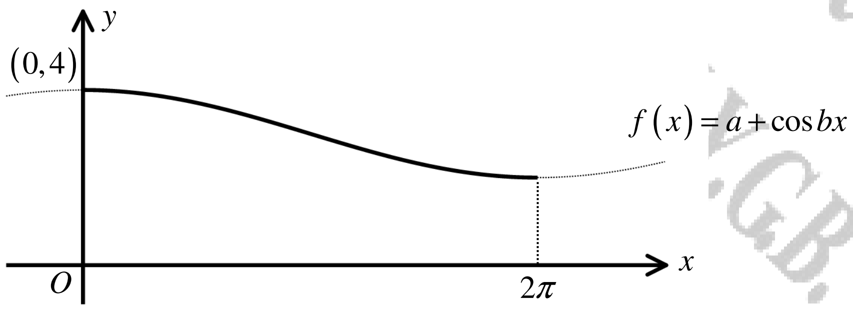
$= \frac{1}{2}\sin 2\theta\cos 2\theta$ $\quad\left[\sin 2\theta \equiv 2\sin\theta\cos\theta\right]$

$= \frac{1}{4}\times 2\sin 2\theta\cos 2\theta$

$= \frac{1}{4}\sin 4\theta$ $\quad \sin 4\theta = \sin(2\times 2\theta)$, $\sin 4\theta = 2\sin 2\theta\cos 2\theta$

$A = \frac{1}{4}$, $k = 4$

Question 188 (****)



The figure above shows the graph of the function

$$f(x) = a + \cos bx,\ 0 \le x \le 2\pi,$$

where a and b are non zero constants.

The stationary points $(0,4)$ and $(2\pi,2)$ are the endpoints of the graph.

a) State the range of $f(x)$ and hence find the value of a and the value of b.

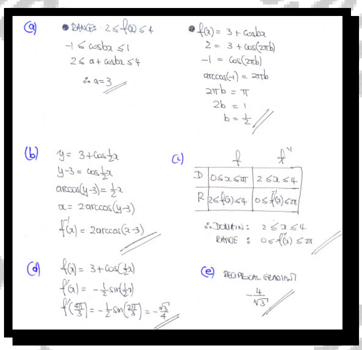
b) Find an expression for $f^{-1}(x)$, the inverse function of $f(x)$.

c) State the domain and range of $f^{-1}(x)$.

d) Find the gradient at the point on $f(x)$ with coordinates $\left(\frac{4\pi}{3}, \frac{5}{2}\right)$.

e) State the gradient at the point on $f^{-1}(x)$ with coordinates $\left(\frac{5}{2}, \frac{4\pi}{3}\right)$.

$\boxed{}$, $\boxed{2 \le f(x) \le 4}$, $\boxed{a = 3,\ b = \frac{1}{2}}$, $\boxed{f^{-1}(x) = 2\arccos(x-3)}$, $\boxed{2 \le x \le 4}$, $\boxed{0 \le f^{-1}(x) \le 2\pi}$, $\boxed{-\frac{\sqrt{3}}{4}}$, $\boxed{-\frac{4}{\sqrt{3}}}$

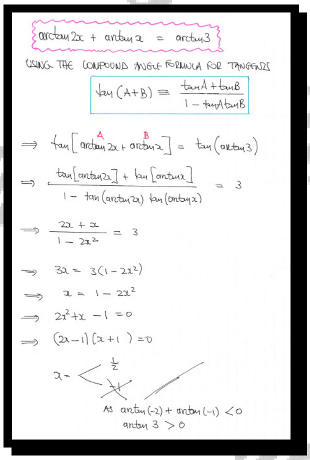


Question 189 (****)

Solve the following trigonometric equation.

$$\arctan 2x + \arctan x = \arctan 3 , \quad x \in \mathbb{R} .$$

$\boxed{}$, $\boxed{x = \frac{1}{2}}$



Question 190 (****)

The function f is defined below.

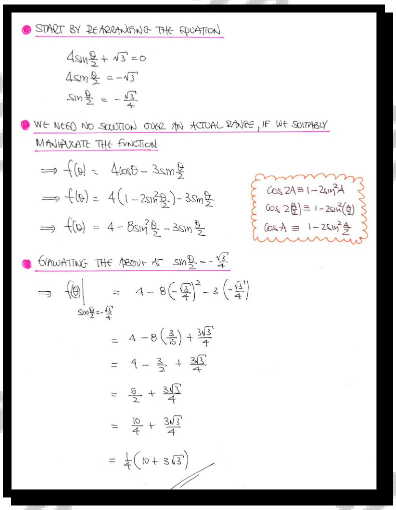
$$f(x) \equiv 4\cos x - 3\sin\left(\tfrac{1}{2}x\right), \quad x \in \mathbb{R}.$$

Show that if θ satisfies the equation

$$4\sin\left(\tfrac{1}{2}\theta\right) + \sqrt{3} = 0,$$

then $f(\theta) = \frac{1}{4}\left(a + b\sqrt{3}\right)$, where a and b are integers to be found.

$\boxed{\frac{1}{4}\left(10+3\sqrt{3}\right)}$



START BY REARRANGING THE EQUATION

$$4\sin\frac{\theta}{2} + \sqrt{3} = 0$$
$$4\sin\frac{\theta}{2} = -\sqrt{3}$$
$$\sin\frac{\theta}{2} = -\frac{\sqrt{3}}{4}$$

WE NEED NO SOLUTION OVER AN ACTUAL RANGE, IF WE SUITABLY MANIPULATE THE FUNCTION

$$\Rightarrow f(\theta) = 4\cos\theta - 3\sin\frac{\theta}{2}$$
$$\Rightarrow f(\theta) = 4\left(1 - 2\sin^2\frac{\theta}{2}\right) - 3\sin\frac{\theta}{2}$$
$$\Rightarrow f(\theta) = 4 - 8\sin^2\frac{\theta}{2} - 3\sin\frac{\theta}{2}$$

$\cos 2A \equiv 1 - 2\sin^2 A$
$\cos 2\left(\frac{A}{2}\right) \equiv 1 - 2\sin^2\left(\frac{A}{2}\right)$
$\cos A \equiv 1 - 2\sin^2\frac{A}{2}$

EVALUATING THE ABOVE AT $\sin\frac{\theta}{2} = -\frac{\sqrt{3}}{4}$

$$\Rightarrow f(\theta)\Big|_{\sin\frac{\theta}{2} = -\frac{\sqrt{3}}{4}} = 4 - 8\left(-\frac{\sqrt{3}}{4}\right)^2 - 3\left(-\frac{\sqrt{3}}{4}\right)$$
$$= 4 - 8\left(\frac{3}{16}\right) + \frac{3\sqrt{3}}{4}$$
$$= 4 - \frac{3}{2} + \frac{3\sqrt{3}}{4}$$
$$= \frac{5}{2} + \frac{3\sqrt{3}}{4}$$
$$= \frac{10}{4} + \frac{3\sqrt{3}}{4}$$
$$= \frac{1}{4}\left(10 + 3\sqrt{3}\right)$$

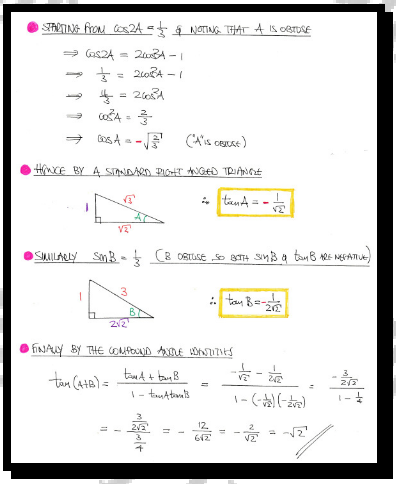
Question 191 (****)

The obtuse angles A and A, satisfy the following relationships.

$$\cos 2A = \sin B = \frac{1}{3}.$$

Determine the exact value of $\tan(A+B)$.

, $\tan(A+B) = -\sqrt{2}$

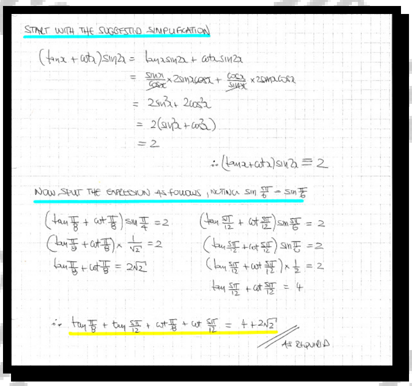


Question 192 (****)

Simplify $(\tan x + \cot x)\sin 2x$ and hence prove that

$$\tan\left(\frac{1}{8}\pi\right) + \tan\left(\frac{5}{12}\pi\right) + \cot\left(\frac{1}{8}\pi\right) + \cot\left(\frac{5}{12}\pi\right) = 4 + 2\sqrt{2}.$$

SYN-, proof



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HARD QUESTIONS

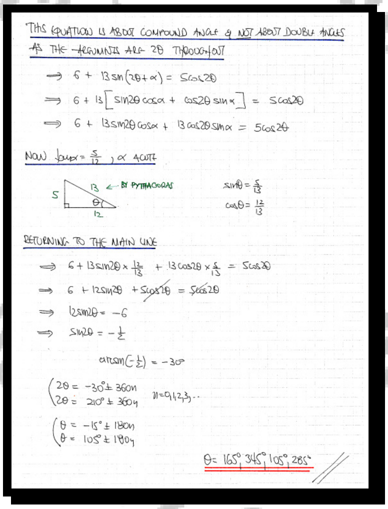
Question 1 (****+)

Find the solutions of the trigonometric equation

$$6+13\sin(2\theta+\alpha)^\circ = 5\cos 2\theta^\circ,\ 0 \le \theta < 360$$

where $\tan\alpha^\circ = \frac{5}{12}$, $0<\alpha<90$.

, $\boxed{\theta = 105^\circ, 165^\circ, 285^\circ, 345^\circ}$



THIS EQUATION IS ABOUT COMPOUND ANGLE & NOT ABOUT DOUBLE ANGLES AS THE ARGUMENTS ARE 2θ THROUGHOUT

$\Rightarrow 6 + 13\sin(2\theta+\alpha) = 5\cos 2\theta$

$\Rightarrow 6 + 13[\sin 2\theta\cos\alpha + \cos 2\theta \sin\alpha] = 5\cos 2\theta$

$\Rightarrow 6 + 13\sin 2\theta\cos\alpha + 13\cos 2\theta\sin\alpha = 5\cos 2\theta$

NOW $\tan\alpha = \frac{5}{12}$, α ACUTE

$\sin\theta = \frac{5}{13}$

$\cos\theta = \frac{12}{13}$

RETURNING TO THE MAIN LINE

$\Rightarrow 6 + 13\sin 2\theta \times \frac{12}{13} + 13\cos 2\theta \times \frac{5}{13} = 5\cos 2\theta$

$\Rightarrow 6 + 12\sin 2\theta + 5\cos 2\theta = 5\cos 2\theta$

$\Rightarrow 12\sin 2\theta = -6$

$\Rightarrow \sin 2\theta = -\frac{1}{2}$

$\arcsin(-\frac{1}{2}) = -30^\circ$

$2\theta = -30^\circ \pm 360n$, $2\theta = 210^\circ \pm 360n$, $n = 0,1,2,3,\ldots$

$\theta = -15^\circ \pm 180n$, $\theta = 105^\circ \pm 180n$

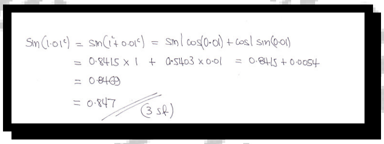
$\theta = 165^\circ, 345^\circ, 105^\circ, 285^\circ$

Question 2 (****+)

It is given that $\sin 1^c \approx 0.8415$ and $\cos 1^c \approx 0.5403$.

Show that $\sin(1.01^c) = 0.847$, correct to three decimal places.

, proof



$\sin(1.01^c) = \sin(1^c + 0.01^c) = \sin 1\cos(0.01) + \cos 1 \sin(0.01)$

$= 0.8415 \times 1 + 0.5403 \times 0.01 = 0.8415 + 0.0054$

$= 0.8469$

$= 0.847$ (3 sf)

Question 3 (****+)

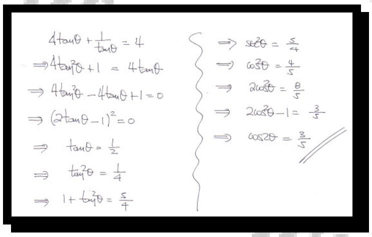
It is given that θ satisfies the equation

$$4\tan\theta + \cot\theta = 4.$$

Show clearly that

$$\cos 2\theta = \frac{3}{5}.$$

proof

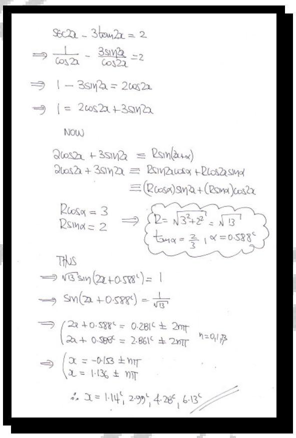


Question 4 (****+)

Find in radians, correct to two decimal places, the solutions of the trigonometric equation

$$\sec 2x - 3\tan 2x = 2, \quad 0 \le x < 2\pi.$$

$x \approx 1.14^{c},\ 2.99^{c},\ 4.28^{c},\ 6.13^{c}$



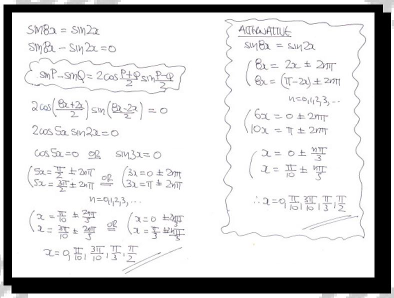
Question 5 (****+)

Solve in radians the trigonometric equation

$$\sin 8x = \sin 2x, \ 0 \le x < \frac{\pi}{2},$$

giving the answers in terms of π.

$$\boxed{x = 0, \ \frac{\pi}{10}, \ \frac{3\pi}{10}, \ \frac{\pi}{3}, \ \frac{\pi}{2}}$$

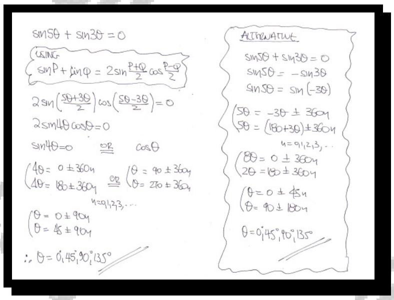


Question 6 (****+)

Solve in degrees the trigonometric equation

$$\sin 5\theta + \sin 3\theta = 0, \ 0° \le \theta < 180°.$$

$$\boxed{\theta = 0°, \ 45°, \ 90°, \ 135°}$$



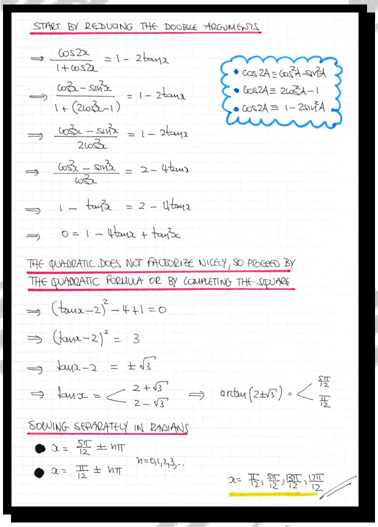
Question 7 (****+)

Solve the following trigonometric equation

$$\frac{\cos 2x}{1+\cos 2x} = 1 - 2\tan x, \quad 0 \le x < 2\pi,$$

giving the answers in terms of π.

$\boxed{}$, $\boxed{x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}}$

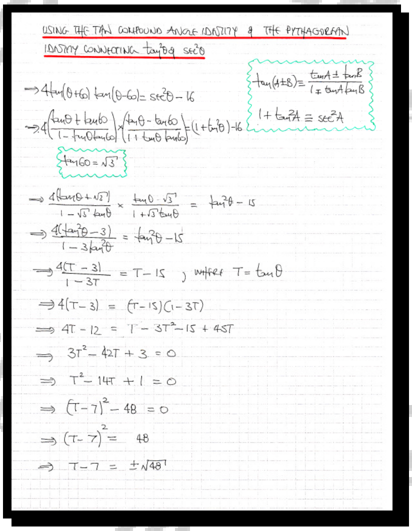


Question 8 (****+)

Solve in **degrees** the trigonometric equation

$$4\tan(\theta+60)\tan(\theta-60)=\sec^2\theta-16,\ 0°\le\theta<180°.$$

$\boxed{}$, $\boxed{\theta=15°,\ 75°,\ 105°,\ 165°}$


USING THE TAN COMPOUND ANGLE IDENTITY & THE PYTHAGOREAN IDENTITY CONNECTING $\tan^2\theta$ & $\sec^2\theta$

$\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$, $\quad 1+\tan^2 A\equiv\sec^2 A$

$\Rightarrow 4\tan(\theta+60)\tan(\theta-60)=\sec^2\theta-16$

$\Rightarrow 4\left(\dfrac{\tan\theta+\tan60}{1-\tan\theta\tan60}\right)\left(\dfrac{\tan\theta-\tan60}{1+\tan\theta\tan60}\right)=(1+\tan^2\theta)-16$

$\tan60=\sqrt{3}$

$\Rightarrow \dfrac{4(\tan\theta+\sqrt{3})}{1-\sqrt{3}\tan\theta}\times\dfrac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta}=\tan^2\theta-15$

$\Rightarrow \dfrac{4(\tan^2\theta-3)}{1-3\tan^2\theta}=\tan^2\theta-15$

$\Rightarrow \dfrac{4(T-3)}{1-3T}=T-15$, where $T=\tan^2\theta$

$\Rightarrow 4(T-3)=(T-15)(1-3T)$

$\Rightarrow 4T-12=T-3T^2-15+45T$

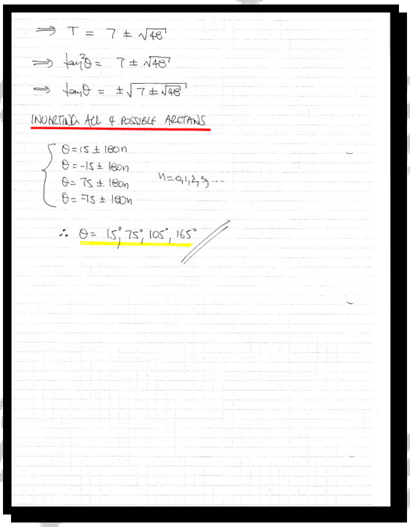
$\Rightarrow 3T^2-42T+3=0$

$\Rightarrow T^2-14T+1=0$

$\Rightarrow (T-7)^2-48=0$

$\Rightarrow (T-7)^2=48$

$\Rightarrow T-7=\pm\sqrt{48}$


$\Rightarrow T=7\pm\sqrt{48}$

$\Rightarrow \tan^2\theta=7\pm\sqrt{48}$

$\Rightarrow \tan\theta=\pm\sqrt{7\pm\sqrt{48}}$

INVERTING ALL 4 POSSIBLE ARCTANS

$\theta=15\pm180n$

$\theta=-15\pm180n$

$\theta=75\pm180n$

$\theta=-75\pm180n$

$n=0,1,2,3\ldots$

$\therefore\ \theta=15°,75°,105°,165°$

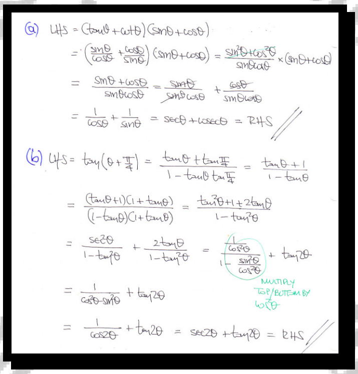
Question 9 (****+)

Prove the validity of each of the following trigonometric identities.

a) $(\tan\theta + \cot\theta)(\sin\theta + \cot\theta) \equiv \sec\theta + \operatorname{cosec}\theta$.

b) $\tan\left(\theta + \dfrac{\pi}{4}\right) \equiv \sec 2\theta + \tan 2\theta$.

proof



(a) $\text{LHS} = (\tan\theta + \cot\theta)(\sin\theta + \cos\theta)$

$= \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)(\sin\theta + \cos\theta) = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \times (\sin\theta + \cos\theta)$

$= \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} = \frac{\sin\theta}{\sin\theta\cos\theta} + \frac{\cos\theta}{\sin\theta\cos\theta}$

$= \frac{1}{\cos\theta} + \frac{1}{\sin\theta} = \sec\theta + \operatorname{cosec}\theta = \text{RHS}$ //

(b) $\text{LHS} = \tan\left(\theta + \frac{\pi}{4}\right) = \frac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\theta\tan\frac{\pi}{4}} = \frac{\tan\theta + 1}{1 - \tan\theta}$

$= \frac{(\tan\theta + 1)(1 + \tan\theta)}{(1 - \tan\theta)(1 + \tan\theta)} = \frac{\tan^2\theta + 1 + 2\tan\theta}{1 - \tan^2\theta}$

$= \frac{\sec^2\theta}{1 - \tan^2\theta} + \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{\frac{1}{\cos^2\theta}}{1 - \frac{\sin^2\theta}{\cos^2\theta}} + \tan 2\theta$

MULTIPLY TOP/BOTTOM BY $\cos^2\theta$

$= \frac{1}{\cos^2\theta - \sin^2\theta} + \tan 2\theta$

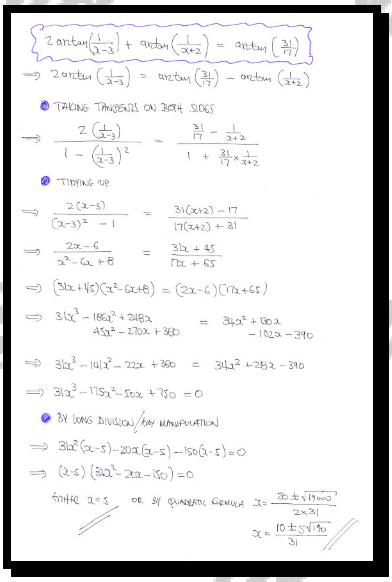
$= \frac{1}{\cos 2\theta} + \tan 2\theta = \sec 2\theta + \tan 2\theta = \text{RHS}$ //

Question 10 (****+)

$$2\arctan\left[\frac{1}{x-3}\right]+\arctan\left[\frac{1}{x+2}\right]=\arctan\left[\frac{31}{17}\right].$$

Show that $x=5$ is one of the solutions of the above trigonometric equation and find, in exact surd form, the other two solutions.

$$\boxed{x=\frac{10\pm 5\sqrt{190}}{31}}$$



$$2\arctan\left(\frac{1}{x-3}\right)+\arctan\left(\frac{1}{x+2}\right)=\arctan\left(\frac{31}{17}\right)$$

$\Rightarrow 2\arctan\left(\frac{1}{x-3}\right)=\arctan\left(\frac{31}{17}\right)-\arctan\left(\frac{1}{x+2}\right)$

- TAKING TANGENTS ON BOTH SIDES

$\Rightarrow \dfrac{2\left(\frac{1}{x-3}\right)}{1-\left(\frac{1}{x-3}\right)^2}=\dfrac{\frac{31}{17}-\frac{1}{x+2}}{1+\frac{31}{17}\times\frac{1}{x+2}}$

- TIDYING UP

$\Rightarrow \dfrac{2(x-3)}{(x-3)^2-1}=\dfrac{31(x+2)-17}{17(x+2)+31}$

$\Rightarrow \dfrac{2x-6}{x^2-6x+8}=\dfrac{31x+45}{17x+65}$

$\Rightarrow (31x+45)(x^2-6x+8)=(2x-6)(17x+65)$

$\Rightarrow 31x^3-186x^2+248x+45x^2-270x+360=34x^2+130x-102x-390$

$\Rightarrow 31x^3-141x^2-22x+360=34x^2+28x-390$

$\Rightarrow 31x^3-175x^2-50x+750=0$

- BY LONG DIVISION/ANY MANIPULATION

$\Rightarrow 31x^2(x-5)-20x(x-5)-150(x-5)=0$

$\Rightarrow (x-5)(31x^2-20x-150)=0$

EITHER $x=5$ OR BY QUADRATIC FORMULA $x=\dfrac{20\pm\sqrt{19000}}{2\times 31}$

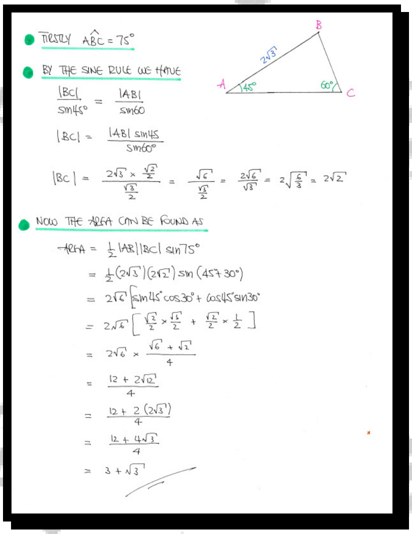
$x=\dfrac{10\pm5\sqrt{190}}{31}$

Question 11 (**+) Non Calculator**

A triangle, ABC has $|AB| = 2\sqrt{3}$ cm, $\measuredangle BAC = 45°$ and $\measuredangle ACB = 60°$.

Determine, in exact simplified surd form, the area of this triangle

area $= 3 + \sqrt{3}$



Question 12 (****+)

It is given that

$$\sin^2\left(\theta+\frac{\pi}{4}\right)-\sin^2\left(\theta-\frac{\pi}{4}\right)\equiv \sin 2\theta .$$

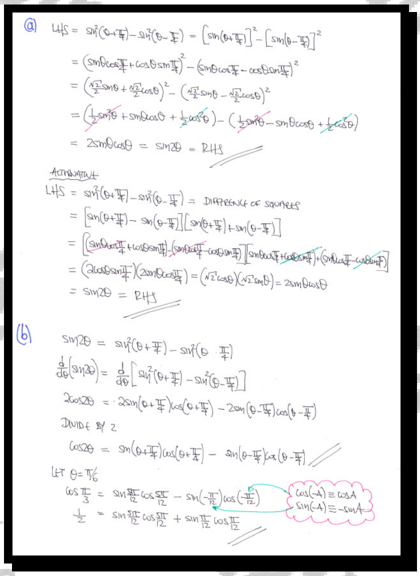
a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, show that …

i. … $\sin\left(\theta+\frac{\pi}{4}\right)\cos\left(\theta+\frac{\pi}{4}\right)-\sin\left(\theta-\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right)\equiv \cos 2\theta .$

ii. … $\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{5\pi}{12}\right)+\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)=\frac{1}{2} .$

proof



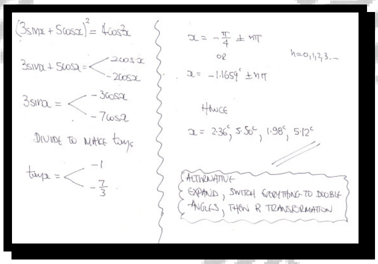
Question 13 (****+)

Solve the trigonometric equation

$$(3\sin x + 5\cos x)^2 = 4\cos^2 x, \text{ for } 0 \le x < 2\pi,$$

giving the answers correct to three significant figures.

$$\boxed{x \approx 1.98^c,\ 2.36^c,\ 5.12^c,\ 5.50^c}$$



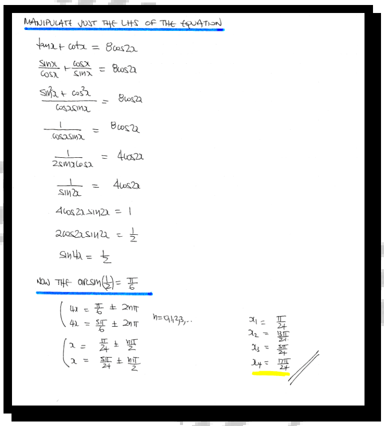
Question 14 (****+)

Solve the following trigonometric equation

$$\tan x + \cot x = 8\cos 2x, \quad 0 \le x < \pi,$$

where x is measured in radians.

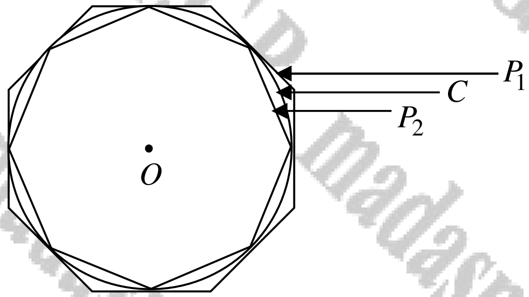
$$\boxed{x = \frac{1}{24}\pi,\ \frac{5}{24}\pi,\ \frac{13}{24}\pi,\ \frac{17}{24}\pi}$$



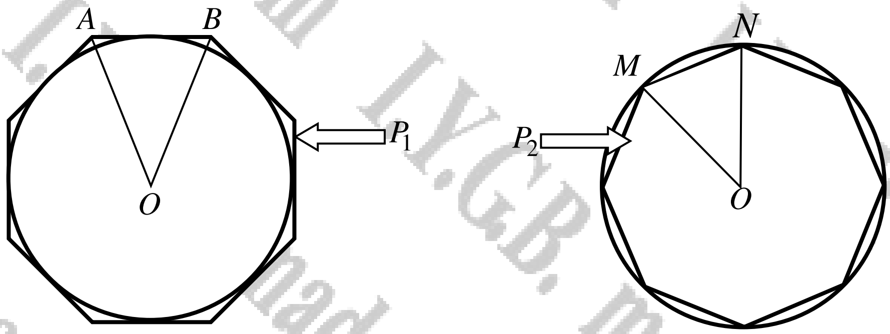
Question 15 (****+)

The figure below shows a regular octagon P_1. A circle C is inscribed inside P_1 and another regular octagon P_2 is inscribed inside the circle C.

The three objects have a common centre at O.



The circle C has a radius of 1 unit. The points A and B are consecutive vertices of P_1, and the points M and N are consecutive vertices of P_2.



a) By considering the triangle OAB, show that the perimeter of the octagon P_1 is $16\tan\frac{\pi}{8}$.

b) Use the triangle OMN in a similar fashion to show that the perimeter of the octagon P_2 is $16\sin\frac{\pi}{8}$.

[continues overleaf]

[continues from previous page]

c) Use a standard identity for $\cos 2\theta$ to show that

$$\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}.$$

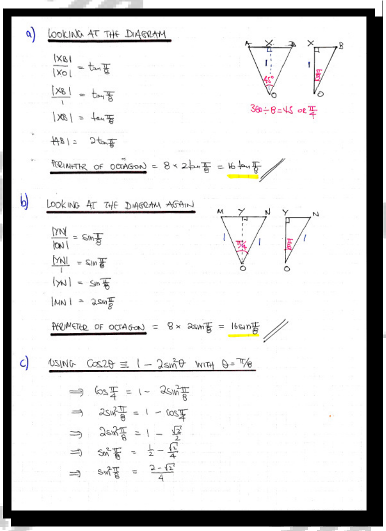
d) Show further that

$$\tan\frac{\pi}{8} = -1+\sqrt{2}.$$

e) Deduce from the results obtained so far that

$$3.06 < \pi < 3.31.$$

[ ] , [proof]



a) LOOKING AT THE DIAGRAM

$\frac{|XB|}{|XO|} = \tan\frac{\pi}{8}$

$\frac{|XB|}{1} = \tan\frac{\pi}{8}$

$|XB| = \tan\frac{\pi}{8}$

$|AB| = 2\tan\frac{\pi}{8}$

PERIMETER OF OCTAGON $= 8 \times 2\tan\frac{\pi}{8} = 16\tan\frac{\pi}{8}$

$360 \div 8 = 45$ or $\frac{\pi}{4}$

b) LOOKING AT THE DIAGRAM AGAIN

$\frac{|YN|}{|ON|} = \sin\frac{\pi}{8}$

$\frac{|YN|}{1} = \sin\frac{\pi}{8}$

$|YN| = \sin\frac{\pi}{8}$

$|MN| = 2\sin\frac{\pi}{8}$

PERIMETER OF OCTAGON $= 8 \times 2\sin\frac{\pi}{8} = 16\sin\frac{\pi}{8}$

c) USING $\cos 2\theta \equiv 1 - 2\sin^2\theta$ WITH $\theta = \pi/8$

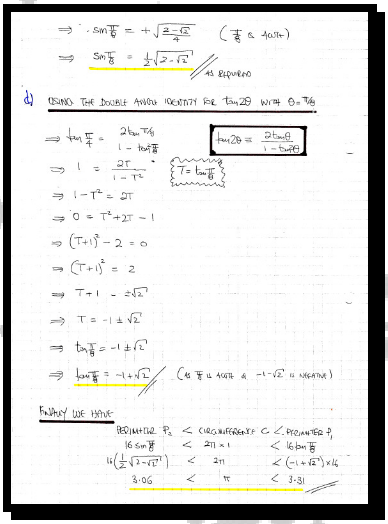
$\Rightarrow \cos\frac{\pi}{4} = 1 - 2\sin^2\frac{\pi}{8}$

$\Rightarrow 2\sin^2\frac{\pi}{8} = 1 - \cos\frac{\pi}{4}$

$\Rightarrow 2\sin^2\frac{\pi}{8} = 1 - \frac{\sqrt{2}}{2}$

$\Rightarrow \sin^2\frac{\pi}{8} = \frac{1}{2} - \frac{\sqrt{2}}{4}$

$\Rightarrow \sin^2\frac{\pi}{8} = \frac{2-\sqrt{2}}{4}$



$\Rightarrow \sin\frac{\pi}{8} = +\sqrt{\frac{2-\sqrt{2}}{4}}$ ($\frac{\pi}{8}$ IS ACUTE)

$\Rightarrow \sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$ AS REQUIRED

d) USING THE DOUBLE ANGLE IDENTITY FOR $\tan 2\theta$ WITH $\theta = \pi/8$

$\tan 2\theta \equiv \frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \tan\frac{\pi}{4} = \frac{2\tan\pi/8}{1-\tan^2\frac{\pi}{8}}$

$\Rightarrow 1 = \frac{2T}{1-T^2}$ $\quad T = \tan\frac{\pi}{8}$

$\Rightarrow 1 - T^2 = 2T$

$\Rightarrow 0 = T^2 + 2T - 1$

$\Rightarrow (T+1)^2 - 2 = 0$

$\Rightarrow (T+1)^2 = 2$

$\Rightarrow T + 1 = \pm\sqrt{2}$

$\Rightarrow T = -1 \pm \sqrt{2}$

$\Rightarrow \tan\frac{\pi}{8} = -1 \pm \sqrt{2}$

$\Rightarrow \tan\frac{\pi}{8} = -1 + \sqrt{2}$ (AS $\frac{\pi}{8}$ IS ACUTE & $-1-\sqrt{2}$ IS NEGATIVE)

FINALLY WE HAVE

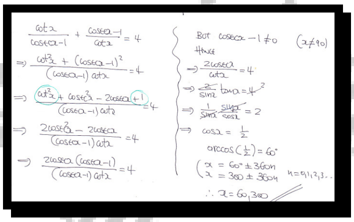
PERIMETER P_2	$<$ CIRCUMFERENCE C	$<$ PERIMETER P_1
$16\sin\frac{\pi}{8}$	$< 2\pi \times 1$	$< 16\tan\frac{\pi}{8}$
$16\left(\frac{1}{2}\sqrt{2-\sqrt{2}}\right)$	$< 2\pi$	$< (-1+\sqrt{2}) \times 16$
3.06	$< \pi$	< 3.31

Question 16 (**+)**

Solve the following trigonometric equation

$$\frac{\cot x}{\operatorname{cosec} x - 1} + \frac{\operatorname{cosec} x - 1}{\cot x} = 4, \quad 0° \le x < 360°.$$

$x = 60°,\ 300°$



$\frac{\cot x}{\operatorname{cosec} x - 1} + \frac{\operatorname{cosec} x - 1}{\cot x} = 4$

$\Rightarrow \frac{\cot^2 x + (\operatorname{cosec} x - 1)^2}{(\operatorname{cosec} x - 1)\cot x} = 4$

$\Rightarrow \frac{\cot^2 x + \operatorname{cosec}^2 x - 2\operatorname{cosec} x + 1}{(\operatorname{cosec} x - 1)\cot x} = 4$

$\Rightarrow \frac{2\operatorname{cosec}^2 x - 2\operatorname{cosec} x}{(\operatorname{cosec} x - 1)\cot x} = 4$

$\Rightarrow \frac{2\operatorname{cosec} x(\operatorname{cosec} x - 1)}{(\operatorname{cosec} x - 1)\cot x} = 4$

But $\operatorname{cosec} x - 1 \neq 0$ $(x \neq 90)$ thus

$\Rightarrow \frac{2\operatorname{cosec} x}{\cot x} = 4$

$\Rightarrow \frac{2}{\sin x}\tan x = 4$

$\Rightarrow \frac{1}{\sin x}\frac{\sin x}{\cos x} = 2$

$\Rightarrow \cos x = \frac{1}{2}$

$\arccos\left(\frac{1}{2}\right) = 60°$

$x = 60° \pm 360n$, $x = 300 \pm 360n$, $n = 0,1,2,3,\ldots$

$\therefore x = 60, 300$

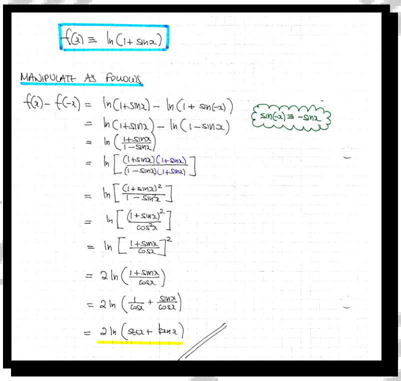
Question 17 (**+)**

$$f(x) = \ln(1 + \sin x), \ \sin x \neq \pm 1.$$

Show clearly that

$$f(x) - f(-x) = 2\ln(\sec x + \tan x).$$

proof



$f(x) = \ln(1+\sin x)$

MANIPULATE AS FOLLOWS

$f(x) - f(-x) = \ln(1+\sin x) - \ln(1+\sin(-x))$ $\quad \sin(-x) = -\sin x$

$= \ln(1+\sin x) - \ln(1-\sin x)$

$= \ln\left(\frac{1+\sin x}{1-\sin x}\right)$

$= \ln\left[\frac{(1+\sin x)(1+\sin x)}{(1-\sin x)(1+\sin x)}\right]$

$= \ln\left[\frac{(1+\sin x)^2}{1-\sin^2 x}\right]$

$= \ln\left[\frac{(1+\sin x)^2}{\cos^2 x}\right]$

$= \ln\left[\frac{1+\sin x}{\cos x}\right]^2$

$= 2\ln\left(\frac{1+\sin x}{\cos x}\right)$

$= 2\ln\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)$

$= 2\ln(\sec x + \tan x)$

Question 18 (****+)

It is given that

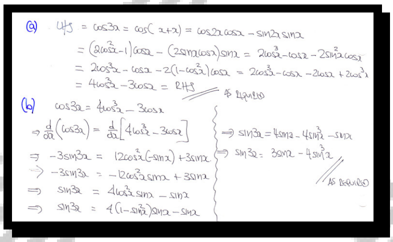
$$\cos 3x \equiv 4\cos^3 x - 3\cos x .$$

a) Prove the validity of the above trigonometric identity.

b) By differentiating both sides of the above identity with respect to x, show that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x .$$

proof



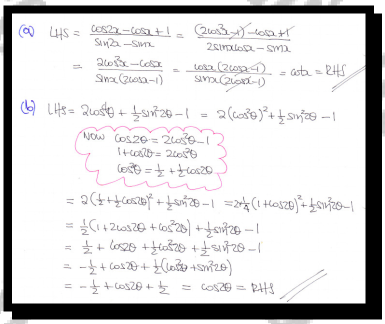
Question 19 (****+)

Prove the validity of each of the following trigonometric identities.

a) $\dfrac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} \equiv \cot x$.

b) $2\cos^4 \theta + \frac{1}{2}\sin^2 2\theta - 1 \equiv \cos 2\theta$.

proof



Question 20 (****+)

It is given that

$$\frac{\tan x}{\sec x - 1} - \frac{\sec x - 1}{\tan x} \equiv 2\cot x\,, \quad \theta \neq \frac{n\pi}{2}\,,\ n \in \mathbb{Z}\,.$$

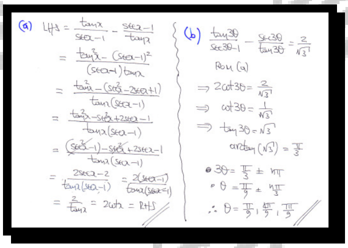
a) Prove the validity of the above trigonometric identity.

b) Hence solve the trigonometric equation

$$\frac{\tan 3\theta}{\sec 3\theta - 1} - \frac{\sec 3\theta - 1}{\tan 3\theta} = \frac{2}{\sqrt{3}}\,, \quad 0 \leq \theta < \pi\,,$$

giving the answers in terms of π.

$x = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}$

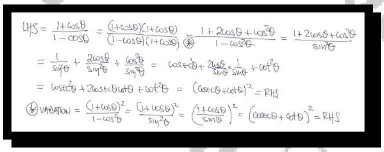


Question 21 (****+)

Prove the validity of the following trigonometric identity.

$$\frac{1 + \cos\theta}{1 - \cos\theta} \equiv \left(\operatorname{cosec}\theta + \cot\theta\right)^2.$$

proof



Question 22 (****+)

It is given that

$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta .$$

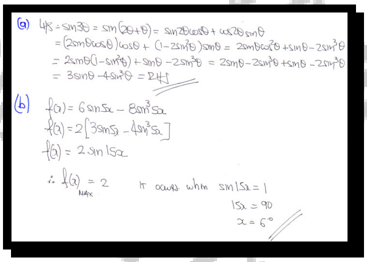
a) Prove the validity of the above trigonometric identity.

b) State the maximum value of

$$f(x) = 6\sin(5x) - 8\sin^3(5x), \; x \in \mathbb{R},$$

and determine, in degrees, the smallest positive value of x which produces this maximum value.

$\boxed{f_{\max}(x) = 2}$, $\boxed{x = 6°}$


(a) $LHS = \sin 3\theta = \sin(2\theta+\theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$
$= (2\sin\theta\cos\theta)\cos\theta + (1-2\sin^2\theta)\sin\theta = 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$
$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta = 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$
$= 3\sin\theta - 4\sin^3\theta = RHS$

(b) $f(x) = 6\sin 5x - 8\sin^3 5x$
$f(x) = 2[3\sin 5x - 4\sin^3 5x]$
$f(x) = 2\sin 15x$

$\therefore f(x)_{MAX} = 2$ It occurs when $\sin 15x = 1$
$15x = 90$
$x = 6°$

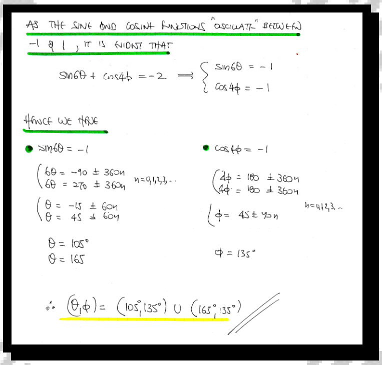
Question 23 (****+)

The obtuse angles θ and φ satisfy the equation

$$\sin 6\theta° + \cos 4\varphi° = -2 .$$

Find the possible values of θ and φ.

$(\theta,\varphi) = (105°,135°) \cup (165°,135°)$


AS THE SINE AND COSINE FUNCTIONS "OSCILLATE" BETWEEN −1 & 1, IT IS EVIDENT THAT

$\sin 6\theta + \cos 4\phi = -2 \implies \begin{cases} \sin 6\theta = -1 \\ \cos 4\phi = -1 \end{cases}$

HENCE WE HAVE

- $\sin 6\theta = -1$

$6\theta = -90 \pm 360n$, $6\theta = 270 \pm 360n$, $n = 0,1,2,3,\ldots$

$\theta = -15 \pm 60n$, $\theta = 45 \pm 60n$

$\theta = 105°$, $\theta = 165$

- $\cos 4\phi = -1$

$4\phi = 180 \pm 360n$, $4\phi = 180 \pm 360n$, $n = 0,1,2,3,\ldots$

$\phi = 45 \pm 90n$

$\phi = 135°$

$\therefore (\theta,\phi) = (105°,135°) \cup (165°,135°)$

Question 24 (****+)

It is given that

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

a) Prove the validity of the above trigonometric identity by using the compound angle formulae for $\sin(A+B)$ and $\cos(A+B)$.

a) Deduce an exact simplified expression for $\tan\left(\theta - \frac{\pi}{3}\right)$, in terms of $\tan\theta$.

b) Solve, for $0 \le \theta < 2\pi$, the trigonometric equation

$$\tan\theta - \sqrt{3} = \left(1 + \sqrt{3}\tan\theta\right)\tan(2\pi - \theta),$$

giving the answers in terms of π.

$\boxed{\tan\left(\theta - \frac{\pi}{3}\right) = \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta}}$, $\boxed{\theta = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}}$

(a) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$ (Divide top & bottom by $\cos A\cos B$)

$= \frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

(b) $\tan\left(\theta - \frac{\pi}{3}\right) = \frac{\tan\theta - \tan\frac{\pi}{3}}{1 + \tan\theta\tan\frac{\pi}{3}} = \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta}$

(c) $\tan\theta - \sqrt{3} = (1 + \sqrt{3}\tan\theta)\tan(2\pi - \theta)$

$\Rightarrow \tan\theta - \sqrt{3} = (1 + \sqrt{3}\tan\theta) \times \frac{\tan 2\pi - \tan\theta}{1 + \tan 2\pi\tan\theta}$

$\Rightarrow \tan\theta - \sqrt{3} = (1 + \sqrt{3}\tan\theta) \times (-\tan\theta)$

$\Rightarrow \tan\theta - \sqrt{3} = -\tan\theta - \sqrt{3}\tan^2\theta$

$\Rightarrow \sqrt{3}\tan^2\theta + 2\tan\theta - \sqrt{3} = 0$

$\Rightarrow \tan^2\theta + \frac{2}{\sqrt{3}}\tan\theta - 1 = 0$

$\Rightarrow \left(\tan\theta + \frac{1}{\sqrt{3}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2 - 1 = 0$

$\Rightarrow \left(\tan\theta + \frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3}$

$\Rightarrow \tan\theta + \frac{1}{\sqrt{3}} = \pm\frac{2}{\sqrt{3}}$

$\Rightarrow \tan\theta = -\frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{3}}$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ or $-\frac{3}{\sqrt{3}} = -\sqrt{3}$

• $\arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$ • $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$

• $\theta = \frac{\pi}{6} \pm n\pi$ • $\theta = -\frac{\pi}{3} \pm n\pi$

$n = 0, 1, 2, 3 \ldots$

$\theta = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{3}$

ALTERNATIVE

$\tan\theta - \sqrt{3} = (1 + \sqrt{3}\tan\theta)\tan(2\pi - \theta)$

$\frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta} = \tan(2\pi - \theta)$

$\tan\left(\theta - \frac{\pi}{3}\right) = \tan(2\pi - \theta)$

$\left(\theta - \frac{\pi}{3}\right) = (2\pi - \theta) \pm n\pi$

$2\theta = \frac{7\pi}{3} \pm n\pi$

$\theta = \frac{7}{6}\pi \pm \frac{n\pi}{2}$

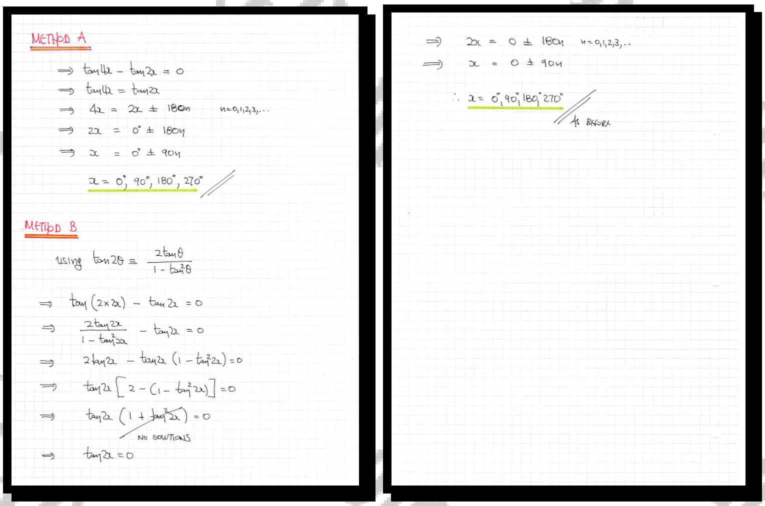
$\theta_1 = \frac{7\pi}{6}$, $\theta_2 = \frac{2\pi}{3}$, $\theta_3 = \frac{\pi}{6}$, $\theta_4 = \frac{5\pi}{3}$ ($-\frac{\pi}{2}$, $-\frac{\pi}{2}$, $+\frac{\pi}{2}$)

Question 25 (****+)

Solve the following trigonometric equation

$$\tan 4x - \tan 2x = 0, \quad 0° \le x < 360°.$$

$\boxed{}$, $\boxed{x = 0°,\ 90°,\ 180°,\ 270°}$



Question 26 (****+)

It is given that

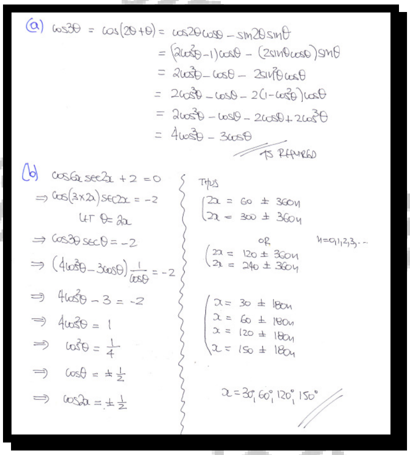
$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta .$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve in degrees the trigonometric equation

$$2 + \cos 6x \sec 2x = 0 , \ 0° \le x < 180° .$$

$\boxed{x = 30°, \ 60°, \ 120°, \ 150°}$



(a) $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$
$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$
$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$
$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$
$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$
$= 4\cos^3\theta - 3\cos\theta$
AS REQUIRED

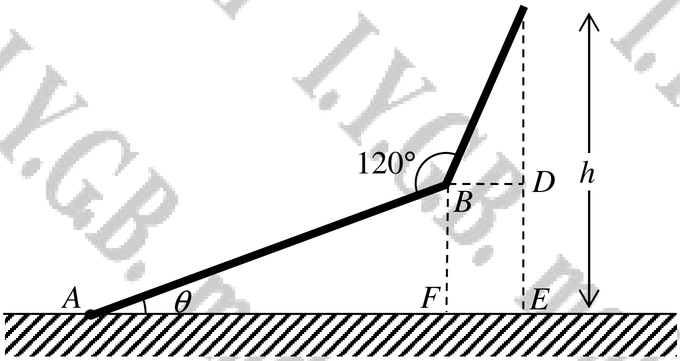
(b) $\cos 6x \sec 2x + 2 = 0$
$\Rightarrow \cos(3 \times 2x)\sec 2x = -2$
LET $\theta = 2x$
$\Rightarrow \cos 3\theta \sec\theta = -2$
$\Rightarrow (4\cos^3\theta - 3\cos\theta)\frac{1}{\cos\theta} = -2$
$\Rightarrow 4\cos^2\theta - 3 = -2$
$\Rightarrow 4\cos^2\theta = 1$
$\Rightarrow \cos^2\theta = \frac{1}{4}$
$\Rightarrow \cos\theta = \pm\frac{1}{2}$
$\Rightarrow \cos 2x = \pm\frac{1}{2}$

THUS
$2x = 60 \pm 360n$
$2x = 300 \pm 360n$
OR $\quad n = 0, 1, 2, 3, \ldots$
$2x = 120 \pm 360n$
$2x = 240 \pm 360n$

$x = 30 \pm 180n$
$x = 60 \pm 180n$
$x = 120 \pm 180n$
$x = 150 \pm 180n$

$x = 30°, 60°, 120°, 150°$

Question 27 (****+)



The figure above shows a rigid rod ABC where AB is 6 metres, BC is 4 metres and the angle ABC is $120°$. The rod is hinged at A so it can be rotated in a vertical plane forming an angle $\theta°$ with the horizontal ground.

Let h metres be the height of the point C from the horizontal ground.

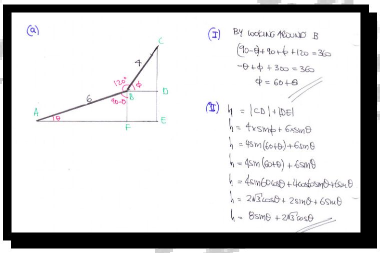
a) Show clearly that …

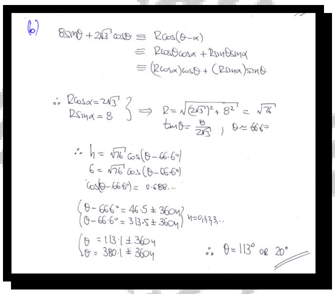
i. … $\measuredangle DBC = \theta° + 60°$.

ii. … $h = 8\sin\theta + 2\sqrt{3}\cos\theta$.

b) By expressing h in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$, find to the nearest degree, the values of θ when $h = 6$.

$\boxed{\theta \approx 20°,\ 113°}$



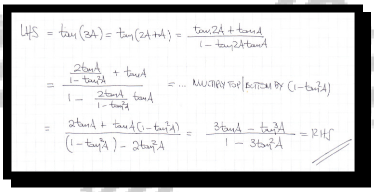


Question 28 (****+)

By considering the expansion of $\tan(2A+A)$, show clearly that

$$\tan 3A \equiv \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}.$$

proof

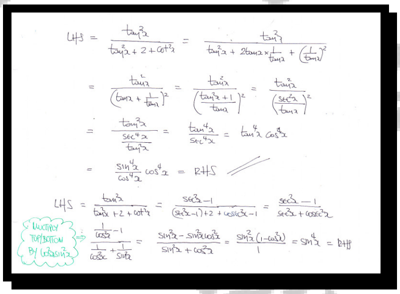


Question 29 (****+)

Show clearly that

$$\frac{\tan^2 x}{\tan^2 x + 2 + \cot^2 x} \equiv \sin^4 x.$$

proof



Question 30 (****+)

It is given that

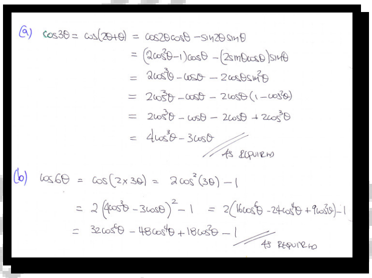
$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta .$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, prove that

$$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 .$$

proof



(a) $\cos 3\theta = \cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$
$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$
$= 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta$
$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta)$
$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$
$= 4\cos^3\theta - 3\cos\theta$ // as required

(b) $\cos 6\theta = \cos(2\times 3\theta) = 2\cos^2(3\theta) - 1$
$= 2(4\cos^3\theta - 3\cos\theta)^2 - 1 = 2(16\cos^6\theta - 24\cos^4\theta + 9\cos^2\theta) - 1$
$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$ // as required

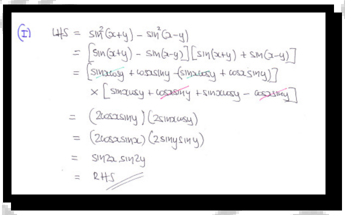
Question 31 (****+)

Prove the validity of each of the following trigonometric identities.

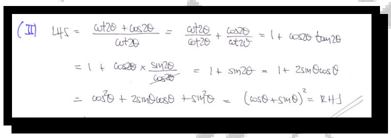
a) $\sin^2(x+y) - \sin^2(x-y) \equiv \sin 2x \sin 2y$.

b) $\dfrac{\cot 2\theta + \cos 2\theta}{\cot 2\theta} \equiv (\cos\theta + \sin\theta)^2$.

proof



(I) LHS $= \sin^2(x+y) - \sin^2(x-y)$
$= [\sin(x+y) - \sin(x-y)][\sin(x+y) + \sin(x-y)]$
$= [\sin x\cos y + \cos x\sin y - (\sin x\cos y - \cos x\sin y)]$
$\times [\sin x\cos y + \cos x\sin y + \sin x\cos y - \cos x\sin y]$
$= (2\cos x\sin y)(2\sin x\cos y)$
$= (2\cos x\sin x)(2\sin y\cos y)$
$= \sin 2x\sin 2y$
$=$ RHS



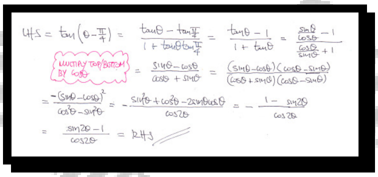
(II) LHS $= \dfrac{\cot 2\theta + \cos 2\theta}{\cot 2\theta} = \dfrac{\cot 2\theta}{\cot 2\theta} + \dfrac{\cos 2\theta}{\cot 2\theta} = 1 + \cos 2\theta\tan 2\theta$
$= 1 + \cos 2\theta \times \dfrac{\sin 2\theta}{\cos 2\theta} = 1 + \sin 2\theta = 1 + 2\sin\theta\cos\theta$
$= \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta = (\cos\theta + \sin\theta)^2 =$ RHS

Question 32 (****+)

Show clearly that

$$\tan\left(\theta-\frac{\pi}{4}\right)\equiv\frac{\sin 2\theta-1}{\cos 2\theta}.$$

proof



Question 33 (****+)

It is given that

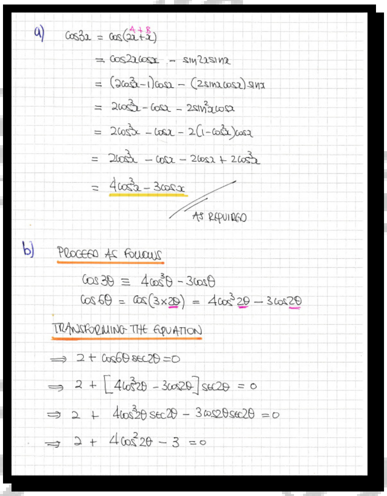
$$\cos 3x\equiv 4\cos^3 x-3\cos x.$$

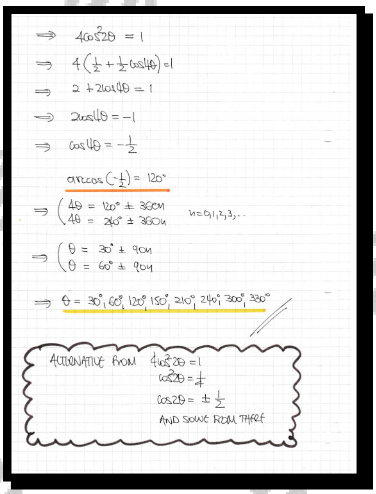
a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise solve the trigonometric equation

$$2+\cos 6\theta\sec 2\theta=0,\ 0°\le\theta<360°.$$

, $\theta=30°,\ 60°,\ 120°,\ 150°,\ 210°,\ 240°,\ 300°,\ 330°$





Question 34 (****+)

It is given that

$$(\cos x+\sin x)(1-\sin x\cos x)\equiv\cos^3 x+\sin^3 x.$$

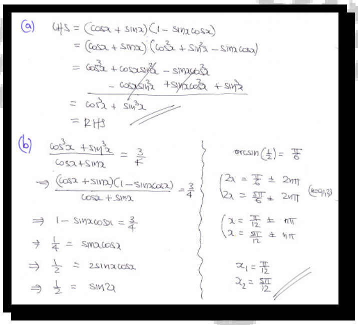
a) Prove the validity of the above trigonometric identity.

b) Hence find, in terms of π, the solutions of the trigonometric equation

$$\frac{\cos^3 x+\sin^3 x}{\cos x+\sin x}=\frac{3}{4},\quad 0\le x<\pi,$$

giving the answers in terms of π.

$$\boxed{x=\frac{\pi}{12},\ \frac{5\pi}{12}}$$



Question 35 (****+)

It is given that

$$\cos(x+36)^\circ = \sin(x-54)^\circ .$$

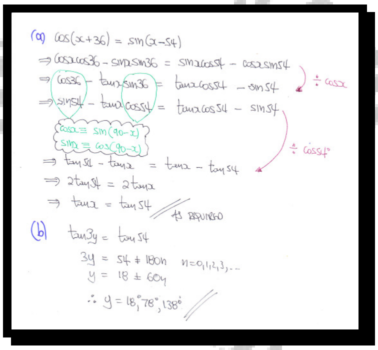
a) Show clearly without a calculating aid that the above trigonometric equation is equivalent to

$$\tan x^\circ = \tan 54^\circ .$$

b) Hence solve the trigonometric equation

$$\cos(3y+36)^\circ = \sin(3y-54)^\circ ,\ 0 \le y < 180 .$$

$\boxed{y = 18,\ \ 78,\ 138}$



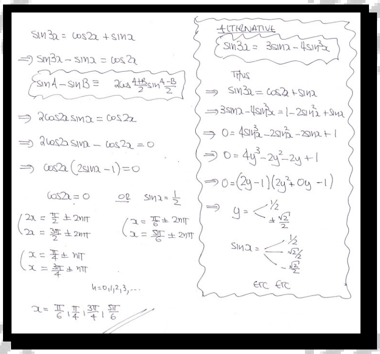
Question 36 (****+)

Solve the trigonometric equation

$$\sin 3x = \cos 2x + \sin x, \quad \text{for } 0 \le \theta < \pi,$$

giving the answers in terms of π.

$$\boxed{x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}}$$



$\sin 3x = \cos 2x + \sin x$

$\Rightarrow \sin 3x - \sin x = \cos 2x$

$\sin A - \sin B \equiv 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\Rightarrow 2\cos 2x \sin x = \cos 2x$

$\Rightarrow 2\cos 2x \sin x - \cos 2x = 0$

$\Rightarrow \cos 2x(2\sin x - 1) = 0$

$\cos 2x = 0$ or $\sin x = \frac{1}{2}$

$2x = \frac{\pi}{2} \pm 2n\pi$, $2x = \frac{3\pi}{2} \pm 2n\pi$; $x = \frac{\pi}{6} \pm 2n\pi$, $x = \frac{5\pi}{6} \pm 2n\pi$

$x = \frac{\pi}{4} \pm n\pi$, $x = \frac{3\pi}{4} \pm n\pi$

$n = 0, 1, 2, 3, \ldots$

$x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}$

ALTERNATIVE

$\sin 3x = 3\sin x - 4\sin^3 x$

Thus

$\Rightarrow \sin 3x = \cos 2x + \sin x$

$\Rightarrow 3\sin x - 4\sin^3 x = 1 - 2\sin^2 x + \sin x$

$\Rightarrow 0 = 4\sin^3 x - 2\sin^2 x - 2\sin x + 1$

$\Rightarrow 0 = 4y^3 - 2y^2 - 2y + 1$

$\Rightarrow 0 = (2y-1)(2y^2 + 0y - 1)$

$\Rightarrow y = \frac{1}{2}, \pm\frac{\sqrt{2}}{2}$

$\sin x = \frac{1}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$

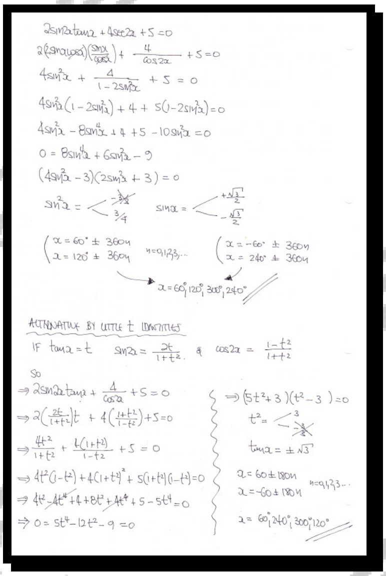
etc etc

Question 37 (****+)

Solve the trigonometric equation

$$2\sin 2x \tan x + 4\sec 2x + 5 = 0, \quad 0° \le x < 360°.$$

$$\boxed{x = 60°, 120°, 240°, 300°}$$



$2\sin 2x \tan x + 4\sec 2x + 5 = 0$

$2(2\sin x\cos x)\left(\frac{\sin x}{\cos x}\right) + \frac{4}{\cos 2x} + 5 = 0$

$4\sin^2 x + \frac{4}{1 - 2\sin^2 x} + 5 = 0$

$4\sin^2 x(1 - 2\sin^2 x) + 4 + 5(1 - 2\sin^2 x) = 0$

$4\sin^2 x - 8\sin^4 x + 4 + 5 - 10\sin^2 x = 0$

$0 = 8\sin^4 x + 6\sin^2 x - 9$

$(4\sin^2 x - 3)(2\sin^2 x + 3) = 0$

$\sin^2 x = \frac{3}{4}$ (~~$-\frac{3}{2}$~~) ; $\sin x = +\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$

$x = 60° \pm 360n$, $x = 120° \pm 360n$; $n = 0, 1, 2, 3, \ldots$; $x = -60° \pm 360n$, $x = 240° \pm 360n$

$x = 60°, 120°, 300°, 240°$

ALTERNATIVE BY LITTLE t IDENTITIES

IF $\tan x = t$, $\sin 2x = \frac{2t}{1+t^2}$ & $\cos 2x = \frac{1-t^2}{1+t^2}$

So

$\Rightarrow 2\sin 2x \tan x + \frac{4}{\cos 2x} + 5 = 0$

$\Rightarrow 2\left(\frac{2t}{1+t^2}\right)t + 4\left(\frac{1+t^2}{1-t^2}\right) + 5 = 0$

$\Rightarrow \frac{4t^2}{1+t^2} + \frac{4(1+t^2)}{1-t^2} + 5 = 0$

$\Rightarrow 4t^2(1-t^2) + 4(1+t^2)^2 + 5(1+t^2)(1-t^2) = 0$

$\Rightarrow 4t^2 - 4t^4 + 4 + 8t^2 + 4t^4 + 5 - 5t^4 = 0$

$\Rightarrow 0 = 5t^4 - 12t^2 - 9 = 0$

$\Rightarrow (5t^2 + 3)(t^2 - 3) = 0$

$t^2 = 3$ (~~$-\frac{3}{5}$~~)

$\tan x = \pm\sqrt{3}$

$x = 60 \pm 180n$, $x = -60 \pm 180n$; $n = 0, 1, 2, 3, \ldots$

$x = 60°, 240°, 300°, 120°$

Question 38 (****+)

It is given that

$$\tan\theta + \tan\varphi = 3,$$

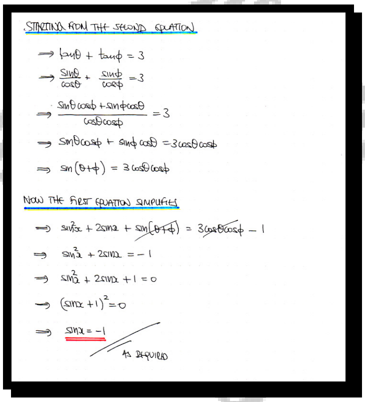
$$\sin^2 x + 2\sin x + \sin(\theta+\varphi) = 3\cos\theta\cos\varphi - 1,$$

for $x \in \mathbb{R}$, $\theta \in \mathbb{R}$, $\varphi \in \mathbb{R}$.

Show that the above relationships imply that

$$\sin x = -1.$$

proof



STARTING FROM THE SECOND EQUATION

$\Rightarrow \tan\theta + \tan\phi = 3$

$\Rightarrow \frac{\sin\theta}{\cos\theta} + \frac{\sin\phi}{\cos\phi} = 3$

$\Rightarrow \frac{\sin\theta\cos\phi + \sin\phi\cos\theta}{\cos\theta\cos\phi} = 3$

$\Rightarrow \sin\theta\cos\phi + \sin\phi\cos\theta = 3\cos\theta\cos\phi$

$\Rightarrow \sin(\theta+\phi) = 3\cos\theta\cos\phi$

NOW THE FIRST EQUATION SIMPLIFIES

$\Rightarrow \sin^2 x + 2\sin x + \sin(\theta+\phi) = 3\cos\theta\cos\phi - 1$

$\Rightarrow \sin^2 x + 2\sin x = -1$

$\Rightarrow \sin^2 x + 2\sin x + 1 = 0$

$\Rightarrow (\sin x + 1)^2 = 0$

$\Rightarrow \sin x = -1$

AS REQUIRED

Question 39 (****+)

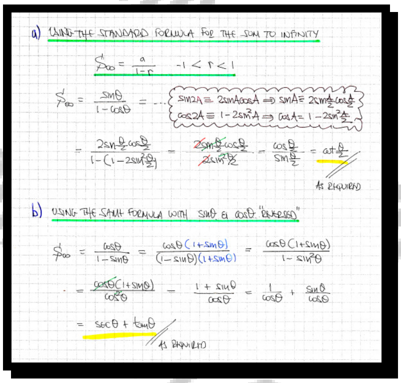
A geometric progression has first term $\sin\theta$ and common ratio $\cos\theta$.

a) Given the value of θ is such so that the progression converges, show that its sum to infinity is $\cot\dfrac{\theta}{2}$.

A different geometric progression has first term $\cos\theta$ and common ratio $\sin\theta$.

b) Given the value of θ is such so that this progression also converges, show that its sum to infinity is $\sec\theta + \tan\theta$.

[ ], [proof]



Question 40 (****+)

Is given that

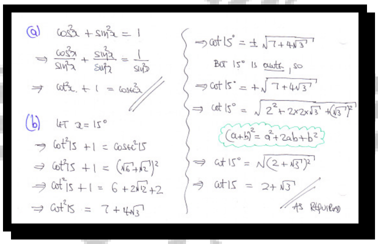
- $\cos^2 x + \sin^2 x \equiv 1$.
- $\operatorname{cosec} 15° = \sqrt{6} + \sqrt{2}$.

Use these facts **only** to show that

a) $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$.

b) $\cot 15° = 2 + \sqrt{3}$.

proof



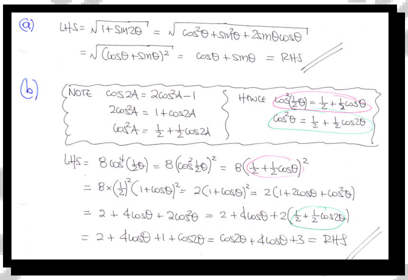
Question 41 (****+)

Prove the validity of each of the following trigonometric identities.

a) $\sqrt{1 + \sin 2\theta} \equiv \cos\theta + \sin\theta$.

b) $8\cos^4\left(\frac{1}{2}\theta\right) = \cos 2\theta + 4\cos\theta + 3$.

proof



Question 42 **(****+)**

A relationship between x and y is given by the equations

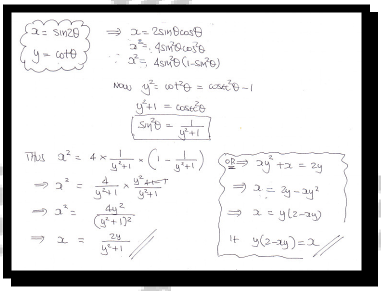
$$x = \sin 2\theta, \quad 0 < \theta < \pi,$$

$$y = \cot\theta, \quad 0 < \theta < \pi.$$

Use trigonometric identities to show that

$$y(2 - xy) = x.$$

proof

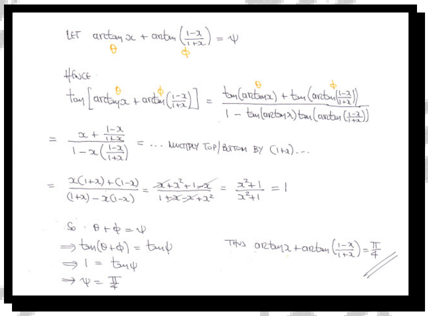


Question 43 **(****+)**

Show clearly that

$$\arctan x + \arctan\left(\frac{1-x}{1+x}\right) = \frac{\pi}{4}.$$

proof



Question 44 (****+)

It is given that

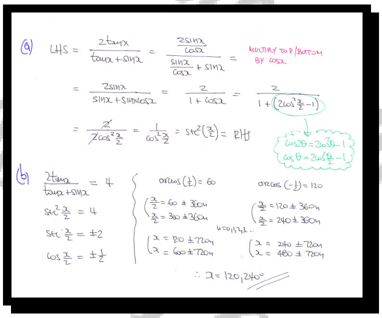
$$\frac{2\tan x}{\tan x+\sin x}\equiv \sec^{2}\left(\frac{x}{2}\right), \quad x\neq n\pi,\ n\in\mathbb{Z}.$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the trigonometric equation

$$\frac{2\tan x}{\tan x+\sin x}=4, \quad 0\le x<360°.$$

$\boxed{x=120°,240°}$

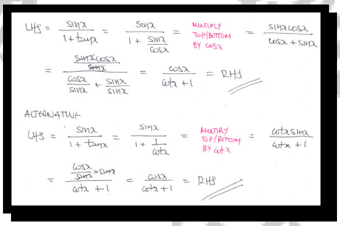


Question 45 (****+)

Prove the validity of the following trigonometric identity

$$\frac{\sin x}{1+\tan x}\equiv\frac{\cos x}{1+\cot x}.$$

$\boxed{\text{proof}}$



Question 46 (****+)

Let $t = \tan \frac{x}{2}$.

a) Show clearly that …

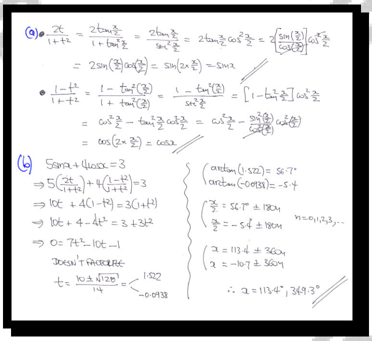
i. … $\sin x = \dfrac{2t}{1+t^2}$,

ii. … $\cos x = \dfrac{1-t^2}{1+t^2}$.

b) Use these results to solve the trigonometric equation

$$5\sin x + 4\cos x = 3,\ 0° \le x < 360°.$$

$x \approx 113.4°,\ 349.3°$



Question 47 (****+)

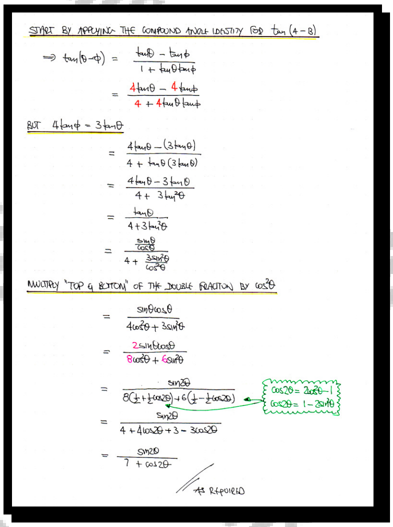
It is given that for $\theta \in \mathbb{R}$, $\varphi \in \mathbb{R}$

$$3\tan\theta = 4\tan\varphi .$$

Show that the above relationship implies that

$$\tan(\theta - \varphi) = \frac{\sin 2\theta}{7 + \cos 2\theta} .$$

[ ], [proof]



START BY APPLYING THE COMPOUND ANGLE IDENTITY FOR $\tan(A-B)$

$$\Rightarrow \tan(\theta-\phi) = \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi} = \frac{4\tan\theta - 4\tan\phi}{4 + 4\tan\theta\tan\phi}$$

BUT $4\tan\phi = 3\tan\theta$

$$= \frac{4\tan\theta - (3\tan\theta)}{4 + \tan\theta(3\tan\theta)} = \frac{4\tan\theta - 3\tan\theta}{4 + 3\tan^2\theta} = \frac{\tan\theta}{4+3\tan^2\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{4 + \frac{3\sin^2\theta}{\cos^2\theta}}$$

MULTIPLY "TOP & BOTTOM" OF THE DOUBLE FRACTION BY $\cos^2\theta$

$$= \frac{\sin\theta\cos\theta}{4\cos^2\theta + 3\sin^2\theta} = \frac{2\sin\theta\cos\theta}{8\cos^2\theta + 6\sin^2\theta}$$

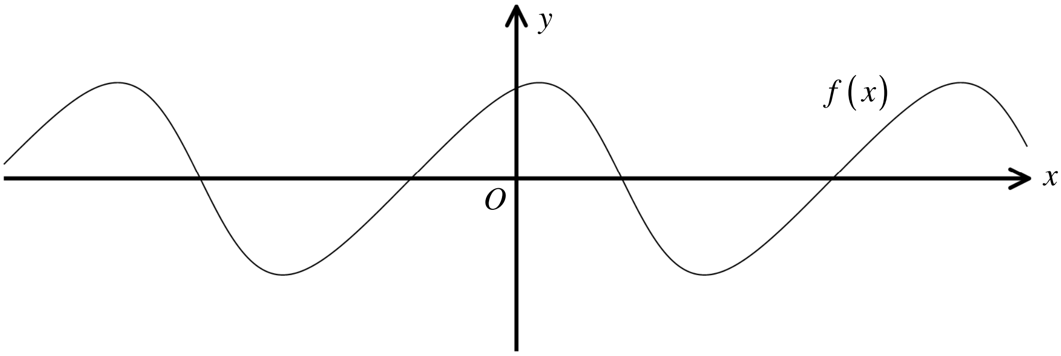
$$= \frac{\sin 2\theta}{8\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right) + 6\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)}$$

$\cos 2\theta = 2\cos^2\theta - 1$
$\cos 2\theta = 1 - 2\sin^2\theta$

$$= \frac{\sin 2\theta}{4 + 4\cos 2\theta + 3 - 3\cos 2\theta} = \frac{\sin 2\theta}{7 + \cos 2\theta}$$

// AS REQUIRED

Question 48 (****+)



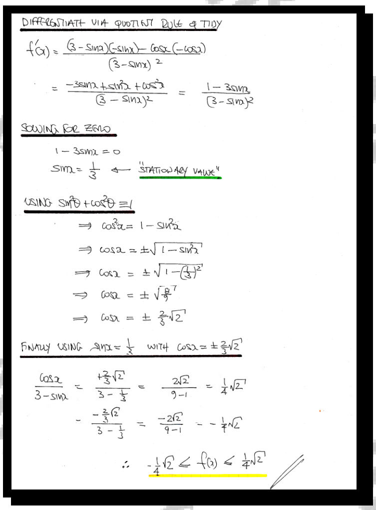
The figure above shows part of the graph of the curve with equation

$$f(x) = \frac{\cos x}{3 - \sin x}, \quad x \in \mathbb{R}.$$

Use differentiation to show that

$$-\tfrac{1}{4}\sqrt{2} \le f(x) \le \tfrac{1}{4}\sqrt{2}.$$

[ ], [proof]



DIFFERENTIATE VIA QUOTIENT RULE & TIDY

$f'(x) = \frac{(3-\sin x)(-\sin x) - \cos x(-\cos x)}{(3-\sin x)^2}$

$= \frac{-3\sin x + \sin^2 x + \cos^2 x}{(3-\sin x)^2} = \frac{1 - 3\sin x}{(3-\sin x)^2}$

SOLVING FOR ZERO

$1 - 3\sin x = 0$

$\sin x = \frac{1}{3}$ ← "STATIONARY VALUE"

USING $\sin^2\theta + \cos^2\theta \equiv 1$

$\Rightarrow \cos^2 x = 1 - \sin^2 x$

$\Rightarrow \cos x = \pm\sqrt{1 - \sin^2 x}$

$\Rightarrow \cos x = \pm\sqrt{1 - \left(\frac{1}{3}\right)^2}$

$\Rightarrow \cos x = \pm\sqrt{\frac{8}{9}}$

$\Rightarrow \cos x = \pm\frac{2}{3}\sqrt{2}$

FINALLY USING $\sin x = \frac{1}{3}$ WITH $\cos x = \pm\frac{2}{3}\sqrt{2}$

$\frac{\cos x}{3 - \sin x} = \frac{+\frac{2}{3}\sqrt{2}}{3 - \frac{1}{3}} = \frac{2\sqrt{2}}{9-1} = \frac{1}{4}\sqrt{2}$

$\frac{-\frac{2}{3}\sqrt{2}}{3 - \frac{1}{3}} = \frac{-2\sqrt{2}}{9-1} = -\frac{1}{4}\sqrt{2}$

$\therefore -\frac{1}{4}\sqrt{2} \le f(x) \le \frac{1}{4}\sqrt{2}$

Question 49 (****+)

It is given that

$$\tan 3x \equiv \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}, \quad x \neq \frac{n\pi}{3}, \quad n = 0,1,2,3,\ldots$$

a) Use the above identity to express $\cot 3x$ in terms of $\cot x$.

b) Show clearly that

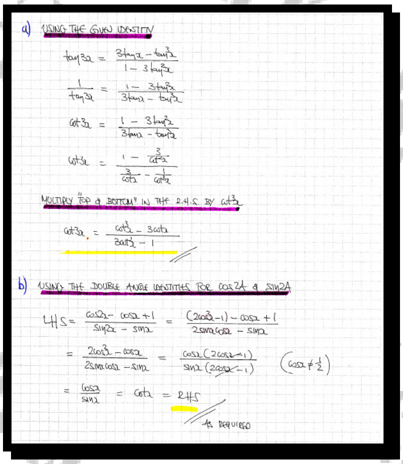
$$\frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} \equiv \cot x, \quad \cos x \neq \frac{1}{2}.$$

a) Hence, or otherwise, given that $\cos 3x \neq \frac{1}{2}$ solve the trigonometric equation

$$\cos 6x + \sin 6x - \cos 3x - \sin 3x + 1 = 0,$$

for $0 < x < \pi$, giving the answers in terms of π.

$\boxed{}$, $\boxed{x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}}$



a) USING THE GIVEN IDENTITY

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

$$\frac{1}{\tan 3x} = \frac{1 - 3\tan^2 x}{3\tan x - \tan^3 x}$$

$$\cot 3x = \frac{1 - 3\tan^2 x}{3\tan x - \tan^3 x}$$

$$\cot 3x = \frac{1 - \frac{3}{\cot^2 x}}{\frac{3}{\cot x} - \frac{1}{\cot^3 x}}$$

MULTIPLY "TOP & BOTTOM" IN THE R.H.S. BY $\cot^3 x$

$$\cot 3x = \frac{\cot^3 x - 3\cot x}{3\cot^2 x - 1}$$

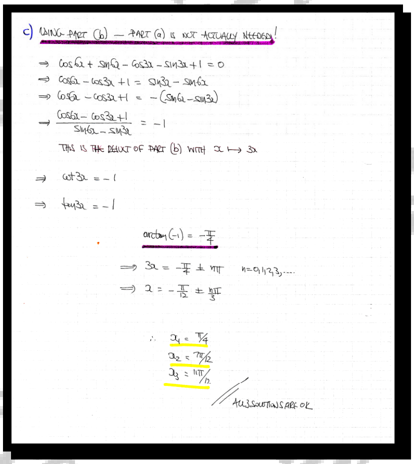
b) USING THE DOUBLE ANGLE IDENTITIES FOR $\cos 2A$ & $\sin 2A$

$$\text{LHS} = \frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} = \frac{(2\cos^2 x - 1) - \cos x + 1}{2\sin x\cos x - \sin x}$$

$$= \frac{2\cos^2 x - \cos x}{2\sin x\cos x - \sin x} = \frac{\cos x(2\cos x - 1)}{\sin x(2\cos x - 1)} \quad \left(\cos x \neq \tfrac{1}{2}\right)$$

$$= \frac{\cos x}{\sin x} = \cot x = \text{RHS}$$

AS REQUIRED



c) USING PART (b) — PART (a) IS NOT ACTUALLY NEEDED!

$$\Rightarrow \cos 6x + \sin 6x - \cos 3x - \sin 3x + 1 = 0$$

$$\Rightarrow \cos 6x - \cos 3x + 1 = \sin 3x - \sin 6x$$

$$\Rightarrow \cos 6x - \cos 3x + 1 = -(\sin 6x - \sin 3x)$$

$$\Rightarrow \frac{\cos 6x - \cos 3x + 1}{\sin 6x - \sin 3x} = -1$$

THIS IS THE RESULT OF PART (b) WITH $x \mapsto 3x$

$$\Rightarrow \cot 3x = -1$$

$$\Rightarrow \tan 3x = -1$$

$$\arctan(-1) = -\frac{\pi}{4}$$

$$\Rightarrow 3x = -\frac{\pi}{4} \pm n\pi \quad n = 0,1,2,3,\ldots$$

$$\Rightarrow x = -\frac{\pi}{12} \pm \frac{n\pi}{3}$$

$$\therefore x_1 = \pi/4, \quad x_2 = 7\pi/12, \quad x_3 = 11\pi/12$$

ALL 3 SOLUTIONS ARE OK

Question 50 (****+)

$$f(x)=\frac{\sin 3x}{\cos x}+\frac{\cos 3x}{\sin x}, \quad x\in\mathbb{R}, \quad x\neq\frac{n\pi}{2}, \; n\in\mathbb{Z}.$$

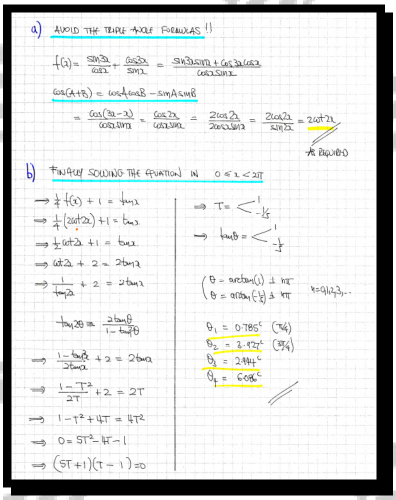
a) Show clearly that

$$f(x)\equiv 2\cot 2x .$$

b) Solve the trigonometric equation

$$\frac{1}{4}f(x)+1=\tan x, \quad 0\le x<2\pi .$$

$\boxed{}$, $\boxed{x=0.785^{c},\ 2.94^{c},\ 3.93^{c},\ 6.09^{c}}$



a) AVOID THE TRIPLE ANGLE FORMULAS !!

$f(x)=\frac{\sin 3x}{\cos x}+\frac{\cos 3x}{\sin x}=\frac{\sin 3x\sin x+\cos 3x\cos x}{\cos x\sin x}$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$=\frac{\cos(3x-x)}{\cos x\sin x}=\frac{\cos 2x}{\cos x\sin x}=\frac{2\cos 2x}{2\cos x\sin x}=\frac{2\cos 2x}{\sin 2x}=2\cot 2x$ // AS REQUIRED

b) FINALLY SOLVING THE EQUATION IN $0\le x<2\pi$

$\Rightarrow \frac{1}{4}f(x)+1=\tan x$

$\Rightarrow \frac{1}{4}(2\cot 2x)+1=\tan x$

$\Rightarrow \frac{1}{2}\cot 2x+1=\tan x$

$\Rightarrow \cot 2x+2=2\tan x$

$\Rightarrow \frac{1}{\tan 2x}+2=2\tan x$

$\tan 2\theta\equiv\frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \frac{1-\tan^2 x}{2\tan x}+2=2\tan x$

$\Rightarrow \frac{1-T^2}{2T}+2=2T$

$\Rightarrow 1-T^2+4T=4T^2$

$\Rightarrow 0=5T^2-4T-1$

$\Rightarrow (5T+1)(T-1)=0$

$\Rightarrow T=1$ or $-\frac{1}{5}$

$\Rightarrow \tan\theta=1$ or $-\frac{1}{5}$

$\theta=\arctan(1)\pm n\pi$, $\theta=\arctan\left(-\frac{1}{5}\right)\pm n\pi$, $n=0,1,2,3,\dots$

$\theta_1=0.785^c\ (\pi/4)$

$\theta_2=3.927^c\ (3\pi/4)$

$\theta_3=2.944^c$

$\theta_4=6.086^c$ //

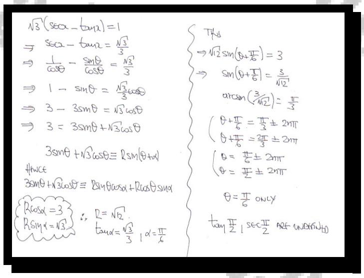
Question 51 (****+)

Solve the trigonometric equation

$$\sqrt{3}\left(\sec x - \tan x\right) = 1, \quad 0 \le x \le 2\pi ,$$

giving the answers in terms of π.

$$\boxed{x = \frac{\pi}{6}}$$

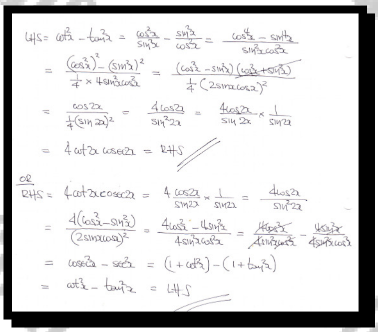


Question 52 (****+)

Prove the validity of the following trigonometric identity

$$\cot^2 x - \tan^2 x \equiv 4\cot 2x \operatorname{cosec} 2x .$$

proof



Question 53 (****+)

Let $t = \tan\frac{x}{2}$.

a) Show clearly that …

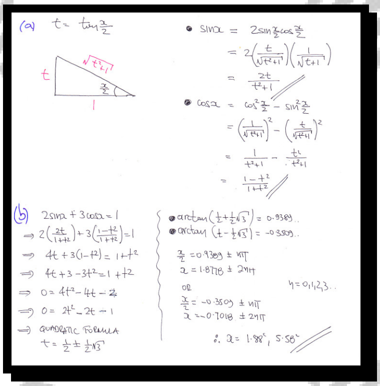
i. … $\sin x = \dfrac{2t}{1+t^2}$,

ii. … $\cos x = \dfrac{1-t^2}{1+t^2}$.

b) Use these results to solve the trigonometric equation

$$2\sin x + 3\cos x = 1, \; 0 \le x < 2\pi .$$

$x \approx 1.88^{c}, \; 5.58^{c}$



Question 54 (****+)

$$f(x) = \sec x, \qquad 0 \le x < \frac{\pi}{2} \cup \frac{\pi}{2} < x \le \pi .$$

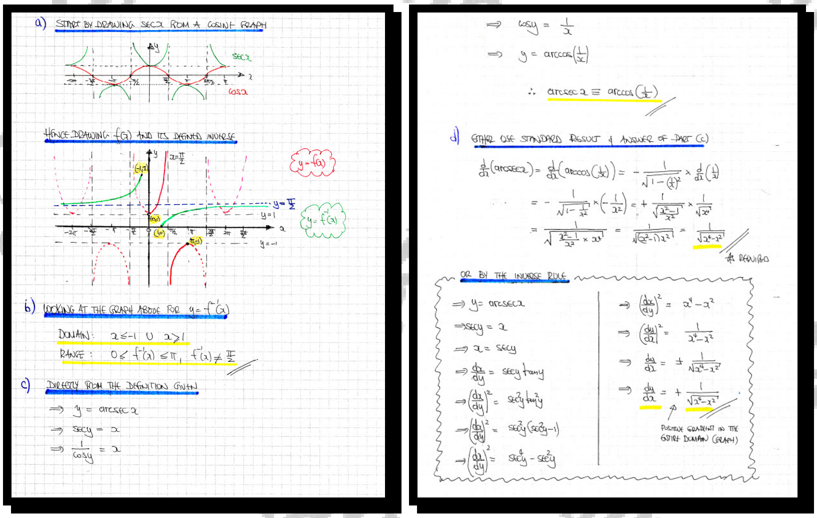
a) Sketch in the same diagram the graphs of $f(x)$ and $f^{-1}(x) = \operatorname{arcsec} x$.

b) State the domain and range of $f^{-1}(x) = \operatorname{arcsec} x$.

c) Show clearly that $\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right)$.

d) Show further that $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{\sqrt{x^4 - x^2}}$.

$\boxed{}$, $\boxed{\text{domain: } x \le -1 \cup x \ge 1}$, $\boxed{\text{range: } 0 \le f^{-1}(x) \le \pi,\ f^{-1}(x) \ne \frac{\pi}{2}}$



Question 55 (****+)

It is given that

$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta \, .$$

a) Prove the validity of the above trigonometric identity.

b) By differentiating both sides of the above identity with respect to θ, show that

$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta \, .$$

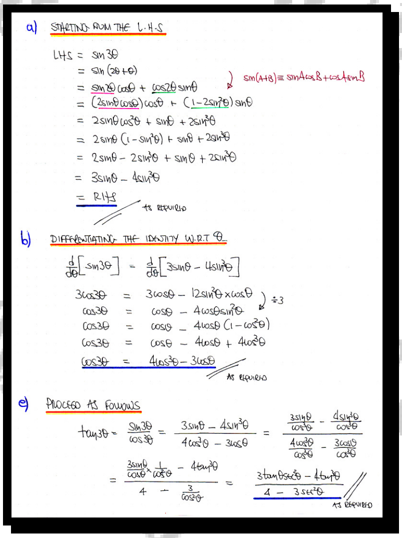
c) Hence show that

$$\tan 3\theta \equiv \frac{3\tan\theta\sec^2\theta - 4\tan^3\theta}{4 - 3\sec^2\theta} \, .$$

d) Deduce that

$$\tan 3\theta \equiv \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \, .$$

proof



a) STARTING FROM THE L.H.S

$LHS = \sin 3\theta$
$= \sin(2\theta + \theta)$
$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$ — $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$
$= (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta$
$= 2\sin\theta\cos^2\theta + \sin\theta + 2\sin^3\theta$
$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta + 2\sin^3\theta$
$= 2\sin\theta - 2\sin^3\theta + \sin\theta + 2\sin^3\theta$
$= 3\sin\theta - 4\sin^3\theta$
$= RHS$

AS REQUIRED

b) DIFFERENTIATING THE IDENTITY W.R.T θ

$\frac{d}{d\theta}[\sin 3\theta] = \frac{d}{d\theta}[3\sin\theta - 4\sin^3\theta]$
$3\cos 3\theta = 3\cos\theta - 12\sin^2\theta \times \cos\theta$ — $\div 3$
$\cos 3\theta = \cos\theta - 4\cos\theta\sin^2\theta$
$\cos 3\theta = \cos\theta - 4\cos\theta(1 - \cos^2\theta)$
$\cos 3\theta = \cos\theta - 4\cos\theta + 4\cos^3\theta$
$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

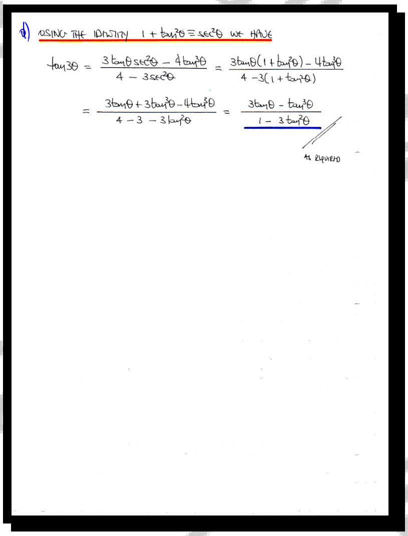
AS REQUIRED

c) PROCEED AS FOLLOWS

$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta} = \frac{\frac{3\sin\theta}{\cos^3\theta} - \frac{4\sin^3\theta}{\cos^3\theta}}{\frac{4\cos^3\theta}{\cos^3\theta} - \frac{3\cos\theta}{\cos^3\theta}}$

$= \frac{\frac{3\sin\theta}{\cos\theta} \times \frac{1}{\cos^2\theta} - 4\tan^3\theta}{4 - \frac{3}{\cos^2\theta}} = \frac{3\tan\theta\sec^2\theta - 4\tan^3\theta}{4 - 3\sec^2\theta}$

AS REQUIRED



d) USING THE IDENTITY $1 + \tan^2\theta \equiv \sec^2\theta$ WE HAVE

$\tan 3\theta = \frac{3\tan\theta\sec^2\theta - 4\tan^3\theta}{4 - 3\sec^2\theta} = \frac{3\tan\theta(1 + \tan^2\theta) - 4\tan^3\theta}{4 - 3(1 + \tan^2\theta)}$

$= \frac{3\tan\theta + 3\tan^3\theta - 4\tan^3\theta}{4 - 3 - 3\tan^2\theta} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

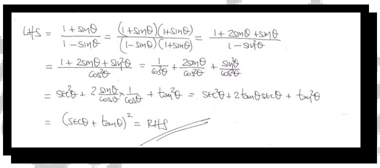
AS REQUIRED

Question 56 (****+)

Prove the validity of the trigonometric identity

$$\frac{1+\sin\theta}{1-\sin\theta} \equiv (\sec\theta + \tan\theta)^2 .$$

proof



$$\text{LHS} = \frac{1+\sin\theta}{1-\sin\theta} = \frac{(1+\sin\theta)(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} = \frac{1+2\sin\theta+\sin\theta}{1-\sin^2\theta}$$

$$= \frac{1+2\sin\theta+\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \sec^2\theta + 2\frac{\sin\theta}{\cos\theta}\times\frac{1}{\cos\theta} + \tan^2\theta = \sec^2\theta + 2\tan\theta\sec\theta + \tan^2\theta$$

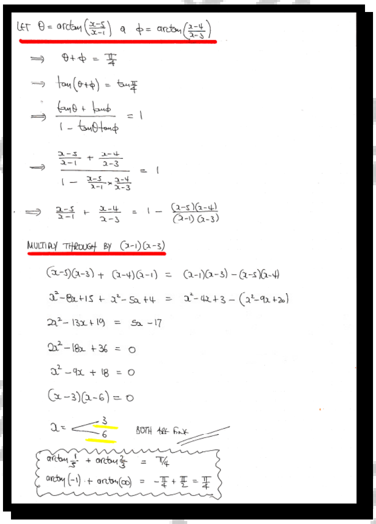
$$= (\sec\theta + \tan\theta)^2 = \text{RHS}$$

Question 57 (****+)

Solve the following trigonometric equation.

$$\arctan\left(\frac{x-5}{x-1}\right) + \arctan\left(\frac{x-4}{x-3}\right) = \frac{\pi}{4}, \quad x \in \mathbb{R} .$$

$x=3 \cup x=6$



LET $\theta = \arctan\left(\frac{x-5}{x-1}\right)$ & $\phi = \arctan\left(\frac{x-4}{x-3}\right)$

$$\Rightarrow \theta + \phi = \frac{\pi}{4}$$

$$\Rightarrow \tan(\theta+\phi) = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = 1$$

$$\Rightarrow \frac{\frac{x-5}{x-1} + \frac{x-4}{x-3}}{1 - \frac{x-5}{x-1}\times\frac{x-4}{x-3}} = 1$$

$$\Rightarrow \frac{x-5}{x-1} + \frac{x-4}{x-3} = 1 - \frac{(x-5)(x-4)}{(x-1)(x-3)}$$

MULTIPLY THROUGH BY $(x-1)(x-3)$

$$(x-5)(x-3) + (x-4)(x-1) = (x-1)(x-3) - (x-5)(x-4)$$

$$x^2-8x+15 + x^2-5x+4 = x^2-4x+3 - (x^2-9x+20)$$

$$2x^2-13x+19 = 5x-17$$

$$2x^2-18x+36 = 0$$

$$x^2-9x+18 = 0$$

$$(x-3)(x-6) = 0$$

$x = 3$ or 6 BOTH ARE FINE

$$\arctan\frac{1}{5} + \arctan\frac{2}{3} = \pi/4$$

$$\arctan(-1) + \arctan(\infty) = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$$

Question 58 (****+)

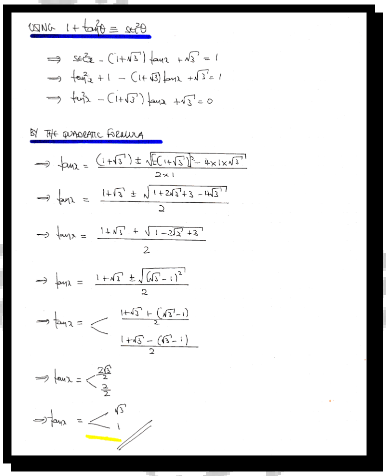
Given that

$$\sec^2 x - \left(1+\sqrt{3}\right)\tan x + \sqrt{3} = 1,$$

show that either $\tan x = 1$ or $\tan x = \sqrt{3}$.

Detailed workings must be shown in this question.

[ ], [proof]



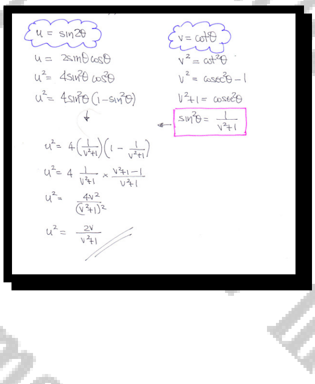
Question 59 (****+)

It is given that

$$u = \sin 2\theta \ , \quad v = \cot\theta \,.$$

Use trigonometric identities to find a simplified expression for u^2 in terms of v.

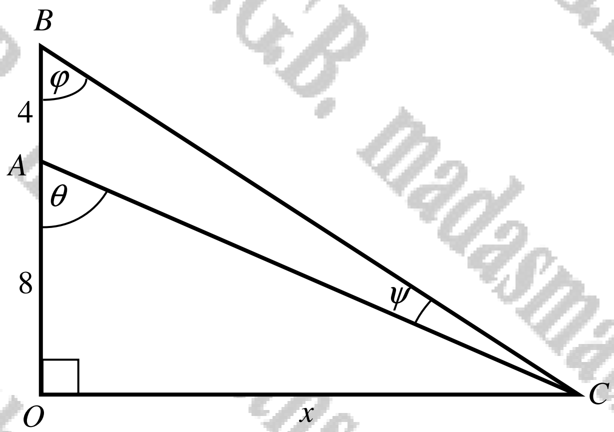
$$u^2 = \frac{2v}{v^2+1}$$



Question 60 (****+)

The diagram below shows a right angled triangle OBC where $|OC| = x$ and the point A on OB so that $|OA| = 8$, $|AB| = 4$.

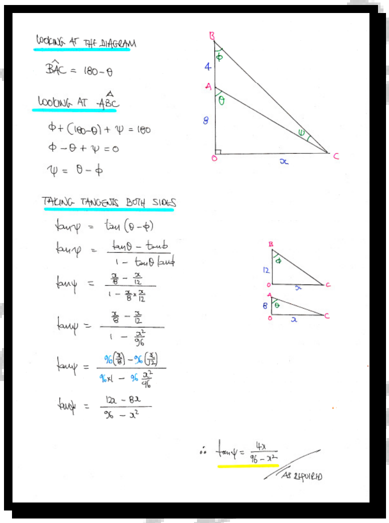
The angles OAC, OBC and ACB are denoted by θ, φ and ψ respectively.



By considering a relationship between the angles θ, φ and ψ, show that

$$\tan\psi = \frac{4x}{96 + x^2}.$$

proof



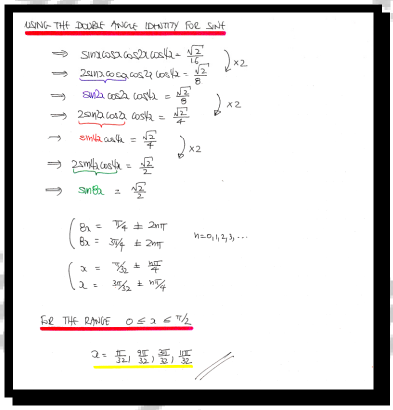
Question 61 (****+)

Solve the trigonometric equation

$$\sin x \cos x \cos 2x \cos 4x = \frac{\sqrt{2}}{16}, \quad 0 \le x \le \frac{\pi}{2},$$

giving the answers in terms of π.

$\boxed{}$, $\boxed{x = \frac{\pi}{32}, \frac{3\pi}{32}, \frac{9\pi}{32}, \frac{11\pi}{32}}$



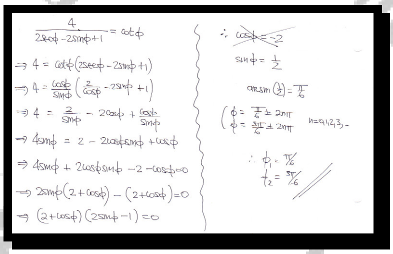
Question 62 (****+)

Solve the trigonometric equation

$$\frac{4}{2\sec\varphi - 2\sin\varphi + 1} = \cot\varphi, \quad 0 < \varphi < 2\pi, \quad \varphi \ne \pi,$$

giving the answers in terms of π.

$\boxed{\varphi = \frac{\pi}{6}, \frac{5\pi}{6}}$

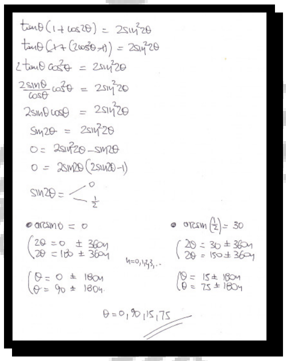


Question 63 (****+)

Solve the trigonometric equation

$$\tan\theta(1+\cos 2\theta) = 2\sin^2 2\theta\,, \qquad 0 \le \theta \le 90°\,.$$

$\boxed{\theta = 0°, 15°, 75°, 90°}$

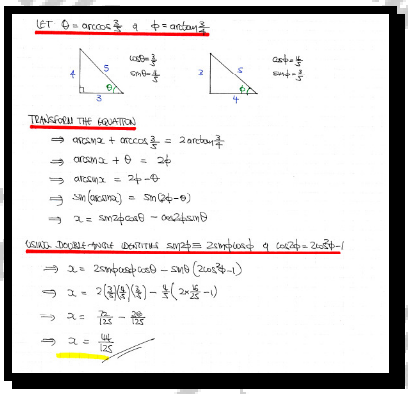


Question 64 (****+)

Solve the trigonometric equation

$$\arcsin x + \arccos\frac{3}{5} = 2\arctan\frac{3}{4}\,.$$

$\boxed{}\,, \boxed{x = \frac{44}{125}}$



Question 65 (****+)

It is given that

$$\frac{\cot x}{\operatorname{cosec} x - 1} - \frac{\cos x}{1 + \sin x} \equiv 2\tan x, \quad x \neq 180°n, \; n \in \mathbb{Z}.$$

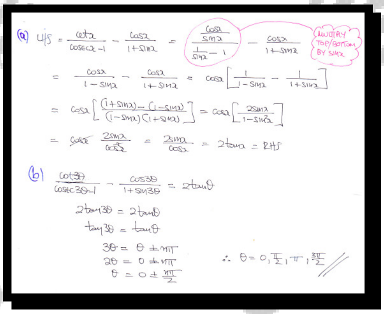
a) Prove the validity of the above trigonometric identity.

b) Hence solve the trigonometric equation

$$\frac{\cot 3\theta}{\operatorname{cosec} 3\theta - 1} - \frac{\cos 3\theta}{1 + \sin 3\theta} = 2\tan\theta, \quad 0 \leq \theta < 2\pi,$$

giving the answers in terms of π.

$$\boxed{\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}}$$


(a) $\text{LHS} = \frac{\cot x}{\operatorname{cosec} x - 1} - \frac{\cos x}{1+\sin x} = \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x} - 1} - \frac{\cos x}{1+\sin x}$ MULTIPLY TOP/BOTTOM BY $\sin x$

$= \frac{\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x} = \cos x\left[\frac{1}{1-\sin x} - \frac{1}{1+\sin x}\right]$

$= \cos x\left[\frac{(1+\sin x)-(1-\sin x)}{(1-\sin x)(1+\sin x)}\right] = \cos x\left[\frac{2\sin x}{1-\sin^2 x}\right]$

$= \cos x \frac{2\sin x}{\cos^2 x} = \frac{2\sin x}{\cos x} = 2\tan x = \text{RHS}$

(b) $\frac{\cot 3\theta}{\operatorname{cosec} 3\theta - 1} - \frac{\cos 3\theta}{1+\sin 3\theta} = 2\tan\theta$

$2\tan 3\theta = 2\tan\theta$

$\tan 3\theta = \tan\theta$

$3\theta = \theta \pm n\pi$

$2\theta = 0 \pm n\pi$

$\theta = 0 \pm \frac{n\pi}{2}$

$\therefore \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

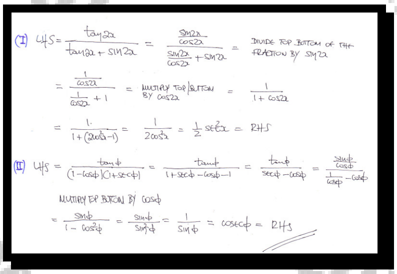
Question 66 (****+)

Prove the validity of the following trigonometric identities.

i. $$\frac{\tan 2x}{\tan 2x + \sin 2x} \equiv \frac{1}{2}\sec^2 x .$$

ii. $$\frac{\tan \varphi}{(1-\cos\varphi)(1+\sec\varphi)} \equiv \operatorname{cosec}\varphi .$$

proof



(I) $\text{LHS} = \frac{\tan 2x}{\tan 2x + \sin 2x} = \frac{\frac{\sin 2x}{\cos 2x}}{\frac{\sin 2x}{\cos 2x} + \sin 2x} =$ DIVIDE TOP BOTTOM OF THE FRACTION BY $\sin 2x$

$= \frac{\frac{1}{\cos 2x}}{\frac{1}{\cos 2x} + 1} =$ MULTIPLY TOP/BOTTOM BY $\cos 2x$ $= \frac{1}{1+\cos 2x}$

$= \frac{1}{1+(2\cos^2 x - 1)} = \frac{1}{2\cos^2 x} = \frac{1}{2}\sec^2 x = \text{RHS}$

(II) $\text{LHS} = \frac{\tan\phi}{(1-\cos\phi)(1+\sec\phi)} = \frac{\tan\phi}{1+\sec\phi-\cos\phi-1} = \frac{\tan\phi}{\sec\phi-\cos\phi} = \frac{\frac{\sin\phi}{\cos\phi}}{\frac{1}{\cos\phi}-\cos\phi}$

MULTIPLY TOP BOTTOM BY $\cos\phi$

$= \frac{\sin\phi}{1-\cos^2\phi} = \frac{\sin\phi}{\sin^2\phi} = \frac{1}{\sin\phi} = \operatorname{cosec}\phi = \text{RHS}$

Question 67 (****+)

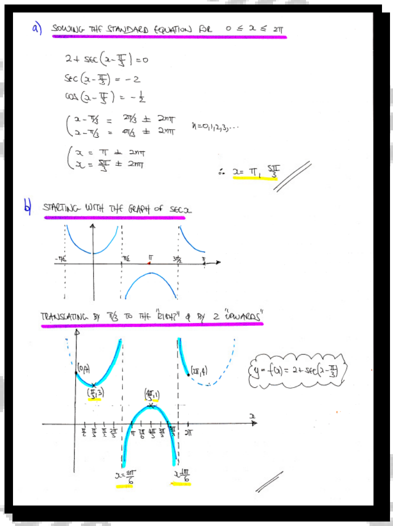
$$f(x) = 2 + \sec\left(x - \frac{\pi}{3}\right), \quad x \in \mathbb{R}, \quad 0 \le x \le 2\pi .$$

a) Solve the equation $f(x) = 0$.

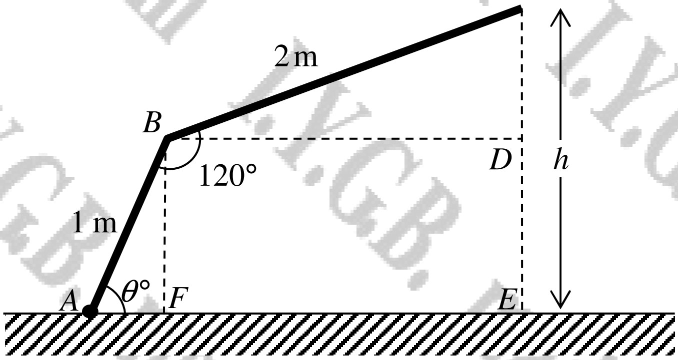
b) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any stationary points, the coordinates of any x or y intercepts and equations of the vertical asymptotes.

MP2-L , $x = \pi, \frac{5\pi}{3}$



Question 68 (****+)



The figure above shows a rigid rod ABC where AB is 1 metre, BC is 2 metres and the angle ABC is $120°$. The rod is hinged at A so it can be rotated in a vertical plane forming an angle $\theta°$ with the horizontal ground.

Let h metres be the height of the point C from the horizontal ground.

a) Show that …

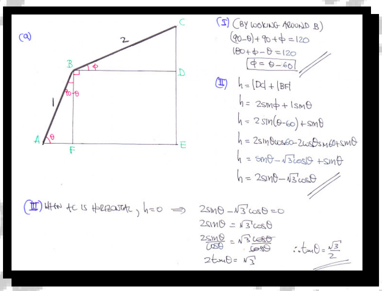
i. … $\measuredangle DBC = \theta° - 60°$.

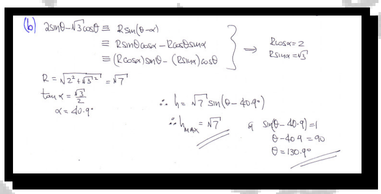
ii. … $h = 2\sin\theta - \sqrt{3}\cos\theta$.

iii. … when AC is horizontal, $\tan\theta = \dfrac{\sqrt{3}}{2}$.

b) By expressing h in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$, find the maximum value of h and the value of θ when h takes this maximum value.

$\boxed{}$, $\boxed{h_{max} = \sqrt{7}}$, $\boxed{\theta \approx 130.9°}$





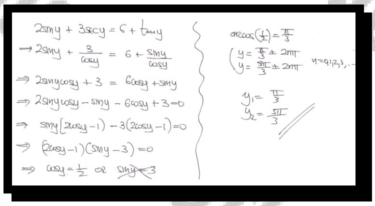
Question 69 **(****+)**

Solve the trigonometric equation

$$2\sin y + 3\sec y = 6 + \tan y, \quad 0 \le y < 2\pi,$$

giving the answers in terms of π.

$$\boxed{y = \frac{\pi}{3}, \frac{5\pi}{3}}$$



Question 70 (****+)

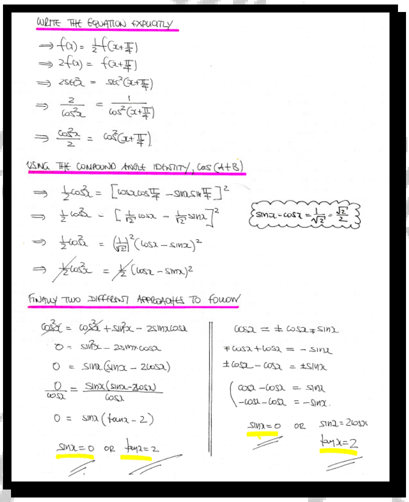
$$f(x)=\sec^2 x,\ x\in\mathbb{R},\ x\neq\frac{\pi}{2}(2n+1),\ n\in\mathbb{N}.$$

Show that if

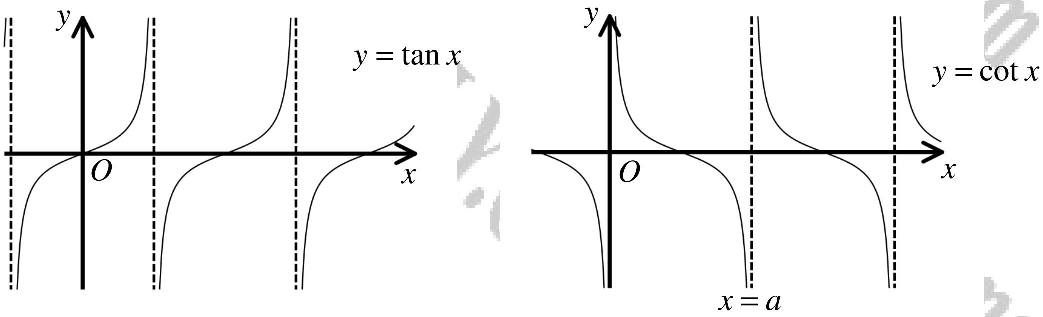
$$f(x)=\frac{1}{2}f\left(x+\frac{\pi}{4}\right),$$

then either $\sin x=0$ or $\tan x=2$.

[ ] , [proof]



Question 71 (****+)



The diagrams above shows part of the graphs of $y = \tan x$ and $y = \cot x$.

a) Sketch the graph of $y_1 = -\tan(-x)$ and hence write a simplified expression for y_1 in terms of $\tan x$.

The graph of $\cot x$ has vertical asymptotes and the equation of one of them is labelled in the diagram as $x = a$.

b) State the value of a.

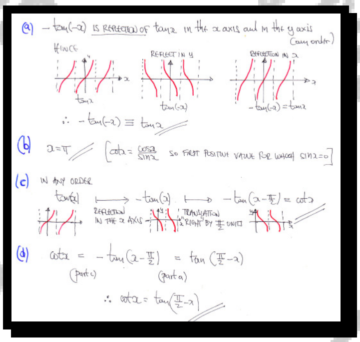
The graph of $\cot x$ can be obtained from the graph of $\tan x$ by a series of two geometric transformations which can be carried out in any order.

c) Describe the two geometric transformations.

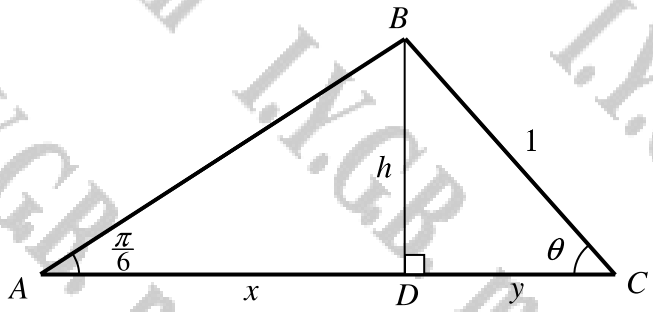
d) Deduce using valid arguments that

$$\cot x = \tan\left(\frac{\pi}{2} - x\right).$$

$\boxed{a = \pi}$, $\boxed{\text{reflection in the } x \text{ axis/translation to the "right" by } \frac{\pi}{2} \text{ units}}$



Question 72 (****+)



The figure above shows a triangle ABC, where $\measuredangle BAC = \frac{\pi}{6}$, $\measuredangle BCA = \theta$ and $|BC| = 1$.

The straight line segment BD, labelled as h, is perpendicular to AC.

Let $AD = x$ and $DC = y$.

a) By expressing h in terms of θ, and x in terms of h, show that

$$x + y = \sqrt{3}\sin\theta + \cos\theta,$$

and hence deduce that the area of the triangle ABC is given by

$$\sin\theta\sin\left(\theta + \frac{\pi}{6}\right).$$

b) By using the trigonometric identities for

$$\cos\left[\theta + \left(\theta + \frac{\pi}{6}\right)\right] \quad \text{and} \quad \cos\left[\theta - \left(\theta + \frac{\pi}{6}\right)\right],$$

write a simplified expression for the area of the triangle ABC.

[continues overleaf]

[continued from overleaf]

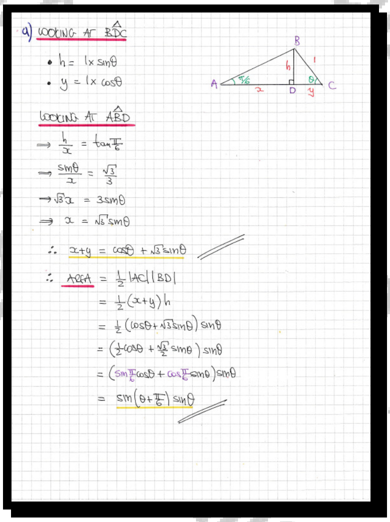
The value of θ can vary.

c) By using part **(b)**, deduce that the maximum value of the area of the triangle ABC is

$$\frac{1}{4}\left(2+\sqrt{3}\right)$$

and this maximum value occurs when $\theta = \frac{5\pi}{12}$.

$$\boxed{}, \boxed{\sin\theta\sin\left(\theta+\frac{\pi}{6}\right) = \frac{\sqrt{3}}{4} - \frac{1}{2}\cos\left(2\theta+\frac{\pi}{6}\right)}$$


a) LOOKING AT $B\hat{D}C$

- $h = 1 \times \sin\theta$
- $y = 1 \times \cos\theta$

LOOKING AT $A\hat{B}D$

$\Rightarrow \frac{h}{x} = \tan\frac{\pi}{6}$

$\Rightarrow \frac{\sin\theta}{x} = \frac{\sqrt{3}}{3}$

$\Rightarrow \sqrt{3}x = 3\sin\theta$

$\Rightarrow x = \sqrt{3}\sin\theta$

$\therefore x+y = \cos\theta + \sqrt{3}\sin\theta$

$\therefore$ AREA $= \frac{1}{2}|AC||BD|$

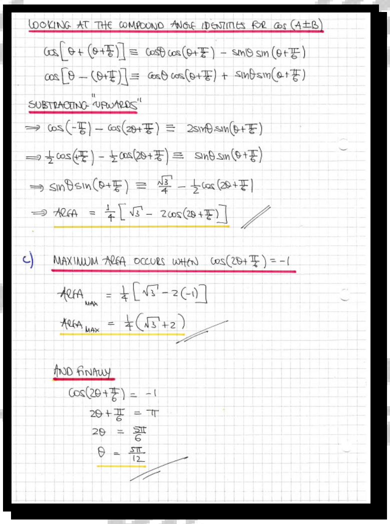
$= \frac{1}{2}(x+y)h$

$= \frac{1}{2}(\cos\theta + \sqrt{3}\sin\theta)\sin\theta$

$= \left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right)\sin\theta$

$= \left(\sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta\right)\sin\theta$

$= \sin\left(\theta+\frac{\pi}{6}\right)\sin\theta$


LOOKING AT THE COMPOUND ANGLE IDENTITIES FOR $\cos(A \pm B)$

$\cos\left[\theta + \left(\theta+\frac{\pi}{6}\right)\right] \equiv \cos\theta\cos\left(\theta+\frac{\pi}{6}\right) - \sin\theta\sin\left(\theta+\frac{\pi}{6}\right)$

$\cos\left[\theta - \left(\theta+\frac{\pi}{6}\right)\right] \equiv \cos\theta\cos\left(\theta+\frac{\pi}{6}\right) + \sin\theta\sin\left(\theta+\frac{\pi}{6}\right)$

SUBTRACTING "UPWARDS"

$\Rightarrow \cos\left(-\frac{\pi}{6}\right) - \cos\left(2\theta+\frac{\pi}{6}\right) \equiv 2\sin\theta\sin\left(\theta+\frac{\pi}{6}\right)$

$\Rightarrow \frac{1}{2}\cos\left(+\frac{\pi}{6}\right) - \frac{1}{2}\cos\left(2\theta+\frac{\pi}{6}\right) \equiv \sin\theta\sin\left(\theta+\frac{\pi}{6}\right)$

$\Rightarrow \sin\theta\sin\left(\theta+\frac{\pi}{6}\right) \equiv \frac{\sqrt{3}}{4} - \frac{1}{2}\cos\left(2\theta+\frac{\pi}{6}\right)$

$\Rightarrow$ AREA $= \frac{1}{4}\left[\sqrt{3} - 2\cos\left(2\theta+\frac{\pi}{6}\right)\right]$

c) MAXIMUM AREA OCCURS WHEN $\cos\left(2\theta+\frac{\pi}{6}\right) = -1$

$\text{AREA}_{\text{MAX}} = \frac{1}{4}\left[\sqrt{3} - 2(-1)\right]$

$\text{AREA}_{\text{MAX}} = \frac{1}{4}\left(\sqrt{3}+2\right)$

AND FINALLY

$\cos\left(2\theta+\frac{\pi}{6}\right) = -1$

$2\theta + \frac{\pi}{6} = \pi$

$2\theta = \frac{5\pi}{6}$

$\theta = \frac{5\pi}{12}$

Question 73 (****+)

The height of tide, h meters, in a harbour on a certain day can be modelled by

$$h(t) = 10 + \sqrt{3}\sin(30t)° + \cos(30t)°,\ 0 \le t \le 12,$$

where t is the time in hours since midnight.

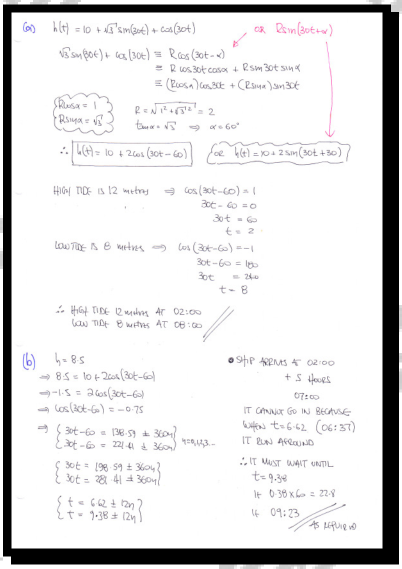
a) Find the time when the high tide and the low tide occur during the morning hours of that day and state the corresponding depth of water in the harbour at these times.

The depth of water in this harbour needs to be at least 8.5 metres for a boat to dock

A boat arrives outside the harbour at high tide and needs five hours to unload.

b) Show that the boat has to wait until $09:23$ to enter the harbour.

[ ], [high tide of 12 metres at 02:00], [low tide of 8 metres at 08:00]



(a) $h(t) = 10 + \sqrt{3}\sin(30t) + \cos(30t)$ OR $R\sin(30t+\alpha)$

$\sqrt{3}\sin(30t) + \cos(30t) \equiv R\cos(30t-\alpha)$
$\equiv R\cos 30t\cos\alpha + R\sin 30t\sin\alpha$
$\equiv (R\cos\alpha)\cos 30t + (R\sin\alpha)\sin 30t$

$R\cos\alpha = 1$, $R\sin\alpha = \sqrt{3}$; $R = \sqrt{1^2+\sqrt{3}^2} = 2$; $\tan\alpha = \sqrt{3} \Rightarrow \alpha = 60°$

$\therefore h(t) = 10 + 2\cos(30t-60)$ (or $h(t) = 10 + 2\sin(30t+30)$)

HIGH TIDE IS 12 metres $\Rightarrow \cos(30t-60) = 1$, $30t-60 = 0$, $30t = 60$, $t = 2$

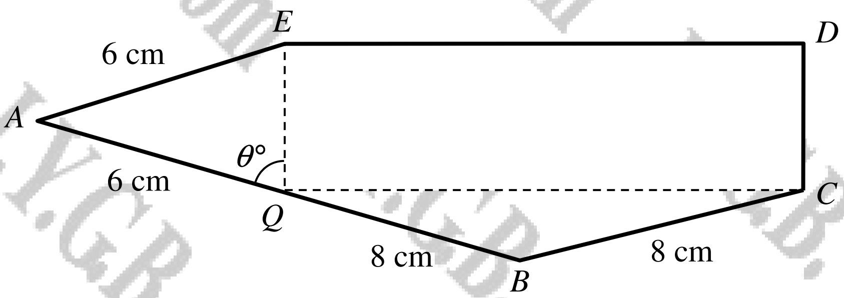
LOW TIDE IS 8 metres $\Rightarrow \cos(30t-60) = -1$, $30t-60 = 180$, $30t = 240$, $t = 8$

$\therefore$ HIGH TIDE 12 metres AT 02:00
LOW TIDE 8 metres AT 08:00

(b) $h = 8.5$
$\Rightarrow 8.5 = 10 + 2\cos(30t-60)$
$\Rightarrow -1.5 = 2\cos(30t-60)$
$\Rightarrow \cos(30t-60) = -0.75$
$\Rightarrow 30t-60 = 138.59 \pm 360n$, $30t-60 = 221.41 \pm 360n$, $n = 0,1,2,3,\ldots$
$30t = 198.59 \pm 360n$, $30t = 281.41 \pm 360n$
$t = 6.62 \pm 12n$, $t = 9.38 \pm 12n$

• SHIP ARRIVES AT 02:00
+ 5 HOURS
07:00
IT CANNOT GO IN BECAUSE WHEN $t = 6.62$ (06:37) IT RUN AGROUND
$\therefore$ IT MUST WAIT UNTIL $t = 9.38$
ie $0.38 \times 60 = 22.8$
ie 09:23 AS REQUIRED

Question 74 (****+)



The figure above shows an irregular pentagon $ABCDE$. The lengths of AB, BC and AE are 14 cm, 8 cm and 6 cm respectively.

The point Q lies on AB so that AQ is 6 cm and QB is 8 cm. The point D is then constructed so that $QEDC$ is a rectangle.

Let the angle AQE be $\theta°$ and assume that $\theta°$ can vary.

a) Given that P cm and R cm^2 are the perimeter and the area of the pentagon respectively, show that …

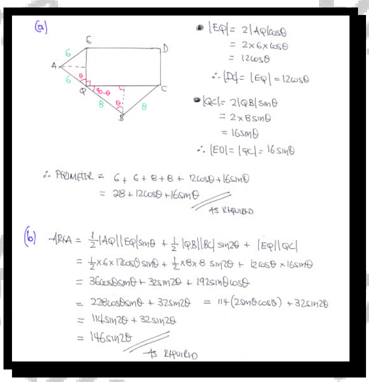
i. … $P = 28 + 12\cos\theta + 16\sin\theta$.

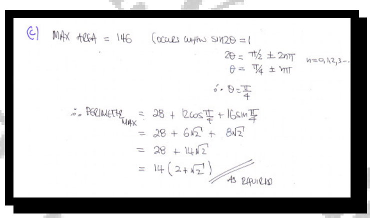
ii. … $R = 146\sin 2\theta$.

b) Hence show that when the pentagon has a maximum area

$$P = 14\left(2+\sqrt{2}\right) \text{ cm}^2 .$$

proof





Question 75 (****+)

Let $t = \tan\left(\frac{x}{2}\right)$.

a) Show that …

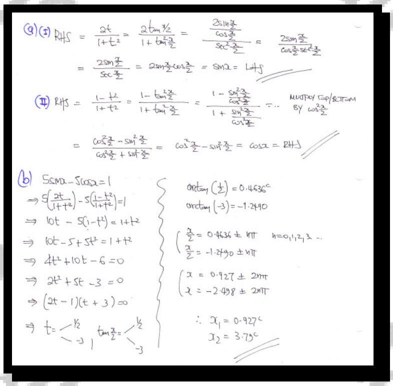
i. … $\sin x = \frac{2t}{1+t^2}$,

ii. … $\cos x = \frac{1-t^2}{1+t^2}$.

b) Use these results to solve the trigonometric equation

$$5\sin x - 5\cos x = 1, \ 0 \le x < 2\pi .$$

$x = 3.79^c, 0.927^c$



Question 76 (****+)

The functions f and g are defined by

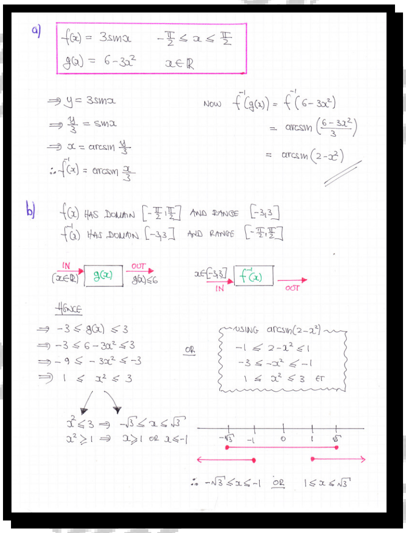
$$f(x) \equiv 3\sin x, \ x \in \mathbb{R}, \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$g(x) \equiv 6 - 3x^2, \ x \in \mathbb{R}.$$

a) Find an expression for $f^{-1}g(x)$.

b) Determine the domain of $f^{-1}g(x)$.

$\boxed{}$, $\boxed{f^{-1}g(x) = \arcsin\left(2 - x^2\right)}$, $\boxed{-\sqrt{3} \le x \le -1 \ \text{ or } \ 1 \le x \le \sqrt{3}}$

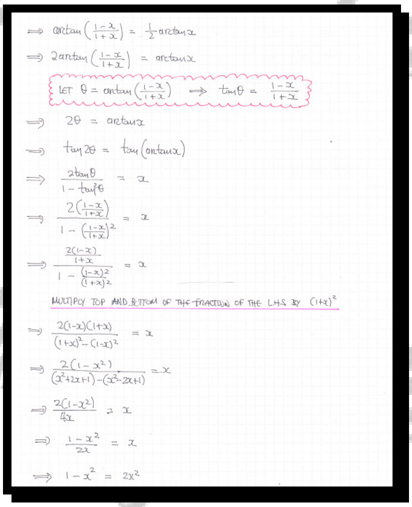


Question 77 (****+)

Find the solution of the equation

$$\arctan\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\arctan x .$$

SP-C , $x=\frac{\sqrt{3}}{3}$



$\Longrightarrow \arctan\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\arctan x$

$\Longrightarrow 2\arctan\left(\frac{1-x}{1+x}\right)=\arctan x$

LET $\theta=\arctan\left(\frac{1-x}{1+x}\right) \Longrightarrow \tan\theta=\frac{1-x}{1+x}$

$\Longrightarrow 2\theta=\arctan x$

$\Longrightarrow \tan 2\theta=\tan(\arctan x)$

$\Longrightarrow \frac{2\tan\theta}{1-\tan^2\theta}=x$

$\Longrightarrow \frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}=x$

$\Longrightarrow \frac{\frac{2(1-x)}{1+x}}{1-\frac{(1-x)^2}{(1+x)^2}}=x$

MULTIPLY TOP AND BOTTOM OF THE FRACTION OF THE LHS BY $(1+x)^2$

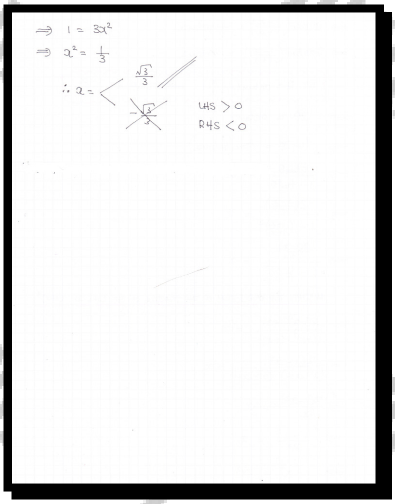
$\Longrightarrow \frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2}=x$

$\Longrightarrow \frac{2(1-x^2)}{(x^2+2x+1)-(x^2-2x+1)}=x$

$\Longrightarrow \frac{2(1-x^2)}{4x}=x$

$\Longrightarrow \frac{1-x^2}{2x}=x$

$\Longrightarrow 1-x^2=2x^2$



$\Longrightarrow 1=3x^2$

$\Longrightarrow x^2=\frac{1}{3}$

$\therefore x = \frac{\sqrt{3}}{3}$ or ~~$-\frac{\sqrt{3}}{3}$~~ (LHS >0, RHS <0)

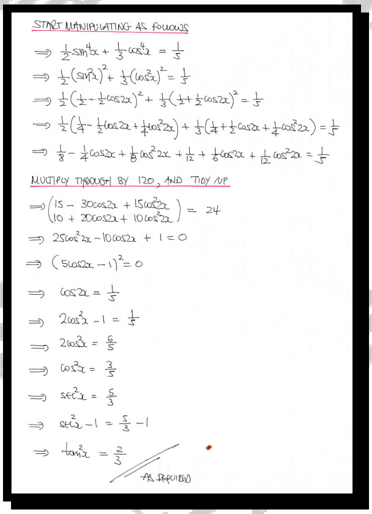
Question 78 (****+)

$$\frac{1}{2}\sin^4 x + \frac{1}{3}\cos^4 x = \frac{1}{5}.$$

Show that the above trigonometric equation is equivalent to

$$\tan^2 x = \frac{2}{3}.$$

[ ], [proof]



START MANIPULATING AS FOLLOWS

$\Rightarrow \frac{1}{2}\sin^4 x + \frac{1}{3}\cos^4 x = \frac{1}{5}$

$\Rightarrow \frac{1}{2}(\sin^2 x)^2 + \frac{1}{3}(\cos^2 x)^2 = \frac{1}{5}$

$\Rightarrow \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2 + \frac{1}{3}\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^2 = \frac{1}{5}$

$\Rightarrow \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\right) + \frac{1}{3}\left(\frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\right) = \frac{1}{5}$

$\Rightarrow \frac{1}{8} - \frac{1}{4}\cos 2x + \frac{1}{8}\cos^2 2x + \frac{1}{12} + \frac{1}{6}\cos 2x + \frac{1}{12}\cos^2 2x = \frac{1}{5}$

MULTIPLY THROUGH BY 120, AND TIDY UP

$\Rightarrow \begin{pmatrix} 15 - 30\cos 2x + 15\cos^2 2x \\ 10 + 20\cos 2x + 10\cos^2 2x \end{pmatrix} = 24$

$\Rightarrow 25\cos^2 2x - 10\cos 2x + 1 = 0$

$\Rightarrow (5\cos 2x - 1)^2 = 0$

$\Rightarrow \cos 2x = \frac{1}{5}$

$\Rightarrow 2\cos^2 x - 1 = \frac{1}{5}$

$\Rightarrow 2\cos^2 x = \frac{6}{5}$

$\Rightarrow \cos^2 x = \frac{3}{5}$

$\Rightarrow \sec^2 x = \frac{5}{3}$

$\Rightarrow \sec^2 x - 1 = \frac{5}{3} - 1$

$\Rightarrow \tan^2 x = \frac{2}{3}$

AS REQUIRED

Question 79 **(****+)**

A relationship is defined as

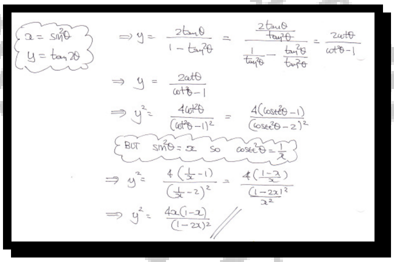
$$x = \sin^2 \theta, \quad 0 \le \theta < \frac{\pi}{4}.$$

$$y = \tan 2\theta, \quad 0 \le \theta < \frac{\pi}{4}.$$

Use trigonometric identities to show that

$$y^2 = \frac{4x(1-x)}{(1-2x)^2}.$$

proof



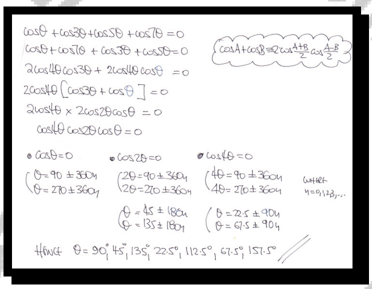
Question 80 **(****+)**

Find the solutions of the trigonometric equation

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0,$$

for which $0 \le \theta < 180°$.

$\theta = 22.5°, 45°, 67.5°, 90°, 112.5°, 135°, 157.5°$



Question 81 (****+)

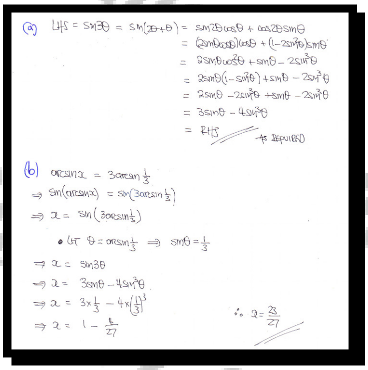
$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta .$$

a) Prove the validity of the above trigonometric identity.

b) Hence or otherwise solve the trigonometric equation

$$\arcsin x = 3\arcsin\left(\frac{1}{3}\right).$$

, $x = \frac{23}{27}$


(a) LHS $= \sin 3\theta = \sin(2\theta+\theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$
$= (2\sin\theta\cos\theta)\cos\theta + (1-2\sin^2\theta)\sin\theta$
$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$
$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$
$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$
$= 3\sin\theta - 4\sin^3\theta$
$=$ RHS // as required

(b) $\arcsin x = 3\arcsin\frac{1}{3}$
$\Rightarrow \sin(\arcsin x) = \sin(3\arcsin\frac{1}{3})$
$\Rightarrow x = \sin(3\arcsin\frac{1}{3})$
• Let $\theta = \arcsin\frac{1}{3} \Rightarrow \sin\theta = \frac{1}{3}$
$\Rightarrow x = \sin 3\theta$
$\Rightarrow x = 3\sin\theta - 4\sin^3\theta$
$\Rightarrow x = 3\times\frac{1}{3} - 4\times\left(\frac{1}{3}\right)^3$
$\Rightarrow x = 1 - \frac{4}{27}$
$\therefore x = \frac{23}{27}$

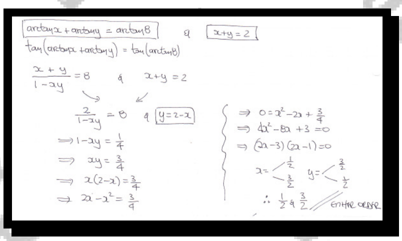
Question 82 (****+)

Solve the simultaneous equations

$$\arctan x + \arctan y = \arctan 8$$

$$x + y = 2 .$$

, $x = \frac{1}{2}, y = \frac{3}{2}$, in either order


$\arctan x + \arctan y = \arctan 8$ & $x+y=2$
$\tan(\arctan x + \arctan y) = \tan(\arctan 8)$
$\frac{x+y}{1-xy} = 8$ & $x+y=2$
$\frac{2}{1-xy} = 8$ & $y = 2-x$
$\Rightarrow 1-xy = \frac{1}{4}$
$\Rightarrow xy = \frac{3}{4}$
$\Rightarrow x(2-x) = \frac{3}{4}$
$\Rightarrow 2x - x^2 = \frac{3}{4}$
$\Rightarrow 0 = x^2 - 2x + \frac{3}{4}$
$\Rightarrow 4x^2 - 8x + 3 = 0$
$\Rightarrow (2x-3)(2x-1) = 0$
$x = \frac{1}{2}, \frac{3}{2}$; $y = \frac{3}{2}, \frac{1}{2}$
$\therefore \frac{1}{2}$ & $\frac{3}{2}$ // either order

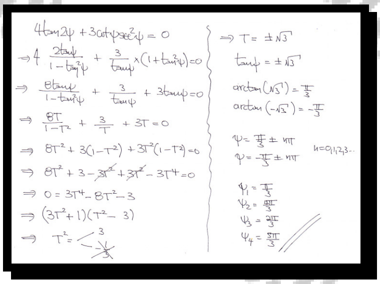
Question 83 (****+)

Solve the trigonometric equation

$$4\tan 2\psi + 3\cot\psi \sec^2\psi = 0, \quad 0 \le \psi < 2\pi,$$

giving the answers in terms of π.

$$\boxed{\psi = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$$

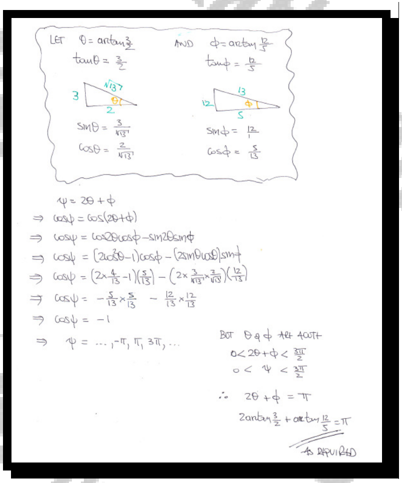


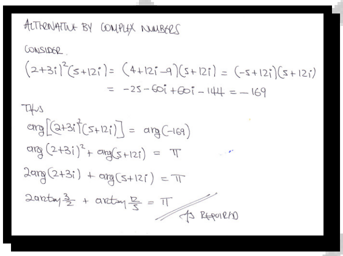
Question 84 (****+)

Show clearly that

$$2\arctan\left(\frac{3}{2}\right) + \arctan\left(\frac{12}{5}\right) = \pi.$$

V, proof



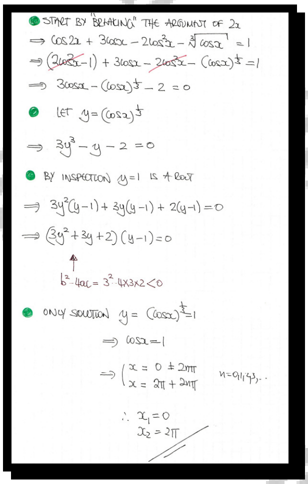


Question 85 (****+)

Find, in terms of π, the solutions of the trigonometric equation

$$\cos 2x + 3\cos x - 2\cos^2 x - \sqrt[3]{\cos x} = 1, \quad 0 \le x < 4\pi .$$

$x = 0, 2\pi$



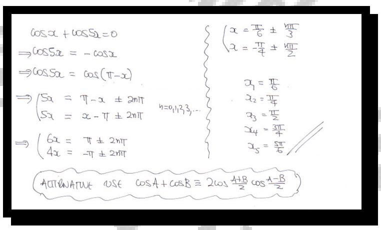
Question 86 (****+)

Solve the trigonometric equation

$$\cos x + \cos 5x = 0, \qquad 0 \le \theta < \pi ,$$

giving the answers in terms of π.

$$x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$$



Question 87 (****+)

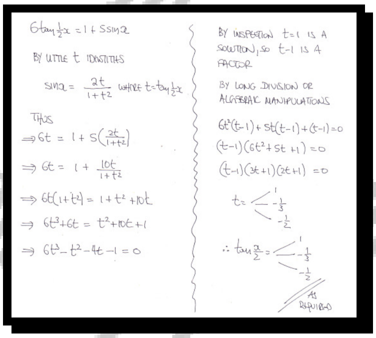
Use the substitution $t = \tan\frac{1}{2}x$ to show that if

$$6\tan\frac{1}{2}x = 1 + 5\sin x,$$

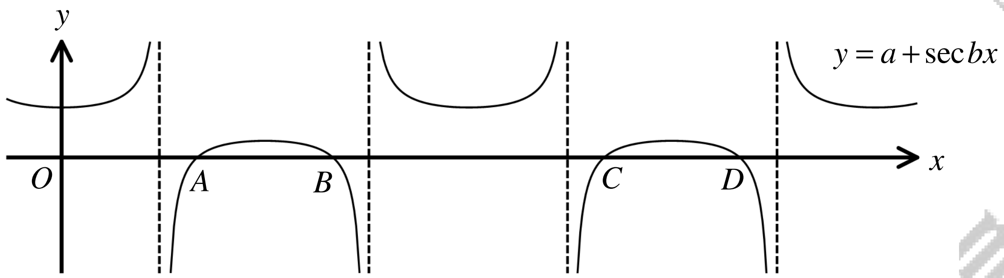
then the three possible values of $\tan\frac{1}{2}x$ are

$$1, \; -\frac{1}{2} \text{ or } -\frac{1}{3}.$$

[ ] , [proof]



Question 88 (****+)



The figure above shows part of the graph of

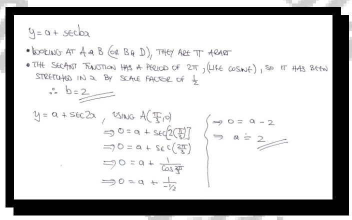
$$y = a + \sec bx,$$

where a and b are positive constants.

The points A, B, C and D are the x intercepts of the graph, with respective coordinates $\left(\frac{\pi}{3},0\right)$, $\left(\frac{2\pi}{3},0\right)$, $\left(\frac{4\pi}{3},0\right)$ and $\left(\frac{5\pi}{3},0\right)$.

Determine the value of a and the value of b.

$\boxed{a=2}$, $\boxed{b=2}$



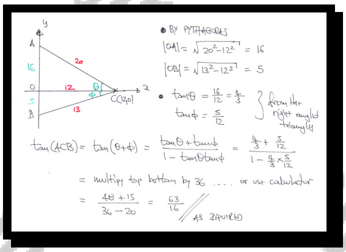
Question 89 (****+)

The point A lies on the y axis above the origin O and the point B lies on the y axis below the origin O.

The point $C(12,0)$ is at a distance of 20 units from A and at a distance of 13 units from B.

By considering the tangent ratios of $\measuredangle OCA$ and $\measuredangle OCB$, show that the tangent of the angle ACB is exactly $\frac{63}{16}$.

[ ], proof

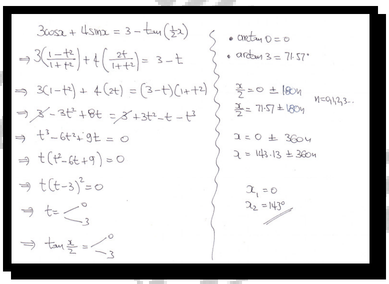


Question 90 (****+)

By using the substitution $t = \tan\left(\frac{1}{2}x\right)$ solve the trigonometric equation

$$3\cos x + 4\sin x = 3 - \tan\left(\frac{1}{2}x\right), \quad 0 \le x < 360° .$$

$x = 0°, 143.1°$



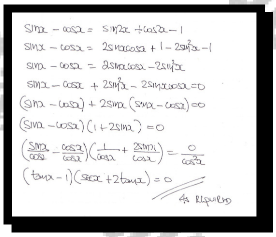
Question 91 (****+)

$$\sin x - \cos x = \sin 2x + \cos 2x - 1, \quad \cos x \neq 0,$$

Show that the above trigonometric equation is equivalent to

$$(\tan x - 1)(\sec x + 2\tan x) = 0.$$

proof

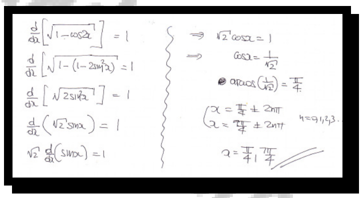


Question 92 (****+)

Solve the trigonometric equation

$$\frac{d}{dx}\left(\sqrt{1-\cos 2x}\right) = 1, \ 0 \leq x < 2\pi.$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$



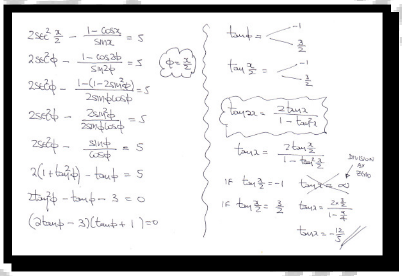
Question 93 (****+)

Given that

$$2\sec^2\left(\frac{x}{2}\right)-\frac{1-\cos x}{\sin x}=5,$$

find the **finite** value of $\tan x$.

$$\boxed{\tan x=-\frac{12}{5}}$$

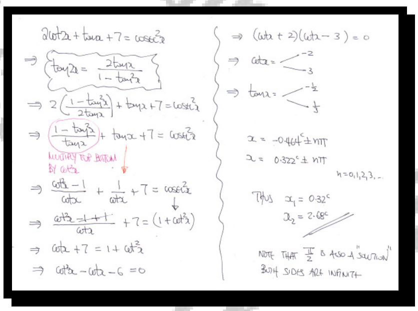


Question 94 (****+)

$$2\cot 2x+\tan x+7=\operatorname{cosec}^2 x,\ \ 0\le x<\pi.$$

Given that $x\neq\frac{\pi}{2}$, find the solutions of the above trigonometric equation, giving the answers in radians correct to two decimal places.

$$\boxed{x=0.32^{\text{c}},2.68^{\text{c}}}$$

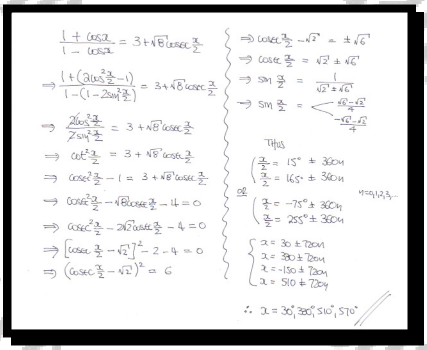


Question 95 (****+)

$$\frac{1+\cos x}{1-\cos x} = 3+\sqrt{8}\,\text{cosec}\left(\frac{x}{2}\right),\ 0 \le x < 720 .$$

Find the solutions of the above trigonometric equation, giving the answers in **degrees**.

$$\boxed{x = 30°, 330°, 510°, 570°}$$



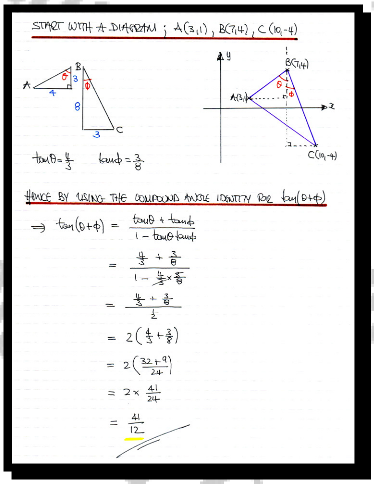
Question 96 (****+)

A triangle has vertices at the points with coordinates $A(3,1)$, $B(7,4)$ and $C(10,-4)$.

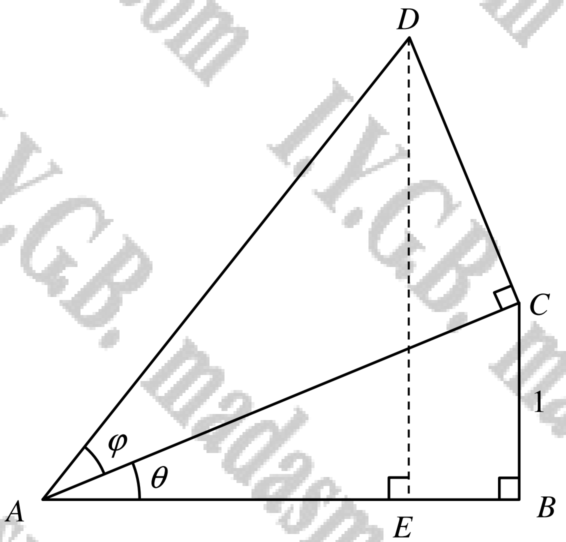
The acute angle θ is defined as the angle formed between AB and the straight line which is parallel to the y axis and passes through B.

Find the value of $\tan\theta$ and hence show that $\tan(\measuredangle ABC) = \frac{41}{12}$.

$\boxed{}$, $\boxed{\tan\theta = \frac{4}{3}}$



Question 97 (****+)



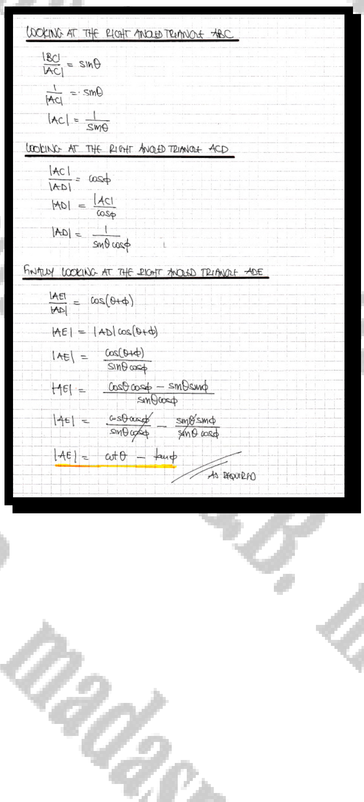
The figure above shows two right angles triangles ABC and ACD. The angles CAB and DAC are denoted by θ and φ, respectively.

The length of BC is 1.

The point E lies on AB so that the angle AED is $90°$.

Show clearly that the length of AE is $\cot\theta - \tan\varphi$.

[ ], proof



LOOKING AT THE RIGHT ANGLED TRIANGLE ABC

$\frac{|BC|}{|AC|} = \sin\theta$

$\frac{1}{|AC|} = \sin\theta$

$|AC| = \frac{1}{\sin\theta}$

LOOKING AT THE RIGHT ANGLED TRIANGLE ACD

$\frac{|AC|}{|AD|} = \cos\phi$

$|AD| = \frac{|AC|}{\cos\phi}$

$|AD| = \frac{1}{\sin\theta\cos\phi}$

FINALLY LOOKING AT THE RIGHT ANGLED TRIANGLE ADE

$\frac{|AE|}{|AD|} = \cos(\theta+\phi)$

$|AE| = |AD|\cos(\theta+\phi)$

$|AE| = \frac{\cos(\theta+\phi)}{\sin\theta\cos\phi}$

$|AE| = \frac{\cos\theta\cos\phi - \sin\theta\sin\phi}{\sin\theta\cos\phi}$

$|AE| = \frac{\cos\theta\cos\phi}{\sin\theta\cos\phi} - \frac{\sin\theta\sin\phi}{\sin\theta\cos\phi}$

$|AE| = \cot\theta - \tan\phi$

AS REQUIRED

Question 98 (****+)

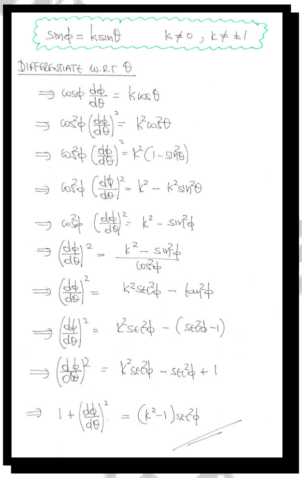
It is given that

$$\sin\varphi = k\sin\theta\,,\ k \neq 0\,,\ k \neq \pm 1\,.$$

Show, by a detailed method, that

$$1+\left(\frac{d\varphi}{d\theta}\right)^2 = \left(k^2-1\right)\sec^2\varphi$$

[ ], proof


$\sin\phi = k\sin\theta \qquad k\neq 0\,,\ k\neq\pm 1$

DIFFERENTIATE W.R.T θ

$\Rightarrow \cos\phi\frac{d\phi}{d\theta} = k\cos\theta$

$\Rightarrow \cos^2\phi\left(\frac{d\phi}{d\theta}\right)^2 = k^2\cos^2\theta$

$\Rightarrow \cos^2\phi\left(\frac{d\phi}{d\theta}\right)^2 = k^2(1-\sin^2\theta)$

$\Rightarrow \cos^2\phi\left(\frac{d\phi}{d\theta}\right)^2 = k^2 - k^2\sin^2\theta$

$\Rightarrow \cos^2\phi\left(\frac{d\phi}{d\theta}\right)^2 = k^2 - \sin^2\phi$

$\Rightarrow \left(\frac{d\phi}{d\theta}\right)^2 = \frac{k^2-\sin^2\phi}{\cos^2\phi}$

$\Rightarrow \left(\frac{d\phi}{d\theta}\right)^2 = k^2\sec^2\phi - \tan^2\phi$

$\Rightarrow \left(\frac{d\phi}{d\theta}\right)^2 = k^2\sec^2\phi - (\sec^2\phi - 1)$

$\Rightarrow \left(\frac{d\phi}{d\theta}\right)^2 = k^2\sec^2\phi - \sec^2\phi + 1$

$\Rightarrow 1+\left(\frac{d\phi}{d\theta}\right)^2 = (k^2-1)\sec^2\phi$

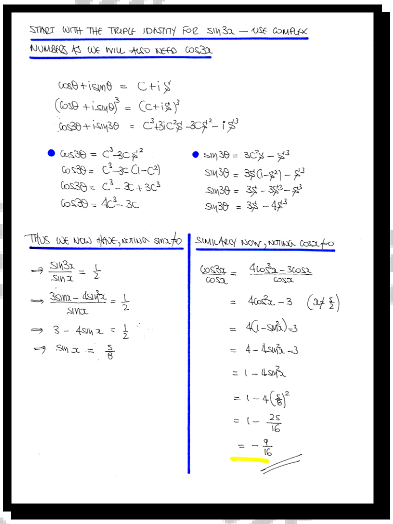
Question 99 (****+)

It is given that if $x \in \left(0, \frac{1}{2}\pi\right)$, then

$$\frac{\sin 3x}{\sin x} = \frac{1}{2}.$$

Determine the exact value of $\dfrac{\cos 3x}{\cos x}$.

$\square$, $\boxed{-\frac{9}{16}}$


START WITH THE TRIPLE IDENTITY FOR $\sin 3x$ — USE COMPLEX NUMBERS AS WE WILL ALSO NEED $\cos 3x$

$\cos\theta + i\sin\theta = C + iS$

$(\cos\theta + i\sin\theta)^3 = (C+iS)^3$

$\cos 3\theta + i\sin 3\theta = C^3 + 3iC^2S - 3CS^2 - iS^3$

- $\cos 3\theta = C^3 - 3CS^2$
  $\cos 3\theta = C^3 - 3C(1-C^2)$
  $\cos 3\theta = C^3 - 3C + 3C^3$
  $\cos 3\theta = 4C^3 - 3C$

- $\sin 3\theta = 3C^2S - S^3$
  $\sin 3\theta = 3S(1-S^2) - S^3$
  $\sin 3\theta = 3S - 3S^3 - S^3$
  $\sin 3\theta = 3S - 4S^3$

THUS WE NOW HAVE, NOTING $\sin x \neq 0$

$\Rightarrow \frac{\sin 3x}{\sin x} = \frac{1}{2}$

$\Rightarrow \frac{3\sin x - 4\sin^3 x}{\sin x} = \frac{1}{2}$

$\Rightarrow 3 - 4\sin^2 x = \frac{1}{2}$

$\Rightarrow \sin^2 x = \frac{5}{8}$

SIMILARLY NOW, NOTING $\cos x \neq 0$

$\frac{\cos 3x}{\cos x} = \frac{4\cos^3 x - 3\cos x}{\cos x}$

$= 4\cos^2 x - 3 \quad \left(x \neq \frac{\pi}{2}\right)$

$= 4(1-\sin^2 x) - 3$

$= 4 - 4\sin^2 x - 3$

$= 1 - 4\sin^2 x$

$= 1 - 4\left(\frac{5}{8}\right)$

$= 1 - \frac{25}{16}$

$= -\frac{9}{16}$

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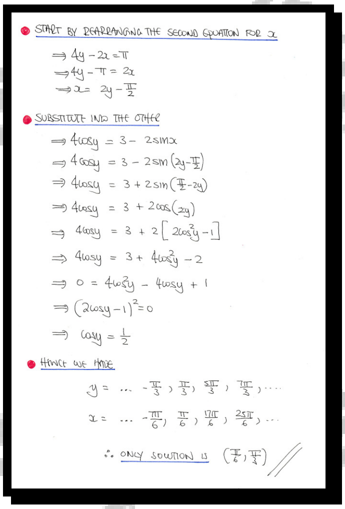
ENRICHMENT QUESTIONS

Question 1 (*****)

Use trigonometric algebra to find the solution of the following simultaneous equations, in the intervals $0 \le x < 2\pi$, $0 \le y < 2\pi$.

$$4\cos y = 3 - 2\sin x \qquad \text{and} \qquad 4y - 2x = \pi .$$

$\boxed{}$, $\boxed{x = \frac{1}{6}\pi, \quad y = \frac{1}{3}\pi}$



START BY REARRANGING THE SECOND EQUATION FOR x

$\Rightarrow 4y - 2x = \pi$
$\Rightarrow 4y - \pi = 2x$
$\Rightarrow x = 2y - \frac{\pi}{2}$

SUBSTITUTE INTO THE OTHER

$\Rightarrow 4\cos y = 3 - 2\sin x$
$\Rightarrow 4\cos y = 3 - 2\sin\left(2y - \frac{\pi}{2}\right)$
$\Rightarrow 4\cos y = 3 + 2\sin\left(\frac{\pi}{2} - 2y\right)$
$\Rightarrow 4\cos y = 3 + 2\cos(2y)$
$\Rightarrow 4\cos y = 3 + 2\left[2\cos^2 y - 1\right]$
$\Rightarrow 4\cos y = 3 + 4\cos^2 y - 2$
$\Rightarrow 0 = 4\cos^2 y - 4\cos y + 1$
$\Rightarrow (2\cos y - 1)^2 = 0$
$\Rightarrow \cos y = \frac{1}{2}$

HENCE WE HAVE

$y = \ldots -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \ldots$

$x = \ldots -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \ldots$

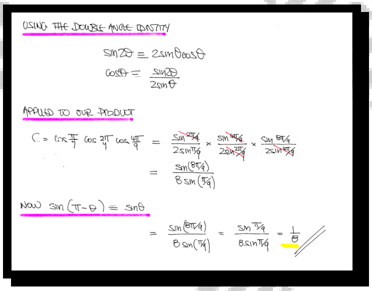
$\therefore$ ONLY SOLUTION IS $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

Question 2 (*****)

$$C = \cos\left(\tfrac{1}{9}\pi\right)\cos\left(\tfrac{2}{9}\pi\right)\cos\left(\tfrac{4}{9}\pi\right)$$

Use the **sine double angle identity** for sine to show that $C = \dfrac{1}{8}$.

X , proof


USING THE DOUBLE ANGLE IDENTITY

$$\sin 2\theta \equiv 2\sin\theta\cos\theta$$

$$\cos\theta \equiv \frac{\sin 2\theta}{2\sin\theta}$$

APPLIED TO OUR PRODUCT

$$C = \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{\sin 2\pi/9}{2\sin\pi/9} \times \frac{\sin 4\pi/9}{2\sin 2\pi/9} \times \frac{\sin 8\pi/9}{2\sin 4\pi/9}$$

$$= \frac{\sin(8\pi/9)}{8\sin(\pi/9)}$$

NOW $\sin(\pi - \theta) = \sin\theta$

$$= \frac{\sin(8\pi/9)}{8\sin(\pi/9)} = \frac{\sin\pi/9}{8\sin\pi/9} = \frac{1}{8}$$

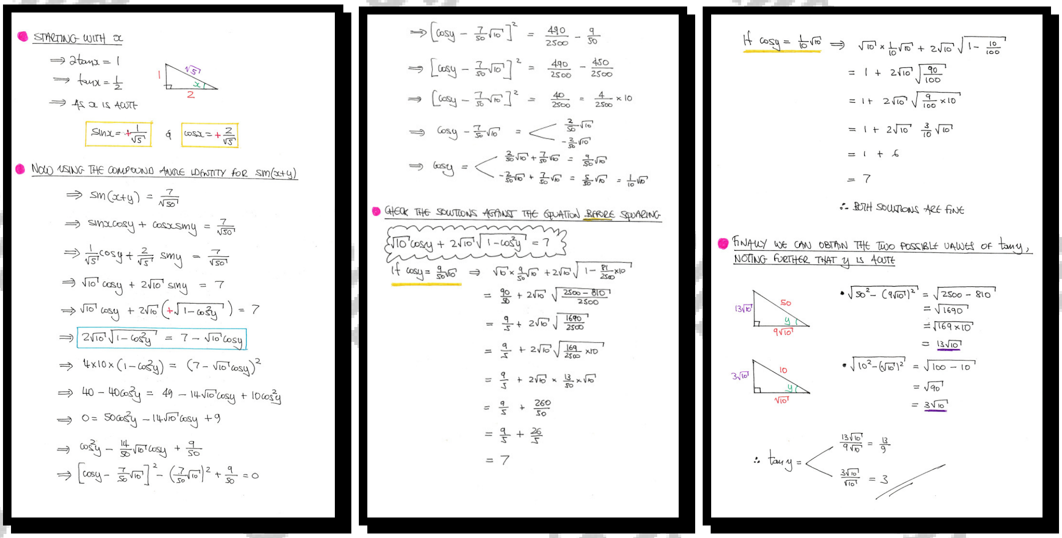
Question 3 (*****)

The acute angles x and y, satisfy the following relationships.

$$2\tan x = 1 \qquad \text{and} \qquad \sin(x+y) = \frac{7}{\sqrt{50}}.$$

Determine the possible values of $\tan y$.

$$\boxed{}, \quad \boxed{\tan y = \begin{cases} 3 \\ \frac{19}{3} \end{cases}}$$



Question 4 (*****)

Two circles, C_1 and C_2, have respective radii of 4 units and 1 unit and are touching each other externally at the point A.

The coordinates axes are tangents to C_1, whose centre P lies in the first quadrant.

The x axis is a tangent to C_2, whose centre Q also lies in the first quadrant.

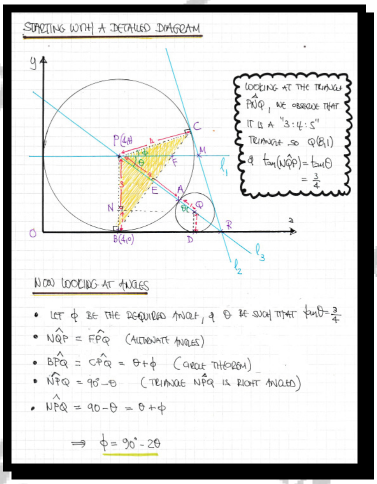
The straight line l_1 is parallel to the x axis and passes through P.

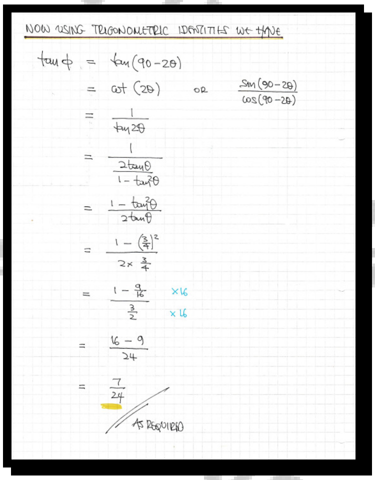
The straight line l_2 has negative gradient and is a common tangent to C_1 and C_2, touching C_1 at the point C.

The acute angle formed by PC and l_1 is denoted by φ.

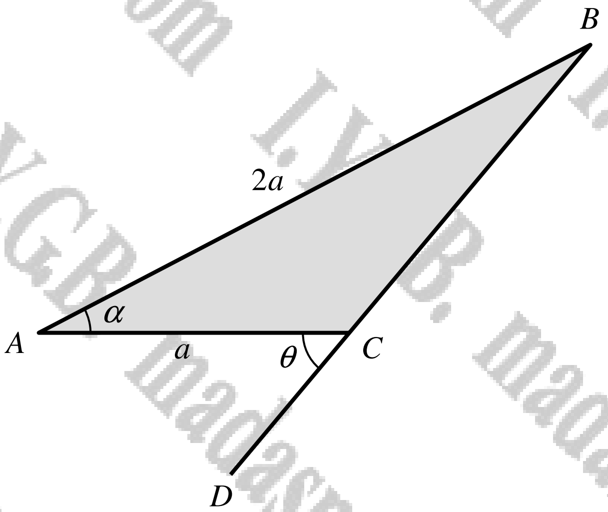
Show that $\tan\varphi = \dfrac{7}{24}$.

[ ], proof





Question 5 (*****)



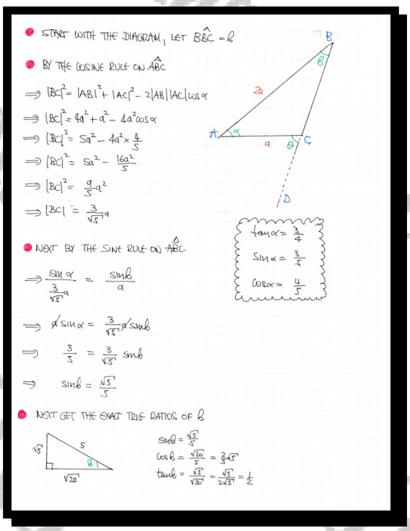
The figure above shows a triangle ABC, where $|AB| = a$ and $|AC| = 2a$.

The angle BAC is α, where $\tan\alpha = \frac{3}{4}$.

The side BC is extended to the point D so that the angle ACD is denoted by θ.

Show clearly that $\theta = \arctan 2$.

[ ], proof



START WITH THE DIAGRAM, LET $B\hat{B}C = \beta$

BY THE COSINE RULE ON $A\hat{B}C$

$\Rightarrow |BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC|\cos\alpha$

$\Rightarrow |BC|^2 = 4a^2 + a^2 - 4a^2\cos\alpha$

$\Rightarrow |BC|^2 = 5a^2 - 4a^2 \times \frac{4}{5}$

$\Rightarrow |BC|^2 = 5a^2 - \frac{16a^2}{5}$

$\Rightarrow |BC|^2 = \frac{9}{5}a^2$

$\Rightarrow |BC| = \frac{3}{\sqrt{5}}a$

$\tan\alpha = \frac{3}{4}$, $\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$

NEXT BY THE SINE RULE ON $A\hat{B}C$

$\Rightarrow \frac{\sin\alpha}{\frac{3}{\sqrt{5}}a} = \frac{\sin\beta}{a}$

$\Rightarrow a\sin\alpha = \frac{3}{\sqrt{5}}a\sin\beta$

$\Rightarrow \frac{3}{5} = \frac{3}{\sqrt{5}}\sin\beta$

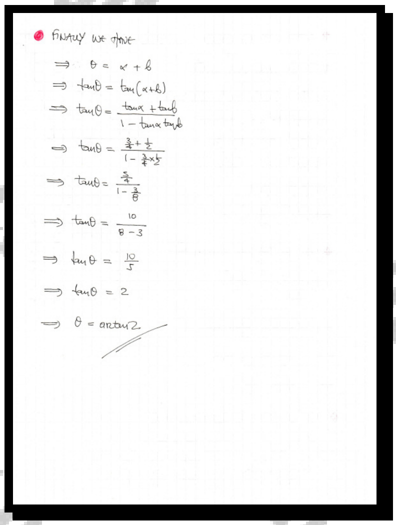
$\Rightarrow \sin\beta = \frac{\sqrt{5}}{5}$

NEXT GET THE EXACT TRIG RATIOS OF β

$\sin\beta = \frac{\sqrt{5}}{5}$

$\cos\beta = \frac{\sqrt{20}}{5} = \frac{2}{5}\sqrt{5}$

$\tan\beta = \frac{\sqrt{5}}{\sqrt{20}} = \frac{\sqrt{5}}{2\sqrt{5}} = \frac{1}{2}$



FINALLY WE HAVE

$\Rightarrow \theta = \alpha + \beta$

$\Rightarrow \tan\theta = \tan(\alpha+\beta)$

$\Rightarrow \tan\theta = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\Rightarrow \tan\theta = \frac{\frac{3}{4} + \frac{1}{2}}{1 - \frac{3}{4}\times\frac{1}{2}}$

$\Rightarrow \tan\theta = \frac{\frac{5}{4}}{1 - \frac{3}{8}}$

$\Rightarrow \tan\theta = \frac{10}{8-3}$

$\Rightarrow \tan\theta = \frac{10}{5}$

$\Rightarrow \tan\theta = 2$

$\Rightarrow \theta = \arctan 2$

Question 6 (*****)

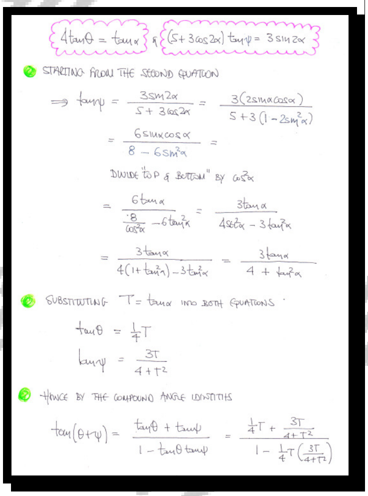
The acute angles θ, ψ and α satisfy the following equations.

$$4\tan\theta = \tan\alpha$$

$$(5+3\cos 2\alpha)\tan\psi = 3\sin 2\alpha .$$

Express $\theta + \psi$, in terms of α.

$\boxed{}$, $\boxed{\theta + \psi = \alpha}$



$4\tan\theta = \tan\alpha$ & $(5+3\cos 2\alpha)\tan\psi = 3\sin 2\alpha$

STARTING FROM THE SECOND EQUATION

$$\Rightarrow \tan\psi = \frac{3\sin 2\alpha}{5+3\cos 2\alpha} = \frac{3(2\sin\alpha\cos\alpha)}{5+3(1-2\sin^2\alpha)} = \frac{6\sin\alpha\cos\alpha}{8-6\sin^2\alpha} =$$

DIVIDE "TOP & BOTTOM" BY $\cos^2\alpha$

$$= \frac{6\tan\alpha}{\frac{8}{\cos^2\alpha} - 6\tan^2\alpha} = \frac{3\tan\alpha}{4\sec^2\alpha - 3\tan^2\alpha}$$

$$= \frac{3\tan\alpha}{4(1+\tan^2\alpha) - 3\tan^2\alpha} = \frac{3\tan\alpha}{4+\tan^2\alpha}$$

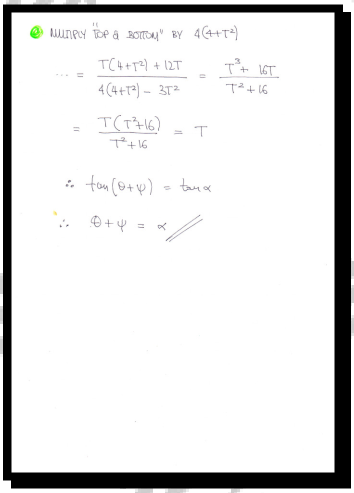
SUBSTITUTING $T = \tan\alpha$ INTO BOTH EQUATIONS

$$\tan\theta = \tfrac{1}{4}T$$

$$\tan\psi = \frac{3T}{4+T^2}$$

HENCE BY THE COMPOUND ANGLE IDENTITIES

$$\tan(\theta+\psi) = \frac{\tan\theta + \tan\psi}{1-\tan\theta\tan\psi} = \frac{\frac{1}{4}T + \frac{3T}{4+T^2}}{1 - \frac{1}{4}T\left(\frac{3T}{4+T^2}\right)}$$



MULTIPLY "TOP & BOTTOM" BY $4(4+T^2)$

$$\dots = \frac{T(4+T^2)+12T}{4(4+T^2)-3T^2} = \frac{T^3+16T}{T^2+16}$$

$$= \frac{T(T^2+16)}{T^2+16} = T$$

$$\therefore \tan(\theta+\psi) = \tan\alpha$$

$$\therefore \theta + \psi = \alpha$$

Question 7 (*****)

The acute angles θ and φ satisfy the following equations

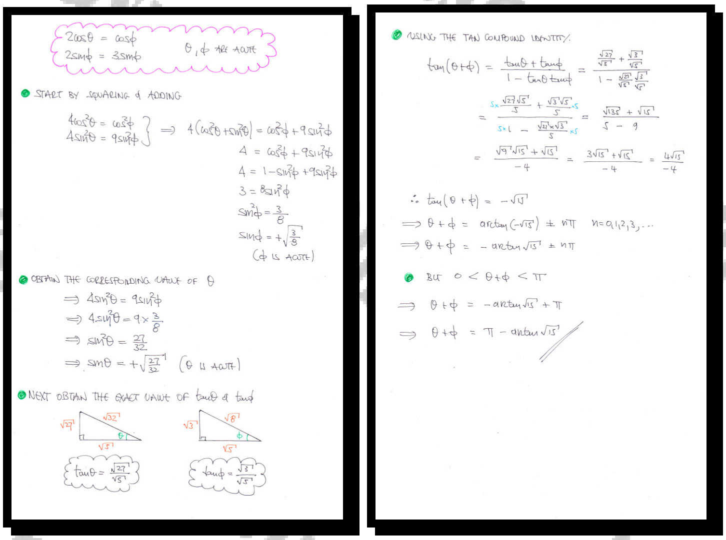
$$2\cos\theta = \cos\varphi$$

$$2\sin\theta = 3\sin\varphi .$$

Show clearly that

$$\theta + \varphi = \pi - \arctan\sqrt{15}$$

[ ], [proof]

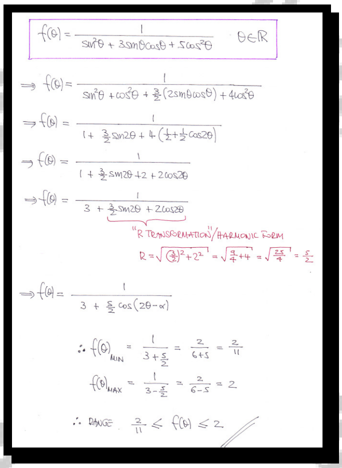


Question 8 (*****)

Determine the range of the following function

$$f(\theta) \equiv \frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}, \quad \theta \in \mathbb{R}$$

$\boxed{}$, $\boxed{\frac{2}{11} \le f(\theta) \le 2}$



$$f(\theta) = \frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta} \quad \theta \in \mathbb{R}$$

$$\Rightarrow f(\theta) = \frac{1}{\sin^2\theta + \cos^2\theta + \frac{3}{2}(2\sin\theta\cos\theta) + 4\cos^2\theta}$$

$$\Rightarrow f(\theta) = \frac{1}{1 + \frac{3}{2}\sin 2\theta + 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)}$$

$$\Rightarrow f(\theta) = \frac{1}{1 + \frac{3}{2}\sin 2\theta + 2 + 2\cos 2\theta}$$

$$\Rightarrow f(\theta) = \frac{1}{3 + \frac{3}{2}\sin 2\theta + 2\cos 2\theta}$$

"R TRANSFORMATION"/HARMONIC FORM

$$R = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2} = \sqrt{\frac{9}{4} + 4} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

$$\Rightarrow f(\theta) = \frac{1}{3 + \frac{5}{2}\cos(2\theta - \alpha)}$$

$$\therefore f(\theta)_{MIN} = \frac{1}{3 + \frac{5}{2}} = \frac{2}{6+5} = \frac{2}{11}$$

$$f(\theta)_{MAX} = \frac{1}{3 - \frac{5}{2}} = \frac{2}{6-5} = 2$$

$\therefore$ RANGE $\frac{2}{11} \le f(\theta) \le 2$

Question 9 (*****)

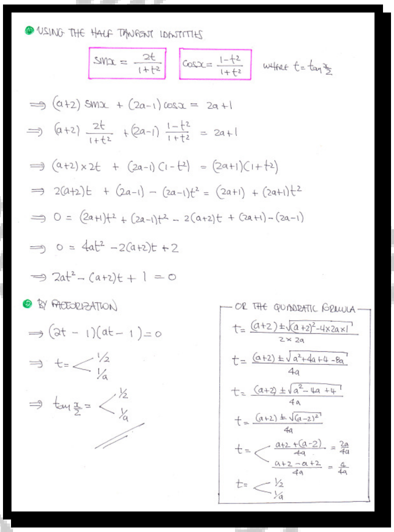
It is given that

$$(a+2)\sin x + (2a-1)\cos x = 2a+1,$$

where a is a non zero constant.

Find the exact value of $\tan\left(\frac{1}{2}x\right)$, giving the answer in terms of a, where appropriate.

$\boxed{}$, $\boxed{\tan\left(\frac{1}{2}x\right)=\frac{1}{2}}$, $\boxed{\tan\left(\frac{1}{2}x\right)=\frac{1}{a}}$

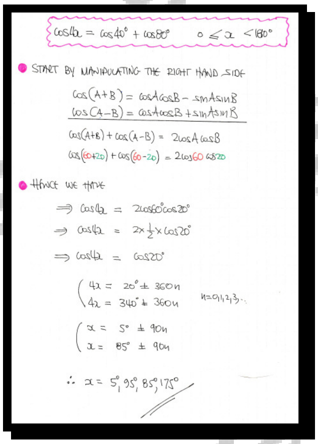


Question 10 (*****)

Solve the following trigonometric equation.

$$\cos 4x° = \cos 40° + \cos 80°, \qquad 0° \le x \le 180°.$$

[ ] , $x = 5°, 85°, 95° \ 175°$



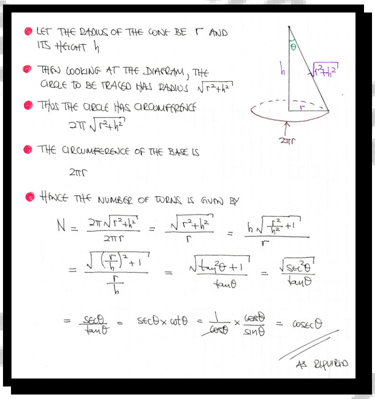
Question 11 (*****)

A right circular cone, of radius r and semi-vertical angle θ, lies with one of its generators in contact with a horizontal surface.

The cone is then rolled on the horizontal surface with its vertex at rest, so that the rolling circumference of its base completes a full circle on the surface, while the cone completes N revolutions about its own axis.

Show that $N = \operatorname{cosec}\theta$.

SP- , proof



Question 12 (*****)

Solve the following trigonometric equation.

$$2\sqrt{3}\sin\left(x+\frac{7\pi}{12}\right)=3\operatorname{cosec}\left(x+\frac{5\pi}{12}\right), \qquad 0\le x\le 2\pi .$$

$\boxed{}$, $\boxed{x=\frac{\pi}{12},\ \frac{11\pi}{12},\ \frac{13\pi}{12}\ \frac{23\pi}{12}}$

$2\sqrt{3}\sin\left(x+\frac{7\pi}{12}\right) = 3\operatorname{cosec}\left(x+\frac{5\pi}{12}\right), \quad 0\leqslant x<2\pi$

$\Rightarrow 2\sqrt{3}\sin\left(x+\frac{7\pi}{12}\right) = \frac{3}{\sin\left(x+\frac{5\pi}{12}\right)}$

$\Rightarrow 2\sin\left(x+\frac{7\pi}{12}\right)\sin\left(x+\frac{5\pi}{12}\right) = \frac{3}{\sqrt{3}}$

NOW DERIVE AN IDENTITY BASED ON THE COMPOUND ANGLE IDENTITIES

$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$
$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$ } SUBTRACT "UPWARDS"

$\cos(A-B) - \cos(A+B) \equiv 2\sin A\sin B$

$\Rightarrow \cos\left[\left(x+\frac{7\pi}{12}\right)-\left(x-\frac{5\pi}{12}\right)\right] - \cos\left[\left(x+\frac{7\pi}{12}\right)+\left(x+\frac{5\pi}{12}\right)\right] = \frac{3\sqrt{3}}{\sqrt{3}\sqrt{3}}$

$\Rightarrow \cos\frac{\pi}{6} - \cos(2x+\pi) = \sqrt{3}$

$\Rightarrow \frac{\sqrt{3}}{2} - \left[\cos 2x\cos\pi - \sin 2x\sin\pi\right] = \sqrt{3}$

$\Rightarrow \frac{\sqrt{3}}{2} + \cos 2x = \sqrt{3}$

$\Rightarrow \cos 2x = \frac{\sqrt{3}}{2}$

$\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

$\begin{cases} 2x = \frac{\pi}{6} \pm 2n\pi \\ 2x = \frac{11\pi}{6} \pm 2n\pi \end{cases}$ $\quad n=0,1,2,3,\ldots$

$\begin{cases} x = \frac{\pi}{12} \pm n\pi \\ x = \frac{11\pi}{12} \pm n\pi \end{cases}$

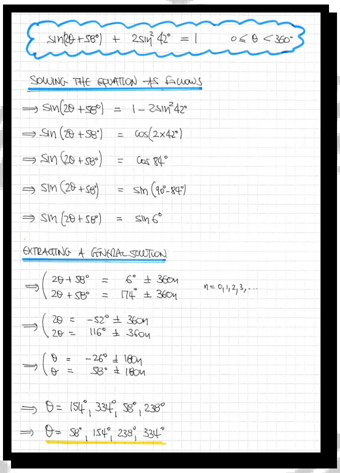
$x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

Question 13 (*****)

Solve the following trigonometric equation.

$$\sin(2\theta + 58)° + 2\sin^2(42°) = 1, \quad 0 \le \theta < 360.$$

$\boxed{}$, $\boxed{\theta = \{58, 154, 238, 334\}}$



$\sin(2\theta + 58°) + 2\sin^2 42° = 1 \qquad 0 \le \theta < 360°$

SOLVING THE EQUATION AS FOLLOWS

$\Rightarrow \sin(2\theta + 58°) = 1 - 2\sin^2 42°$

$\Rightarrow \sin(2\theta + 58°) = \cos(2 \times 42°)$

$\Rightarrow \sin(2\theta + 58°) = \cos 84°$

$\Rightarrow \sin(2\theta + 58°) = \sin(90° - 84°)$

$\Rightarrow \sin(2\theta + 58°) = \sin 6°$

EXTRACTING A GENERAL SOLUTION

$\Rightarrow \begin{cases} 2\theta + 58° = 6° \pm 360n \\ 2\theta + 58° = 174° \pm 360n \end{cases} \quad n = 0, 1, 2, 3, \dots$

$\Rightarrow \begin{cases} 2\theta = -52° \pm 360n \\ 2\theta = 116° \pm 360n \end{cases}$

$\Rightarrow \begin{cases} \theta = -26° \pm 180n \\ \theta = 58° \pm 180n \end{cases}$

$\Rightarrow \theta = 154°, 334°, 58°, 238°$

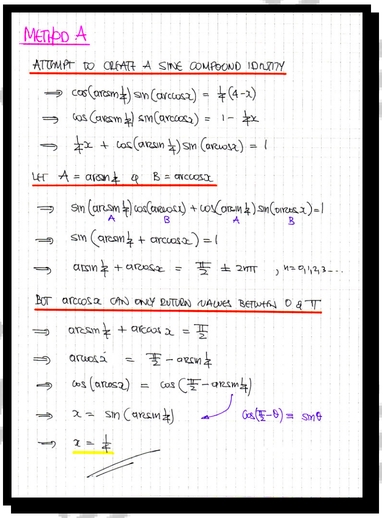
$\Rightarrow \theta = 58°, 154°, 238°, 334°$

Question 14 (*****)

Solve the following trigonometric equation

$$\cos\left(\arcsin\tfrac{1}{4}\right)\sin\left(\arccos x\right)=\tfrac{1}{4}(4-x)\ ,\quad x\in\mathbb{R}\,.$$

[ ] , $x=\frac{1}{4}$



METHOD A

ATTEMPT TO CREATE A SINE COMPOUND IDENTITY

$\Rightarrow \cos\left(\arcsin\frac{1}{4}\right)\sin\left(\arccos x\right)=\frac{1}{4}(4-x)$

$\Rightarrow \cos\left(\arcsin\frac{1}{4}\right)\sin\left(\arccos x\right)=1-\frac{1}{4}x$

$\Rightarrow \frac{1}{4}x+\cos\left(\arcsin\frac{1}{4}\right)\sin\left(\arccos x\right)=1$

LET $A=\arcsin\frac{1}{4}$ & $B=\arccos x$

$\Rightarrow \sin\left(\arcsin\frac{1}{4}\right)\cos\left(\arccos x\right)+\cos\left(\arcsin\frac{1}{4}\right)\sin\left(\arccos x\right)=1$

$\Rightarrow \sin\left(\arcsin\frac{1}{4}+\arccos x\right)=1$

$\Rightarrow \arcsin\frac{1}{4}+\arccos x=\frac{\pi}{2}\pm 2n\pi\ ,\ n=0,1,2,3\ldots$

BUT $\arccos x$ CAN ONLY RETURN VALUES BETWEEN 0 & π

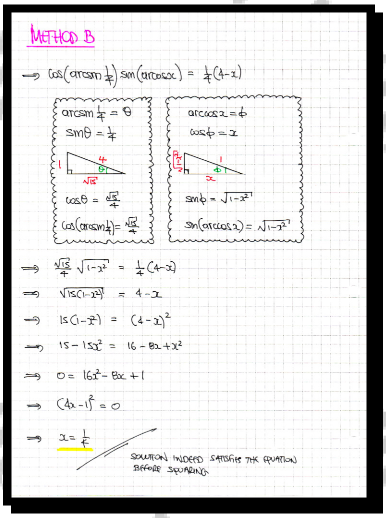
$\Rightarrow \arcsin\frac{1}{4}+\arccos x=\frac{\pi}{2}$

$\Rightarrow \arccos x=\frac{\pi}{2}-\arcsin\frac{1}{4}$

$\Rightarrow \cos\left(\arccos x\right)=\cos\left(\frac{\pi}{2}-\arcsin\frac{1}{4}\right)$

$\Rightarrow x=\sin\left(\arcsin\frac{1}{4}\right)$ $\quad \cos\left(\frac{\pi}{2}-\theta\right)\equiv\sin\theta$

$\Rightarrow x=\frac{1}{4}$



METHOD B

$\Rightarrow \cos\left(\arcsin\frac{1}{4}\right)\sin\left(\arccos x\right)=\frac{1}{4}(4-x)$

$\arcsin\frac{1}{4}=\theta$, $\sin\theta=\frac{1}{4}$, $\cos\theta=\frac{\sqrt{15}}{4}$, $\cos\left(\arcsin\frac{1}{4}\right)=\frac{\sqrt{15}}{4}$

$\arccos x=\phi$, $\cos\phi=x$, $\sin\phi=\sqrt{1-x^2}$, $\sin\left(\arccos x\right)=\sqrt{1-x^2}$

$\Rightarrow \frac{\sqrt{15}}{4}\sqrt{1-x^2}=\frac{1}{4}(4-x)$

$\Rightarrow \sqrt{15(1-x^2)}=4-x$

$\Rightarrow 15(1-x^2)=(4-x)^2$

$\Rightarrow 15-15x^2=16-8x+x^2$

$\Rightarrow 0=16x^2-8x+1$

$\Rightarrow (4x-1)^2=0$

$\Rightarrow x=\frac{1}{4}$

SOLUTION INDEED SATISFIES THE EQUATION BEFORE SQUARING

Question 15 (*****)

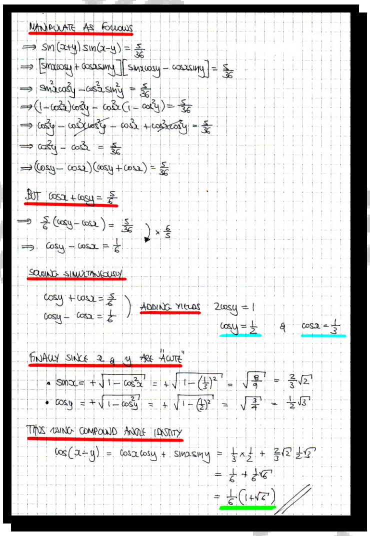
It is given that $0 < x < \frac{1}{2}\pi$ and $0 < y < \frac{1}{2}\pi$.

It is further given that

$$\sin(x+y)\sin(x-y) = \frac{5}{36} \quad \text{and} \quad \cos x + \cos y = \frac{5}{6}.$$

Show that $\cos(x-y) = \dfrac{1+\sqrt{n}}{n}$, where n is a positive integer to be found.

$\boxed{}$, $\boxed{n=6}$



Question 16 (*****)

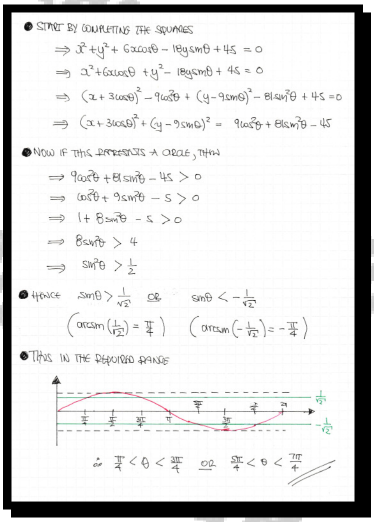
A curve in the x-y plane has equation

$$x^2 + y^2 + 6x\cos\theta - 18y\sin\theta + 45 = 0,$$

where θ is a parameter such that $0 \le \theta < 2\pi$.

Given that curve represents a circle determine the range of possible values of θ.

$\left\{\frac{1}{4}\pi < \theta < \frac{3}{4}\pi\right\} \cup \left\{\frac{5}{4}\pi < \theta < \frac{7}{4}\pi\right\}$



- START BY COMPLETING THE SQUARES

$\Rightarrow x^2+y^2+6x\cos\theta-18y\sin\theta+45=0$

$\Rightarrow x^2+6x\cos\theta+y^2-18y\sin\theta+45=0$

$\Rightarrow (x+3\cos\theta)^2-9\cos^2\theta+(y-9\sin\theta)^2-81\sin^2\theta+45=0$

$\Rightarrow (x+3\cos\theta)^2+(y-9\sin\theta)^2=9\cos^2\theta+81\sin^2\theta-45$

- NOW IF THIS REPRESENTS A CIRCLE, THEN

$\Rightarrow 9\cos^2\theta+81\sin^2\theta-45>0$

$\Rightarrow \cos^2\theta+9\sin^2\theta-5>0$

$\Rightarrow 1+8\sin^2\theta-5>0$

$\Rightarrow 8\sin^2\theta>4$

$\Rightarrow \sin^2\theta>\frac{1}{2}$

- HENCE $\sin\theta>\frac{1}{\sqrt{2}}$ OR $\sin\theta<-\frac{1}{\sqrt{2}}$

$\left(\arcsin\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}\right)$ $\left(\arcsin\left(-\frac{1}{\sqrt{2}}\right)=-\frac{\pi}{4}\right)$

- THUS IN THE REQUIRED RANGE

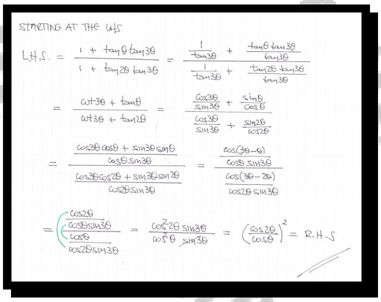
$\therefore \frac{\pi}{4}<\theta<\frac{3\pi}{4}$ OR $\frac{5\pi}{4}<\theta<\frac{7\pi}{4}$

Question 17 (*****)

Prove the validity of the following trigonometric identity.

$$\frac{1+\tan\theta\tan 3\theta}{1+\tan 2\theta\tan 3\theta} \equiv \frac{\cos^2 2\theta}{\cos^2\theta}.$$

proof



Question 18 (*****)

A surveyor views the top of a building, of height h, at an angle of elevation α.

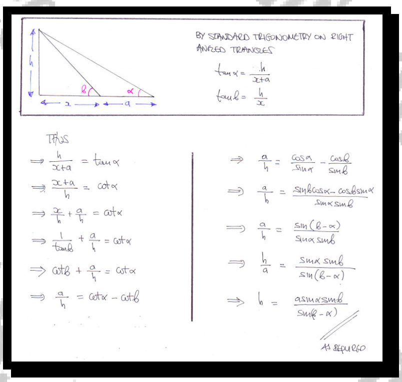
The surveyor walks a distance a, directly towards the building.

From this new position he views the top of the building at an angle of elevation β.

Show that

$$h = \frac{a\sin\alpha\sin\beta}{\sin(\beta-\alpha)}.$$

proof

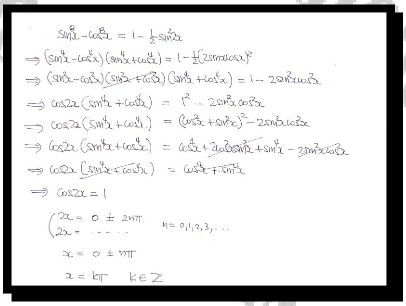


Question 19 (*****)

$$\sin^8 x - \cos^8 x = 1 - \frac{1}{2}\sin^2 2x .$$

Use trigonometric identities to show that the general solution of the above equation is $x = k\pi$, $k \in \mathbb{Z}$.

proof



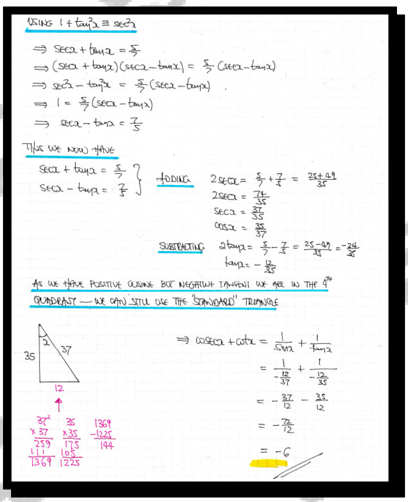
Question 20 (*****)

It is given that x is a solution of the following equation.

$$\sec x + \tan x = \frac{5}{7} .$$

Without solving the above equation for x, find the value of $\operatorname{cosec} x + \cot x$.

V , , -6



Question 21 (*****)

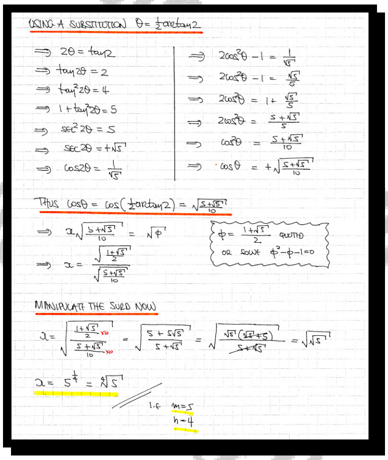
The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

Solve the equation

$$x\cos\left(\tfrac{1}{2}\arctan 2\right) = \sqrt{\phi}\,, \quad x \in \mathbb{R}\ .$$

Give the answer in the form $\sqrt[n]{m}$, where m and n are positive integers.

V, [ ], $x = \sqrt[4]{5}$



USING A SUBSTITUTION $\theta = \frac{1}{2}\arctan 2$

$\Rightarrow 2\theta = \tan 2$
$\Rightarrow \tan 2\theta = 2$
$\Rightarrow \tan^2 2\theta = 4$
$\Rightarrow 1 + \tan^2 2\theta = 5$
$\Rightarrow \sec^2 2\theta = 5$
$\Rightarrow \sec 2\theta = +\sqrt{5}$
$\Rightarrow \cos 2\theta = \frac{1}{\sqrt{5}}$

$\Rightarrow 2\cos^2\theta - 1 = \frac{1}{\sqrt{5}}$
$\Rightarrow 2\cos^2\theta - 1 = \frac{\sqrt{5}}{5}$
$\Rightarrow 2\cos^2\theta = 1 + \frac{\sqrt{5}}{5}$
$\Rightarrow 2\cos^2\theta = \frac{5+\sqrt{5}}{5}$
$\Rightarrow \cos^2\theta = \frac{5+\sqrt{5}}{10}$
$\Rightarrow \cos\theta = +\sqrt{\frac{5+\sqrt{5}}{10}}$

THUS $\cos\theta = \cos\left(\frac{1}{2}\arctan 2\right) = \sqrt{\frac{5+\sqrt{5}}{10}}$

$\Rightarrow x\sqrt{\frac{5+\sqrt{5}}{10}} = \sqrt{\phi}$

$\Rightarrow x = \frac{\sqrt{\frac{1+\sqrt{5}}{2}}}{\sqrt{\frac{5+\sqrt{5}}{10}}}$

$\phi = \frac{1+\sqrt{5}}{2}$ QUOTED OR SOLVE $\phi^2 - \phi - 1 = 0$

MANIPULATE THE SURD NOW

$x = \sqrt{\frac{\frac{1+\sqrt{5}}{2} \times 10}{\frac{5+\sqrt{5}}{10} \times 10}} = \sqrt{\frac{5+5\sqrt{5}}{5+\sqrt{5}}} = \sqrt{\frac{\sqrt{5}(\sqrt{5}+5)}{5+\sqrt{5}}} = \sqrt{\sqrt{5}}$

$x = 5^{\frac{1}{4}} = \sqrt[4]{5}$

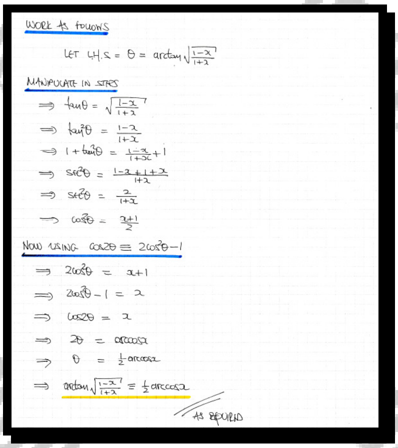
i.e. $m = 5$
$n = 4$

Question 22 (*****)

Prove that

$$\arctan\left[\sqrt{\frac{1-x}{1+x}}\right] \equiv \frac{1}{2}\arccos x\ .$$

V, SP X-V, proof



WORK AS FOLLOWS

LET L.H.S $= \theta = \arctan\sqrt{\frac{1-x}{1+x}}$

MANIPULATE IN STEPS

$\Rightarrow \tan\theta = \sqrt{\frac{1-x}{1+x}}$

$\Rightarrow \tan^2\theta = \frac{1-x}{1+x}$

$\Rightarrow 1+\tan^2\theta = \frac{1-x}{1+x}+1$

$\Rightarrow \sec^2\theta = \frac{1-x+1+x}{1+x}$

$\Rightarrow \sec^2\theta = \frac{2}{1+x}$

$\Rightarrow \cos^2\theta = \frac{x+1}{2}$

NOW USING $\cos 2\theta \equiv 2\cos^2\theta - 1$

$\Rightarrow 2\cos^2\theta = x+1$

$\Rightarrow 2\cos^2\theta - 1 = x$

$\Rightarrow \cos 2\theta = x$

$\Rightarrow 2\theta = \arccos x$

$\Rightarrow \theta = \frac{1}{2}\arccos x$

$\Rightarrow \arctan\sqrt{\frac{1-x}{1+x}} \equiv \frac{1}{2}\arccos x$

AS REQUIRED

Question 23 (*****)

The functions f and g are defined in the largest possible domain by the equations

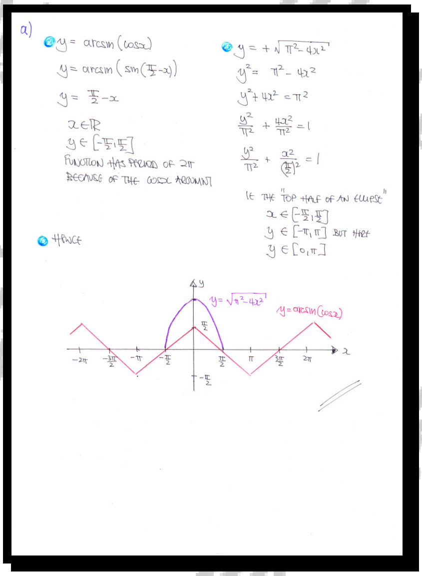
$$f(x) = \arcsin(\cos x) \quad \text{and} \quad g(x) = \sqrt{\pi^2 - 4x^2}\,.$$

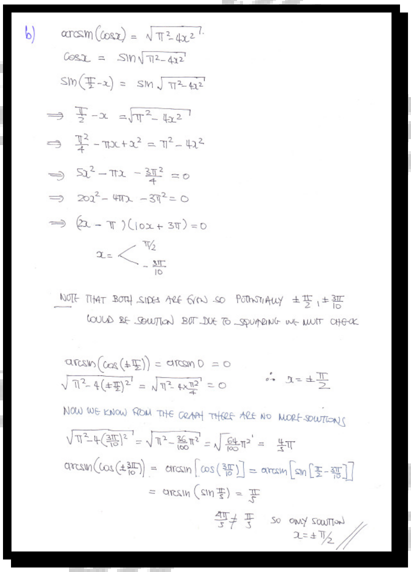
a) Sketch the graphs of f and g on the same set of axes.

b) Use an algebraic method to solve the equation

$$\arcsin(\cos x) = \sqrt{\pi^2 - 4x^2}\,.$$

$$\boxed{x = \pm\frac{\pi}{2}}$$





Question 24 (*****) **non calculator**

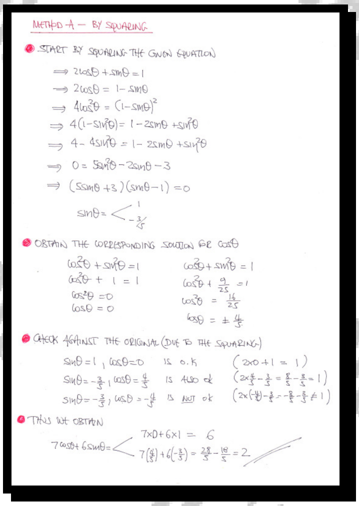
It is given that

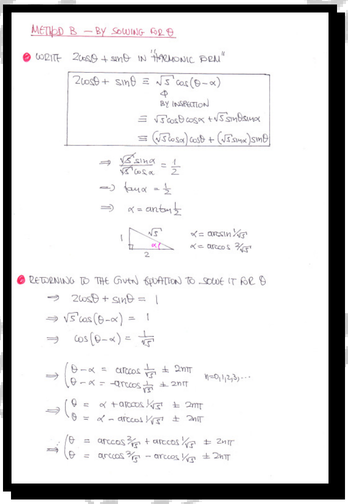
$$2\cos\theta + \sin\theta = 1 .$$

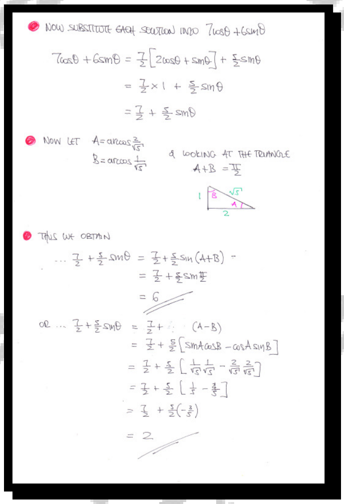
Determine the possible values of

$$7\cos\theta + 6\sin\theta .$$

$\boxed{}$, $\boxed{7\cos\theta + 6\sin\theta = 2, \quad 7\cos\theta + 6\sin\theta = 6}$







Question 25 (*****)

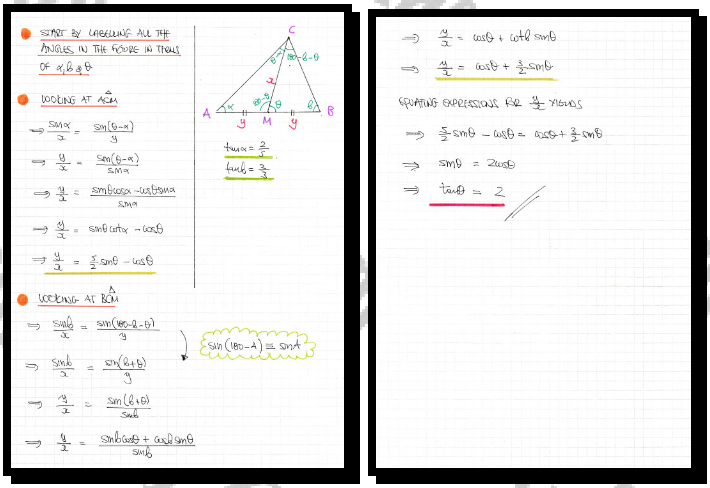
The point M is the midpoint of AB, on a triangle ABC.

Given further that

$$\tan[\measuredangle CAM] = \frac{2}{5} \quad \text{and} \quad \tan[\measuredangle CBM] = \frac{2}{3},$$

Use trigonometric identities to find the value of $\tan[\measuredangle CMB]$.

$\boxed{}$, $\boxed{\tan[\measuredangle CMB] = 2}$



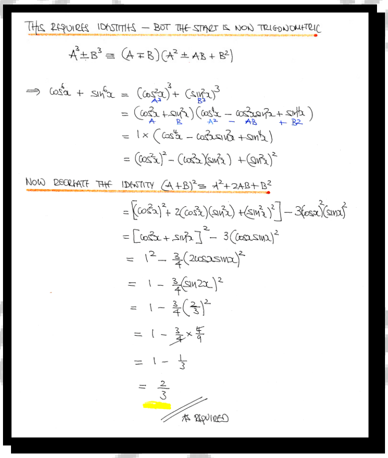
Question 26 (*****)

Show, with detailed workings, that if $\sin 2x = \frac{2}{3}$ then

$$\cos^6 x + \sin^6 x$$

also equals to $\frac{2}{3}$.

V, □, proof



THIS REQUIRES IDENTITIES – BUT THE START IS NON TRIGONOMETRIC

$$A^3 \pm B^3 \equiv (A \mp B)(A^2 \pm AB + B^2)$$

$$\begin{aligned} \Rightarrow \cos^6 x + \sin^6 x &= (\cos^2 x)^3 + (\sin^2 x)^3 \\ &= (\cos^2 x + \sin^2 x)(\cos^4 x - \cos^2 x \sin^2 x + \sin^4 x) \\ &= 1 \times (\cos^4 x - \cos^2 x \sin^2 x + \sin^4 x) \\ &= (\cos^2 x)^2 - (\cos^2 x)(\sin^2 x) + (\sin^2 x)^2 \end{aligned}$$

NOW RECREATE THE IDENTITY $(A+B)^2 \equiv A^2 + 2AB + B^2$

$$\begin{aligned} &= \left[(\cos^2 x)^2 + 2(\cos^2 x)(\sin^2 x) + (\sin^2 x)^2\right] - 3(\cos x)^2(\sin x)^2 \\ &= \left[\cos^2 x + \sin^2 x\right]^2 - 3(\cos x \sin x)^2 \\ &= 1^2 - \tfrac{3}{4}(2\cos x \sin x)^2 \\ &= 1 - \tfrac{3}{4}(\sin 2x)^2 \\ &= 1 - \tfrac{3}{4}\left(\tfrac{2}{3}\right)^2 \\ &= 1 - \tfrac{3}{4} \times \tfrac{4}{9} \\ &= 1 - \tfrac{1}{3} \\ &= \tfrac{2}{3} \end{aligned}$$

AS REQUIRED

Question 27 (*****)

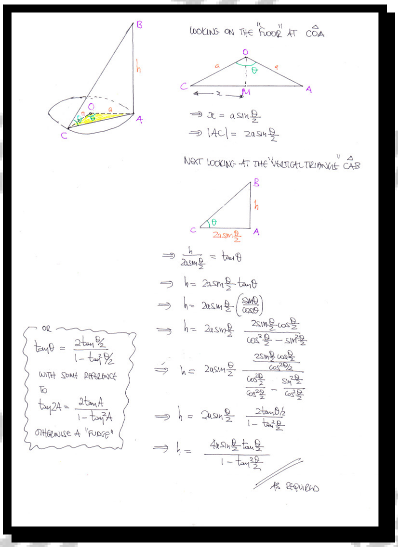
A pole AB, of height h, is standing vertically on level horizontal ground with A on the circumference of a circle of radius a, centred at the point O.

The point C is another point on the circumference of this circle so that $\measuredangle COA = \theta$ and $\measuredangle ACB = \theta$.

Use a detailed method to show that

$$h = \frac{4a\sin\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)}{1-\tan^2\left(\frac{1}{2}\theta\right)}.$$

[ ], proof



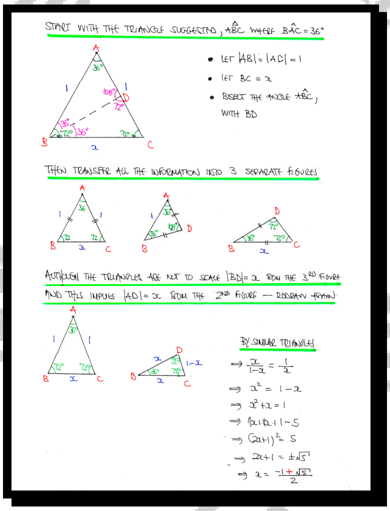
Question 28 (*****)

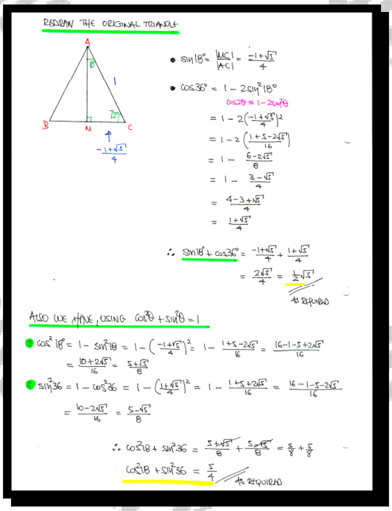
A triangle has angles of $36°$, $72°$ and $72°$.

By suitably partitioning this triangle and using similar triangles, show that

$$\sin 18° + \cos 36° = \frac{1}{2}\sqrt{5} \quad \text{and} \quad \sin^2 36° + \cos^2 18° = \frac{5}{4}$$

$\boxed{}$, $\boxed{-\frac{9}{16}}$



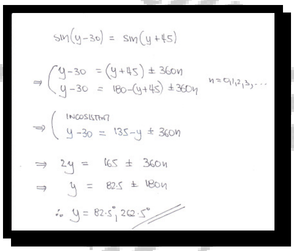


Question 329 (*****) **non calculator**

Solve the trigonometric equation

$$\sin(y-30) = \sin(y-45), \quad 0 \le y < 360°.$$

$\boxed{y = 82.5°, 262.5°}$

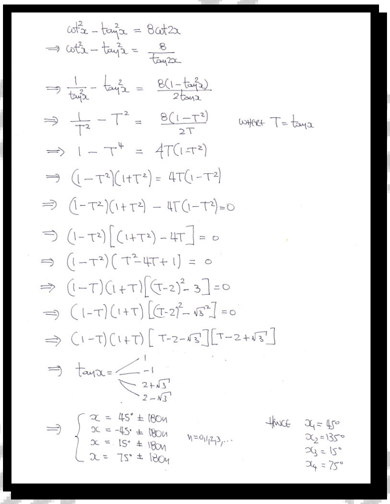


Question 30 (*****)

$$\cot^2 x - \tan^2 x = 8\cot 2x, \ 0 \le x < 180.$$

Find the solutions of the above trigonometric equation, giving the answers in **degrees**.

$$\boxed{x = 15°, 45°, 75°, 135°}$$



$\cot^2 x - \tan^2 x = 8\cot 2x$

$\Rightarrow \cot^2 x - \tan^2 x = \frac{8}{\tan 2x}$

$\Rightarrow \frac{1}{\tan^2 x} - \tan^2 x = \frac{8(1-\tan^2 x)}{2\tan x}$

$\Rightarrow \frac{1}{T^2} - T^2 = \frac{8(1-T^2)}{2T}$ where $T = \tan x$

$\Rightarrow 1 - T^4 = 4T(1-T^2)$

$\Rightarrow (1-T^2)(1+T^2) = 4T(1-T^2)$

$\Rightarrow (1-T^2)(1+T^2) - 4T(1-T^2) = 0$

$\Rightarrow (1-T^2)\left[(1+T^2) - 4T\right] = 0$

$\Rightarrow (1-T^2)(T^2-4T+1) = 0$

$\Rightarrow (1-T)(1+T)\left[(T-2)^2 - 3\right] = 0$

$\Rightarrow (1-T)(1+T)\left[(T-2)^2 - \sqrt{3}^2\right] = 0$

$\Rightarrow (1-T)(1+T)\left[T-2-\sqrt{3}\right]\left[T-2+\sqrt{3}\right]$

$\Rightarrow \tan x = 1, -1, 2+\sqrt{3}, 2-\sqrt{3}$

$\Rightarrow \begin{cases} x = 45° \pm 180n \\ x = -45° \pm 180n \\ x = 15° \pm 180n \\ x = 75° \pm 180n \end{cases}$ $n = 0, 1, 2, 3, \dots$

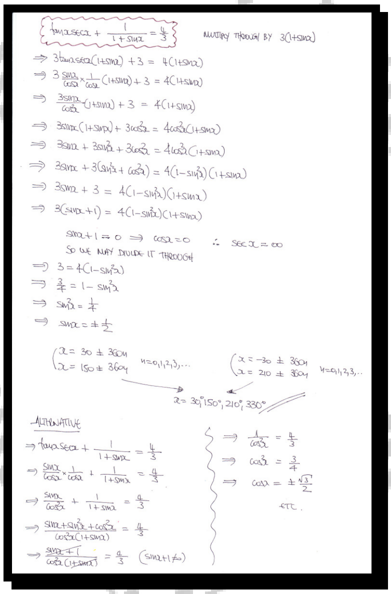
Hence $x_1 = 45°$, $x_2 = 135°$, $x_3 = 15°$, $x_4 = 75°$

Question 31 (*****)

Solve the following trigonometric equation for $0 \le x < 360°$

$$\tan x \sec x + \frac{1}{1+\sin x} = \frac{4}{3}.$$

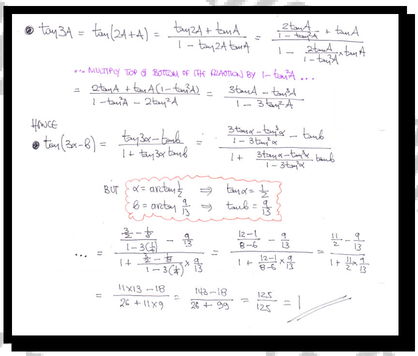
$x = 30°,\ 150°,\ 210°,\ 330°$



Question 32 (*****) **non calculator**

Given that $\alpha = \arctan\frac{1}{2}$ and $\beta = \arctan\frac{9}{13}$, find the value of $\tan(3\alpha - \beta)$.

$\tan(3\alpha - \beta) = 1$



Question 33 (*****)

It is given that for some value of the constant k

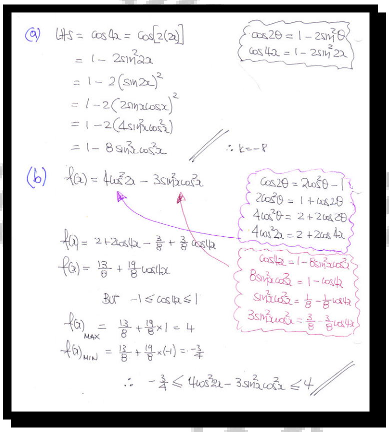
$$\cos 4x \equiv 1 + k\sin^2 x\cos^2 x .$$

a) Determine the value of k .

b) Hence, or otherwise, show clearly that for $x \in \mathbb{R}$

$$-\frac{3}{4} \le 4\cos^2 2x - 3\sin^2 x\cos^2 x \le 4 .$$

$\boxed{}$, $\boxed{k = -8}$


(a) LHS $= \cos 4x = \cos[2(2x)]$
$= 1 - 2\sin^2 2x$
$= 1 - 2(\sin 2x)^2$
$= 1 - 2(2\sin x\cos x)^2$
$= 1 - 2(4\sin^2 x\cos^2 x)$
$= 1 - 8\sin^2 x\cos^2 x$ $\therefore k = -8$

$\cos 2\theta = 1 - 2\sin^2\theta$
$\cos 4x = 1 - 2\sin^2 2x$

(b) $f(x) = 4\cos^2 2x - 3\sin^2 x\cos^2 x$

$\cos 2\theta = 2\cos^2\theta - 1$
$2\cos^2\theta = 1 + \cos 2\theta$
$4\cos^2\theta = 2 + 2\cos 2\theta$
$4\cos^2 2x = 2 + 2\cos 4x$

$\cos 4x = 1 - 8\sin^2 x\cos^2 x$
$8\sin^2 x\cos^2 x = 1 - \cos 4x$
$\sin^2 x\cos^2 x = \frac{1}{8} - \frac{1}{8}\cos 4x$
$3\sin^2 x\cos^2 x = \frac{3}{8} - \frac{3}{8}\cos 4x$

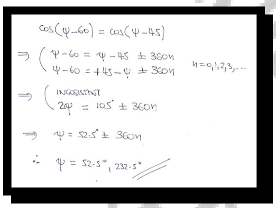
$f(x) = 2 + 2\cos 4x - \frac{3}{8} + \frac{3}{8}\cos 4x$
$f(x) = \frac{13}{8} + \frac{19}{8}\cos 4x$
BUT $-1 \le \cos 4x \le 1$
$f(x)_{MAX} = \frac{13}{8} + \frac{19}{8}\times 1 = 4$
$f(x)_{MIN} = \frac{13}{8} + \frac{19}{8}\times(-1) = -\frac{3}{4}$
$\therefore -\frac{3}{4} \le 4\cos^2 2x - 3\sin^2 x\cos^2 x \le 4$

Question 34 (*****) **non calculator**

Solve the following trigonometric equation

$$\cos(\psi - 60) = \cos(\psi - 45), \quad 0 \le \psi < 360° .$$

$\boxed{\psi = 52.5°, 232.5°}$

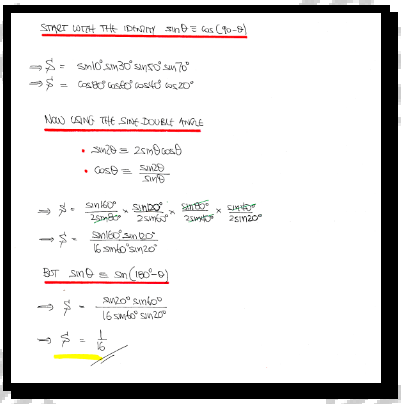

$\cos(\psi - 60) = \cos(\psi - 45)$
$\Rightarrow \psi - 60 = \psi - 45 \pm 360n$, $\psi - 60 = +45 - \psi \pm 360n$ $n = 0, 1, 2, 3, \ldots$
$\Rightarrow$ INCONSISTENT, $2\psi = 105° \pm 360n$
$\Rightarrow \psi = 52.5° \pm 360n$
$\therefore \psi = 52.5°, 232.5°$

Question 35 (*****)

$$S = \sin(10°)\sin(30°)\sin(50°)\sin(70°)$$

Use the **sine double angle identity** for sine to show that $S = \frac{1}{16}$.

V , X , proof


START WITH THE IDENTITY $\sin\theta \equiv \cos(90-\theta)$

$\Rightarrow S = \sin 10° \sin 30° \sin 50° \sin 70°$

$\Rightarrow S = \cos 80° \cos 60° \cos 40° \cos 20°$

NOW USING THE SINE DOUBLE ANGLE

- $\sin 2\theta \equiv 2\sin\theta\cos\theta$
- $\cos\theta \equiv \frac{\sin 2\theta}{\sin\theta}$

$\Rightarrow S = \frac{\sin 160°}{2\sin 80°} \times \frac{\sin 120°}{2\sin 60°} \times \frac{\sin 80°}{2\sin 40°} \times \frac{\sin 40°}{2\sin 20°}$

$\Rightarrow S = \frac{\sin 160° \sin 120°}{16 \sin 60° \sin 20°}$

BUT $\sin\theta \equiv \sin(180°-\theta)$

$\Rightarrow S = \frac{\sin 20° \sin 60°}{16 \sin 60° \sin 20°}$

$\Rightarrow S = \frac{1}{16}$

Question 36 (*****)

Solve the trigonometric equation

$$(\cos 4x + \cos x)^2 + (\sin 4x + \sin x)^2 = 2\sqrt{3}\sin 3x \, , \; 0 \le x < \pi \, ,$$

giving the answers in terms of π.

$\boxed{}$, $\boxed{\frac{\pi}{9}, \frac{\pi}{3}, \frac{7\pi}{9}}$

START BY EXPANDING THE LHS

$\Rightarrow (\cos 4x + \cos x)^2 + (\sin 4x + \sin x)^2 = 2\sqrt{3}\sin 3x$

$\Rightarrow \left\{ \begin{array}{l} \cos^2 4x + 2\cos 4x \cos x + \cos^2 x \\ \sin^2 4x + 2\sin 4x \sin x + \sin^2 x \end{array} \right\} = 2\sqrt{3}\sin 3x$

$\Rightarrow 1 + 2[\cos 4x \cos x + \sin 4x \sin x] + 1 = 2\sqrt{3}\sin 3x$

$\Rightarrow 1 + 2\cos(4x - x) + 1 = 2\sqrt{3}\sin 3x$

$\Rightarrow 2 + 2\cos 3x = 2\sqrt{3}\sin 3x$

$\Rightarrow 1 + \cos 3x = \sqrt{3}\sin 3x$

$\Rightarrow \sqrt{3}\sin 3x - \cos 3x = 1$

NOW WE MAY PROCEED BY AN "R TRANSFORMATION, OR AS FOLLOWS

$\Rightarrow \frac{\sqrt{3}}{2}\sin 3x - \frac{1}{2}\cos 3x = \frac{1}{2}$

$\Rightarrow \cos\frac{\pi}{6}\sin 3x - \sin\frac{\pi}{6}\cos 3x = \frac{1}{2}$

$\Rightarrow \sin\left(3x - \frac{\pi}{6}\right) = \frac{1}{2}$

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

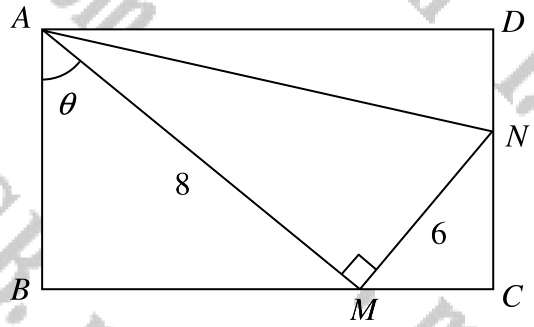
$\Rightarrow \left(\begin{array}{l} 3x - \frac{\pi}{6} = \frac{\pi}{6} \pm 2n\pi \\ 3x - \frac{\pi}{6} = \frac{5\pi}{6} \pm 2n\pi \end{array} \right. \quad n = 0, 1, 2, 3, \ldots$

$\Rightarrow \left(\begin{array}{l} 3x = \frac{\pi}{3} \pm 2n\pi \\ 3x = \pi \pm 2n\pi \end{array} \right.$

$\Rightarrow \left(\begin{array}{l} x = \frac{\pi}{9} \pm \frac{2n\pi}{3} \\ x = \frac{\pi}{3} \pm \frac{2n\pi}{3} \end{array} \right.$ ← $x = 20° \pm 120°n$, $x = 60° \pm 120°n$ ↓ $x = 20, 140, 60$

$\therefore x = \frac{\pi}{9}, \frac{\pi}{3}, \frac{7\pi}{9}$ //

Question 37 (*****)



The figure above shows a rectangle $ABCD$.

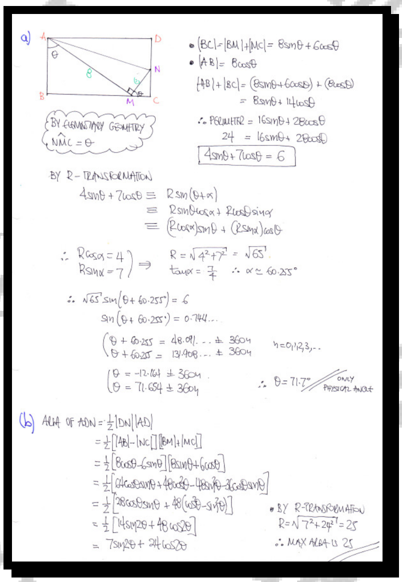
The points M and N lie on BC and CD respectively.

The angle AMN is $90°$, $|AM| = 8$ and $|MN| = 6$. The angle BAM is denoted by θ.

a) Given that the perimeter of the rectangle $ABCD$ is fixed at 24 units, determine the possible value(s) of θ.

b) Given instead that the perimeter of the rectangle $ABCD$ can vary, determine the largest possible area of the **triangle** ADN.

$\boxed{\theta \approx 71.7°}$, $\boxed{\text{area}_{\text{max}} = 25}$

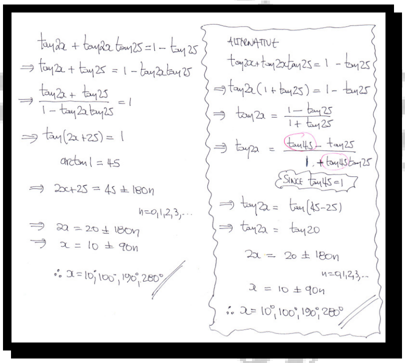


Question 38 **(*****) non calculator**

$$\tan 2x° + \tan 2x° \tan 25° = 1 - \tan 25°, \ \ 0 \le x < 360 .$$

Find the solutions of the above trigonometric equation, giving the answers in **degrees**.

$$\boxed{x = 10°, 100°, 190°, 280°}$$

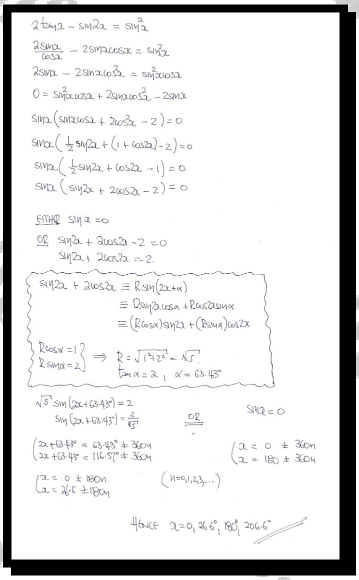


Question 39 **(*****)**

$$2\tan x - \sin 2x = \sin^2 x, \ \ 0 \le x < 360 .$$

Find the solutions of the above trigonometric equation, giving the answers in **degrees**.

$$\boxed{x = 0°, 26.6°, 180°, 206.6°}$$

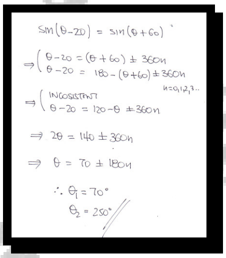


Question 40 **(*****) non calculator**

Solve the following trigonometric equation

$$\sin(\theta - 20) = \sin(\theta + 60), \quad 0 \le \theta < 360°.$$

$\boxed{\theta = 70°,\ 250°}$



$\sin(\theta-20) = \sin(\theta+60)$

$\Rightarrow \begin{cases} \theta - 20 = (\theta + 60) \pm 360n \\ \theta - 20 = 180 - (\theta+60) \pm 360n \end{cases}$ $n = 0,1,2,3 \ldots$

$\Rightarrow \begin{cases} \text{INCOSISTANT} \\ \theta - 20 = 120 - \theta \pm 360n \end{cases}$

$\Rightarrow 2\theta = 140 \pm 360n$

$\Rightarrow \theta = 70 \pm 180n$

$\therefore \theta_1 = 70°$

$\theta_2 = 250°$

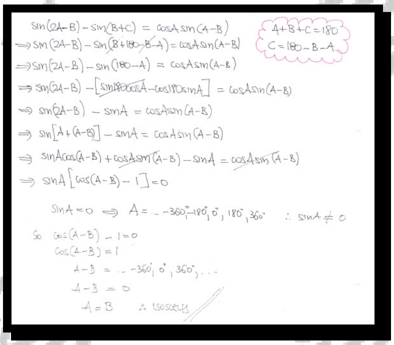
Question 41 **(*****)**

The three angles in a triangle ABC satisfy the relationship

$$\sin(2A - B) - \sin(B + C) = \cos A \sin(A - B).$$

Show that the triangle ABC is isosceles.

$\boxed{\text{proof}}$



$A+B+C=180$

$C = 180 - B - A$

$\sin(2A-B) - \sin(B+C) = \cos A \sin(A-B)$

$\Rightarrow \sin(2A-B) - \sin(B+180-B-A) = \cos A \sin(A-B)$

$\Rightarrow \sin(2A-B) - \sin(180-A) = \cos A \sin(A-B)$

$\Rightarrow \sin(2A-B) - [\sin 180 \cos A - \cos 180 \sin A] = \cos A \sin(A-B)$

$\Rightarrow \sin(2A-B) - \sin A = \cos A \sin(A-B)$

$\Rightarrow \sin[A + (A-B)] - \sin A = \cos A \sin(A-B)$

$\Rightarrow \sin A \cos(A-B) + \cos A \sin(A-B) - \sin A = \cos A \sin(A-B)$

$\Rightarrow \sin A[\cos(A-B) - 1] = 0$

$\sin A = 0 \Rightarrow A = \ldots -360°, -180°, 0°, 180°, 360° \quad \therefore \sin A \neq 0$

So $\cos(A-B) - 1 = 0$

$\cos(A-B) = 1$

$A - B = \ldots -360°, 0°, 360°, \ldots$

$A - B = 0$

$A = B \quad \therefore$ isosceles

Question 42 (*****)

It is given that

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B .$$

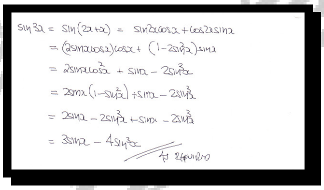
Use the above trigonometric identity to show that

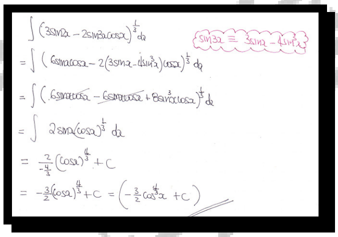
$$\sin 3x \equiv 3\sin x - 4\sin^3 x ,$$

and hence find

$$\int \sqrt[3]{3\sin 2x - 2\sin 3x \cos x} \, dx .$$

$$\boxed{-\frac{3}{2}\sin^{\frac{4}{3}} x + C}$$





Question 43 (*****)

It is given that a, b and c are consecutive terms of an arithmetic progression.

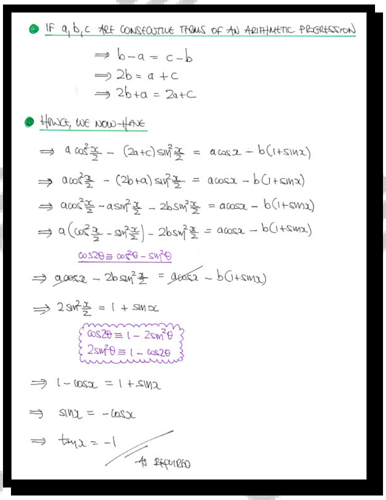
It is further given that

$$a\cos^2\frac{x}{2}-(2a+c)\sin^2\frac{x}{2}=a\cos x-b(1+\sin x),\ x\in\mathbb{R}.$$

Show clearly that

$$\tan x=-1.$$

[ ], [proof]



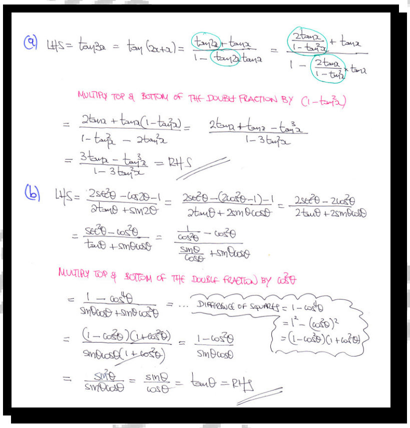
Question 44 (*****)

Prove the validity of each of the following trigonometric identities.

a) $\tan 3x \equiv \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

b) $\dfrac{2\sec^2\theta - \cos 2\theta - 1}{2\tan\theta + \sin 2\theta} \equiv \tan\theta$.

proof



(a) $\text{LHS} = \tan 3x = \tan(2x + x) = \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x} = \dfrac{\frac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \frac{2\tan x}{1-\tan^2 x} \times \tan x}$

MULTIPLY TOP & BOTTOM OF THE DOUBLE FRACTION BY $(1 - \tan^2 x)$

$= \dfrac{2\tan x + \tan x(1-\tan^2 x)}{1 - \tan^2 x - 2\tan^2 x} = \dfrac{2\tan x + \tan x - \tan^3 x}{1 - 3\tan^2 x}$

$= \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} = \text{RHS}$

(b) $\text{LHS} = \dfrac{2\sec^2\theta - \cos 2\theta - 1}{2\tan\theta + \sin 2\theta} = \dfrac{2\sec^2\theta - (2\cos^2\theta - 1) - 1}{2\tan\theta + 2\sin\theta\cos\theta} = \dfrac{2\sec^2\theta - 2\cos^2\theta}{2\tan\theta + 2\sin\theta\cos\theta}$

$= \dfrac{\sec^2\theta - \cos^2\theta}{\tan\theta + \sin\theta\cos\theta} = \dfrac{\frac{1}{\cos^2\theta} - \cos^2\theta}{\frac{\sin\theta}{\cos\theta} + \sin\theta\cos\theta}$

MULTIPLY TOP & BOTTOM OF THE DOUBLE FRACTION BY $\cos^2\theta$

$= \dfrac{1 - \cos^4\theta}{\sin\theta\cos\theta + \sin\theta\cos^3\theta} = \ldots$ DIFFERENCE OF SQUARES $= 1 - \cos^4\theta = 1^2 - (\cos^2\theta)^2 = (1-\cos^2\theta)(1+\cos^2\theta)$

$= \dfrac{(1-\cos^2\theta)(1+\cos^2\theta)}{\sin\theta\cos\theta(1+\cos^2\theta)} = \dfrac{1-\cos^2\theta}{\sin\theta\cos\theta}$

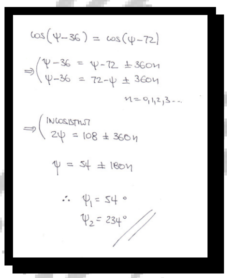
$= \dfrac{\sin^2\theta}{\sin\theta\cos\theta} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta = \text{RHS}$

Question 45 (*****) **non calculator**

Solve the trigonometric equation

$$\cos(\psi - 36) = \cos(\psi - 72), \quad 0 \le \psi < 360°.$$

$\psi = 54°, 234°$



$\cos(\psi - 36) = \cos(\psi - 72)$

$\Rightarrow \begin{cases} \psi - 36 = \psi - 72 \pm 360n \\ \psi - 36 = 72 - \psi \pm 360n \end{cases}$ $n = 0, 1, 2, 3 \ldots$

$\Rightarrow \begin{cases} \text{INCONSISTENT} \\ 2\psi = 108 \pm 360n \end{cases}$

$\psi = 54 \pm 180n$

$\therefore \psi_1 = 54°$

$\psi_2 = 234°$

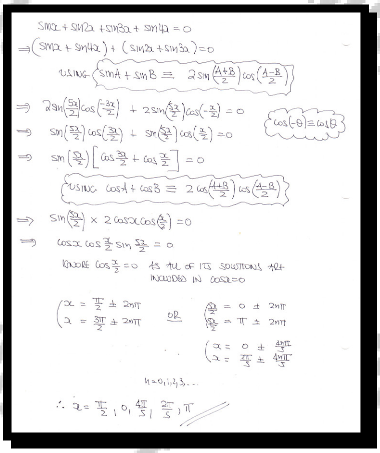
Question 46 (*****)

Solve the trigonometric equation

$$\sin x + \sin 2x + \sin 3x + \sin 4x = 0, \quad 0 \le \theta \le \pi,$$

giving the answers in terms of π.

$$\boxed{x = 0, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{4\pi}{5}, \pi}$$


$\sin x + \sin 2x + \sin 3x + \sin 4x = 0$

$\Rightarrow (\sin x + \sin 4x) + (\sin 2x + \sin 3x) = 0$

USING $\sin A + \sin B \equiv 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\Rightarrow 2\sin\left(\frac{5x}{2}\right)\cos\left(\frac{-3x}{2}\right) + 2\sin\left(\frac{5x}{2}\right)\cos\left(-\frac{x}{2}\right) = 0$

$\cos(-\theta) \equiv \cos\theta$

$\Rightarrow \sin\left(\frac{5x}{2}\right)\cos\left(\frac{3x}{2}\right) + \sin\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right) = 0$

$\Rightarrow \sin\left(\frac{5x}{2}\right)\left[\cos\frac{3x}{2} + \cos\frac{x}{2}\right] = 0$

USING $\cos A + \cos B \equiv 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\Rightarrow \sin\left(\frac{5x}{2}\right) \times 2\cos x\cos\left(\frac{x}{2}\right) = 0$

$\Rightarrow \cos x \cos\frac{x}{2}\sin\frac{5x}{2} = 0$

IGNORE $\cos\frac{x}{2} = 0$ AS ALL OF ITS SOLUTIONS ARE INCLUDED IN $\cos x = 0$

$x = \frac{\pi}{2} \pm 2n\pi$, $x = \frac{3\pi}{2} \pm 2n\pi$ OR $\frac{5x}{2} = 0 \pm 2n\pi$, $\frac{5x}{2} = \pi \pm 2n\pi$

$x = 0 \pm \frac{4n\pi}{5}$, $x = \frac{2\pi}{5} \pm \frac{4n\pi}{5}$

$n = 0, 1, 2, 3, \ldots$

$\therefore x = \frac{\pi}{2}, 0, \frac{4\pi}{5}, \frac{2\pi}{5}, \pi$

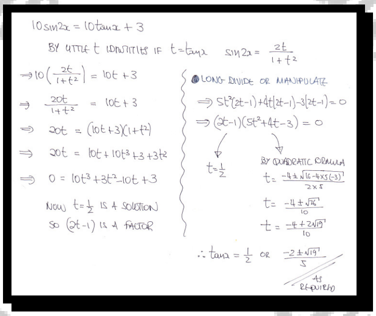
Question 47 (*****)

Use the substitution $t = \tan x$ to show that if

$$10\sin 2x = 3 + 10\tan x,$$

then either $\tan x = \frac{1}{2}$ or $\tan x = \frac{-2 \pm \sqrt{19}}{5}$.

proof



$10\sin 2x = 10\tan x + 3$

BY LITTLE t IDENTITIES IF $t = \tan x$ $\quad \sin 2x = \frac{2t}{1+t^2}$

$\Rightarrow 10\left(\frac{2t}{1+t^2}\right) = 10t + 3$

$\Rightarrow \frac{20t}{1+t^2} = 10t + 3$

$\Rightarrow 20t = (10t+3)(1+t^2)$

$\Rightarrow 20t = 10t + 10t^3 + 3 + 3t^2$

$\Rightarrow 0 = 10t^3 + 3t^2 - 10t + 3$

NOW $t = \frac{1}{2}$ IS A SOLUTION SO $(2t-1)$ IS A FACTOR

- LONG DIVIDE OR MANIPULATE

$\Rightarrow 5t^2(2t-1) + 4t(2t-1) - 3(2t-1) = 0$

$\Rightarrow (2t-1)(5t^2+4t-3) = 0$

$t = \frac{1}{2}$

BY QUADRATIC FORMULA

$t = \frac{-4 \pm \sqrt{16 - 4\times5(-3)}}{2\times5}$

$t = \frac{-4 \pm \sqrt{76}}{10}$

$t = \frac{-4 + 2\sqrt{19}}{10}$

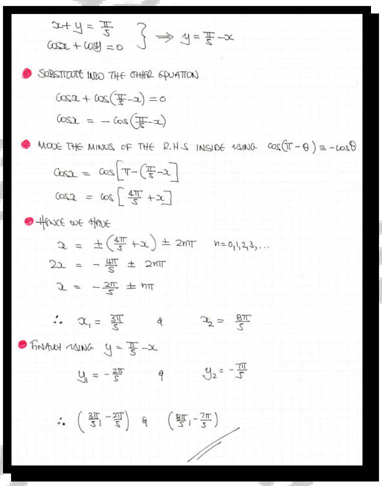
$\therefore \tan x = \frac{1}{2}$ OR $\frac{-2 \pm \sqrt{19}}{5}$ AS REQUIRED

Question 48 (*****)

Solve the following simultaneous equations.

$$x + y = \frac{1}{5}\pi, \quad \cos x + \cos y = 0, \quad 0 \le x < 2\pi.$$

$\boxed{}$, $\boxed{(x, y) = \left(\frac{3}{5}\pi, -\frac{2}{5}\pi\right) = \left(\frac{8}{5}\pi, -\frac{7}{5}\pi\right)}$



$x + y = \frac{\pi}{5}$, $\cos x + \cos y = 0$ $\Rightarrow y = \frac{\pi}{5} - x$

- SUBSTITUTE INTO THE OTHER EQUATION

$\cos x + \cos\left(\frac{\pi}{5} - x\right) = 0$

$\cos x = -\cos\left(\frac{\pi}{5} - x\right)$

- MOVE THE MINUS OF THE R.H.S INSIDE USING $\cos(\pi - \theta) \equiv -\cos\theta$

$\cos x = \cos\left[\pi - \left(\frac{\pi}{5} - x\right)\right]$

$\cos x = \cos\left[\frac{4\pi}{5} + x\right]$

- HENCE WE HAVE

$x = \pm\left(\frac{4\pi}{5} + x\right) \pm 2n\pi \quad n = 0,1,2,3,\ldots$

$2x = -\frac{4\pi}{5} \pm 2n\pi$

$x = -\frac{2\pi}{5} \pm n\pi$

$\therefore x_1 = \frac{3\pi}{5}$ & $x_2 = \frac{8\pi}{5}$

- FINALLY USING $y = \frac{\pi}{5} - x$

$y_1 = -\frac{2\pi}{5}$ & $y_2 = -\frac{7\pi}{5}$

$\therefore \left(\frac{3\pi}{5}, -\frac{2\pi}{5}\right)$ & $\left(\frac{8\pi}{5}, -\frac{7\pi}{5}\right)$

Question 49 (*****)

$$f(x) \equiv \frac{1-\sin 2x}{\sin x - \cos x}, \quad x \in \mathbb{R}, \ \sin x \neq \cos x .$$

a) Show clearly that

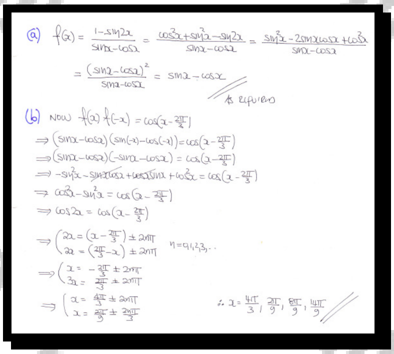
$$f(x) \equiv \sin x - \cos x .$$

b) Solve the equation

$$f(x)\,f(-x) = \cos\left(x - \frac{2\pi}{3}\right), \quad 0 \le x < 2\pi ,$$

giving the answers in terms of π.

$$\boxed{x = \frac{2\pi}{9}, \frac{8\pi}{9}, \frac{4\pi}{3}, \frac{14\pi}{9}}$$



(a) $f(x) = \frac{1-\sin 2x}{\sin x - \cos x} = \frac{\cos^2 x + \sin^2 x - \sin 2x}{\sin x - \cos x} = \frac{\sin^2 x - 2\sin x\cos x + \cos^2 x}{\sin x - \cos x}$

$= \frac{(\sin x - \cos x)^2}{\sin x - \cos x} = \sin x - \cos x$ // As required

(b) Now $f(x)f(-x) = \cos(x - \frac{2\pi}{3})$

$\Rightarrow (\sin x - \cos x)(\sin(-x) - \cos(-x)) = \cos(x - \frac{2\pi}{3})$

$\Rightarrow (\sin x - \cos x)(-\sin x - \cos x) = \cos(x - \frac{2\pi}{3})$

$\Rightarrow -\sin^2 x - \sin x\cos x + \cos x\sin x + \cos^2 x = \cos(x - \frac{2\pi}{3})$

$\Rightarrow \cos^2 x - \sin^2 x = \cos(x - \frac{2\pi}{3})$

$\Rightarrow \cos 2x = \cos(x - \frac{2\pi}{3})$

$\Rightarrow \begin{cases} 2x = (x - \frac{2\pi}{3}) \pm 2n\pi \\ 2x = (\frac{2\pi}{3} - x) \pm 2n\pi \end{cases}$ $n = 0, 1, 2, 3, \ldots$

$\Rightarrow \begin{cases} x = -\frac{2\pi}{3} \pm 2n\pi \\ 3x = \frac{2\pi}{3} \pm 2n\pi \end{cases}$

$\Rightarrow \begin{cases} x = \frac{4\pi}{3} \pm 2n\pi \\ x = \frac{2\pi}{9} \pm \frac{2n\pi}{3} \end{cases}$

$\therefore x = \frac{4\pi}{3}, \frac{2\pi}{9}, \frac{8\pi}{9}, \frac{14\pi}{9}$ //

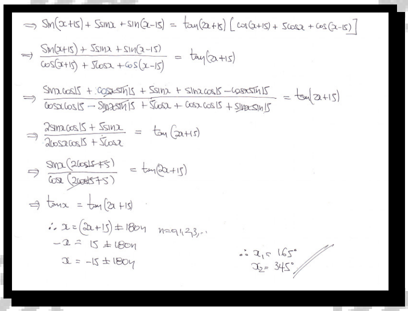
Question 50 (*****)

Find the solutions of the trigonometric equation

$$\sin(x+15)+5\sin x+\sin(x-15)=\left[\cos(x+15)+5\cos x+\cos(x-15)\right]\tan(2x+15),$$

in the range $0° \le x < 360°$.

$x = 165°, 345°$

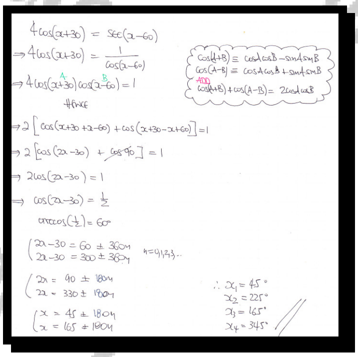


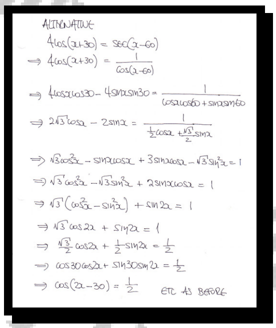
Question 51 (*****)

Solve the trigonometric equation

$$4\cos(x+30°)=\sec(x-60°), \quad 0 \le x < 360°.$$

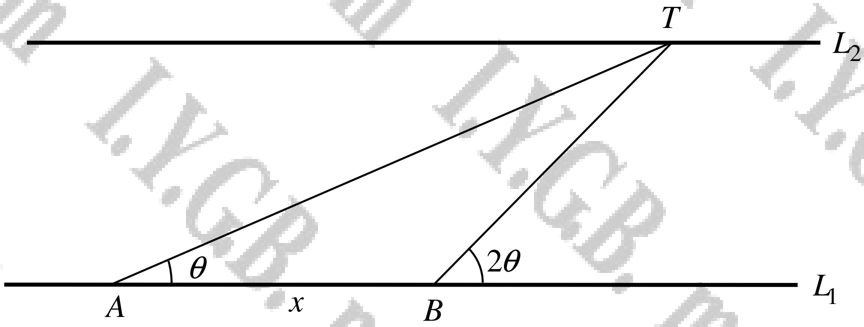
$x = 45°, 165°, 225°, 345°$





Question 52 (*****)

In the following question you may not use the sine or the cosine rule.



The figure above shows the plan of a river whose banks are modelled as straight parallel lines L_1 and L_2.

The points A and B lie on L_1, so that $|AB| = x$.

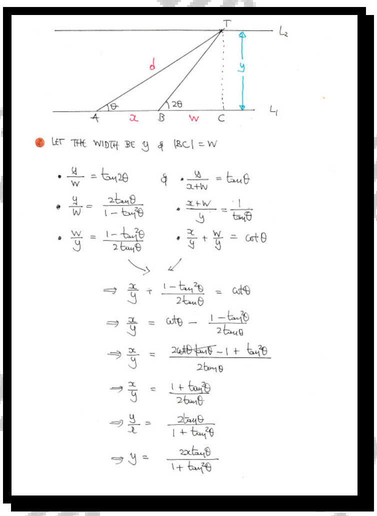
A tree is positioned at the point T on L_2, so that AT and BT subtend angles of θ and 2θ, respectively.

The tree located at T has height h. The angle of elevation of the top of the tree as viewed from A is θ.

Show that

$$h = 2x\sin\theta .$$

[ ], [proof]



LET THE WIDTH BE y & $|BC| = w$

- $\frac{y}{w} = \tan 2\theta$ & $\frac{y}{x+w} = \tan\theta$
- $\frac{y}{w} = \frac{2\tan\theta}{1-\tan^2\theta}$; $\frac{x+w}{y} = \frac{1}{\tan\theta}$
- $\frac{w}{y} = \frac{1-\tan^2\theta}{2\tan\theta}$; $\frac{x}{y} + \frac{w}{y} = \cot\theta$

$\Rightarrow \frac{x}{y} + \frac{1-\tan^2\theta}{2\tan\theta} = \cot\theta$

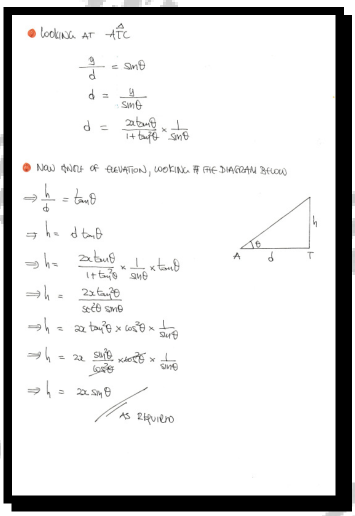
$\Rightarrow \frac{x}{y} = \cot\theta - \frac{1-\tan^2\theta}{2\tan\theta}$

$\Rightarrow \frac{x}{y} = \frac{2\cot\theta\tan\theta - 1 + \tan^2\theta}{2\tan\theta}$

$\Rightarrow \frac{x}{y} = \frac{1+\tan^2\theta}{2\tan\theta}$

$\Rightarrow \frac{y}{x} = \frac{2\tan\theta}{1+\tan^2\theta}$

$\Rightarrow y = \frac{2x\tan\theta}{1+\tan^2\theta}$



LOOKING AT $\hat{ATC}$

$\frac{y}{d} = \sin\theta$

$d = \frac{y}{\sin\theta}$

$d = \frac{2x\tan\theta}{1+\tan^2\theta} \times \frac{1}{\sin\theta}$

NOW ANGLE OF ELEVATION, LOOKING AT THE DIAGRAM BELOW

$\Rightarrow \frac{h}{d} = \tan\theta$

$\Rightarrow h = d\tan\theta$

$\Rightarrow h = \frac{2x\tan\theta}{1+\tan^2\theta} \times \frac{1}{\sin\theta} \times \tan\theta$

$\Rightarrow h = \frac{2x\tan^2\theta}{\sec^2\theta\sin\theta}$

$\Rightarrow h = 2x\tan^2\theta \times \cos^2\theta \times \frac{1}{\sin\theta}$

$\Rightarrow h = 2x\frac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta \times \frac{1}{\sin\theta}$

$\Rightarrow h = 2x\sin\theta$

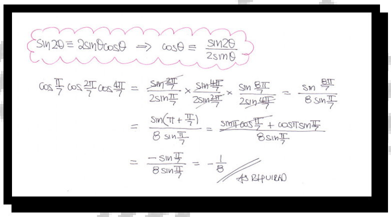
AS REQUIRED

Question 53 (*****)

By using the trigonometric identity for $\sin 2\theta$, or otherwise, show clearly that

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}=-\frac{1}{8}.$$

proof

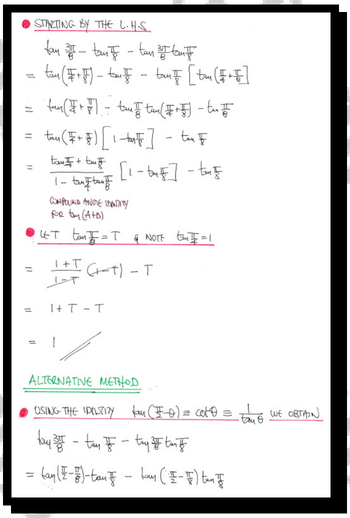


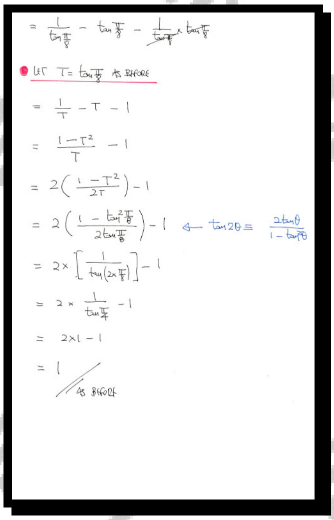
Question 54 (*****)

Show clearly that

$$\tan\frac{3\pi}{8}-\tan\frac{\pi}{8}-\tan\frac{3\pi}{8}\tan\frac{\pi}{8}=1.$$

proof



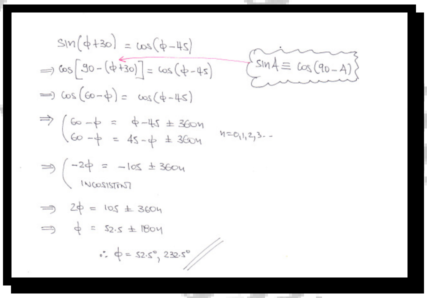


Question 55 **(*****)** **non calculator**

Solve the trigonometric equation

$$\sin(\varphi + 30) = \cos(\varphi - 45), \quad 0 \le \varphi < 360° .$$

$\boxed{\varphi = 52.5°,\ 232.5°}$



Question 56 (*****)

Eliminate θ from the following pair of equation.

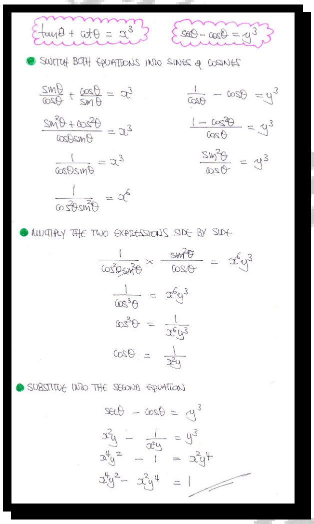
$$\tan\theta + \cot\theta = x^3$$

$$\sec\theta - \cos\theta = y^3$$

Write the answer in the form

$$f(x, y) = 1 .$$

SP-M , $x^4y^2 - y^4x^2 = 1$



Question 57 (*****)

It is given that

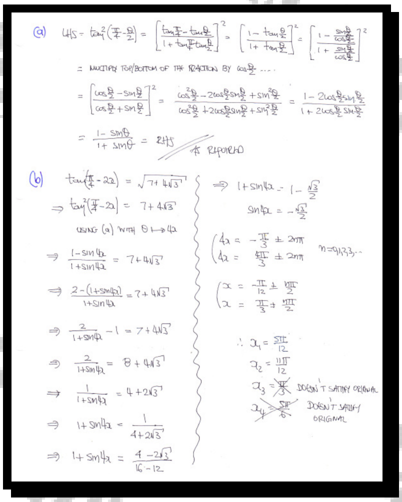
$$\tan^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right) \equiv \frac{1-\sin\theta}{1+\sin\theta}, \quad x \neq \frac{\pi}{2}(4n+3),\ n \in \mathbb{Z}.$$

a) Prove the validity of the above trigonometric identity

b) Hence solve the equation

$$\tan\left(\frac{\pi}{4}-2x\right) = \sqrt{7+4\sqrt{3}}\ ,\ 0 \le x < \pi .$$

$$\boxed{x = \frac{5\pi}{12}, \frac{11\pi}{6}}$$



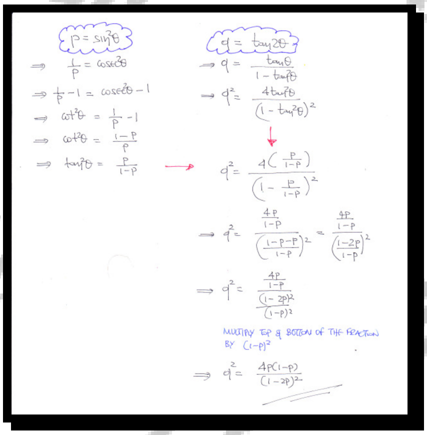
Question 58 (*****)

It is given that

$$p = \sin^2 \theta \ , \quad q = \tan 2\theta \ .$$

Use trigonometric identities to find a simplified expression for q^2 in terms of p.

$$\boxed{q^2 = \frac{4p(1-p)}{(1-2p)^2}}$$

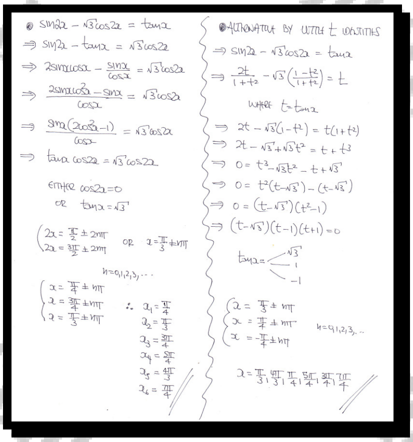


Question 659 (*****)

$$\sin 2x - \sqrt{3}\cos 2x = \tan x ,\ 0 \le x < 2\pi .$$

Find the solutions of the above trigonometric equation, giving the answers terms of π.

$$\boxed{x = \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{7\pi}{4}}$$

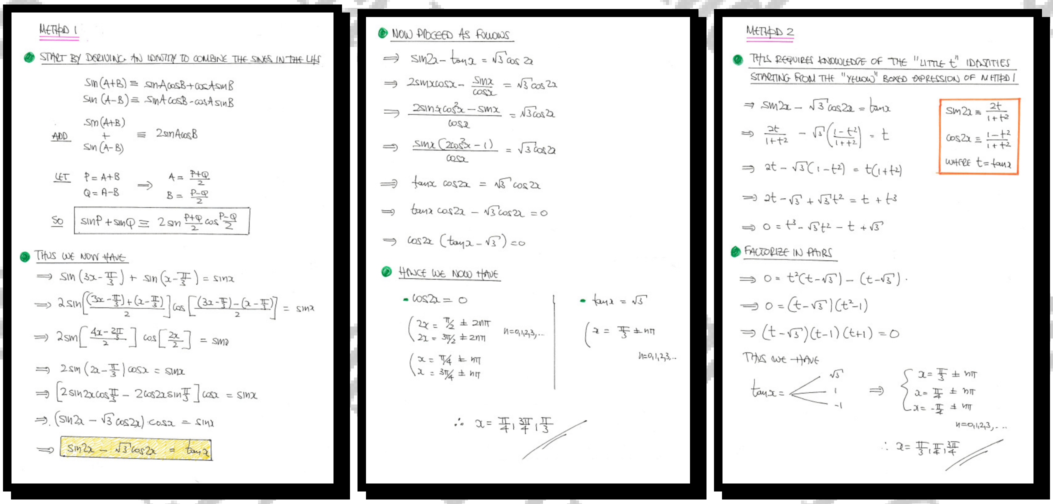


Question 60 (*****)

$$\sin\left(3x-\frac{\pi}{3}\right)+\sin\left(x-\frac{\pi}{3}\right)=\sin x\,, \quad 0 \le x < \pi\,.$$

Determine the solutions of the above trigonometric equation, giving the answers in terms of π.

$\boxed{}$, $\boxed{x=\frac{\pi}{4},\frac{\pi}{3},\frac{3\pi}{4}}$



Question 61 (*****)

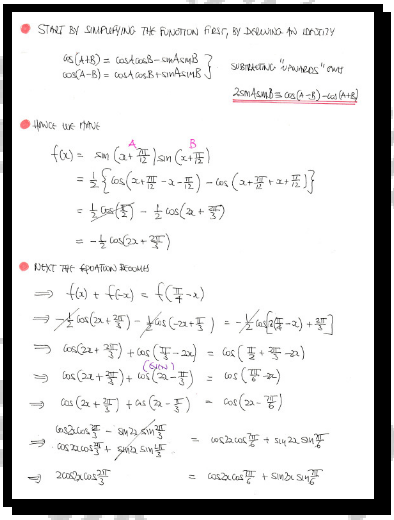
The function f is defined as

$$f(x)=\sin\left(x+\frac{7\pi}{12}\right)\sin\left(x+\frac{\pi}{12}\right),\quad 0\le x<2\pi$$

Solve the equation

$$f(x)+f(-x)=f\left(\frac{\pi}{4}-x\right).$$

$\square$, $\boxed{x=\frac{\pi}{24},\frac{13\pi}{24},\frac{25\pi}{24},\frac{37\pi}{24}}$



START BY SIMPLIFYING THE FUNCTION FIRST, BY DERIVING AN IDENTITY

$\cos(A+B)=\cos A\cos B-\sin A\sin B$
$\cos(A-B)=\cos A\cos B+\sin A\sin B$ } SUBTRACTING "UPWARDS" GIVES

$2\sin A\sin B\equiv\cos(A-B)-\cos(A+B)$

HENCE WE HAVE

$f(x)=\sin\left(x+\frac{7\pi}{12}\right)\sin\left(x+\frac{\pi}{12}\right)$

$=\frac{1}{2}\left\{\cos\left(x+\frac{7\pi}{12}-x-\frac{\pi}{12}\right)-\cos\left(x+\frac{7\pi}{12}+x+\frac{\pi}{12}\right)\right\}$

$=\frac{1}{2}\cos\left(\frac{\pi}{2}\right)-\frac{1}{2}\cos\left(2x+\frac{2\pi}{3}\right)$

$=-\frac{1}{2}\cos\left(2x+\frac{2\pi}{3}\right)$

NEXT THE EQUATION BECOMES

$\Rightarrow f(x)+f(-x)=f\left(\frac{\pi}{4}-x\right)$

$\Rightarrow -\frac{1}{2}\cos\left(2x+\frac{2\pi}{3}\right)-\frac{1}{2}\cos\left(-2x+\frac{\pi}{3}\right)=-\frac{1}{2}\cos\left[2\left(\frac{\pi}{4}-x\right)+\frac{2\pi}{3}\right]$

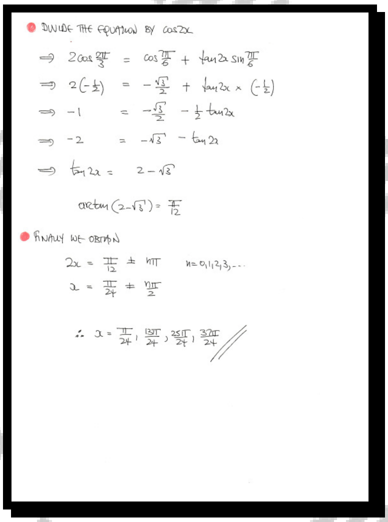
$\Rightarrow \cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(\frac{\pi}{3}-2x\right)=\cos\left(\frac{\pi}{2}+\frac{2\pi}{3}-2x\right)$

$\Rightarrow \cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x-\frac{\pi}{3}\right)=\cos\left(\frac{7\pi}{6}-2x\right)$ (EVEN)

$\Rightarrow \cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x-\frac{\pi}{3}\right)=\cos\left(2x-\frac{7\pi}{6}\right)$

$\Rightarrow \cos 2x\cos\frac{2\pi}{3}-\sin 2x\sin\frac{2\pi}{3}+\cos 2x\cos\frac{2\pi}{3}+\sin 2x\sin\frac{2\pi}{3}=\cos 2x\cos\frac{7\pi}{6}+\sin 2x\sin\frac{7\pi}{6}$

$\Rightarrow 2\cos 2x\cos\frac{2\pi}{3}=\cos 2x\cos\frac{7\pi}{6}+\sin 2x\sin\frac{7\pi}{6}$



DIVIDE THE EQUATION BY $\cos 2x$

$\Rightarrow 2\cos\frac{2\pi}{3}=\cos\frac{7\pi}{6}+\tan 2x\sin\frac{7\pi}{6}$

$\Rightarrow 2\left(-\frac{1}{2}\right)=-\frac{\sqrt{3}}{2}+\tan 2x\times\left(-\frac{1}{2}\right)$

$\Rightarrow -1=-\frac{\sqrt{3}}{2}-\frac{1}{2}\tan 2x$

$\Rightarrow -2=-\sqrt{3}-\tan 2x$

$\Rightarrow \tan 2x=2-\sqrt{3}$

$\arctan(2-\sqrt{3})=\frac{\pi}{12}$

FINALLY WE OBTAIN

$2x=\frac{\pi}{12}\pm n\pi \qquad n=0,1,2,3,\dots$

$x=\frac{\pi}{24}\pm\frac{n\pi}{2}$

$\therefore x=\frac{\pi}{24},\frac{13\pi}{24},\frac{25\pi}{24},\frac{37\pi}{24}$

Question 62 (*****)

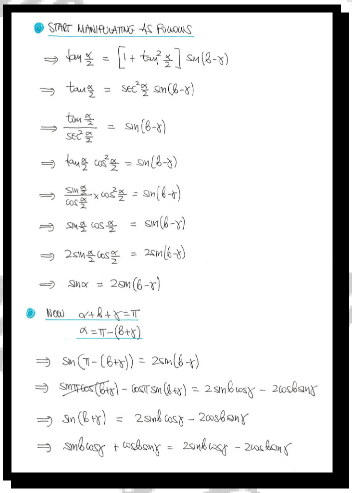
It is given that the three angles of a triangle α, β and γ satisfy the relationship

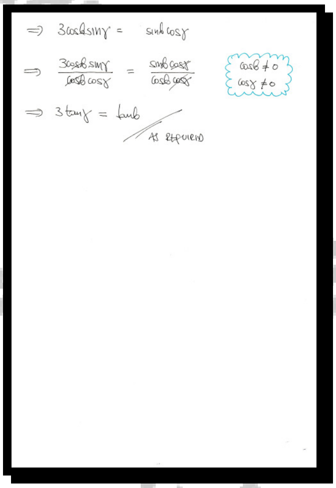
$$\tan\frac{\alpha}{2} = \left(1+\tan^2\frac{\alpha}{2}\right)\sin(\beta-\gamma).$$

Assuming that the triangle is not right angled, show that

$$3\tan\gamma = \tan\beta.$$

[ ], [proof]



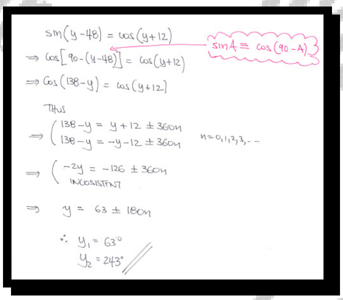


Question 63 (*****) **non calculator**

Solve the trigonometric equation

$$\sin(y-48) = \cos(y+12), \quad 0 \le y < 360°.$$

$y = 63°, 243°$



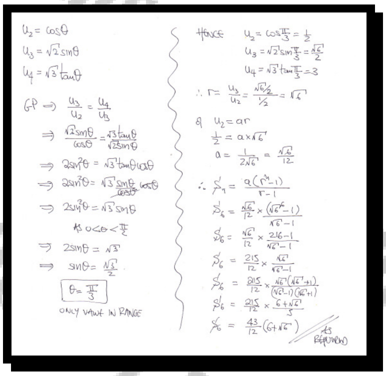
Question 64 (*****)

The 2^{nd}, 3^{rd} and 4^{th} terms of a geometric series are $\cos\theta$, $\sqrt{2}\sin\theta$ and $\sqrt{3}\tan\theta$, respectively, where $0 < \theta < \frac{\pi}{2}$.

Show clearly that the sum of the first 6 terms of the series is

$$\frac{43}{12}\left(6+\sqrt{6}\right).$$

proof



Question 65 (*****)

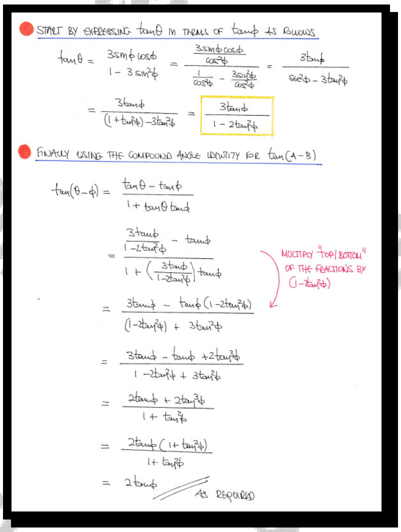
It is given that θ and φ satisfy the relationship

$$\tan\theta = \frac{3\sin\varphi\cos\varphi}{1-3\sin^2\varphi}.$$

Show clearly that

$$\tan(\theta-\varphi) = 2\tan\varphi.$$

[ ], [proof]


START BY EXPRESSING $\tan\theta$ IN TERMS OF $\tan\phi$ AS FOLLOWS

$$\tan\theta = \frac{3\sin\phi\cos\phi}{1-3\sin^2\phi} = \frac{\frac{3\sin\phi\cos\phi}{\cos^2\phi}}{\frac{1}{\cos^2\phi}-\frac{3\sin^2\phi}{\cos^2\phi}} = \frac{3\tan\phi}{\sec^2\phi-3\tan^2\phi}$$

$$= \frac{3\tan\phi}{(1+\tan^2\phi)-3\tan^2\phi} = \boxed{\frac{3\tan\phi}{1-2\tan^2\phi}}$$

FINALLY USING THE COMPOUND ANGLE IDENTITY FOR $\tan(A-B)$

$$\tan(\theta-\phi) = \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$$

$$= \frac{\frac{3\tan\phi}{1-2\tan^2\phi}-\tan\phi}{1+\left(\frac{3\tan\phi}{1-2\tan^2\phi}\right)\tan\phi}$$

MULTIPLY "TOP/BOTTOM" OF THE FRACTIONS BY $(1-2\tan^2\phi)$

$$= \frac{3\tan\phi-\tan\phi(1-2\tan^2\phi)}{(1-2\tan^2\phi)+3\tan^2\phi}$$

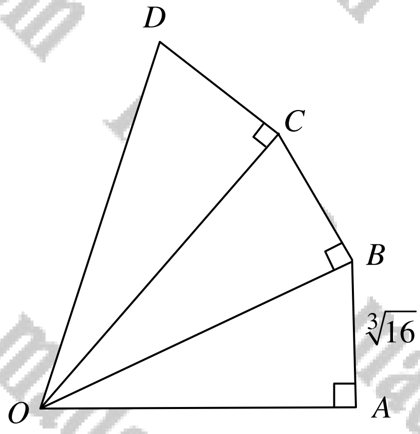
$$= \frac{3\tan\phi-\tan\phi+2\tan^3\phi}{1-2\tan^2\phi+3\tan^2\phi}$$

$$= \frac{2\tan\phi+2\tan^3\phi}{1+\tan^2\phi}$$

$$= \frac{2\tan\phi(1+\tan^2\phi)}{1+\tan^2\phi}$$

$$= 2\tan\phi$$ AS REQUIRED

Question 66 (*****)

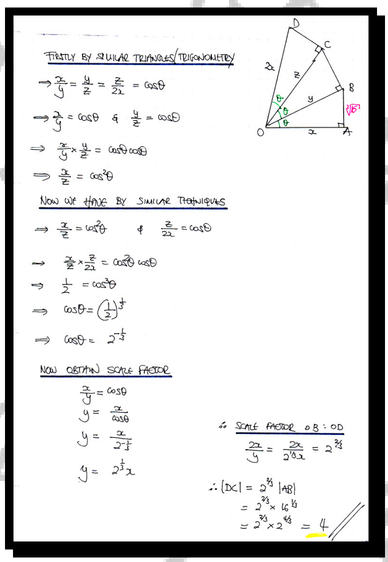


The figure above shows three right angle triangles, OAB, OBC and OCD.

It is given that $\measuredangle AOB = \measuredangle BOC = \measuredangle COD$ and $|OD| = 2|OA|$.

Given further that the length of AB is $\sqrt[3]{16}$, determine the length of DC.

$\boxed{}$, $\boxed{|DC| = 4}$

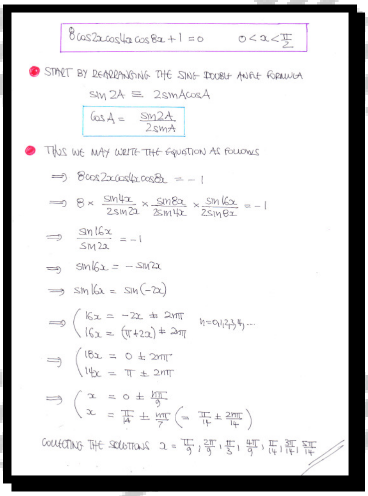


Question 67 (*****)

Solve the trigonometric equation

$$8\cos 2x \cos 4x \cos 8x + 1 = 0, \quad 0 < x < \frac{\pi}{2}.$$

SP , $x = \frac{\pi}{14}, \frac{\pi}{9}, \frac{3\pi}{14}, \frac{2\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{14}, \frac{4\pi}{9}$



$8\cos 2x \cos 4x \cos 8x + 1 = 0 \qquad 0 < x < \frac{\pi}{2}$

- START BY REARRANGING THE SINE DOUBLE ANGLE FORMULA

$\sin 2A \equiv 2\sin A\cos A$

$\cos A = \frac{\sin 2A}{2\sin A}$

- THUS WE MAY WRITE THE EQUATION AS FOLLOWS

$\Rightarrow 8\cos 2x\cos 4x\cos 8x = -1$

$\Rightarrow 8 \times \frac{\sin 4x}{2\sin 2x} \times \frac{\sin 8x}{2\sin 4x} \times \frac{\sin 16x}{2\sin 8x} = -1$

$\Rightarrow \frac{\sin 16x}{\sin 2x} = -1$

$\Rightarrow \sin 16x = -\sin 2x$

$\Rightarrow \sin 16x = \sin(-2x)$

$\Rightarrow \begin{cases} 16x = -2x \pm 2n\pi \\ 16x = (\pi + 2x) \pm 2n\pi \end{cases} \quad n = 0, 1, 2, 3, 4, \dots$

$\Rightarrow \begin{cases} 18x = 0 \pm 2n\pi \\ 14x = \pi \pm 2n\pi \end{cases}$

$\Rightarrow \begin{cases} x = 0 \pm \frac{n\pi}{9} \\ x = \frac{\pi}{14} \pm \frac{n\pi}{7} \left(= \frac{\pi}{14} \pm \frac{2n\pi}{14} \right) \end{cases}$

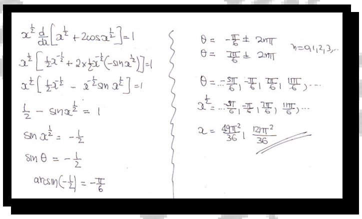
COLLECTING THE SOLUTIONS $x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{\pi}{3}, \frac{4\pi}{9}, \frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}$

Question 68 (*****)

Find, in terms of π, the solutions of the equation

$$\sqrt{x}\frac{d}{dx}\left(\sqrt{x} + 2\cos\sqrt{x}\right) = 1, \quad 0 \le x < 4\pi^2.$$

$x = \frac{49\pi^2}{36}, \frac{121\pi^2}{36}$



$x^{\frac{1}{2}} \frac{d}{dx}\left[x^{\frac{1}{2}} + 2\cos x^{\frac{1}{2}}\right] = 1$

$x^{\frac{1}{2}}\left[\frac{1}{2}x^{-\frac{1}{2}} + 2 \times \frac{1}{2}x^{-\frac{1}{2}}\left(-\sin x^{\frac{1}{2}}\right)\right] = 1$

$x^{\frac{1}{2}}\left[\frac{1}{2}x^{-\frac{1}{2}} - x^{-\frac{1}{2}}\sin x^{\frac{1}{2}}\right] = 1$

$\frac{1}{2} - \sin x^{\frac{1}{2}} = 1$

$\sin x^{\frac{1}{2}} = -\frac{1}{2}$

$\sin\theta = -\frac{1}{2}$

$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\theta = -\frac{\pi}{6} \pm 2n\pi$

$\theta = \frac{7\pi}{6} \pm 2n\pi \quad n = 0, 1, 2, 3, \dots$

$\theta = \dots -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \dots$

$x^{\frac{1}{2}} = \dots -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \dots$

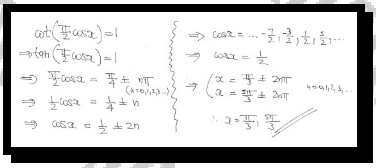
$x = \frac{49\pi^2}{36}, \frac{121\pi^2}{36}$

Question 69 (*****)

Find in terms of π the solutions of the trigonometric equation

$$\cot\left(\frac{\pi}{2}\cos x\right)=1,\ 0 \le x < 2\pi .$$

$$\boxed{x=\frac{\pi}{3},\frac{5\pi}{3}}$$



$\cot\left(\frac{\pi}{2}\cos x\right)=1$

$\Rightarrow \tan\left(\frac{\pi}{2}\cos x\right)=1$

$\Rightarrow \frac{\pi}{2}\cos x = \frac{\pi}{4} \pm n\pi \quad (n=0,1,2,3\ldots)$

$\Rightarrow \frac{1}{2}\cos x = \frac{1}{4} \pm n$

$\Rightarrow \cos x = \frac{1}{2} \pm 2n$

$\Rightarrow \cos x = \ldots -\frac{7}{2}, -\frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \ldots$

$\Rightarrow \cos x = \frac{1}{2}$

$\Rightarrow \begin{cases} x = \frac{\pi}{3} \pm 2n\pi \\ x = \frac{5\pi}{3} \pm 2n\pi \end{cases} \quad n=0,1,2,3,\ldots$

$\therefore x = \frac{\pi}{3}, \frac{5\pi}{3}$

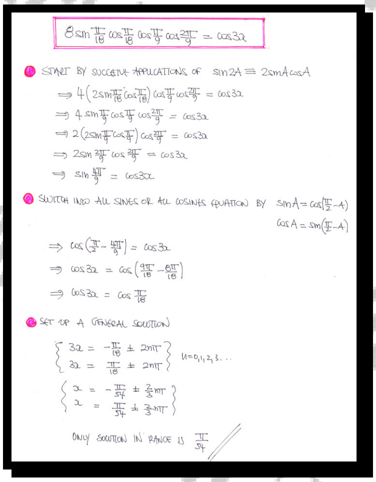
Question 70 (*****)

Solve the trigonometric equation

$$8\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)=\cos 3x, \qquad 0 \le x < \frac{\pi}{2},$$

giving the answer in terms of π.

$$\boxed{}, \boxed{x=\frac{\pi}{54}}$$



$8\sin\frac{\pi}{18}\cos\frac{\pi}{18}\cos\frac{\pi}{9}\cos\frac{2\pi}{9} = \cos 3x$

START BY SUCCESSIVE APPLICATIONS OF $\sin 2A \equiv 2\sin A\cos A$

$\Rightarrow 4\left(2\sin\frac{\pi}{18}\cos\frac{\pi}{18}\right)\cos\frac{\pi}{9}\cos\frac{2\pi}{9} = \cos 3x$

$\Rightarrow 4\sin\frac{\pi}{9}\cos\frac{\pi}{9}\cos\frac{2\pi}{9} = \cos 3x$

$\Rightarrow 2\left(2\sin\frac{\pi}{9}\cos\frac{\pi}{9}\right)\cos\frac{2\pi}{9} = \cos 3x$

$\Rightarrow 2\sin\frac{2\pi}{9}\cos\frac{2\pi}{9} = \cos 3x$

$\Rightarrow \sin\frac{4\pi}{9} = \cos 3x$

SWITCH INTO ALL SINES OR ALL COSINES EQUATION BY $\sin A = \cos\left(\frac{\pi}{2}-A\right)$, $\cos A = \sin\left(\frac{\pi}{2}-A\right)$

$\Rightarrow \cos\left(\frac{\pi}{2}-\frac{4\pi}{9}\right) = \cos 3x$

$\Rightarrow \cos 3x = \cos\left(\frac{9\pi}{18}-\frac{8\pi}{18}\right)$

$\Rightarrow \cos 3x = \cos\frac{\pi}{18}$

SET UP A GENERAL SOLUTION

$\begin{cases} 3x = -\frac{\pi}{18} \pm 2n\pi \\ 3x = \frac{\pi}{18} \pm 2n\pi \end{cases} \quad n=0,1,2,3\ldots$

$\begin{cases} x = -\frac{\pi}{54} \pm \frac{2}{3}n\pi \\ x = \frac{\pi}{54} \pm \frac{2}{3}n\pi \end{cases}$

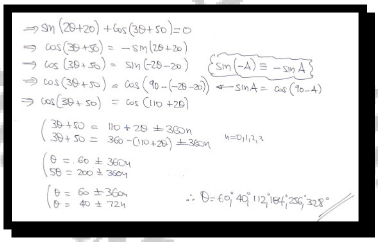
ONLY SOLUTION IN RANGE IS $\frac{\pi}{54}$

Question 71 (*****)

Solve the trigonometric equation

$$\sin\left(2\theta^\circ + 20^\circ\right) + \cos\left(3\theta^\circ + 50^\circ\right) = 0\,,\quad 0 \le \theta \le 360^\circ .$$

$\boxed{\theta = 40^\circ, 60^\circ, 112^\circ, 184^\circ, 256^\circ, 328^\circ}$


$\Rightarrow \sin(2\theta+20) + \cos(3\theta+50) = 0$
$\Rightarrow \cos(3\theta+50) = -\sin(2\theta+20)$
$\Rightarrow \cos(3\theta+50) = \sin(-2\theta-20)$ $\{\sin(-A) \equiv -\sin A\}$
$\Rightarrow \cos(3\theta+50) = \cos(90-(-2\theta-20))$ ← $\sin A = \cos(90-A)$
$\Rightarrow \cos(3\theta+50) = \cos(110+2\theta)$
$3\theta+50 = 110+2\theta \pm 360n$
$3\theta+50 = 360-(110+2\theta) \pm 360n$ $n=0,1,2,3$
$\theta = 60 \pm 360n$
$5\theta = 200 \pm 360n$
$\theta = 60 \pm 360n$
$\theta = 40 \pm 72n$
$\therefore \theta = 60^\circ, 40^\circ, 112^\circ, 184^\circ, 256^\circ, 328^\circ$

Question 72 (*****)

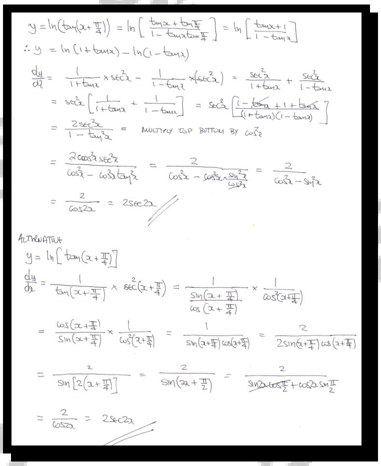
A curve has equation

$$y = \ln\left[\tan\left(x + \frac{\pi}{4}\right)\right],\quad -\frac{\pi}{4} < x < \frac{\pi}{4} .$$

Show clearly that

$$\frac{dy}{dx} = 2\sec 2x .$$

$\boxed{\text{proof}}$



Question 73 (*****)

Prove the validity of each of the following trigonometric identities.

i. $\sin^2\theta\tan\theta+\cos^2\theta\cot\theta+2\sin\theta\cos\theta\equiv\tan\theta+\cot\theta$.

ii. $\sin\theta(1+\tan\theta)+\cos\theta(1+\cot\theta)\equiv\sec\theta+\operatorname{cosec}\theta$.

[illegible], proof

I) L.H.S. $=\sin^2\theta\tan\theta+\cos^2\theta\cot\theta+2\sin\theta\cos\theta$

$=\sin^2\theta\left(\frac{\sin\theta}{\cos\theta}\right)+\cos^2\theta\left(\frac{\cos\theta}{\sin\theta}\right)+2\sin\theta\cos\theta$

$=\frac{\sin^3\theta}{\cos\theta}+2\sin\theta\cos\theta+\frac{\cos^3\theta}{\sin\theta}$

$=\frac{\sin^4\theta+2\sin^2\theta\cos^2\theta+\cos^4\theta}{\cos\theta\sin\theta}$

$=\frac{(\sin^2\theta+\cos^2\theta)^2}{\cos\theta\sin\theta}=\frac{1}{\cos\theta\sin\theta}$

$=\frac{\cos^2\theta+\sin^2\theta}{\cos\theta\sin\theta}=\frac{\cos^2\theta}{\cos\theta\sin\theta}+\frac{\sin^2\theta}{\cos\theta\sin\theta}$

$=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}=\cot\theta+\tan\theta=$ R.H.S.

II) L.H.S. $=\sin\theta(1+\tan\theta)+\cos\theta(1+\cot\theta)$

$=\sin\theta+\sin\theta\tan\theta+\cos\theta+\cos\theta\cot\theta$

$=\sin\theta+\sin\theta\left(\frac{\sin\theta}{\cos\theta}\right)+\cos\theta+\cos\theta\left(\frac{\cos\theta}{\sin\theta}\right)$

$=\sin\theta+\frac{\sin^2\theta}{\cos\theta}+\cos\theta+\frac{\cos^2\theta}{\sin\theta}$

$=\frac{\sin^2\theta\cos\theta+\sin^3\theta+\cos^2\theta\sin\theta+\cos^3\theta}{\cos\theta\sin\theta}$

$=\frac{\sin^2\theta(\cos\theta+\sin\theta)+\cos^2\theta(\cos\theta+\sin\theta)}{\cos\theta\sin\theta}$

$=\frac{(\cos\theta+\sin\theta)(\sin^2\theta+\cos^2\theta)}{\cos\theta\sin\theta}=\frac{\cos\theta+\sin\theta}{\cos\theta\sin\theta}$

$=\frac{\cos\theta}{\cos\theta\sin\theta}+\frac{\sin\theta}{\cos\theta\sin\theta}=\operatorname{cosec}\theta+\sec\theta=$ R.H.S.

Question 74 (*****)

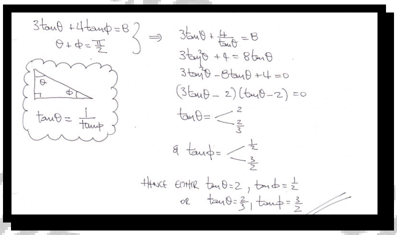
Given the simultaneous equations

$$3\tan\theta + 4\tan\varphi = 8$$

$$\theta + \varphi = \frac{\pi}{2},$$

find the possible values of $\tan\theta$ and $\tan\varphi$.

$$\boxed{[\tan\theta, \tan\varphi] = \left[2, \frac{1}{2}\right] = \left[\frac{2}{3}, \frac{3}{2}\right]}$$



Question 75 (*****)

A relationship between x and y is given by the equations

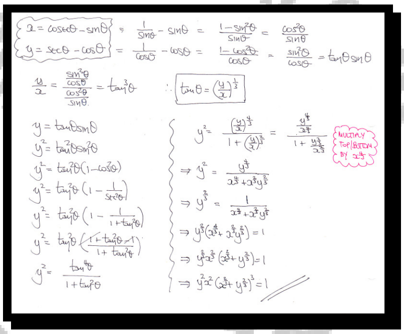
$$x = \operatorname{cosec}\theta - \sin\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$y = \sec\theta - \cos\theta, \quad 0 < \theta < \frac{\pi}{2}.$$

Use trigonometric identities to show that

$$y^2 x^2 \left(x^{\frac{2}{3}} + y^{\frac{2}{3}} \right)^3 = 1.$$

proof



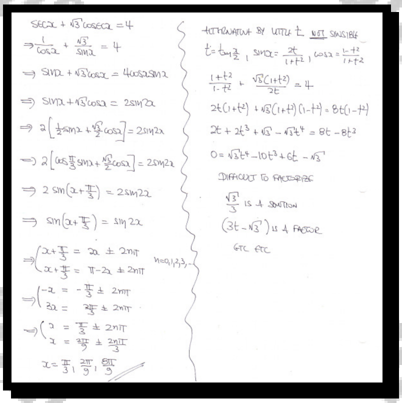
Question 76 (*****)

Solve the trigonometric equation

$$\sec x + \sqrt{3}\operatorname{cosec} x = 4, \quad 0 \le \theta < \pi,$$

giving the answers in terms of π.

$$\boxed{\theta = \frac{2\pi}{9}, \frac{\pi}{3}, \frac{8\pi}{9}}$$



Question 77 (*****)

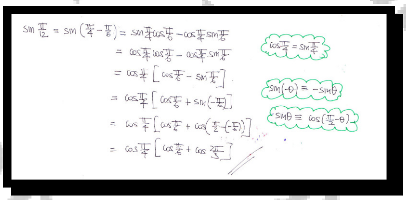
It is given that

$$\cos\left(\frac{\pi}{12}\right) \equiv \cos\left(\frac{\pi}{4}\right)\left[\cos(A\pi) + \cos(B\pi)\right],$$

where A and B are constants such that $0 < A < 1$ and $0 < B < 1$.

Determine the value of A and the value of B.

$$\boxed{A = \frac{1}{6}}, \boxed{B = \frac{2}{3}}$$

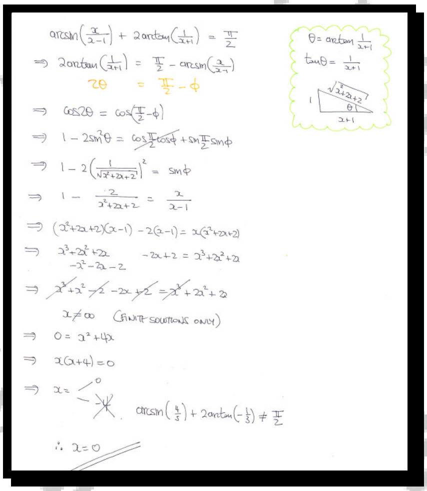


Question 78 (*****)

Find the only finite solution of the trigonometric equation

$$\arcsin\left(\frac{x}{x-1}\right)+2\arctan\left(\frac{1}{x+1}\right)=\frac{\pi}{2}.$$

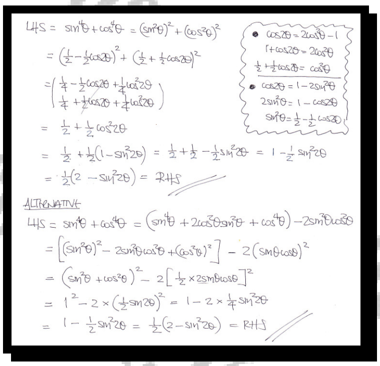
$\boxed{x=0}$



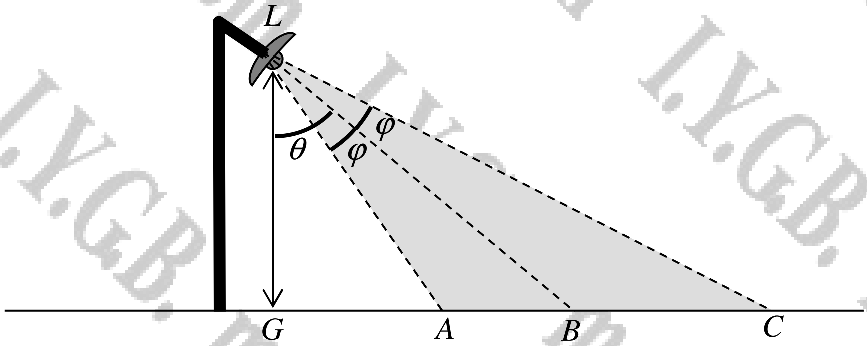
Question 79 (*****)

Prove the validity of the following trigonometric identity

$$\sin^4\theta+\cos^4\theta\equiv\frac{1}{2}\left(2-\sin^2 2\theta\right).$$



Question 80 (*****)



The figure above shows a spotlight L, beaming down on level ground, where L is mounted at a height of 12 metres and the point G is directly below L.

The bulb emits light in the shape of a cone whose axis of symmetry LB is angled at $\theta°$ to the vertical.

The beam is $\varphi°$ wide all the way round the axis of symmetry of the light cone.

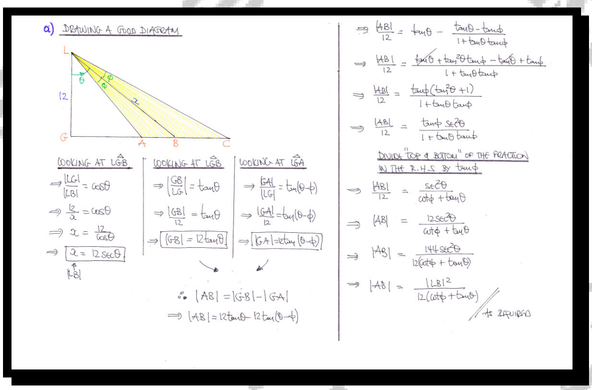
a) Show that the length of AB is

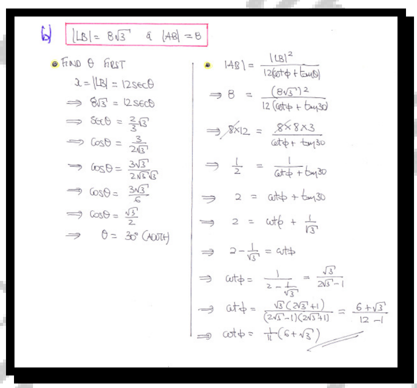
$$|AB| = \frac{12\sec^2\theta}{\cot\varphi + \tan\theta}.$$

The lengths of LG and AB are $8\sqrt{3}$ metres and 8 metres, respectively.

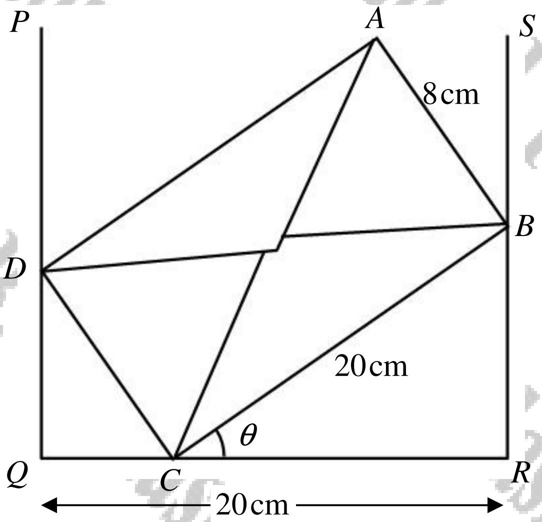
b) Show further that $\tan\varphi = \frac{1}{11}\left(6+\sqrt{3}\right)$.

SP-P , proof





Question 81 (*****)



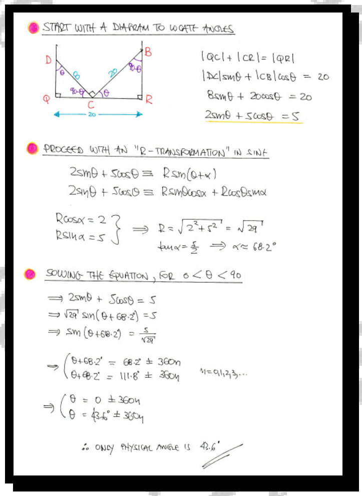
The figure above shows the cross section of a letter inside a filling slot.

The letter $ABCD$ is modelled as a rectangle with $|AB| = 8\text{cm}$ and $|BC| = 20\text{cm}$.

The width of the filling slot QR is also 20 cm and the angle BCR is θ.

Determine the value of θ.

$\boxed{}$, $\boxed{\theta \approx 43.6°}$

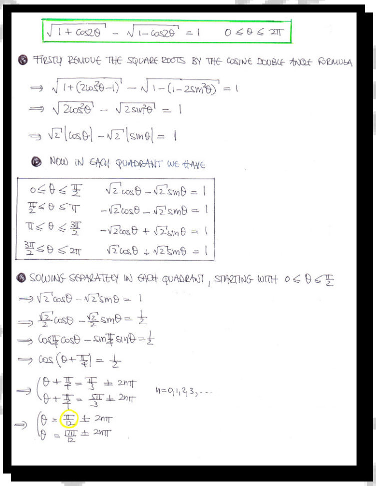


Question 82 (*****)

Solve the following trigonometric equation

$$\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta} = 1, \quad 0 \le \theta < 2\pi .$$

$\boxed{}$, $\boxed{\theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}}$



$\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta} = 1 \qquad 0 \le \theta \le 2\pi$

- FIRSTLY REMOVE THE SQUARE ROOTS BY THE COSINE DOUBLE ANGLE FORMULA

$\Rightarrow \sqrt{1+(2\cos^2\theta - 1)} - \sqrt{1-(1-2\sin^2\theta)} = 1$

$\Rightarrow \sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta} = 1$

$\Rightarrow \sqrt{2}|\cos\theta| - \sqrt{2}|\sin\theta| = 1$

- NOW IN EACH QUADRANT WE HAVE

$0 \le \theta \le \frac{\pi}{2}$	$\sqrt{2}\cos\theta - \sqrt{2}\sin\theta = 1$
$\frac{\pi}{2} \le \theta \le \pi$	$-\sqrt{2}\cos\theta - \sqrt{2}\sin\theta = 1$
$\pi \le \theta \le \frac{3\pi}{2}$	$-\sqrt{2}\cos\theta + \sqrt{2}\sin\theta = 1$
$\frac{3\pi}{2} \le \theta \le 2\pi$	$\sqrt{2}\cos\theta + \sqrt{2}\sin\theta = 1$

- SOLVING SEPARATELY IN EACH QUADRANT, STARTING WITH $0 \le \theta \le \frac{\pi}{2}$

$\Rightarrow \sqrt{2}\cos\theta - \sqrt{2}\sin\theta = 1$

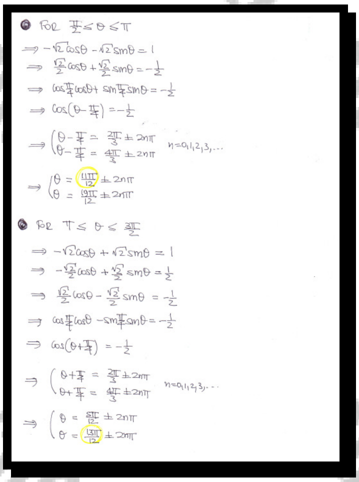
$\Rightarrow \frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta = \frac{1}{2}$

$\Rightarrow \cos\frac{\pi}{4}\cos\theta - \sin\frac{\pi}{4}\sin\theta = \frac{1}{2}$

$\Rightarrow \cos\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2}$

$\Rightarrow \begin{cases} \theta + \frac{\pi}{4} = \frac{\pi}{3} \pm 2n\pi \\ \theta + \frac{\pi}{4} = \frac{5\pi}{3} \pm 2n\pi \end{cases} \quad n = 0,1,2,3,\ldots$

$\Rightarrow \begin{cases} \theta = \frac{\pi}{12} \pm 2n\pi \\ \theta = \frac{17\pi}{12} \pm 2n\pi \end{cases}$



- FOR $\frac{\pi}{2} \le \theta \le \pi$

$\Rightarrow -\sqrt{2}\cos\theta - \sqrt{2}\sin\theta = 1$

$\Rightarrow \frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta = -\frac{1}{2}$

$\Rightarrow \cos\frac{\pi}{4}\cos\theta + \sin\frac{\pi}{4}\sin\theta = -\frac{1}{2}$

$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = -\frac{1}{2}$

$\Rightarrow \begin{cases} \theta - \frac{\pi}{4} = \frac{2\pi}{3} \pm 2n\pi \\ \theta - \frac{\pi}{4} = \frac{4\pi}{3} \pm 2n\pi \end{cases} \quad n = 0,1,2,3,\ldots$

$\Rightarrow \begin{cases} \theta = \frac{11\pi}{12} \pm 2n\pi \\ \theta = \frac{19\pi}{12} \pm 2n\pi \end{cases}$

- FOR $\pi \le \theta \le \frac{3\pi}{2}$

$\Rightarrow -\sqrt{2}\cos\theta + \sqrt{2}\sin\theta = 1$

$\Rightarrow -\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta = \frac{1}{2}$

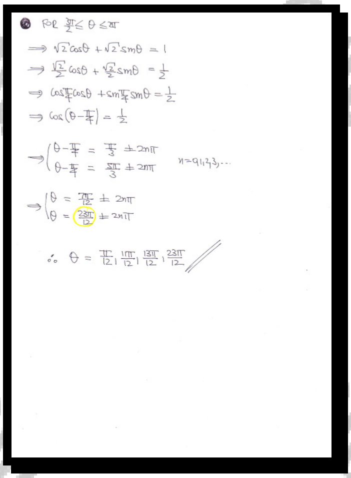
$\Rightarrow \frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta = -\frac{1}{2}$

$\Rightarrow \cos\frac{\pi}{4}\cos\theta - \sin\frac{\pi}{4}\sin\theta = -\frac{1}{2}$

$\Rightarrow \cos\left(\theta + \frac{\pi}{4}\right) = -\frac{1}{2}$

$\Rightarrow \begin{cases} \theta + \frac{\pi}{4} = \frac{2\pi}{3} \pm 2n\pi \\ \theta + \frac{\pi}{4} = \frac{4\pi}{3} \pm 2n\pi \end{cases} \quad n = 0,1,2,3,\ldots$

$\Rightarrow \begin{cases} \theta = \frac{5\pi}{12} \pm 2n\pi \\ \theta = \frac{13\pi}{12} \pm 2n\pi \end{cases}$



- FOR $\frac{3\pi}{2} \le \theta \le 2\pi$

$\Rightarrow \sqrt{2}\cos\theta + \sqrt{2}\sin\theta = 1$

$\Rightarrow \frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta = \frac{1}{2}$

$\Rightarrow \cos\frac{\pi}{4}\cos\theta + \sin\frac{\pi}{4}\sin\theta = \frac{1}{2}$

$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2}$

$\Rightarrow \begin{cases} \theta - \frac{\pi}{4} = \frac{\pi}{3} \pm 2n\pi \\ \theta - \frac{\pi}{4} = \frac{5\pi}{3} \pm 2n\pi \end{cases} \quad n = 0,1,2,3,\ldots$

$\Rightarrow \begin{cases} \theta = \frac{7\pi}{12} \pm 2n\pi \\ \theta = \frac{23\pi}{12} \pm 2n\pi \end{cases}$

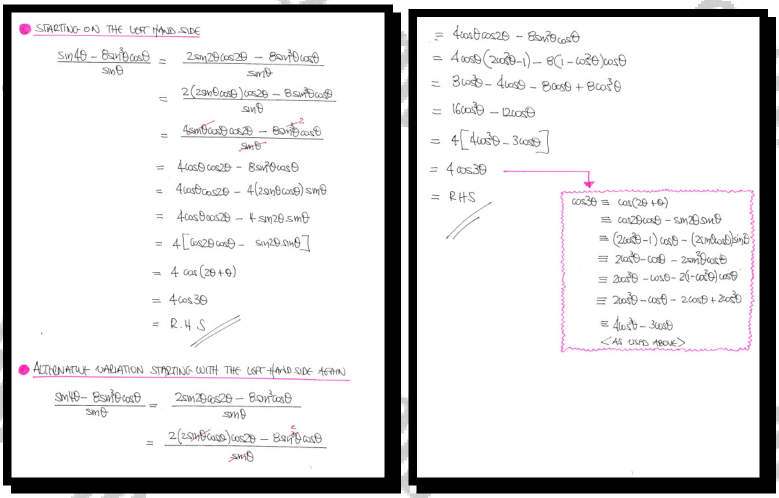
$\therefore \theta = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

Question 83 (*****)

Prove the validity of the following trigonometric identity.

$$\frac{\sin 4\theta - 8\sin^3\theta\cos\theta}{\sin\theta} \equiv 4\cos 3\theta\,, \quad \theta \neq n\pi\,.$$

S K, proof



● STARTING ON THE LEFT HAND SIDE

$$\frac{\sin 4\theta - 8\sin^3\theta\cos\theta}{\sin\theta} = \frac{2\sin 2\theta\cos 2\theta - 8\sin^3\theta\cos\theta}{\sin\theta}$$
$$= \frac{2(2\sin\theta\cos\theta)\cos 2\theta - 8\sin^3\theta\cos\theta}{\sin\theta}$$
$$= \frac{4\sin\theta\cos\theta\cos 2\theta - 8\sin^3\theta\cos\theta}{\sin\theta}$$
$$= 4\cos\theta\cos 2\theta - 8\sin^2\theta\cos\theta$$
$$= 4\cos\theta\cos 2\theta - 4(2\sin\theta\cos\theta)\sin\theta$$
$$= 4\cos\theta\cos 2\theta - 4\sin 2\theta\sin\theta$$
$$= 4[\cos 2\theta\cos\theta - \sin 2\theta\sin\theta]$$
$$= 4\cos(2\theta+\theta)$$
$$= 4\cos 3\theta$$
$$= \text{R.H.S}$$

● ALTERNATIVE VARIATION STARTING WITH THE LEFT HAND SIDE AGAIN

$$\frac{\sin 4\theta - 8\sin^3\theta\cos\theta}{\sin\theta} = \frac{2\sin 2\theta\cos 2\theta - 8\sin^3\theta\cos\theta}{\sin\theta}$$
$$= \frac{2(2\sin\theta\cos\theta)\cos 2\theta - 8\sin^3\theta\cos\theta}{\sin\theta}$$
$$= 4\cos\theta\cos 2\theta - 8\sin^2\theta\cos\theta$$
$$= 4\cos\theta(2\cos^2\theta - 1) - 8(1-\cos^2\theta)\cos\theta$$
$$= 8\cos^3\theta - 4\cos\theta - 8\cos\theta + 8\cos^3\theta$$
$$= 16\cos^3\theta - 12\cos\theta$$
$$= 4[4\cos^3\theta - 3\cos\theta]$$
$$= 4\cos 3\theta$$
$$= \text{RHS}$$

$$\cos 3\theta \equiv \cos(2\theta+\theta)$$
$$\equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$
$$\equiv (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$
$$\equiv 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$
$$\equiv 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$$
$$\equiv 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$
$$\equiv 4\cos^3\theta - 3\cos\theta$$

<AS USED ABOVE>

Question 84 (*****)

The three angles of a triangle are denoted by α, β and γ.

a) Show clearly that …

i. … $\sin(\alpha+\beta)=\sin\gamma$.

ii. … $\sin\left(\dfrac{\alpha+\beta}{2}\right)=\cos\left(\dfrac{\gamma}{2}\right)$.

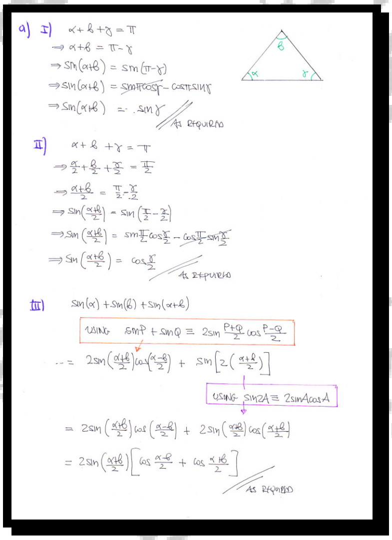
iii. … $\sin\alpha+\sin\beta+\sin(\alpha+\beta)=2\sin\left(\dfrac{\alpha+\beta}{2}\right)\left[\cos\left(\dfrac{\alpha+\beta}{2}\right)+\cos\left(\dfrac{\alpha-\beta}{2}\right)\right]$.

iv. … $\sin\alpha+\sin\beta+\sin\gamma=4\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}$.

b) By using (**iv**) with suitable values for α, β and γ, show that

$$\cos\left(\frac{\pi}{12}\right)=\frac{\sqrt{6}+\sqrt{2}}{4}.$$

proof



a) I) $\alpha+\beta+\gamma=\pi$

$\Rightarrow \alpha+\beta=\pi-\gamma$

$\Rightarrow \sin(\alpha+\beta)=\sin(\pi-\gamma)$

$\Rightarrow \sin(\alpha+\beta)=\sin\pi\cos\gamma-\cos\pi\sin\gamma$

$\Rightarrow \sin(\alpha+\beta)=\sin\gamma$ // AS REQUIRED

II) $\alpha+\beta+\gamma=\pi$

$\Rightarrow \frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=\frac{\pi}{2}$

$\Rightarrow \frac{\alpha+\beta}{2}=\frac{\pi}{2}-\frac{\gamma}{2}$

$\Rightarrow \sin\left(\frac{\alpha+\beta}{2}\right)=\sin\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)$

$\Rightarrow \sin\left(\frac{\alpha+\beta}{2}\right)=\sin\frac{\pi}{2}\cos\frac{\gamma}{2}-\cos\frac{\pi}{2}\sin\frac{\gamma}{2}$

$\Rightarrow \sin\left(\frac{\alpha+\beta}{2}\right)=\cos\frac{\gamma}{2}$ // AS REQUIRED

III) $\sin(\alpha)+\sin(\beta)+\sin(\alpha+\beta)$

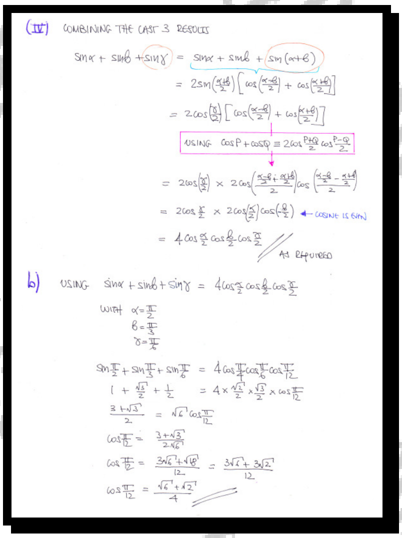
USING $\sin P+\sin Q\equiv 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$

$\dots = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)+\sin\left[2\left(\frac{\alpha+\beta}{2}\right)\right]$

USING $\sin 2A\equiv 2\sin A\cos A$

$=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)+2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

$=2\sin\left(\frac{\alpha+\beta}{2}\right)\left[\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}\right]$ // AS REQUIRED



(IV) COMBINING THE LAST 3 RESULTS

$\sin\alpha+\sin\beta+\sin\gamma=\sin\alpha+\sin\beta+\sin(\alpha+\beta)$

$=2\sin\left(\frac{\alpha+\beta}{2}\right)\left[\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}\right)\right]$

$=2\cos\left(\frac{\gamma}{2}\right)\left[\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}\right)\right]$

USING $\cos P+\cos Q\equiv 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$

$=2\cos\left(\frac{\gamma}{2}\right)\times 2\cos\left(\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta}{2}}{2}\right)\cos\left(\frac{\frac{\alpha-\beta}{2}-\frac{\alpha+\beta}{2}}{2}\right)$

$=2\cos\frac{\gamma}{2}\times 2\cos\left(\frac{\alpha}{2}\right)\cos\left(-\frac{\beta}{2}\right)$ ← COSINE IS EVEN

$=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$ // AS REQUIRED

b) USING $\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

WITH $\alpha=\frac{\pi}{2}$, $\beta=\frac{\pi}{3}$, $\gamma=\frac{\pi}{6}$

$\sin\frac{\pi}{2}+\sin\frac{\pi}{3}+\sin\frac{\pi}{6}=4\cos\frac{\pi}{4}\cos\frac{\pi}{6}\cos\frac{\pi}{12}$

$1+\frac{\sqrt{3}}{2}+\frac{1}{2}=4\times\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}\times\cos\frac{\pi}{12}$

$\frac{3+\sqrt{3}}{2}=\sqrt{6}\cos\frac{\pi}{12}$

$\cos\frac{\pi}{12}=\frac{3+\sqrt{3}}{2\sqrt{6}}$

$\cos\frac{\pi}{12}=\frac{3\sqrt{6}+\sqrt{18}}{12}=\frac{3\sqrt{6}+3\sqrt{2}}{12}$

$\cos\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}$ //

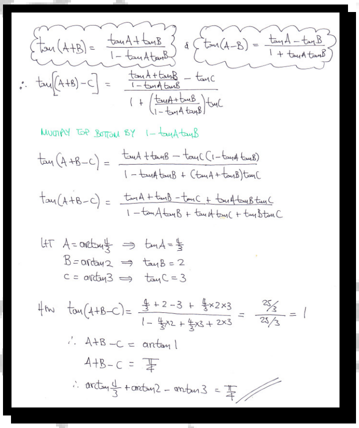
Question 85 (*****)

Simplify, showing all steps in the calculation, the expression

$$\arctan\frac{4}{3} + \arctan 2 - \arctan 3,$$

giving the answer in terms of π.

$$\boxed{\frac{\pi}{4}}$$



$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ & $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$\therefore \tan[(A+B)-C] = \dfrac{\frac{\tan A + \tan B}{1-\tan A \tan B} - \tan C}{1 + \left(\frac{\tan A+\tan B}{1-\tan A\tan B}\right)\tan C}$

MULTIPLY TOP BOTTOM BY $1 - \tan A \tan B$

$\tan(A+B-C) = \dfrac{\tan A + \tan B - \tan C(1-\tan A\tan B)}{1 - \tan A \tan B + (\tan A + \tan B)\tan C}$

$\tan(A+B-C) = \dfrac{\tan A + \tan B - \tan C + \tan A \tan B \tan C}{1 - \tan A\tan B + \tan A \tan C + \tan B \tan C}$

LET $A = \arctan\frac{4}{3} \Rightarrow \tan A = \frac{4}{3}$

$B = \arctan 2 \Rightarrow \tan B = 2$

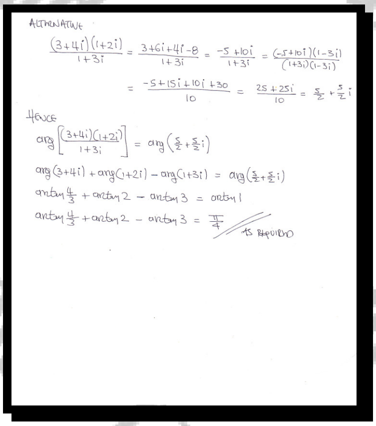
$C = \arctan 3 \Rightarrow \tan C = 3$

THEN $\tan(A+B-C) = \dfrac{\frac{4}{3}+2-3+\frac{4}{3}\times2\times3}{1-\frac{4}{3}\times2+\frac{4}{3}\times3+2\times3} = \dfrac{25/3}{25/3} = 1$

$\therefore A+B-C = \arctan 1$

$A+B-C = \frac{\pi}{4}$

$\therefore \arctan\frac{4}{3} + \arctan 2 - \arctan 3 = \frac{\pi}{4}$



ALTERNATIVE

$\dfrac{(3+4i)(1+2i)}{1+3i} = \dfrac{3+6i+4i-8}{1+3i} = \dfrac{-5+10i}{1+3i} = \dfrac{(-5+10i)(1-3i)}{(1+3i)(1-3i)}$

$= \dfrac{-5+15i+10i+30}{10} = \dfrac{25+25i}{10} = \frac{5}{2}+\frac{5}{2}i$

HENCE

$\arg\left[\dfrac{(3+4i)(1+2i)}{1+3i}\right] = \arg\left(\frac{5}{2}+\frac{5}{2}i\right)$

$\arg(3+4i) + \arg(1+2i) - \arg(1+3i) = \arg\left(\frac{5}{2}+\frac{5}{2}i\right)$

$\arctan\frac{4}{3} + \arctan 2 - \arctan 3 = \arctan 1$

$\arctan\frac{4}{3} + \arctan 2 - \arctan 3 = \frac{\pi}{4}$ AS REQUIRED

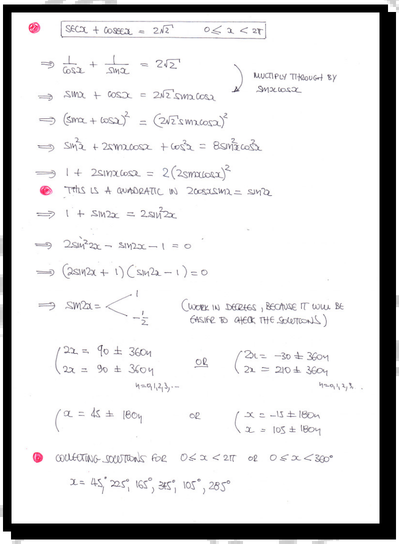
Question 86 (*****)

Solve the trigonometric equation

$$\sec x + \operatorname{cosec} x = 2\sqrt{2}, \qquad 0 \le x < 2\pi,$$

giving the answers in terms of π.

$\boxed{}$, $\boxed{x = \frac{\pi}{4}, \frac{11\pi}{12}, \frac{19\pi}{12}}$



$\sec x + \operatorname{cosec} x = 2\sqrt{2} \qquad 0 \le x < 2\pi$

$\Rightarrow \frac{1}{\cos x} + \frac{1}{\sin x} = 2\sqrt{2}$ (MULTIPLY THROUGH BY $\sin x \cos x$)

$\Rightarrow \sin x + \cos x = 2\sqrt{2}\sin x \cos x$

$\Rightarrow (\sin x + \cos x)^2 = (2\sqrt{2}\sin x \cos x)^2$

$\Rightarrow \sin^2 x + 2\sin x \cos x + \cos^2 x = 8\sin^2 x \cos^2 x$

$\Rightarrow 1 + 2\sin x \cos x = 2(2\sin x \cos x)^2$

THIS IS A QUADRATIC IN $2\cos x \sin x = \sin 2x$

$\Rightarrow 1 + \sin 2x = 2\sin^2 2x$

$\Rightarrow 2\sin^2 2x - \sin 2x - 1 = 0$

$\Rightarrow (2\sin 2x + 1)(\sin 2x - 1) = 0$

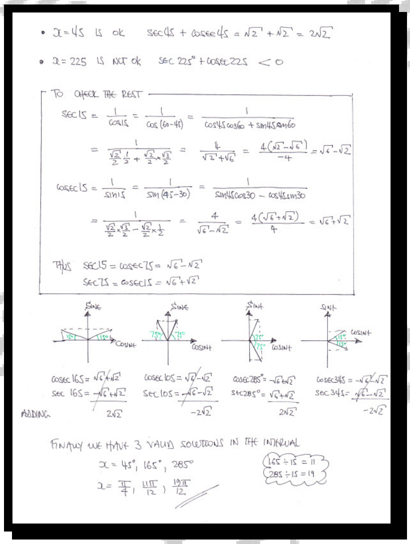
$\Rightarrow \sin 2x = 1$ or $-\frac{1}{2}$ (WORK IN DEGREES, BECAUSE IT WILL BE EASIER TO CHECK THE SOLUTIONS)

$2x = 90 \pm 360n$, $2x = 90 \pm 360n$, $n = 0, 1, 2, 3, \dots$ OR $2x = -30 \pm 360n$, $2x = 210 \pm 360n$, $n = 0, 1, 2, 3, \dots$

$x = 45 \pm 180n$ OR $x = -15 \pm 180n$, $x = 105 \pm 180n$

COLLECTING SOLUTIONS FOR $0 \le x < 2\pi$ OR $0 \le x < 360°$

$x = 45°, 225°, 165°, 345°, 105°, 285°$



• $x = 45$ IS OK $\quad \sec 45 + \operatorname{cosec} 45 = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

• $x = 225$ IS NOT OK $\quad \sec 225° + \operatorname{cosec} 225 < 0$

TO CHECK THE REST

$\sec 15 = \frac{1}{\cos 15} = \frac{1}{\cos(60-45)} = \frac{1}{\cos 45 \cos 60 + \sin 45 \sin 60} = \frac{1}{\frac{\sqrt{2}}{2}\frac{1}{2} + \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}} = \frac{4}{\sqrt{2}+\sqrt{6}} = \frac{4(\sqrt{2}-\sqrt{6})}{-4} = \sqrt{6}-\sqrt{2}$

$\operatorname{cosec} 15 = \frac{1}{\sin 15} = \frac{1}{\sin(45-30)} = \frac{1}{\sin 45 \cos 30 - \cos 45 \sin 30} = \frac{1}{\frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\times\frac{1}{2}} = \frac{4}{\sqrt{6}-\sqrt{2}} = \frac{4(\sqrt{6}+\sqrt{2})}{4} = \sqrt{6}+\sqrt{2}$

THUS $\sec 15 = \operatorname{cosec} 75 = \sqrt{6}-\sqrt{2}$

$\sec 75 = \operatorname{cosec} 15 = \sqrt{6}+\sqrt{2}$

$\operatorname{cosec} 165 = \sqrt{6}+\sqrt{2}$, $\sec 165 = -\sqrt{6}+\sqrt{2}$; ADDING $2\sqrt{2}$

$\operatorname{cosec} 105 = \sqrt{6}-\sqrt{2}$, $\sec 105 = -\sqrt{6}-\sqrt{2}$; $-2\sqrt{2}$

$\operatorname{cosec} 285° = -\sqrt{6}+\sqrt{2}$, $\sec 285° = \sqrt{6}+\sqrt{2}$; $2\sqrt{2}$

$\operatorname{cosec} 345 = -\sqrt{6}-\sqrt{2}$, $\sec 345 = \sqrt{6}-\sqrt{2}$; $-2\sqrt{2}$

FINALLY WE HAVE 3 VALID SOLUTIONS IN THE INTERVAL

$x = 45°, 165°, 285°$ $\quad$ ($165 \div 15 = 11$, $285 \div 15 = 19$)

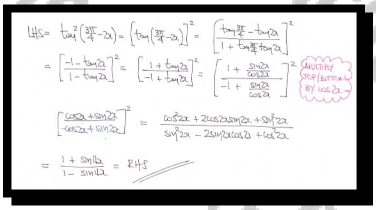
$x = \frac{\pi}{4}, \frac{11\pi}{12}, \frac{19\pi}{12}$

Question 87 (*****)

Prove the validity of the trigonometric identity

$$\tan^2\left(\frac{3\pi}{4} - 2x\right) \equiv \frac{1+\sin 4x}{1-\sin 4x}, \quad x \neq \frac{\pi}{3}(4n+1), \quad n \in \mathbb{Z}$$

$\boxed{\text{proof}}$



LHS $= \tan^2\left(\frac{3\pi}{4}-2x\right) = \left[\tan\left(\frac{3\pi}{4}-2x\right)\right]^2 = \left[\frac{\tan\frac{3\pi}{4} - \tan 2x}{1+\tan\frac{3\pi}{4}\tan 2x}\right]^2$

$= \left[\frac{-1-\tan 2x}{1-\tan 2x}\right]^2 = \left[\frac{1+\tan 2x}{-1+\tan 2x}\right]^2 = \left[\frac{1+\frac{\sin 2x}{\cos 2x}}{-1+\frac{\sin 2x}{\cos 2x}}\right]^2$ (MULTIPLY TOP/BOTTOM BY $\cos 2x$)

$\left[\frac{\cos 2x + \sin 2x}{-\cos 2x + \sin 2x}\right]^2 = \frac{\cos^2 2x + 2\cos 2x \sin 2x + \sin^2 2x}{\sin^2 2x - 2\sin 2x \cos 2x + \cos^2 2x}$

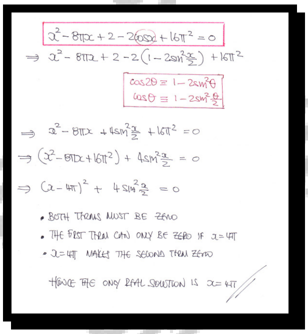
$= \frac{1+\sin 4x}{1-\sin 4x} = $ RHS

Question 88 (*****)

Use algebra to find, in terms of π, the solution of the trigonometric equation

$$x^2 - 8\pi x + 2 - 2\cos x + 16\pi^2 = 0, \quad x \in \mathbb{R}.$$

$\boxed{}$, $\boxed{x = 4\pi}$



$x^2 - 8\pi x + 2 - 2\cos x + 16\pi^2 = 0$

$\Rightarrow x^2 - 8\pi x + 2 - 2\left(1 - 2\sin^2\frac{x}{2}\right) + 16\pi^2$

$\cos 2\theta \equiv 1 - 2\sin^2\theta$

$\cos\theta \equiv 1 - 2\sin^2\frac{\theta}{2}$

$\Rightarrow x^2 - 8\pi x + 4\sin^2\frac{x}{2} + 16\pi^2 = 0$

$\Rightarrow \left(x^2 - 8\pi x + 16\pi^2\right) + 4\sin^2\frac{x}{2} = 0$

$\Rightarrow (x - 4\pi)^2 + 4\sin^2\frac{x}{2} = 0$

- BOTH TERMS MUST BE ZERO
- THE FIRST TERM CAN ONLY BE ZERO IF $x = 4\pi$
- $x = 4\pi$ MAKES THE SECOND TERM ZERO

HENCE THE ONLY REAL SOLUTION IS $x = 4\pi$

Question 89 (*****)

The piecewise continuous function f is given below

$$f(x) \equiv \begin{cases} \sin x^\circ & 0 \le x < 360 \\ \sin 2x^\circ & 360 \le x < 720 \\ \sin 3x^\circ & 720 \le x < 1080 \end{cases}$$

a) Sketch the graph of $f(x)$.

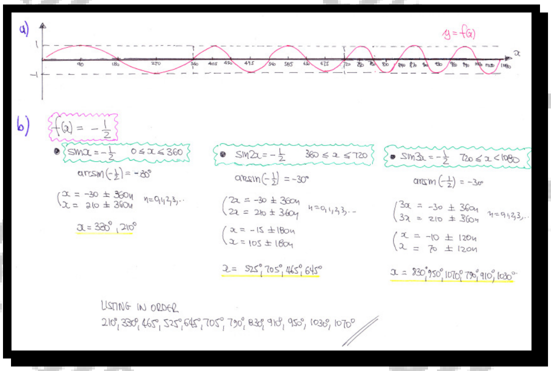
b) Solve the equation …

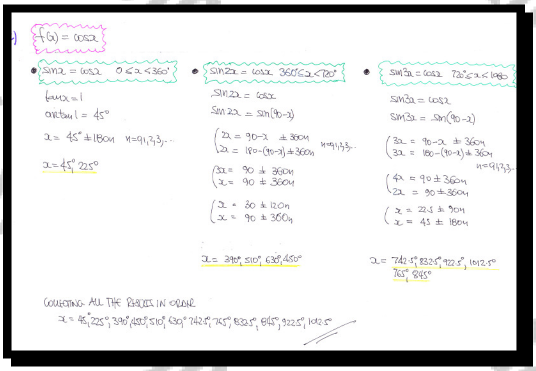
i. … $f(x) = -\frac{1}{2}$.

ii. … $f(x) = \cos x$.

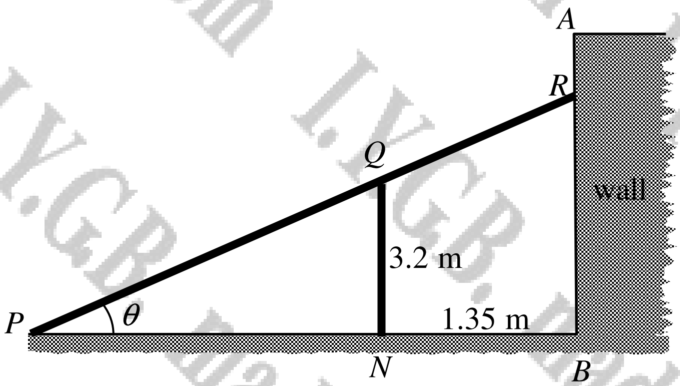
[ ] , $x = 210, 330, 465, 525, 645, 705, 790, 830, 910, 950, 1030, 1070$,

$x = 45, 225, 390, 450, 510, 630, 735, 742.5, 832.5, 915, 922.5, 1012.5$





Question 90 (*****)



The figure above shows the wall AB of a certain structure, which is supported by a straight rigid beam PR, where P is on level ground and R is at some point on the wall.

In order to increase the rigidity of the support, the beam is rested on a steady pole NQ, of height 3.2 metres.

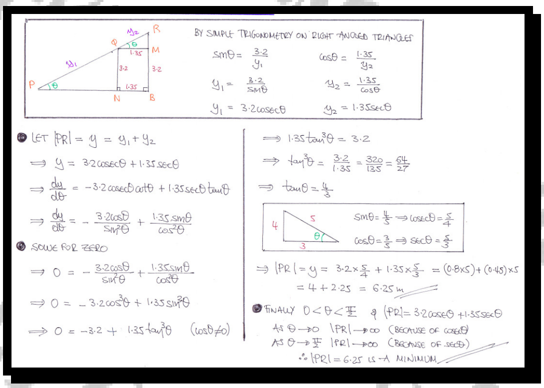
The pole is placed at a distance of 1.35 metres from the bottom of the wall B.

The beam PR is forming an acute angle θ with the horizontal ground PNB.

The angle θ is chosen so that the length of the beam PR, is least.

Determine the least value for the length of the beam PR, assuming that R lies on the wall, fully justifying that this is indeed the minimum value.

[ ], [6.25]



Question 91 (*****)

The functions f and g are defined by

$$f(x) \equiv \cos x, \; x \in \mathbb{R}, \; 0 \le x \le \pi$$

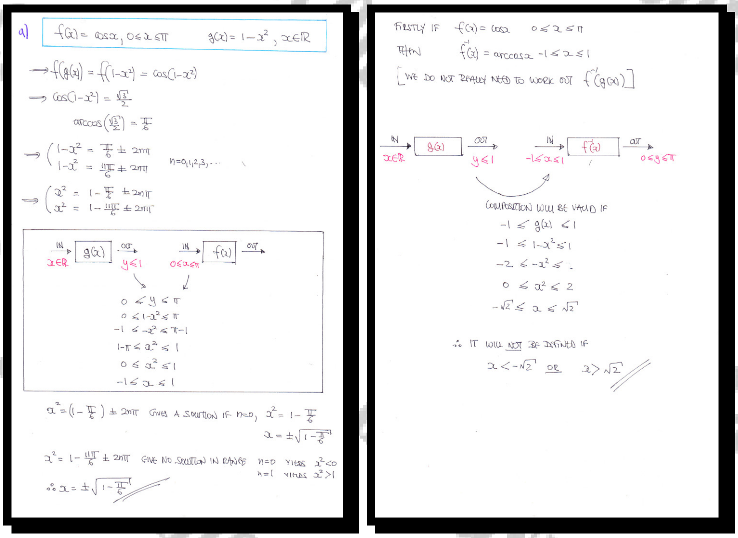
$$g(x) \equiv 1 - x^2, \; x \in \mathbb{R}.$$

a) Solve the equation

$$fg(x) = \frac{1}{2}.$$

b) Determine the values of x for which $f^{-1}g(x)$ is **not** defined.

SP-T, $\boxed{x = \pm\sqrt{1 - \frac{\pi}{6}}}$, $\boxed{x < -\sqrt{2} \text{ or } x > \sqrt{2}}$



Question 92 (*****)

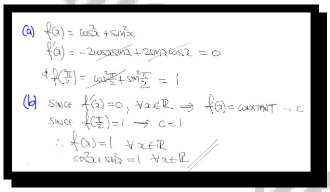
$$f(x) \equiv \cos^2 x + \sin^2 x, \; x \in \mathbb{R}.$$

a) Determine an expression for $f'(x)$ and find the value of $f\left(\frac{1}{2}\pi\right)$.

b) By using the results of part **(a)** only, show that

$$\cos^2 x + \sin^2 x \equiv 1.$$

$\boxed{f'(x) = 0}$, $\boxed{f\left(\frac{1}{2}\pi\right) = 1}$



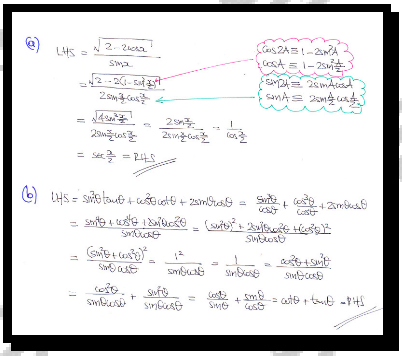
Question 93 (*****)

Prove the validity of each of the following trigonometric identities.

a) $\dfrac{\sqrt{2 - 2\cos x}}{\sin x} \equiv \sec\dfrac{x}{2}$.

b) $\sin^2\theta \tan\theta + \cos^2\theta \cot\theta + 2\sin\theta\cos\theta \equiv \tan\theta + \cot\theta$.

$\boxed{\text{proof}}$

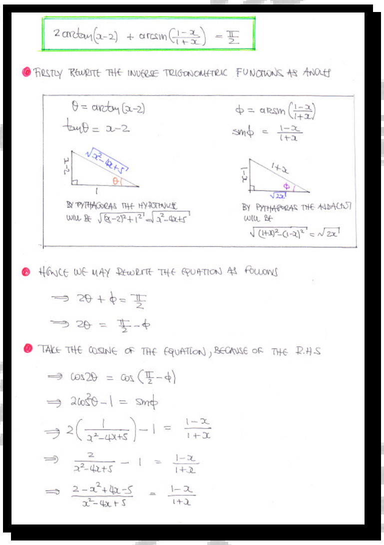


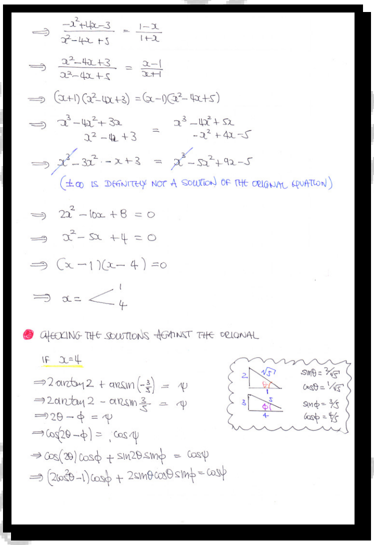
Question 94 (*****)

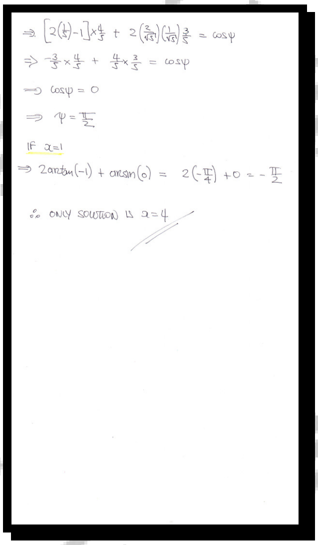
Solve the trigonometric equation

$$2\arctan(x-2)+\arcsin\left(\frac{1-x}{1+x}\right)=\frac{\pi}{2}.$$

$x = 4$







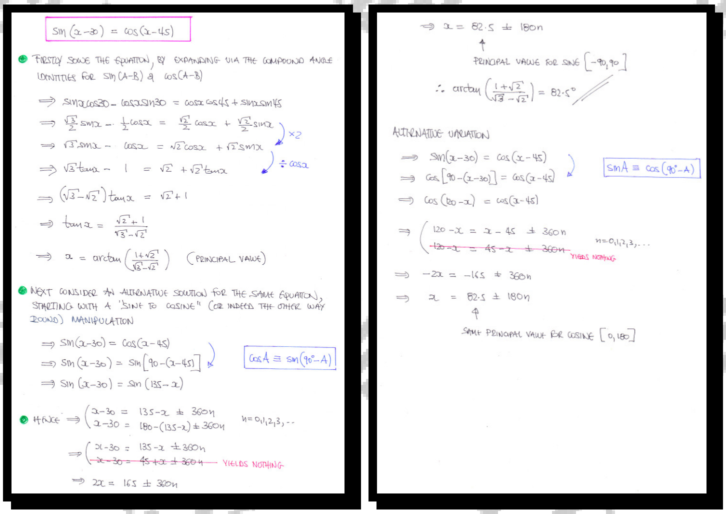
Question 95 (*****)

By considering the solution of trigonometric equation

$$\sin(x-30)^\circ = \cos(x-45)^\circ,$$

find, in degrees, the exact value of $\arctan\left[\dfrac{1+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right]$.

$\boxed{}$, $\boxed{82.5^\circ}$



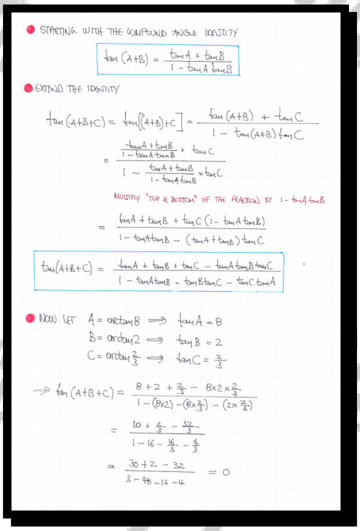
Question 96 (*****)

Simplify, showing all steps in the calculation, the expression

$$\arctan 8 + \arctan 2 + \arctan \frac{2}{3},$$

giving the answer in terms of π.

[ ], $\boxed{\pi}$



- STARTING WITH THE COMPOUND ANGLE IDENTITY

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

- EXTEND THE IDENTITY

$$\tan(A+B+C) = \tan[(A+B)+C] = \frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$$

$$= \frac{\frac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \frac{\tan A + \tan B}{1 - \tan A \tan B} \times \tan C}$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY $1 - \tan A \tan B$

$$= \frac{\tan A + \tan B + \tan C(1 - \tan A \tan B)}{1 - \tan A \tan B - (\tan A + \tan B)\tan C}$$

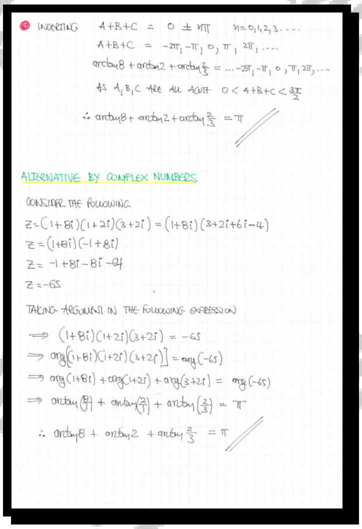
$$\boxed{\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}}$$

- NOW LET $A = \arctan 8 \Rightarrow \tan A = 8$
  $B = \arctan 2 \Rightarrow \tan B = 2$
  $C = \arctan \frac{2}{3} \Rightarrow \tan C = \frac{2}{3}$

$$\Rightarrow \tan(A+B+C) = \frac{8 + 2 + \frac{2}{3} - 8 \times 2 \times \frac{2}{3}}{1 - (8 \times 2) - (8 \times \frac{2}{3}) - (2 \times \frac{2}{3})}$$

$$= \frac{10 + \frac{2}{3} - \frac{32}{3}}{1 - 16 - \frac{16}{3} - \frac{4}{3}}$$

$$= \frac{30 + 2 - 32}{3 - 48 - 16 - 4} = 0$$



- INVERTING $A+B+C = 0 \pm n\pi \quad n = 0, 1, 2, 3 \ldots$

$A+B+C = -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

$\arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \ldots -2\pi, -\pi, 0, \pi, 2\pi, \ldots$

AS A, B, C ARE ALL ACUTE $0 < A+B+C < \frac{3\pi}{2}$

$\therefore \arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \pi$

ALTERNATIVE BY COMPLEX NUMBERS

CONSIDER THE FOLLOWING

$z = (1+8i)(1+2i)(3+2i) = (1+8i)(3+2i+6i-4)$

$z = (1+8i)(-1+8i)$

$z = -1 + 8i - 8i - 64$

$z = -65$

TAKING ARGUMENT IN THE FOLLOWING EXPRESSION

$\Rightarrow (1+8i)(1+2i)(3+2i) = -65$

$\Rightarrow \arg[(1+8i)(1+2i)(3+2i)] = \arg(-65)$

$\Rightarrow \arg(1+8i) + \arg(1+2i) + \arg(3+2i) = \arg(-65)$

$\Rightarrow \arctan\left(\frac{8}{1}\right) + \arctan\left(\frac{2}{1}\right) + \arctan\left(\frac{2}{3}\right) = \pi$

$\therefore \arctan 8 + \arctan 2 + \arctan \frac{2}{3} = \pi$

Question 97 (*****)

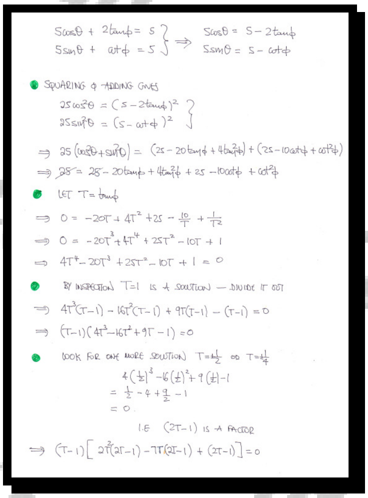
Given that $0 < \theta < \frac{1}{2}\pi$, $0 < \varphi < \frac{1}{2}\pi$, solve the following simultaneous equations.

$$5\cos\theta + 2\tan\varphi = 5 \qquad \text{and} \qquad 5\sin\theta + \cot\varphi = 5 .$$

Give the answers in exact form in terms of inverse trigonometric functions.

S-M, $\boxed{[\theta,\varphi] = \left[\arccos\frac{3}{5}, \arctan 1\right]}$, $\boxed{[\theta,\varphi] = \left[\arccos\frac{4}{5}, \arctan\frac{1}{2}\right]}$,

$\boxed{[\theta,\varphi] = \left[\arccos\left[\frac{1}{10}\left(3+\sqrt{41}\right)\right], \arctan\left[\frac{1}{4}\left(7+\sqrt{41}\right)\right]\right]}$



$5\cos\theta + 2\tan\phi = 5$, $5\sin\theta + \cot\phi = 5$ $\Rightarrow$ $5\cos\theta = 5 - 2\tan\phi$, $5\sin\theta = 5 - \cot\phi$

- SQUARING & ADDING GIVES

$25\cos^2\theta = (5-2\tan\phi)^2$, $25\sin^2\theta = (5-\cot\phi)^2$

$\Rightarrow 25(\cos^2\theta + \sin^2\theta) = (25 - 20\tan\phi + 4\tan^2\phi) + (25 - 10\cot\phi + \cot^2\phi)$

$\Rightarrow 25 = 25 - 20\tan\phi + 4\tan^2\phi + 25 - 10\cot\phi + \cot^2\phi$

- LET $T = \tan\phi$

$\Rightarrow 0 = -20T + 4T^2 + 25 - \frac{10}{T} + \frac{1}{T^2}$

$\Rightarrow 0 = -20T^3 + 4T^4 + 25T^2 - 10T + 1$

$\Rightarrow 4T^4 - 20T^3 + 25T^2 - 10T + 1 = 0$

- BY INSPECTION $T=1$ IS A SOLUTION — DIVIDE IT OUT

$\Rightarrow 4T^3(T-1) - 16T^2(T-1) + 9T(T-1) - (T-1) = 0$

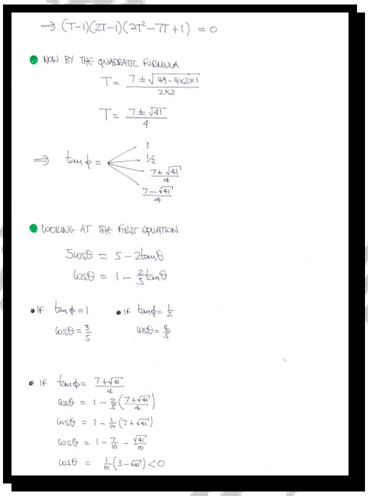
$\Rightarrow (T-1)(4T^3 - 16T^2 + 9T - 1) = 0$

- LOOK FOR ONE MORE SOLUTION $T = \pm\frac{1}{2}$ OR $T = \pm\frac{1}{4}$

$4\left(\frac{1}{2}\right)^3 - 16\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right) - 1 = \frac{1}{2} - 4 + \frac{9}{2} - 1 = 0.$

I.E $(2T-1)$ IS A FACTOR

$\Rightarrow (T-1)\left[2T^2(2T-1) - 7T(2T-1) + (2T-1)\right] = 0$



$\Rightarrow (T-1)(2T-1)(2T^2 - 7T + 1) = 0$

- NOW BY THE QUADRATIC FORMULA

$T = \frac{7 \pm \sqrt{49 - 4\times2\times1}}{2\times2}$

$T = \frac{7 \pm \sqrt{41}}{4}$

$\Rightarrow \tan\phi = 1,\ \frac{1}{2},\ \frac{7+\sqrt{41}}{4},\ \frac{7-\sqrt{41}}{4}$

- LOOKING AT THE FIRST EQUATION

$5\cos\theta = 5 - 2\tan\theta$

$\cos\theta = 1 - \frac{2}{5}\tan\theta$

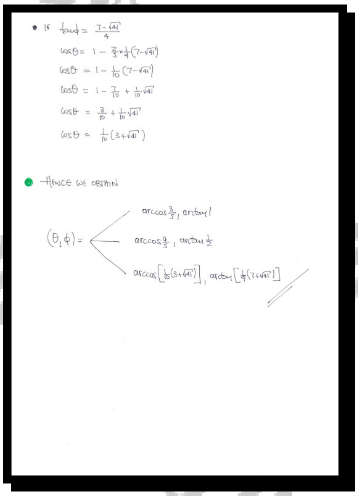
- IF $\tan\phi = 1$: $\cos\theta = \frac{3}{5}$
- IF $\tan\phi = \frac{1}{2}$: $\cos\theta = \frac{4}{5}$
- IF $\tan\phi = \frac{7+\sqrt{41}}{4}$

$\cos\theta = 1 - \frac{2}{5}\left(\frac{7+\sqrt{41}}{4}\right)$

$\cos\theta = 1 - \frac{1}{10}(7+\sqrt{41})$

$\cos\theta = 1 - \frac{7}{10} - \frac{\sqrt{41}}{10}$

$\cos\theta = \frac{1}{10}(3 - \sqrt{41}) < 0$



- IF $\tan\phi = \frac{7-\sqrt{41}}{4}$

$\cos\theta = 1 - \frac{2}{5}\times\frac{1}{4}(7-\sqrt{41})$

$\cos\theta = 1 - \frac{1}{10}(7-\sqrt{41})$

$\cos\theta = 1 - \frac{7}{10} + \frac{1}{10}\sqrt{41}$

$\cos\theta = \frac{3}{10} + \frac{1}{10}\sqrt{41}$

$\cos\theta = \frac{1}{10}(3+\sqrt{41})$

- HENCE WE OBTAIN

$(\theta,\phi) = \arccos\frac{3}{5}, \arctan 1$; $\arccos\frac{4}{5}, \arctan\frac{1}{2}$; $\arccos\left[\frac{1}{10}(3+\sqrt{41})\right], \arctan\left[\frac{1}{4}(7+\sqrt{41})\right]$

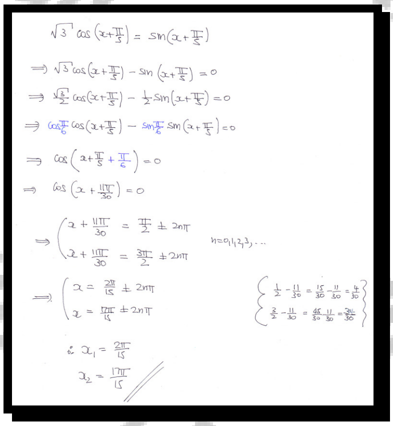
Question 98 (*****)

Solve the trigonometric equation

$$\sqrt{3}\cos\left(x+\frac{\pi}{5}\right)=\sin\left(x+\frac{\pi}{5}\right), \qquad 0 \le \theta < 2\pi ,$$

giving the answers in terms of π.

$\boxed{}$, $\boxed{x=\frac{2\pi}{15},\frac{17\pi}{15}}$



$$\sqrt{3}\cos\left(x+\frac{\pi}{5}\right)=\sin\left(x+\frac{\pi}{5}\right)$$

$$\Rightarrow \sqrt{3}\cos\left(x+\frac{\pi}{5}\right)-\sin\left(x+\frac{\pi}{5}\right)=0$$

$$\Rightarrow \frac{\sqrt{3}}{2}\cos\left(x+\frac{\pi}{5}\right)-\frac{1}{2}\sin\left(x+\frac{\pi}{5}\right)=0$$

$$\Rightarrow \cos\frac{\pi}{6}\cos\left(x+\frac{\pi}{5}\right)-\sin\frac{\pi}{6}\sin\left(x+\frac{\pi}{5}\right)=0$$

$$\Rightarrow \cos\left(x+\frac{\pi}{5}+\frac{\pi}{6}\right)=0$$

$$\Rightarrow \cos\left(x+\frac{11\pi}{30}\right)=0$$

$$\Rightarrow \begin{cases} x+\frac{11\pi}{30}=\frac{\pi}{2}\pm 2n\pi \\ x+\frac{11\pi}{30}=\frac{3\pi}{2}\pm 2n\pi \end{cases} \quad n=0,1,2,3,\ldots$$

$$\Rightarrow \begin{cases} x=\frac{2\pi}{15}\pm 2n\pi \\ x=\frac{17\pi}{15}\pm 2n\pi \end{cases} \quad \begin{cases} \frac{1}{2}-\frac{11}{30}=\frac{15}{30}-\frac{11}{30}=\frac{4}{30} \\ \frac{3}{2}-\frac{11}{30}=\frac{45}{30}-\frac{11}{30}=\frac{34}{30} \end{cases}$$

$$\therefore x_1=\frac{2\pi}{15}$$

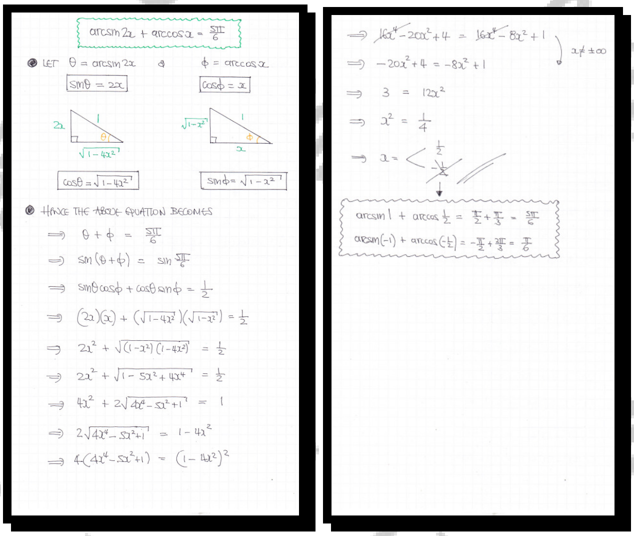
$$x_2=\frac{17\pi}{15}$$

Question 99 (*****)

Solve the following trigonometric equation

$$\arcsin 2x + \arccos x = \frac{5\pi}{6}.$$

S-P , $\boxed{x = \frac{1}{2}}$



$\arcsin 2x + \arccos x = \frac{5\pi}{6}$

- LET $\theta = \arcsin 2x$ & $\phi = \arccos x$

$\sin\theta = 2x$ $\quad$ $\cos\phi = x$

$\cos\theta = \sqrt{1-4x^2}$ $\quad$ $\sin\phi = \sqrt{1-x^2}$

- HENCE THE ABOVE EQUATION BECOMES

$\Rightarrow \theta + \phi = \frac{5\pi}{6}$

$\Rightarrow \sin(\theta+\phi) = \sin\frac{5\pi}{6}$

$\Rightarrow \sin\theta\cos\phi + \cos\theta\sin\phi = \frac{1}{2}$

$\Rightarrow (2x)(x) + \left(\sqrt{1-4x^2}\right)\left(\sqrt{1-x^2}\right) = \frac{1}{2}$

$\Rightarrow 2x^2 + \sqrt{(1-x^2)(1-4x^2)} = \frac{1}{2}$

$\Rightarrow 2x^2 + \sqrt{1-5x^2+4x^4} = \frac{1}{2}$

$\Rightarrow 4x^2 + 2\sqrt{4x^4-5x^2+1} = 1$

$\Rightarrow 2\sqrt{4x^4-5x^2+1} = 1-4x^2$

$\Rightarrow 4(4x^4-5x^2+1) = (1-4x^2)^2$

$\Rightarrow 16x^4 - 20x^2 + 4 = 16x^4 - 8x^2 + 1$ $\quad$ $x \neq \pm\infty$

$\Rightarrow -20x^2 + 4 = -8x^2 + 1$

$\Rightarrow 3 = 12x^2$

$\Rightarrow x^2 = \frac{1}{4}$

$\Rightarrow x = \frac{1}{2}$ ~~$-\frac{1}{2}$~~

$\arcsin 1 + \arccos\frac{1}{2} = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

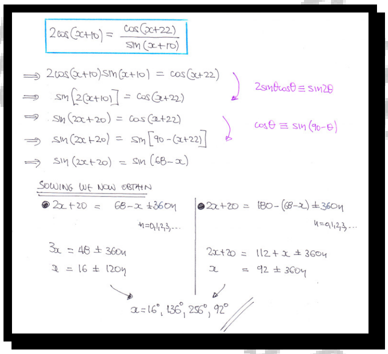
$\arcsin(-1) + \arccos\left(-\frac{1}{2}\right) = -\frac{\pi}{2} + \frac{2\pi}{3} = \frac{\pi}{6}$

Question 100 (*****) **non calculator**

Find, in degrees, the solutions of the trigonometric equation

$$2\cos(x+10)^\circ = \frac{\cos(x+22)^\circ}{\sin(x+10)^\circ}, \quad 0^\circ \le x \le 360^\circ.$$

$\boxed{}$, $\boxed{x = 16^\circ, 92^\circ, 136^\circ, 256^\circ}$



$$2\cos(x+10) = \frac{\cos(x+22)}{\sin(x+10)}$$

$\Rightarrow 2\cos(x+10)\sin(x+10) = \cos(x+22)$

$\Rightarrow \sin[2(x+10)] = \cos(x+22)$ ($2\sin\theta\cos\theta \equiv \sin 2\theta$)

$\Rightarrow \sin(2x+20) = \cos(x+22)$

$\Rightarrow \sin(2x+20) = \sin[90-(x+22)]$ ($\cos\theta \equiv \sin(90-\theta)$)

$\Rightarrow \sin(2x+20) = \sin(68-x)$

SOLVING WE NOW OBTAIN

- $2x+20 = 68-x \pm 360n$, $n=0,1,2,3,\dots$

  $3x = 48 \pm 360n$

  $x = 16 \pm 120n$

- $2x+20 = 180-(68-x) \pm 360n$, $n=0,1,2,3,\dots$

  $2x+20 = 112 + x \pm 360n$

  $x = 92 \pm 360n$

$x = 16^\circ, 136^\circ, 256^\circ, 92^\circ$

Question 101 (*****)

It is given that for $\theta \neq (4k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$,

$$\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) \equiv \tan\theta + \sec\theta .$$

a) Prove the validity of the above trigonometric identity.

b) Hence find a similar expression for $\tan\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$.

You are now given the equation

$$\tan x - \tan(x-\alpha) = 2\tan x ,$$

where α is a constant.

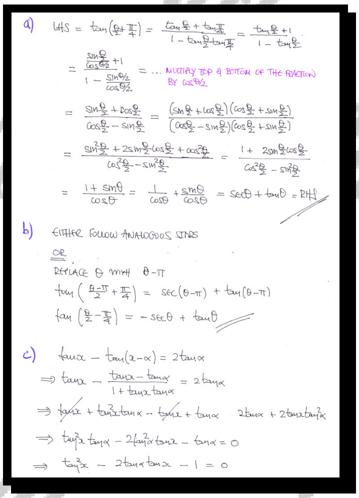
c) Express $\tan x$ in terms of trigonometric functions involving α only.

d) Hence solve the trigonometric equation

$$\tan x - \tan\left(x-\frac{3\pi}{5}\right) = 2\tan\frac{3\pi}{5}, \quad 0 \leq x < 2\pi ,$$

giving the answers in terms of π.

$\boxed{}$, $\boxed{\tan x = \tan\alpha \pm \sec\alpha}$, $\boxed{x = \frac{\pi}{20}, \frac{11\pi}{20}, \frac{21\pi}{20}, \frac{31\pi}{20}}$



a) LHS $= \tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) = \frac{\tan\frac{\theta}{2}+\tan\frac{\pi}{4}}{1-\tan\frac{\theta}{2}\tan\frac{\pi}{4}} = \frac{\tan\frac{\theta}{2}+1}{1-\tan\frac{\theta}{2}}$

$= \frac{\frac{\sin\theta/2}{\cos\theta/2}+1}{1-\frac{\sin\theta/2}{\cos\theta/2}} = \ldots$ MULTIPLY TOP & BOTTOM OF THE FRACTION BY $\cos\theta/2$

$= \frac{\sin\frac{\theta}{2}+\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}} = \frac{(\sin\frac{\theta}{2}+\cos\frac{\theta}{2})(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})}{(\cos\frac{\theta}{2}-\sin\frac{\theta}{2})(\cos\frac{\theta}{2}+\sin\frac{\theta}{2})}$

$= \frac{\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}+\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}} = \frac{1+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}$

$= \frac{1+\sin\theta}{\cos\theta} = \frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta} = \sec\theta+\tan\theta =$ RHS

b) EITHER FOLLOW ANALOGOUS STEPS

OR

REPLACE θ WITH $\theta-\pi$

$\tan\left(\frac{\theta-\pi}{2}+\frac{\pi}{4}\right) = \sec(\theta-\pi)+\tan(\theta-\pi)$

$\tan\left(\frac{\theta}{2}-\frac{\pi}{4}\right) = -\sec\theta+\tan\theta$

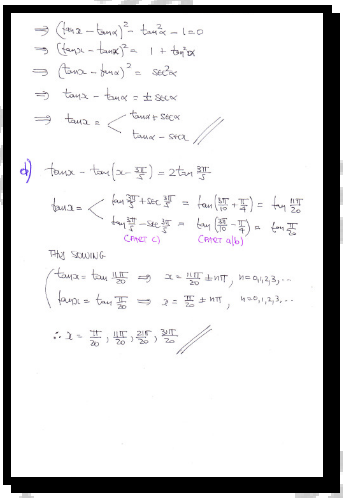
c) $\tan x - \tan(x-\alpha) = 2\tan\alpha$

$\Rightarrow \tan x - \frac{\tan x-\tan\alpha}{1+\tan x\tan\alpha} = 2\tan\alpha$

$\Rightarrow \tan x + \tan^2 x\tan\alpha - \tan x + \tan\alpha \quad 2\tan\alpha + 2\tan x\tan^2\alpha$

$\Rightarrow \tan^2 x\tan\alpha - 2\tan^2\alpha\tan x - \tan\alpha = 0$

$\Rightarrow \tan^2 x - 2\tan\alpha\tan x - 1 = 0$



$\Rightarrow (\tan x-\tan\alpha)^2 - \tan^2\alpha - 1 = 0$

$\Rightarrow (\tan x-\tan\alpha)^2 = 1+\tan^2\alpha$

$\Rightarrow (\tan x-\tan\alpha)^2 = \sec^2\alpha$

$\Rightarrow \tan x - \tan\alpha = \pm\sec\alpha$

$\Rightarrow \tan x = \tan\alpha+\sec\alpha$ or $\tan\alpha-\sec\alpha$

d) $\tan x - \tan\left(x-\frac{3\pi}{5}\right) = 2\tan\frac{3\pi}{5}$

$\tan x = \tan\frac{3\pi}{5}+\sec\frac{3\pi}{5} = \tan\left(\frac{3\pi}{10}+\frac{\pi}{4}\right) = \tan\frac{11\pi}{20}$

$\tan x = \tan\frac{3\pi}{5}-\sec\frac{3\pi}{5} = \tan\left(\frac{3\pi}{10}-\frac{\pi}{4}\right) = \tan\frac{\pi}{20}$

(PART c) (PART a/b)

THUS SOLVING

$\tan x = \tan\frac{11\pi}{20} \Rightarrow x = \frac{11\pi}{20} \pm n\pi, \quad n=0,1,2,3,\ldots$

$\tan x = \tan\frac{\pi}{20} \Rightarrow x = \frac{\pi}{20} \pm n\pi, \quad n=0,1,2,3,\ldots$

$\therefore x = \frac{\pi}{20}, \frac{11\pi}{20}, \frac{21\pi}{20}, \frac{31\pi}{20}$

Question 102 (*****)

It is given that θ and φ satisfy the simultaneous equations

$$\frac{\sin 2\theta}{1+\sin\theta} = 1 - \sin\varphi,$$

$$\theta + \varphi = \pi$$

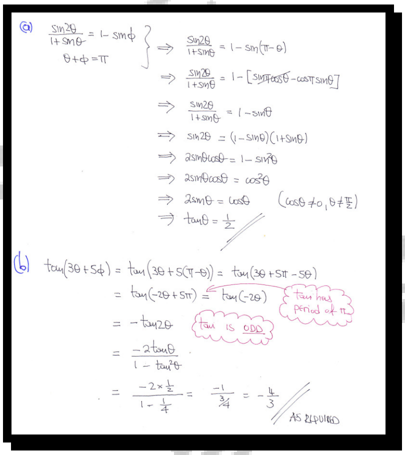
where $0 < \theta < \pi$, $\theta \neq \frac{\pi}{2}$, $0 < \varphi < \pi$.

a) Determine the value of $\tan\theta$.

b) Show clearly that

$$\tan(3\theta + 5\varphi) = -\frac{4}{3}.$$

$\boxed{}$, $\boxed{\tan\theta = \frac{1}{2}}$


(a) $\frac{\sin 2\theta}{1+\sin\theta} = 1-\sin\phi$, $\theta+\phi=\pi$ $\Rightarrow \frac{\sin 2\theta}{1+\sin\theta} = 1-\sin(\pi-\theta)$

$\Rightarrow \frac{\sin 2\theta}{1+\sin\theta} = 1-[\sin\pi\cos\theta - \cos\pi\sin\theta]$

$\Rightarrow \frac{\sin 2\theta}{1+\sin\theta} = 1-\sin\theta$

$\Rightarrow \sin 2\theta = (1-\sin\theta)(1+\sin\theta)$

$\Rightarrow 2\sin\theta\cos\theta = 1-\sin^2\theta$

$\Rightarrow 2\sin\theta\cos\theta = \cos^2\theta$

$\Rightarrow 2\sin\theta = \cos\theta$ $(\cos\theta \neq 0, \theta \neq \frac{\pi}{2})$

$\Rightarrow \tan\theta = \frac{1}{2}$

(b) $\tan(3\theta+5\phi) = \tan(3\theta+5(\pi-\theta)) = \tan(3\theta+5\pi-5\theta)$

$= \tan(-2\theta+5\pi) = \tan(-2\theta)$ (tan has period of π)

$= -\tan 2\theta$ (tan is ODD)

$= \frac{-2\tan\theta}{1-\tan^2\theta}$

$= \frac{-2\times\frac{1}{2}}{1-\frac{1}{4}} = \frac{-1}{\frac{3}{4}} = -\frac{4}{3}$

AS REQUIRED

Question 103 (*****)

The angle θ satisfies the equation

$$\tan\theta \tan 2\theta = \sum_{r=0}^{\infty} 2\cos^r 2\theta .$$

Given that $0 < \theta < \frac{\pi}{2}$, find the exact value of $\tan\theta$.

$\boxed{}$, $\boxed{\tan\theta = 3^{-\frac{1}{4}}}$

$$\boxed{\tan\theta\tan 2\theta = \sum_{r=0}^{\infty} 2\cos^r 2\theta}$$

$\Rightarrow \tan\theta\tan 2\theta = 2\left[1 + \cos 2\theta + \cos^2 2\theta + \cos^3 2\theta + \cos^4 2\theta + \dots\right]$

AS $0 < \theta < \frac{\pi}{2}$, $-1 < \cos 2\theta < 1$

THUS THE G.P ON THE R.H.S CONVERGES TO $S_\infty = \frac{a}{1-r}$

$\Rightarrow \tan\theta\tan 2\theta = 2 \times \frac{1}{1-\cos 2\theta}$

$\Rightarrow \tan\theta\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \frac{2}{1-(1-2\sin^2\theta)}$

$\Rightarrow \frac{2\tan^2\theta}{1-\tan^2\theta} = \frac{2}{2\sin^2\theta}$

$\Rightarrow \frac{2\tan^2\theta}{1-\tan^2\theta} = \operatorname{cosec}^2\theta$

$\Rightarrow \frac{2\tan^2\theta}{1-\tan^2\theta} = 1 + \cot^2\theta$

$\Rightarrow \frac{2\tan^2\theta}{1-\tan^2\theta} = 1 + \frac{1}{\tan^2\theta}$

LET $T = \tan^2\theta$

$\Rightarrow \frac{2T}{1-T} = 1 + \frac{1}{T}$ $\quad) \times T(1-T)$

$\Rightarrow 2T^2 = T(1-T) + (1-T)$

$\Rightarrow 2T^2 = T - T^2 + 1 - T$

$\Rightarrow 3T^2 = 1$

$\Rightarrow T^2 = \frac{1}{3}$

$\Rightarrow (\tan^2\theta)^2 = \frac{1}{3}$

$\Rightarrow \tan^4\theta = \frac{1}{3}$

$\Rightarrow \tan^4\theta = 3^{-1}$

$\Rightarrow \tan\theta = 3^{-\frac{1}{4}}$ // $\quad (\tan\theta > 0$ AS $0 < \theta < \frac{\pi}{2})$

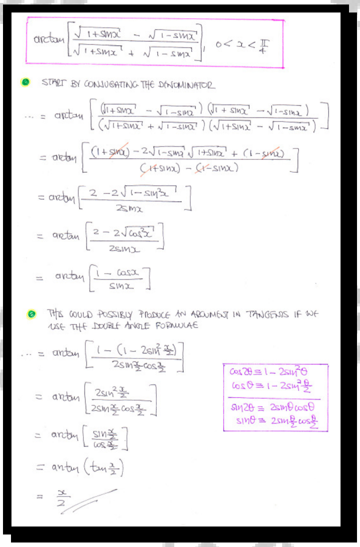
Question 104 (*****)

Use trigonometric algebra to fully simplify

$$\arctan\left[\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right],\ 0<x<\frac{\pi}{4},$$

giving the final answer in terms of x.

$\boxed{}$, $\boxed{\frac{1}{2}x}$



$$\arctan\left[\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right],\quad 0<x<\frac{\pi}{4}$$

- START BY CONJUGATING THE DENOMINATOR

$$\ldots = \arctan\left[\frac{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)}{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)}\right]$$

$$= \arctan\left[\frac{(1+\sin x)-2\sqrt{1-\sin x}\sqrt{1+\sin x}+(1-\sin x)}{(1+\sin x)-(1-\sin x)}\right]$$

$$= \arctan\left[\frac{2-2\sqrt{1-\sin^2 x}}{2\sin x}\right]$$

$$= \arctan\left[\frac{2-2\sqrt{\cos^2 x}}{2\sin x}\right]$$

$$= \arctan\left[\frac{1-\cos x}{\sin x}\right]$$

- THIS COULD POSSIBLY PRODUCE AN ARGUMENT IN TANGENTS IF WE USE THE DOUBLE ANGLE FORMULAE

$$\ldots = \arctan\left[\frac{1-\left(1-2\sin^2\frac{x}{2}\right)}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$$

$$= \arctan\left[\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$$

$$= \arctan\left[\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right]$$

$$= \arctan\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

$\cos 2\theta \equiv 1-2\sin^2\theta$
$\cos\theta \equiv 1-2\sin^2\frac{\theta}{2}$

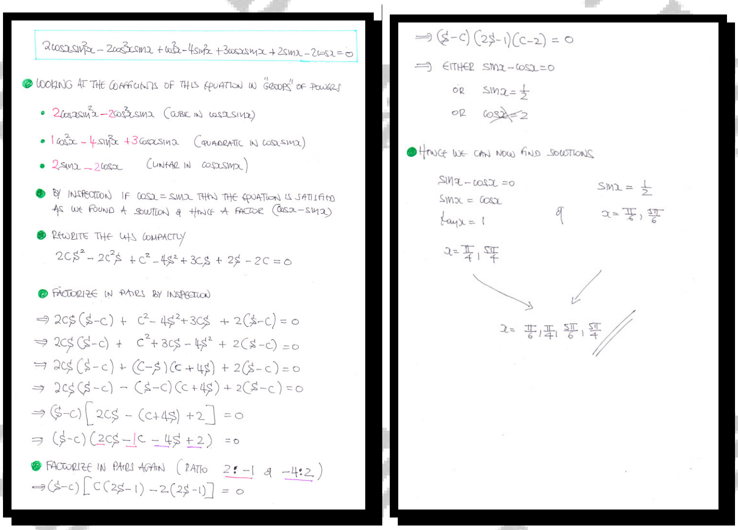
$\sin 2\theta \equiv 2\sin\theta\cos\theta$
$\sin\theta \equiv 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

Question 105 (*****)

Solve the following trigonometric equation, for $0 \le x < 2\pi$.

$$2\cos x \sin^2 x - 2\cos^2 x \sin x + \cos^2 x - 4\sin^2 x + 3\cos x \sin x + 2\sin x - 2\cos x = 0.$$

$\square$, $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}$



$2\cos x\sin^2 x - 2\cos^2 x\sin x + \cos^2 x - 4\sin^2 x + 3\cos x\sin x + 2\sin x - 2\cos x = 0$

- LOOKING AT THE COEFFICIENTS OF THIS EQUATION IN "GROUPS" OF POWERS
  - $2\cos x\sin^2 x - 2\cos^2 x\sin x$ (CUBIC IN $\cos x \sin x$)
  - $1\cos^2 x - 4\sin^2 x + 3\cos x\sin x$ (QUADRATIC IN $\cos x \sin x$)
  - $2\sin x - 2\cos x$ (LINEAR IN $\cos x \sin x$)
- BY INSPECTION IF $\cos x = \sin x$ THEN THE EQUATION IS SATISFIED AS WE FOUND A SOLUTION & HENCE A FACTOR $(\cos x - \sin x)$
- REWRITE THE LHS COMPACTLY

$2CS^2 - 2C^2S + C^2 - 4S^2 + 3CS + 2S - 2C = 0$

- FACTORIZE IN PAIRS BY INSPECTION

$\Rightarrow 2CS(S-C) + C^2 - 4S^2 + 3CS + 2(S-C) = 0$

$\Rightarrow 2CS(S-C) + C^2 + 3CS - 4S^2 + 2(S-C) = 0$

$\Rightarrow 2CS(S-C) + (C-S)(C+4S) + 2(S-C) = 0$

$\Rightarrow 2CS(S-C) - (S-C)(C+4S) + 2(S-C) = 0$

$\Rightarrow (S-C)[2CS - (C+4S) + 2] = 0$

$\Rightarrow (S-C)(2CS - 1C - 4S + 2) = 0$

- FACTORIZE IN PAIRS AGAIN (RATIO $2:-1$ & $-4:2$)

$\Rightarrow (S-C)[C(2S-1) - 2(2S-1)] = 0$

$\Rightarrow (S-C)(2S-1)(C-2) = 0$

$\Rightarrow$ EITHER $\sin x - \cos x = 0$

OR $\sin x = \frac{1}{2}$

OR $\cos x = 2$ (crossed out)

- HENCE WE CAN NOW FIND SOLUTIONS

$\sin x - \cos x = 0$

$\sin x = \cos x$

$\tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

&

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

$x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}$

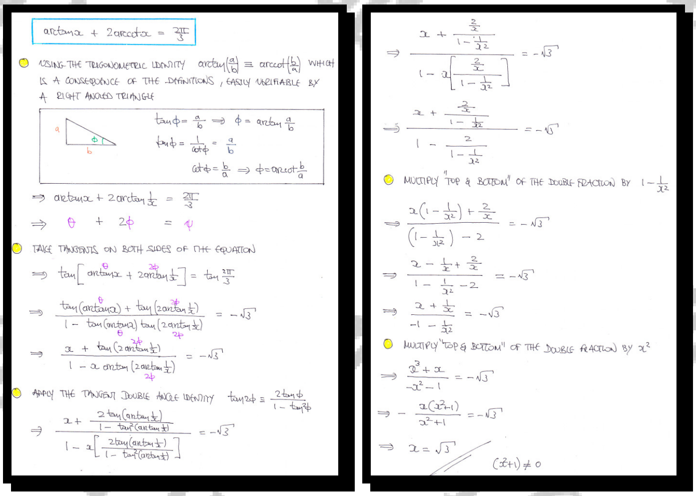
Question 106 (*****)

Use trigonometric algebra to solve the equation

$$\arctan x + 2\,\text{arccot}\, x = \frac{2\pi}{3}.$$

You may assume that $\text{arccot}\, x$ is the inverse function for the part of $\cot x$ for which $0 \le x \le \pi$.

$\boxed{}$, $\boxed{x = \sqrt{3}}$


$$\arctan x + 2\,\text{arccot}\,x = \frac{2\pi}{3}$$

USING THE TRIGONOMETRIC IDENTITY $\arctan\left(\frac{a}{b}\right) \equiv \text{arccot}\left(\frac{b}{a}\right)$ WHICH IS A CONSEQUENCE OF THE DEFINITIONS, EASILY VERIFIABLE BY A RIGHT ANGLED TRIANGLE

$\tan\phi = \frac{a}{b} \Rightarrow \phi = \arctan\frac{a}{b}$

$\tan\phi = \frac{1}{\cot\phi} = \frac{a}{b}$

$\cot\phi = \frac{b}{a} \Rightarrow \phi = \text{arccot}\frac{b}{a}$

$\Rightarrow \arctan x + 2\arctan\frac{1}{x} = \frac{2\pi}{3}$

$\Rightarrow \theta + 2\phi = \psi$

TAKE TANGENTS ON BOTH SIDES OF THE EQUATION

$\Rightarrow \tan\left[\arctan x + 2\arctan\frac{1}{x}\right] = \tan\frac{2\pi}{3}$

$\Rightarrow \dfrac{\tan(\arctan x) + \tan\left(2\arctan\frac{1}{x}\right)}{1 - \tan(\arctan x)\tan\left(2\arctan\frac{1}{x}\right)} = -\sqrt{3}$

$\Rightarrow \dfrac{x + \tan\left(2\arctan\frac{1}{x}\right)}{1 - x\tan\left(2\arctan\frac{1}{x}\right)} = -\sqrt{3}$

APPLY THE TANGENT DOUBLE ANGLE IDENTITY $\tan 2\phi \equiv \frac{2\tan\phi}{1-\tan^2\phi}$

$\Rightarrow \dfrac{x + \dfrac{2\tan\left(\arctan\frac{1}{x}\right)}{1-\tan^2\left(\arctan\frac{1}{x}\right)}}{1 - x\left[\dfrac{2\tan\left(\arctan\frac{1}{x}\right)}{1-\tan^2\left(\arctan\frac{1}{x}\right)}\right]} = -\sqrt{3}$

$\Rightarrow \dfrac{x + \dfrac{\frac{2}{x}}{1-\frac{1}{x^2}}}{1 - x\left[\dfrac{\frac{2}{x}}{1-\frac{1}{x^2}}\right]} = -\sqrt{3}$

$\Rightarrow \dfrac{x + \dfrac{\frac{2}{x}}{1-\frac{1}{x^2}}}{1 - \dfrac{2}{1-\frac{1}{x^2}}} = -\sqrt{3}$

MULTIPLY "TOP & BOTTOM" OF THE DOUBLE FRACTION BY $1-\frac{1}{x^2}$

$\Rightarrow \dfrac{x\left(1-\frac{1}{x^2}\right) + \frac{2}{x}}{\left(1-\frac{1}{x^2}\right) - 2} = -\sqrt{3}$

$\Rightarrow \dfrac{x - \frac{1}{x} + \frac{2}{x}}{1 - \frac{1}{x^2} - 2} = -\sqrt{3}$

$\Rightarrow \dfrac{x + \frac{1}{x}}{-1 - \frac{1}{x^2}} = -\sqrt{3}$

MULTIPLY "TOP & BOTTOM" OF THE DOUBLE FRACTION BY x^2

$\Rightarrow \dfrac{x^3 + x}{-x^2 - 1} = -\sqrt{3}$

$\Rightarrow -\dfrac{x(x^2+1)}{x^2+1} = -\sqrt{3}$

$\Rightarrow x = \sqrt{3}$

$(x^2+1) \neq 0$

Question 107 (*****)

The distinct acute angles θ and φ, $\theta > \varphi$ satisfy the equation

$$f(\theta,\varphi) = g(\theta,\varphi)\tan\varphi,$$

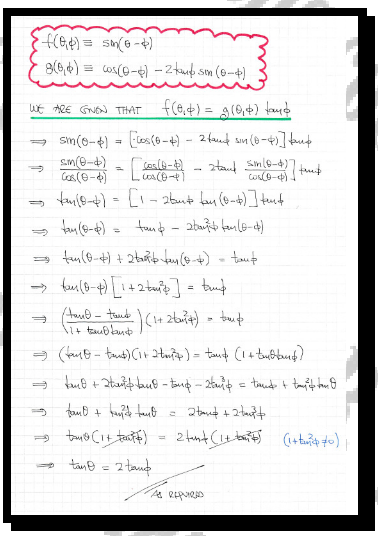
where the functions f and g are defined as

$$f(\theta,\varphi) \equiv \sin(\theta-\varphi) \quad \text{and} \quad g(\theta,\varphi) \equiv \cos(\theta-\varphi) - 2\tan\varphi\sin(\theta-\varphi).$$

Use trigonometric identities to show that

$$\tan\theta = 2\tan\varphi.$$

MP2-Z, proof



$f(\theta,\phi) \equiv \sin(\theta-\phi)$

$g(\theta,\phi) \equiv \cos(\theta-\phi) - 2\tan\phi\sin(\theta-\phi)$

WE ARE GIVEN THAT $f(\theta,\phi) = g(\theta,\phi)\tan\phi$

$\Rightarrow \sin(\theta-\phi) = [\cos(\theta-\phi) - 2\tan\phi\sin(\theta-\phi)]\tan\phi$

$\Rightarrow \frac{\sin(\theta-\phi)}{\cos(\theta-\phi)} = \left[\frac{\cos(\theta-\phi)}{\cos(\theta-\phi)} - 2\tan\phi\frac{\sin(\theta-\phi)}{\cos(\theta-\phi)}\right]\tan\phi$

$\Rightarrow \tan(\theta-\phi) = [1 - 2\tan\phi\tan(\theta-\phi)]\tan\phi$

$\Rightarrow \tan(\theta-\phi) = \tan\phi - 2\tan^2\phi\tan(\theta-\phi)$

$\Rightarrow \tan(\theta-\phi) + 2\tan^2\phi\tan(\theta-\phi) = \tan\phi$

$\Rightarrow \tan(\theta-\phi)[1+2\tan^2\phi] = \tan\phi$

$\Rightarrow \left(\frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}\right)(1+2\tan^2\phi) = \tan\phi$

$\Rightarrow (\tan\theta - \tan\phi)(1+2\tan^2\phi) = \tan\phi(1+\tan\theta\tan\phi)$

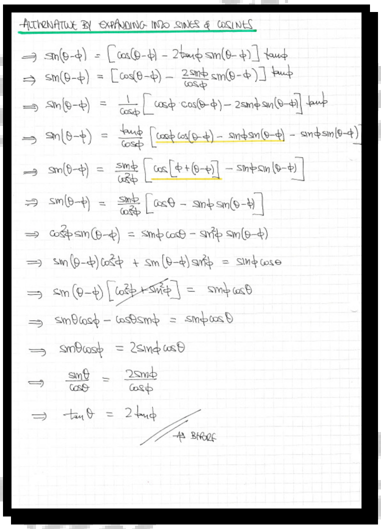
$\Rightarrow \tan\theta + 2\tan^2\phi\tan\theta - \tan\phi - 2\tan^3\phi = \tan\phi + \tan^2\phi\tan\theta$

$\Rightarrow \tan\theta + \tan^2\phi\tan\theta = 2\tan\phi + 2\tan^3\phi$

$\Rightarrow \tan\theta(1+\tan^2\phi) = 2\tan\phi(1+\tan^2\phi)$ $\quad (1+\tan^2\phi \neq 0)$

$\Rightarrow \tan\theta = 2\tan\phi$

AS REQUIRED



ALTERNATIVE BY EXPANDING INTO SINES & COSINES

$\Rightarrow \sin(\theta-\phi) = [\cos(\theta-\phi) - 2\tan\phi\sin(\theta-\phi)]\tan\phi$

$\Rightarrow \sin(\theta-\phi) = \left[\cos(\theta-\phi) - \frac{2\sin\phi}{\cos\phi}\sin(\theta-\phi)\right]\tan\phi$

$\Rightarrow \sin(\theta-\phi) = \frac{1}{\cos\phi}[\cos\phi\cos(\theta-\phi) - 2\sin\phi\sin(\theta-\phi)]\tan\phi$

$\Rightarrow \sin(\theta-\phi) = \frac{\tan\phi}{\cos\phi}[\cos\phi\cos(\theta-\phi) - \sin\phi\sin(\theta-\phi) - \sin\phi\sin(\theta-\phi)]$

$\Rightarrow \sin(\theta-\phi) = \frac{\sin\phi}{\cos^2\phi}[\cos[\phi+(\theta-\phi)] - \sin\phi\sin(\theta-\phi)]$

$\Rightarrow \sin(\theta-\phi) = \frac{\sin\phi}{\cos^2\phi}[\cos\theta - \sin\phi\sin(\theta-\phi)]$

$\Rightarrow \cos^2\phi\sin(\theta-\phi) = \sin\phi\cos\theta - \sin^2\phi\sin(\theta-\phi)$

$\Rightarrow \sin(\theta-\phi)\cos^2\phi + \sin(\theta-\phi)\sin^2\phi = \sin\phi\cos\theta$

$\Rightarrow \sin(\theta-\phi)[\cos^2\phi + \sin^2\phi] = \sin\phi\cos\theta$

$\Rightarrow \sin\theta\cos\phi - \cos\theta\sin\phi = \sin\phi\cos\theta$

$\Rightarrow \sin\theta\cos\phi = 2\sin\phi\cos\theta$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{2\sin\phi}{\cos\phi}$

$\Rightarrow \tan\theta = 2\tan\phi$

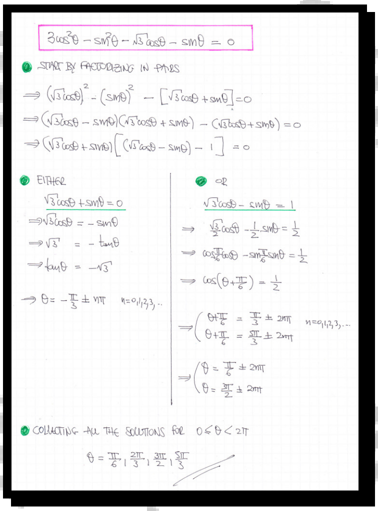
AS BEFORE

Question 108 (*****)

Solve the following trigonometric equation, for $0 \le \theta < 2\pi$.

$$3\cos^2\theta - \sin^2\theta - \sqrt{3}\cos\theta - \sin\theta = 0.$$

$\boxed{}$, $\boxed{\theta = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}}$



$3\cos^2\theta - \sin^2\theta - \sqrt{3}\cos\theta - \sin\theta = 0$

START BY FACTORIZING IN PAIRS

$\Rightarrow (\sqrt{3}\cos\theta)^2 - (\sin\theta)^2 - [\sqrt{3}\cos\theta + \sin\theta] = 0$

$\Rightarrow (\sqrt{3}\cos\theta - \sin\theta)(\sqrt{3}\cos\theta + \sin\theta) - (\sqrt{3}\cos\theta + \sin\theta) = 0$

$\Rightarrow (\sqrt{3}\cos\theta + \sin\theta)\left[(\sqrt{3}\cos\theta - \sin\theta) - 1\right] = 0$

EITHER

$\sqrt{3}\cos\theta + \sin\theta = 0$

$\Rightarrow \sqrt{3}\cos\theta = -\sin\theta$

$\Rightarrow \sqrt{3} = -\tan\theta$

$\Rightarrow \tan\theta = -\sqrt{3}$

$\Rightarrow \theta = -\frac{\pi}{3} \pm n\pi \quad n = 0, 1, 2, 3, \ldots$

OR

$\sqrt{3}\cos\theta - \sin\theta = 1$

$\Rightarrow \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \frac{1}{2}$

$\Rightarrow \cos\frac{\pi}{6}\cos\theta - \sin\frac{\pi}{6}\sin\theta = \frac{1}{2}$

$\Rightarrow \cos\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}$

$\Rightarrow \begin{cases} \theta + \frac{\pi}{6} = \frac{\pi}{3} \pm 2n\pi \\ \theta + \frac{\pi}{6} = \frac{5\pi}{3} \pm 2n\pi \end{cases} \quad n = 0, 1, 2, 3, \ldots$

$\Rightarrow \begin{cases} \theta = \frac{\pi}{6} \pm 2n\pi \\ \theta = \frac{3\pi}{2} \pm 2n\pi \end{cases}$

COLLECTING ALL THE SOLUTIONS FOR $0 \le \theta < 2\pi$

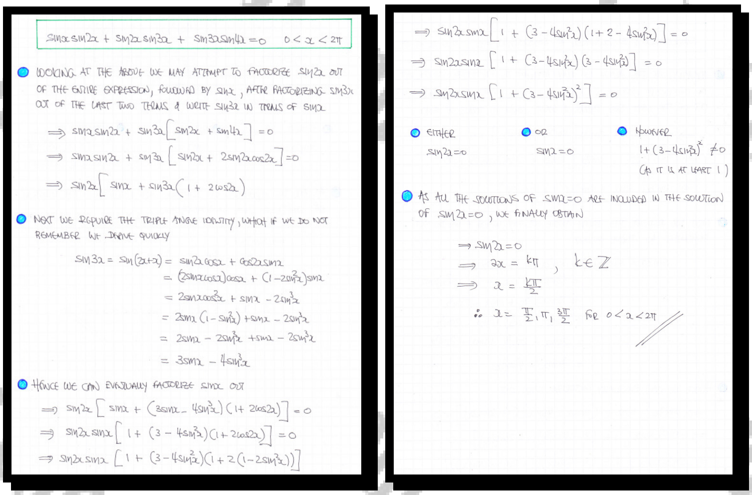
$\theta = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

Question 109 (*****)

Solve the following trigonometric equation, for $0 < x < 2\pi$.

$$\sin x \sin 2x + \sin 2x \sin 3x + \sin 3x \sin 4x = 0.$$

, $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$



$\sin x \sin 2x + \sin 2x \sin 3x + \sin 3x \sin 4x = 0 \qquad 0 < x < 2\pi$

LOOKING AT THE ABOVE WE MAY ATTEMPT TO FACTORIZE $\sin 2x$ OUT OF THE ENTIRE EXPRESSION, FOLLOWED BY $\sin x$, AFTER FACTORIZING $\sin 3x$ OUT OF THE LAST TWO TERMS & WRITE $\sin 3x$ IN TERMS OF $\sin x$

$\Rightarrow \sin x \sin 2x + \sin 3x\left[\sin 2x + \sin 4x\right] = 0$

$\Rightarrow \sin x \sin 2x + \sin 3x\left[\sin 2x + 2\sin 2x \cos 2x\right] = 0$

$\Rightarrow \sin 2x\left[\sin x + \sin 3x(1 + 2\cos 2x)\right.$

NEXT WE REQUIRE THE TRIPLE ANGLE IDENTITY, WHICH IF WE DO NOT REMEMBER WE DERIVE QUICKLY

$\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x$

$= (2\sin x \cos x)\cos x + (1 - 2\sin^2 x)\sin x$

$= 2\sin x \cos^2 x + \sin x - 2\sin^3 x$

$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x$

$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$= 3\sin x - 4\sin^3 x$

HENCE WE CAN EVENTUALLY FACTORIZE $\sin x$ OUT

$\Rightarrow \sin 2x\left[\sin x + (3\sin x - 4\sin^3 x)(1 + 2\cos 2x)\right] = 0$

$\Rightarrow \sin 2x \sin x\left[1 + (3 - 4\sin^2 x)(1 + 2\cos 2x)\right] = 0$

$\Rightarrow \sin 2x \sin x\left[1 + (3 - 4\sin^2 x)(1 + 2(1 - 2\sin^2 x))\right]$

$\Rightarrow \sin 2x \sin x\left[1 + (3 - 4\sin^2 x)(1 + 2 - 4\sin^2 x)\right] = 0$

$\Rightarrow \sin 2x \sin x\left[1 + (3 - 4\sin^2 x)(3 - 4\sin^2 x)\right] = 0$

$\Rightarrow \sin 2x \sin x\left[1 + (3 - 4\sin^2 x)^2\right] = 0$

EITHER $\sin 2x = 0$

OR $\sin x = 0$

HOWEVER $1 + (3 - 4\sin^2 x)^2 \neq 0$ (AS IT IS AT LEAST 1)

AS ALL THE SOLUTIONS OF $\sin x = 0$ ARE INCLUDED IN THE SOLUTION OF $\sin 2x = 0$, WE FINALLY OBTAIN

$\Rightarrow \sin 2x = 0$

$\Rightarrow 2x = k\pi, \quad k \in \mathbb{Z}$

$\Rightarrow x = \frac{k\pi}{2}$

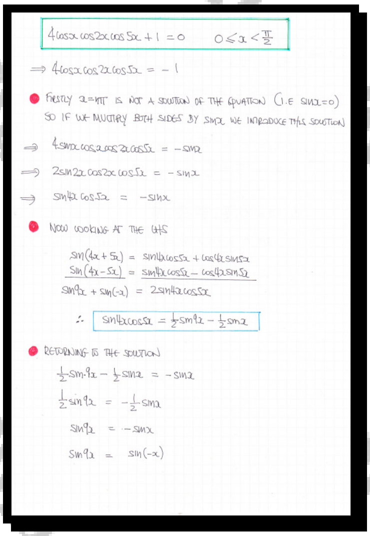
$\therefore x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ FOR $0 < x < 2\pi$

Question 110 (*****)

Solve the following trigonometric equation, for $0 < x < \frac{1}{2}\pi$.

$$4\cos x \cos 2x \cos 5x + 1 = 0.$$

$\boxed{}$, $\boxed{x = \frac{1}{8}\pi, \frac{1}{5}\pi, \frac{3}{8}\pi, \frac{2}{5}\pi}$



$4\cos x\cos 2x\cos 5x + 1 = 0 \qquad 0 \leqslant x < \frac{\pi}{2}$

$\Rightarrow 4\cos x\cos 2x\cos 5x = -1$

- FIRSTLY $x = n\pi$ IS NOT A SOLUTION OF THE EQUATION (I.E $\sin x = 0$) SO IF WE MULTIPLY BOTH SIDES BY $\sin x$ WE INTRODUCE THIS SOLUTION

$\Rightarrow 4\sin x\cos x\cos 2x\cos 5x = -\sin x$

$\Rightarrow 2\sin 2x\cos 2x\cos 5x = -\sin x$

$\Rightarrow \sin 4x\cos 5x = -\sin x$

- NOW LOOKING AT THE LHS

$\sin(4x+5x) = \sin 4x\cos 5x + \cos 4x\sin 5x$

$\sin(4x-5x) = \sin 4x\cos 5x - \cos 4x\sin 5x$

$\sin 9x + \sin(-x) = 2\sin 4x\cos 5x$

$\therefore \boxed{\sin 4x\cos 5x = \frac{1}{2}\sin 9x - \frac{1}{2}\sin x}$

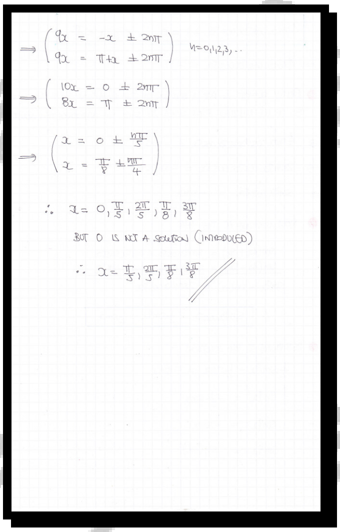
- RETURNING TO THE SOLUTION

$\frac{1}{2}\sin 9x - \frac{1}{2}\sin x = -\sin x$

$\frac{1}{2}\sin 9x = -\frac{1}{2}\sin x$

$\sin 9x = -\sin x$

$\sin 9x = \sin(-x)$



$\Rightarrow \begin{pmatrix} 9x = -x \pm 2n\pi \\ 9x = \pi + x \pm 2n\pi \end{pmatrix} \quad n = 0, 1, 2, 3, \ldots$

$\Rightarrow \begin{pmatrix} 10x = 0 \pm 2n\pi \\ 8x = \pi \pm 2n\pi \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x = 0 \pm \frac{n\pi}{5} \\ x = \frac{\pi}{8} \pm \frac{n\pi}{4} \end{pmatrix}$

$\therefore x = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{8}, \frac{3\pi}{8}$

BUT 0 IS NOT A SOLUTION (INTRODUCED)

$\therefore x = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{\pi}{8}, \frac{3\pi}{8}$

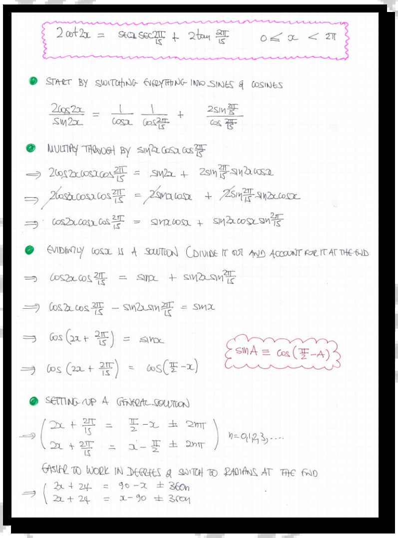
Question 111 (*****)

Solve the following trigonometric equation.

$$2\cot 2x = \sec x \sec\left(\frac{2\pi}{15}\right) + 2\tan\left(\frac{2\pi}{15}\right), \quad 0 \le x < 2\pi .$$

Give the answers in the form $k\pi$, where k is rational.

$$k = \frac{1}{2}, \frac{3}{2}, \frac{11}{90}, \frac{71}{90}, \frac{131}{90}, \frac{41}{30}$$


$2\cot 2x = \sec x \sec\frac{2\pi}{15} + 2\tan\frac{2\pi}{15} \quad 0 \le x < 2\pi$

- START BY SWITCHING EVERYTHING INTO SINES & COSINES

$\frac{2\cos 2x}{\sin 2x} = \frac{1}{\cos x}\frac{1}{\cos\frac{2\pi}{15}} + \frac{2\sin\frac{2\pi}{15}}{\cos\frac{2\pi}{15}}$

- MULTIPLY THROUGH BY $\sin 2x \cos x \cos\frac{2\pi}{15}$

$\Rightarrow 2\cos 2x \cos x \cos\frac{2\pi}{15} = \sin 2x + 2\sin\frac{2\pi}{15}\sin 2x \cos x$

$\Rightarrow 2\cos 2x \cos x \cos\frac{2\pi}{15} = 2\sin x \cos x + 2\sin\frac{2\pi}{15}\sin 2x \cos x$

$\Rightarrow \cos 2x \cos x \cos\frac{2\pi}{15} = \sin x \cos x + \sin 2x \cos x \sin\frac{2\pi}{15}$

- EVIDENTLY $\cos x$ IS A SOLUTION (DIVIDE IT OUT AND ACCOUNT FOR IT AT THE END

$\Rightarrow \cos 2x \cos\frac{2\pi}{15} = \sin x + \sin 2x \sin\frac{2\pi}{15}$

$\Rightarrow \cos 2x \cos\frac{2\pi}{15} - \sin 2x \sin\frac{2\pi}{15} = \sin x$

$\Rightarrow \cos\left(2x + \frac{2\pi}{15}\right) = \sin x$

$\Rightarrow \cos\left(2x + \frac{2\pi}{15}\right) = \cos\left(\frac{\pi}{2} - x\right)$

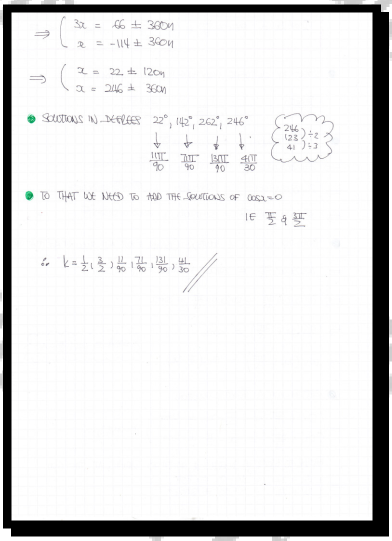
$\sin A \equiv \cos\left(\frac{\pi}{2} - A\right)$

- SETTING UP A GENERAL SOLUTION

$\Rightarrow \begin{pmatrix} 2x + \frac{2\pi}{15} = \frac{\pi}{2} - x \pm 2n\pi \\ 2x + \frac{2\pi}{15} = x - \frac{\pi}{2} \pm 2n\pi \end{pmatrix} \quad n = 0, 1, 2, 3, \dots$

EASIER TO WORK IN DEGREES & SWITCH TO RADIANS AT THE END

$\Rightarrow \begin{cases} 2x + 24 = 90 - x \pm 360n \\ 2x + 24 = x - 90 \pm 360n \end{cases}$


$\Rightarrow \begin{cases} 3x = 66 \pm 360n \\ x = -114 \pm 360n \end{cases}$

$\Rightarrow \begin{cases} x = 22 \pm 120n \\ x = 246 \pm 360n \end{cases}$

- SOLUTIONS IN DEGREES $22°, 142°, 262°, 246°$

$\frac{11\pi}{90}, \frac{71\pi}{90}, \frac{131\pi}{90}, \frac{41\pi}{30}$

246) ÷2
123) ÷3
41

- TO THAT WE NEED TO ADD THE SOLUTIONS OF $\cos x = 0$

IE $\frac{\pi}{2}$ & $\frac{3\pi}{2}$

$\therefore k = \frac{1}{2}, \frac{3}{2}, \frac{11}{90}, \frac{71}{90}, \frac{131}{90}, \frac{41}{30}$

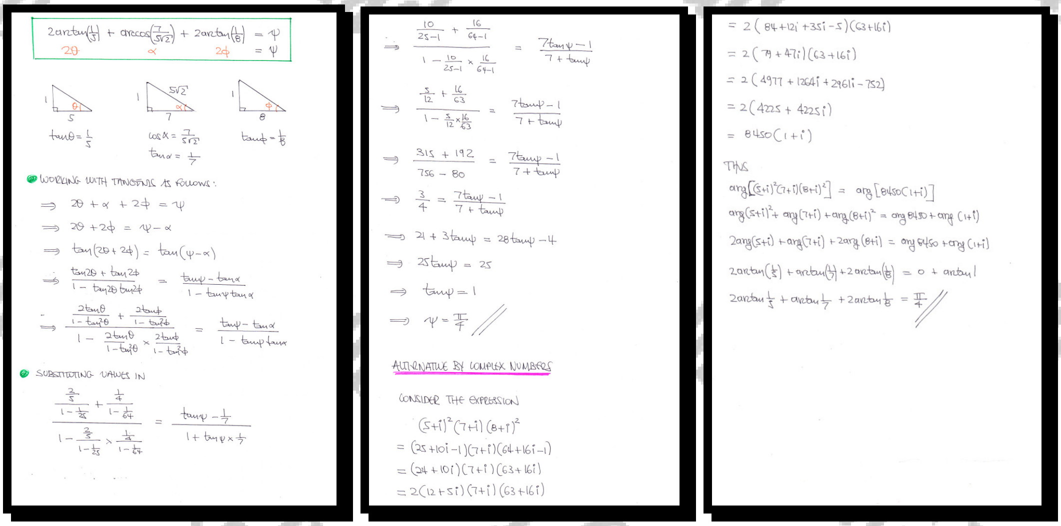
Question 112 (*****)

Use trigonometric algebra to fully simplify

$$2\arctan\left(\frac{1}{5}\right)+\arccos\left(\frac{7}{5\sqrt{2}}\right)+\arctan\left(\frac{1}{8}\right),$$

giving the final answer in terms of π.

$\boxed{}$, $\boxed{\frac{\pi}{4}}$



Question 113 (*****)

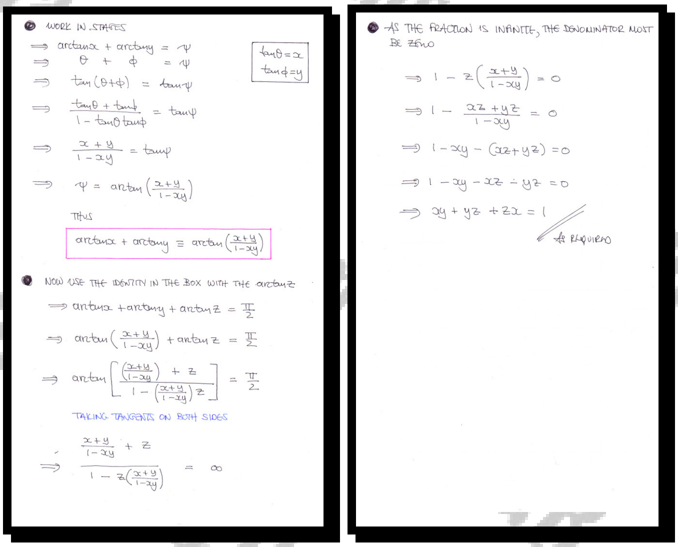
It given that

$$\arctan x + \arctan y + \arctan z = \frac{\pi}{2}.$$

Show that x, y and z satisfy the relationship

$$xy + yz + zx = 1.$$

[ ] , [proof]



• WORK IN STAGES

$\arctan x + \arctan y = \psi$ | $\tan\theta = x$, $\tan\phi = y$

$\Rightarrow \theta + \phi = \psi$

$\Rightarrow \tan(\theta + \phi) = \tan\psi$

$\Rightarrow \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \tan\psi$

$\Rightarrow \frac{x+y}{1-xy} = \tan\psi$

$\Rightarrow \psi = \arctan\left(\frac{x+y}{1-xy}\right)$

THUS

$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$

• NOW USE THE IDENTITY IN THE BOX WITH THE $\arctan z$

$\Rightarrow \arctan x + \arctan y + \arctan z = \frac{\pi}{2}$

$\Rightarrow \arctan\left(\frac{x+y}{1-xy}\right) + \arctan z = \frac{\pi}{2}$

$\Rightarrow \arctan\left[\frac{\left(\frac{x+y}{1-xy}\right) + z}{1 - \left(\frac{x+y}{1-xy}\right)z}\right] = \frac{\pi}{2}$

TAKING TANGENTS ON BOTH SIDES

$\Rightarrow \frac{\frac{x+y}{1-xy} + z}{1 - z\left(\frac{x+y}{1-xy}\right)} = \infty$

• AS THE FRACTION IS INFINITE, THE DENOMINATOR MUST BE ZERO

$\Rightarrow 1 - z\left(\frac{x+y}{1-xy}\right) = 0$

$\Rightarrow 1 - \frac{xz + yz}{1-xy} = 0$

$\Rightarrow 1 - xy - (xz + yz) = 0$

$\Rightarrow 1 - xy - xz - yz = 0$

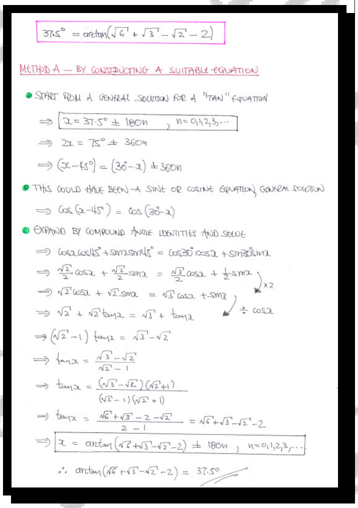
$\Rightarrow xy + yz + zx = 1$ // AS REQUIRED

Question 114 (*****)

Use a fully detailed method to show that

$$\arctan\left[\sqrt{6}+\sqrt{3}-\sqrt{2}-2\right]=37.5° .$$

[ ], [proof]


$37.5° = \arctan(\sqrt{6}+\sqrt{3}-\sqrt{2}-2)$

METHOD A — BY CONSTRUCTING A SUITABLE EQUATION

- START FROM A GENERAL SOLUTION FOR A "TAN" EQUATION

$\Rightarrow x = 37.5° \pm 180n, \quad n = 0,1,2,3,\ldots$

$\Rightarrow 2x = 75° \pm 360n$

$\Rightarrow (x-45°) = (30°-x) \pm 360n$

- THIS COULD HAVE BEEN A SINE OR COSINE EQUATION GENERAL SOLUTION

$\Rightarrow \cos(x-45°) = \cos(30°-x)$

- EXPAND BY COMPOUND ANGLE IDENTITIES AND SOLVE

$\Rightarrow \cos x\cos 45° + \sin x\sin 45° = \cos 30°\cos x + \sin 30°\sin x$

$\Rightarrow \frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x = \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x \quad (\times 2)$

$\Rightarrow \sqrt{2}\cos x + \sqrt{2}\sin x = \sqrt{3}\cos x + \sin x \quad (\div \cos x)$

$\Rightarrow \sqrt{2} + \sqrt{2}\tan x = \sqrt{3} + \tan x$

$\Rightarrow (\sqrt{2}-1)\tan x = \sqrt{3}-\sqrt{2}$

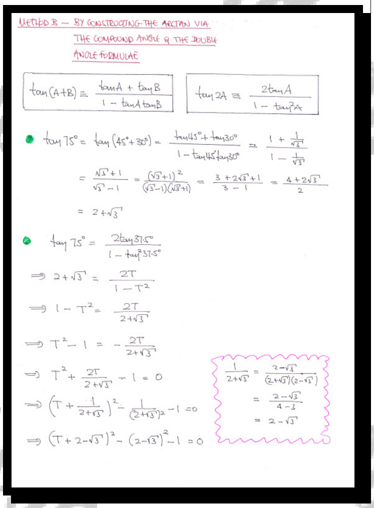
$\Rightarrow \tan x = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$

$\Rightarrow \tan x = \frac{(\sqrt{3}-\sqrt{2})(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$

$\Rightarrow \tan x = \frac{\sqrt{6}+\sqrt{3}-2-\sqrt{2}}{2-1} = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$

$\Rightarrow x = \arctan(\sqrt{6}+\sqrt{3}-\sqrt{2}-2) \pm 180n, \quad n=0,1,2,3,\ldots$

$\therefore \arctan(\sqrt{6}+\sqrt{3}-\sqrt{2}-2) = 37.5°$


METHOD B — BY CONSTRUCTING THE ARCTAN VIA THE COMPOUND ANGLE & THE DOUBLE ANGLE FORMULAE

$\tan(A+B) \equiv \frac{\tan A + \tan B}{1-\tan A\tan B}$ $\qquad$ $\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$

- $\tan 75° = \tan(45°+30°) = \frac{\tan 45° + \tan 30°}{1-\tan 45°\tan 30°} = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$

$= \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{3+2\sqrt{3}+1}{3-1} = \frac{4+2\sqrt{3}}{2}$

$= 2+\sqrt{3}$

- $\tan 75° = \frac{2\tan 37.5°}{1-\tan^2 37.5°}$

$\Rightarrow 2+\sqrt{3} = \frac{2T}{1-T^2}$

$\Rightarrow 1-T^2 = \frac{2T}{2+\sqrt{3}}$

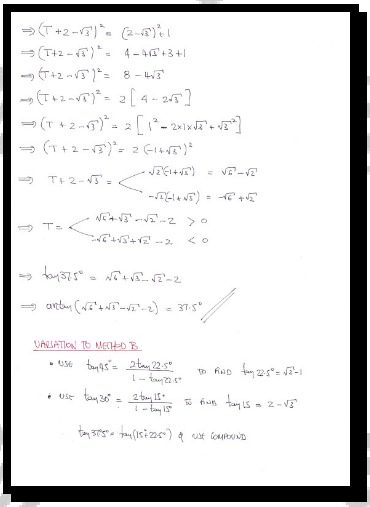
$\Rightarrow T^2-1 = -\frac{2T}{2+\sqrt{3}}$

$\Rightarrow T^2 + \frac{2T}{2+\sqrt{3}} - 1 = 0$

$\Rightarrow \left(T+\frac{1}{2+\sqrt{3}}\right)^2 - \frac{1}{(2+\sqrt{3})^2} - 1 = 0$

$\Rightarrow (T+2-\sqrt{3})^2 - (2-\sqrt{3})^2 - 1 = 0$

$\frac{1}{2+\sqrt{3}} = \frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$


$\Rightarrow (T+2-\sqrt{3})^2 = (2-\sqrt{3})^2+1$

$\Rightarrow (T+2-\sqrt{3})^2 = 4-4\sqrt{3}+3+1$

$\Rightarrow (T+2-\sqrt{3})^2 = 8-4\sqrt{3}$

$\Rightarrow (T+2-\sqrt{3})^2 = 2[4-2\sqrt{3}]$

$\Rightarrow (T+2-\sqrt{3})^2 = 2[1^2 - 2\times1\times\sqrt{3} + \sqrt{3}^2]$

$\Rightarrow (T+2-\sqrt{3})^2 = 2(-1+\sqrt{3})^2$

$\Rightarrow T+2-\sqrt{3} = \sqrt{2}(-1+\sqrt{3}) = \sqrt{6}-\sqrt{2}$ or $-\sqrt{2}(-1+\sqrt{3}) = -\sqrt{6}+\sqrt{2}$

$\Rightarrow T = \sqrt{6}+\sqrt{3}-\sqrt{2}-2 > 0$ or $-\sqrt{6}+\sqrt{3}+\sqrt{2}-2 < 0$

$\Rightarrow \tan 37.5° = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$

$\Rightarrow \arctan(\sqrt{6}+\sqrt{3}-\sqrt{2}-2) = 37.5°$

VARIATION TO METHOD B

- USE $\tan 45° = \frac{2\tan 22.5°}{1-\tan^2 22.5°}$ TO FIND $\tan 22.5° = \sqrt{2}-1$
- USE $\tan 30° = \frac{2\tan 15°}{1-\tan^2 15°}$ TO FIND $\tan 15 = 2-\sqrt{3}$

$\tan 37.5° = \tan(15°+22.5°)$ & USE COMPOUND

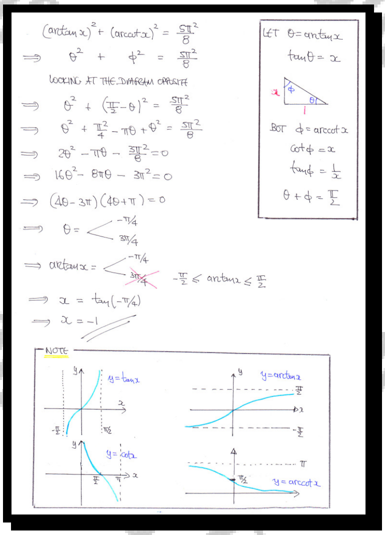
Question 115 (*****)

Use a trigonometric algebra to solve the following equation

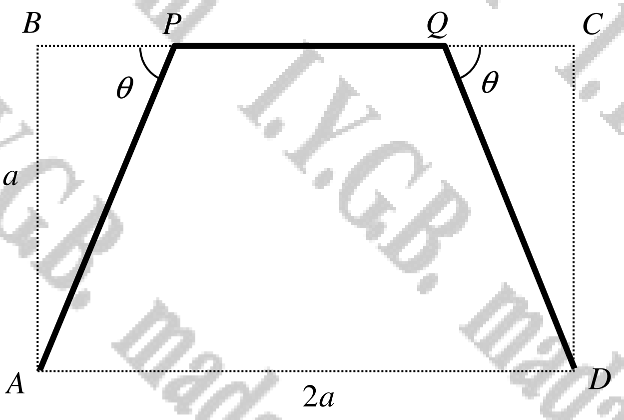
$$(\arctan x)^2 + (\operatorname{arccot} x)^2 = \frac{5\pi^2}{8}.$$

You may assume that $y = \operatorname{arccot} x$ is the inverse function of $y = \cot x$, $0 \le x \le \pi$

$\boxed{}$, $\boxed{x = -1}$



Question 116 (*****)

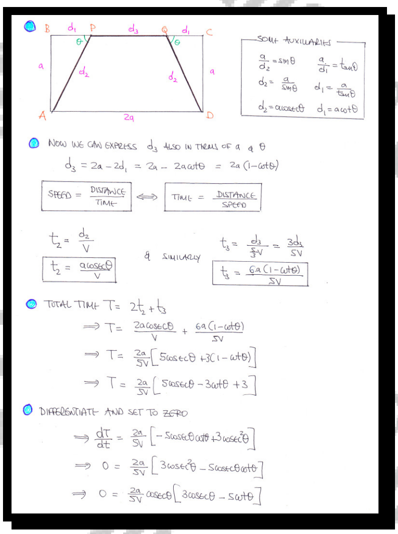


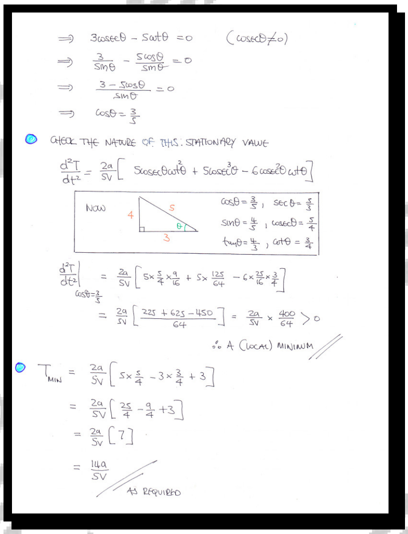
The figure above shows a network $APQD$ inside a rectangle $ABCD$, where $|AB| = a$ and $|BC| = 2a$. The endpoints of the network A and D are fixed. The points P and Q are variable so that they always lie on BC with $|AP| = |QD|$. The angles BPA and CQD are both equal to θ. A particle travels with constant speed v on the sections AP and QD, and with constant speed $\frac{5}{3}v$ on the section PQ.

Let T be the total time for the journey $APQD$.

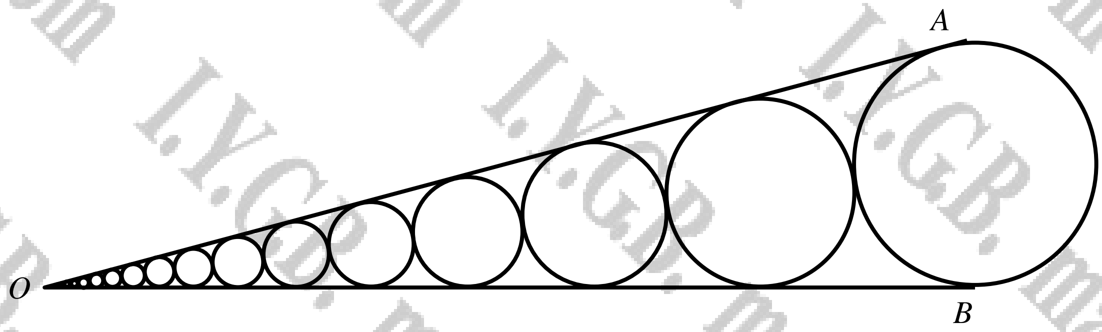
Given that the positions of the points P and Q can be varied as appropriate, show that the minimum value of T is $\dfrac{14a}{5v}$, fully justifying that this is the minimum value.

[ ] , [proof]





Question 117 (*****)

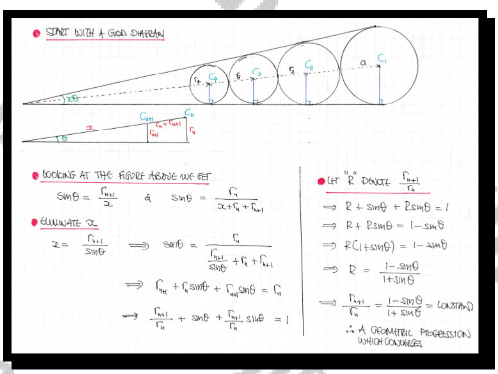


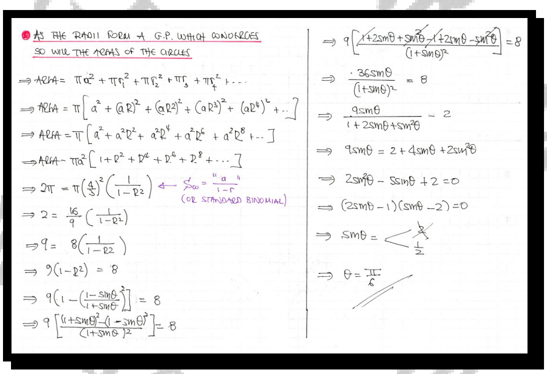
The figure above shows a infinite sequence of circles of decreasing radius, the radius of the larger circle being $\frac{4}{3}$.

The centres of these circles lie on a straight line. The straight lines OA and OB are tangents to every circle in the sequence, the angle AOB denoted by 2θ.

Given that the total area of these circles is 2π, determine the value of θ.

$\boxed{}$, $\boxed{\theta = \frac{1}{6}\pi}$



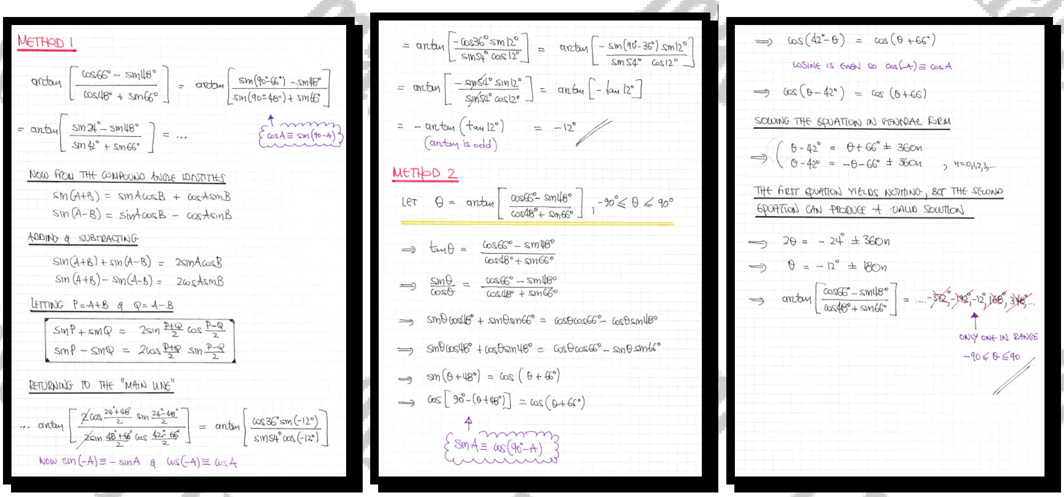


Question 118 (*****)

$$\theta = \arctan\left[\frac{\cos 66° - \sin 48°}{\cos 48° - \sin 66°}\right].$$

Show by detailed working that $\theta = -12°$.

, proof



Question 119 (*****)

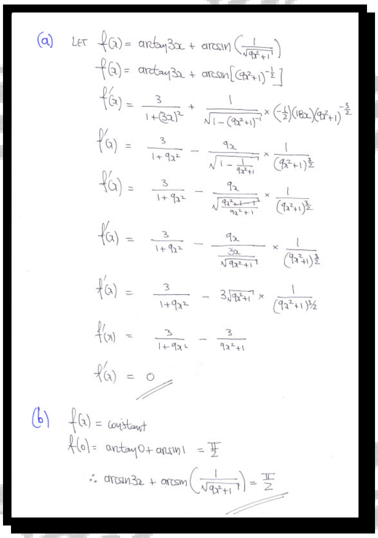
$$f(x) = \arctan(3x) + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right), \quad x \in \mathbb{R}.$$

Show clearly that …

a) … $f'(x) = 0$.

b) … $\arctan(3x) + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right) \equiv k\pi$, stating the value of the constant k.

$$\boxed{k = \frac{1}{2}}$$



(a) LET $f(x) = \arctan 3x + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right)$

$f(x) = \arctan 3x + \arcsin\left[(9x^2+1)^{-\frac{1}{2}}\right]$

$f'(x) = \frac{3}{1+(3x)^2} + \frac{1}{\sqrt{1-(9x^2+1)^{-1}}} \times \left(-\frac{1}{2}\right)(18x)(9x^2+1)^{-\frac{3}{2}}$

$f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\sqrt{1-\frac{1}{9x^2+1}}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$

$f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\sqrt{\frac{9x^2+1-1}{9x^2+1}}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$

$f'(x) = \frac{3}{1+9x^2} - \frac{9x}{\frac{3x}{\sqrt{9x^2+1}}} \times \frac{1}{(9x^2+1)^{\frac{3}{2}}}$

$f'(x) = \frac{3}{1+9x^2} - 3\sqrt{9x^2+1} \times \frac{1}{(9x^2+1)^{3/2}}$

$f'(x) = \frac{3}{1+9x^2} - \frac{3}{9x^2+1}$

$f'(x) = 0$

(b) $f(x) = \text{constant}$

$f(0) = \arctan 0 + \arcsin 1 = \frac{\pi}{2}$

$\therefore \arcsin 3x + \arcsin\left(\frac{1}{\sqrt{9x^2+1}}\right) = \frac{\pi}{2}$

Question 120 (*****)

It is given that

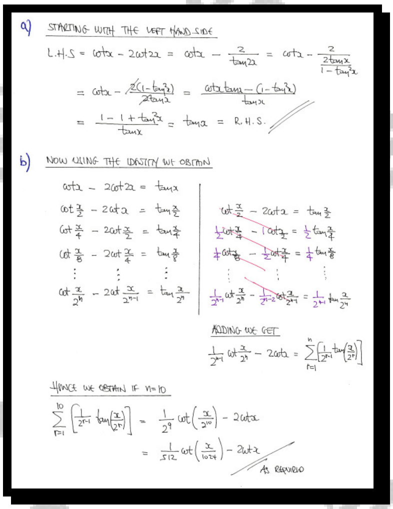
$$\cot x - 2\cot 2x \equiv \tan x \,.$$

a) Prove the validity of the above trigonometric identity.

b) Hence, or otherwise, show that

$$\sum_{r=1}^{10} \frac{1}{2^{r-1}} \tan\left(\frac{x}{2^r}\right) = \frac{1}{512}\cot\left(\frac{x}{1024}\right) - 2\cot x \,.$$

[ ] , [proof]



Question 121 (*****)

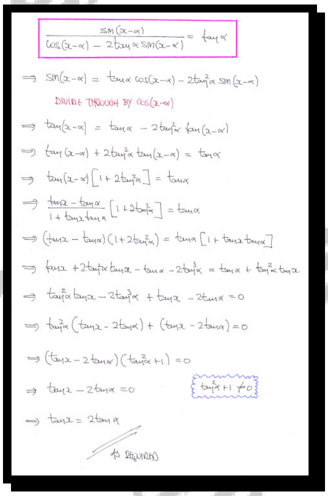
Given the trigonometric equation

$$\frac{\sin(x-\alpha)}{\cos(x-\alpha)-2\tan\alpha\sin(x-\alpha)}=\tan\alpha,$$

show clearly that

$$\tan x = 2\tan\alpha.$$

[ ], [proof]



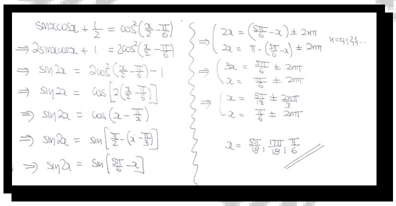
Question 122 **(*****)**

Solve the trigonometric equation

$$\sin x \cos x + \frac{1}{2} = \cos^2\left(\frac{x}{2} - \frac{\pi}{6}\right), \ 0 \le x < \pi,$$

giving the answers in terms of π.

$\boxed{}$, $\boxed{x = \frac{\pi}{6}, \frac{5\pi}{18}, \frac{17\pi}{18}}$



$\sin x\cos x + \frac{1}{2} = \cos^2\left(\frac{x}{2}-\frac{\pi}{6}\right)$

$\Rightarrow 2\sin x\cos x + 1 = 2\cos^2\left(\frac{x}{2}-\frac{\pi}{6}\right)$

$\Rightarrow \sin 2x = 2\cos^2\left(\frac{x}{2}-\frac{\pi}{6}\right) - 1$

$\Rightarrow \sin 2x = \cos\left[2\left(\frac{x}{2}-\frac{\pi}{6}\right)\right]$

$\Rightarrow \sin 2x = \cos\left(x-\frac{\pi}{3}\right)$

$\Rightarrow \sin 2x = \sin\left[\frac{\pi}{2}-\left(x-\frac{\pi}{3}\right)\right]$

$\Rightarrow \sin 2x = \sin\left[\frac{5\pi}{6}-x\right]$

$\Rightarrow \begin{cases} 2x = \left(\frac{5\pi}{6}-x\right) \pm 2n\pi \\ 2x = \pi - \left(\frac{5\pi}{6}-x\right) \pm 2n\pi \end{cases}$ $n = 0,1,2,3\ldots$

$\Rightarrow \begin{cases} 3x = \frac{5\pi}{6} \pm 2n\pi \\ x = \frac{\pi}{6} \pm 2n\pi \end{cases}$

$\Rightarrow \begin{cases} x = \frac{5\pi}{18} \pm \frac{2n\pi}{3} \\ x = \frac{\pi}{6} \pm 2n\pi \end{cases}$

$x = \frac{5\pi}{18}, \frac{17\pi}{18}, \frac{\pi}{6}$

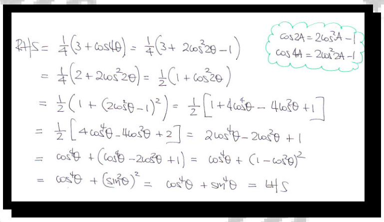
Question 123 (*****)

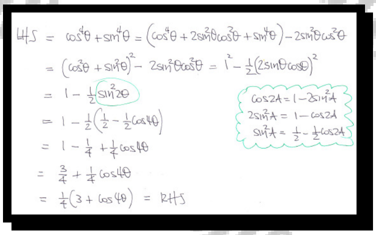
Prove the validity of the following trigonometric identities.

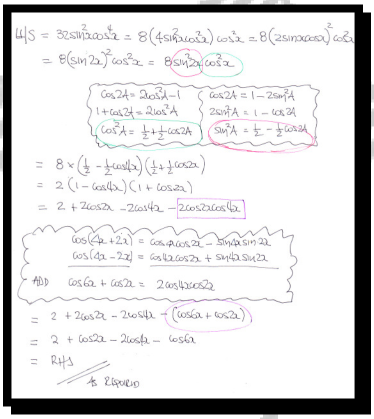
a) $\cos^4\theta + \sin^4\theta \equiv \frac{1}{4}(3+4\cos 4\theta)$.

b) $32\sin^2 x\cos^4 x \equiv 2+\cos 2x - 2\cos 4x - \cos 6x$.

proof





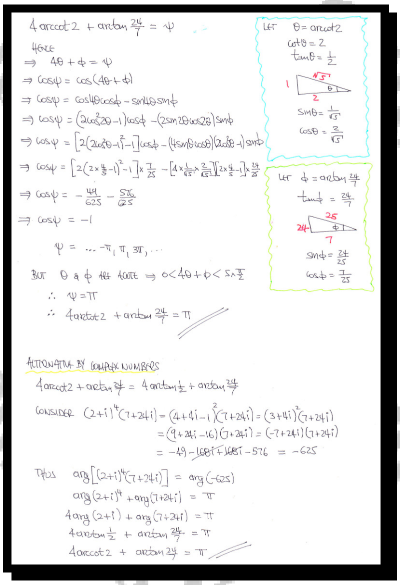


Question 124 (*****)

Show clearly that

$$4\operatorname{arccot} 2 + \arctan\left(\frac{24}{7}\right) = \pi .$$

proof



Question 125 (*****)

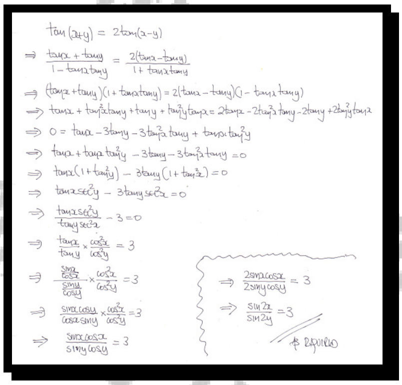
It is given that if $x \neq y$, $x \neq 0$, $y \neq 0$,

$$\tan(x+y) = 2\tan(x-y).$$

Show clearly that

$$\frac{\sin 2x}{\sin 2y} = 3.$$

[ ], [proof]



Question 126 (*****)

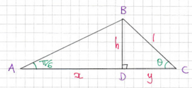
A triangle ABC is such so that $\measuredangle BAC = \frac{1}{6}\pi$ and $|BC| = 1$.

Show that the maximum value of the area of the triangle ABC is

$$\frac{1}{4}\left(2+\sqrt{3}\right).$$

[ ] , [proof]

STARTING WITH A DIAGRAM



LOOKING AT $\overset{\triangle}{BDC}$

- $h = 1 \times \sin\theta \implies \boxed{h = \sin\theta}$
- $y = 1 \times \cos\theta \implies \boxed{y = \cos\theta}$

LOOKING AT $\overset{\triangle}{ABD}$

$\implies \frac{h}{x} = \tan\frac{\pi}{6}$

$\implies \frac{h}{x} = \frac{1}{\sqrt{3}}$

$\implies x = \sqrt{3}h$

$\implies \boxed{x = \sqrt{3}\sin\theta}$

HENCE THE AREA OF THE TRIANGLE $\overset{\triangle}{ABC}$

$\implies \text{AREA} = \frac{1}{2}|AC||BD|$

$\implies \text{AREA} = \frac{1}{2}(x+y)h$

$\implies \text{AREA} = \frac{1}{2}(\sqrt{3}\sin\theta + \cos\theta)\sin\theta$

$\implies \text{AREA} = \left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)\sin\theta$

$\implies \text{AREA} = \left(\cos\frac{\pi}{6}\sin\theta + \sin\frac{\pi}{6}\cos\theta\right)\sin\theta$

$\implies \text{AREA} = \sin\left(\theta+\frac{\pi}{6}\right)\sin\theta$

BY DIFFERENTIATION

$f(\theta) = \sin\theta\sin\left(\theta+\frac{\pi}{6}\right)$

$f'(\theta) = \cos\theta\sin\left(\theta+\frac{\pi}{6}\right) + \sin\theta\cos\left(\theta+\frac{\pi}{6}\right)$

$f'(\theta) = \sin\left(\theta + \left(\theta+\frac{\pi}{6}\right)\right)$

$f'(\theta) = \sin\left(2\theta+\frac{\pi}{6}\right)$

SOLVING FOR ZERO, ONLY LOOKING FOR $\theta > 0$

$2\theta + \frac{\pi}{6} = 0, \pi, 2\pi, \ldots$

$2\theta = \frac{5\pi}{6}, \frac{11\pi}{6}, \ldots$

$\theta = \frac{5\pi}{12}, \frac{11\pi}{12}, \ldots$

HENCE $\text{AREA}_{MAX} = \sin\frac{5\pi}{12}\sin\left(\frac{5\pi}{12}+\frac{\pi}{6}\right)$

$= \sin\frac{5\pi}{12}\sin\frac{7\pi}{12}$

$= \sin\frac{5\pi}{12}\sin\frac{5\pi}{12}$ $\quad \left\{\sin\frac{7\pi}{12} \equiv \sin\frac{5\pi}{12}\right\}$

$= \sin^2\frac{5\pi}{12}$

$= \frac{1}{2} - \frac{1}{2}\cos\left(2\times\frac{5\pi}{12}\right)$ $\quad \left\{\cos 2\theta = 1 - 2\sin^2\theta, \ \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta\right\}$

$= \frac{1}{2} - \frac{1}{2}\cos\frac{5\pi}{6}$

$= \frac{1}{2} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2} + \frac{\sqrt{3}}{4}$

$= \frac{1}{4}\left(2+\sqrt{3}\right)$

ALTERNATIVE WITHOUT DIFFERENTIATION

$\boxed{\text{AREA} = \sin\theta\sin\left(\theta+\frac{\pi}{6}\right)}$

LOOKING AT THE COMPOUND ANGLE IDENTITIES FOR $\cos(A \pm B)$

$\cos\left[\theta + \left(\theta+\frac{\pi}{6}\right)\right] \equiv \cos\theta\cos\left(\theta+\frac{\pi}{6}\right) - \sin\theta\sin\left(\theta+\frac{\pi}{6}\right)$

$\cos\left[\theta - \left(\theta+\frac{\pi}{6}\right)\right] \equiv \cos\theta\cos\left(\theta+\frac{\pi}{6}\right) + \sin\theta\sin\left(\theta+\frac{\pi}{6}\right)$

SUBTRACTING UPWARDS

$\implies \cos\left(-\frac{\pi}{6}\right) - \cos\left(2\theta+\frac{\pi}{6}\right) \equiv 2\sin\theta\sin\left(\theta+\frac{\pi}{6}\right)$

$\implies \frac{1}{2}\cos\left(+\frac{\pi}{6}\right) - \frac{1}{2}\cos\left(2\theta+\frac{\pi}{6}\right) \equiv \sin\theta\sin\left(\theta+\frac{\pi}{6}\right)$

$\implies \sin\theta\sin\left(\theta+\frac{\pi}{6}\right) \equiv \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\cos\left(2\theta+\frac{\pi}{6}\right)$

$\implies \text{AREA} = \frac{\sqrt{3}}{4} - \frac{1}{2}\cos\left(2\theta+\frac{\pi}{6}\right)$

$\implies \text{AREA}_{MAX} = \frac{\sqrt{3}}{4} - \frac{1}{2}(-1)$ $\quad \left\{-1 \leqslant \cos\left(2\theta+\frac{\pi}{6}\right) \leqslant 1\right\}$

$\implies \text{AREA}_{MAX} = \frac{\sqrt{3}}{4} + \frac{1}{2}$

$\implies \text{AREA}_{MAX} = \frac{1}{4}\left(2+\sqrt{3}\right)$

Question 127 (*****)

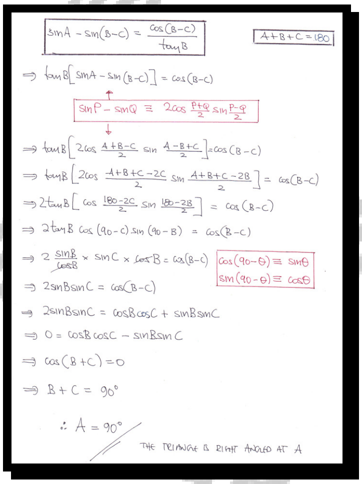
It is given that the angles A, B and C are the three angles of a triangle ABC with $B \neq 90°$.

Given further that

$$\sin A - \sin(B-C) = \frac{\cos(B-C)}{\tan B},$$

show that the triangle ABC is right angled.

[ ], [proof]



$$\sin A - \sin(B-C) = \frac{\cos(B-C)}{\tan B}$$

$A+B+C=180$

$\Rightarrow \tan B\left[\sin A - \sin(B-C)\right] = \cos(B-C)$

$\sin P - \sin Q \equiv 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$\Rightarrow \tan B\left[2\cos\frac{A+B-C}{2}\sin\frac{A-B+C}{2}\right] = \cos(B-C)$

$\Rightarrow \tan B\left[2\cos\frac{A+B+C-2C}{2}\sin\frac{A+B+C-2B}{2}\right] = \cos(B-C)$

$\Rightarrow 2\tan B\left[\cos\frac{180-2C}{2}\sin\frac{180-2B}{2}\right] = \cos(B-C)$

$\Rightarrow 2\tan B\cos(90-C)\sin(90-B) = \cos(B-C)$

$\Rightarrow 2\frac{\sin B}{\cos B} \times \sin C \times \cos B = \cos(B-C)$

$\cos(90-\theta) \equiv \sin\theta$

$\sin(90-\theta) \equiv \cos\theta$

$\Rightarrow 2\sin B\sin C = \cos(B-C)$

$\Rightarrow 2\sin B\sin C = \cos B\cos C + \sin B\sin C$

$\Rightarrow 0 = \cos B\cos C - \sin B\sin C$

$\Rightarrow \cos(B+C) = 0$

$\Rightarrow B + C = 90°$

$\therefore A = 90°$

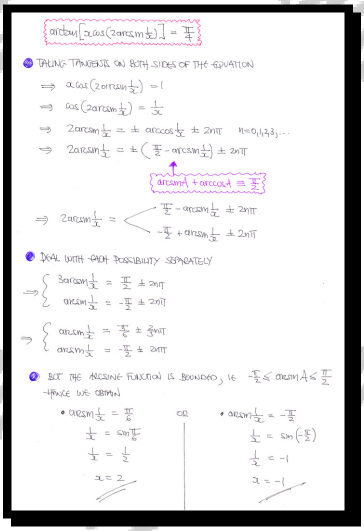
THE TRIANGLE IS RIGHT ANGLED AT A

Question 128 (*****)

Solve the following trigonometric equation

$$\arctan\left[x\cos\left(2\arcsin\frac{1}{x}\right)\right]=\frac{1}{4}\pi .$$

$\boxed{}$, $\boxed{x=-1,\quad x=2}$



$$\arctan\left[x\cos\left(2\arcsin\frac{1}{x}\right)\right]=\frac{\pi}{4}$$

- TAKING TANGENTS ON BOTH SIDES OF THE EQUATION

$\Rightarrow x\cos\left(2\arcsin\frac{1}{x}\right)=1$

$\Rightarrow \cos\left(2\arcsin\frac{1}{x}\right)=\frac{1}{x}$

$\Rightarrow 2\arcsin\frac{1}{x}=\pm\arccos\frac{1}{x}\pm 2n\pi \qquad n=0,1,2,3,\ldots$

$\Rightarrow 2\arcsin\frac{1}{x}=\pm\left(\frac{\pi}{2}-\arcsin\frac{1}{x}\right)\pm 2n\pi$

$\arcsin A+\arccos A\equiv\frac{\pi}{2}$

$\Rightarrow 2\arcsin\frac{1}{x}=\begin{cases}\frac{\pi}{2}-\arcsin\frac{1}{x}\pm 2n\pi\\ -\frac{\pi}{2}+\arcsin\frac{1}{x}\pm 2n\pi\end{cases}$

- DEAL WITH EACH POSSIBILITY SEPARATELY

$\Rightarrow \begin{cases}3\arcsin\frac{1}{x}=\frac{\pi}{2}\pm 2n\pi\\ \arcsin\frac{1}{x}=-\frac{\pi}{2}\pm 2n\pi\end{cases}$

$\Rightarrow \begin{cases}\arcsin\frac{1}{x}=\frac{\pi}{6}\pm \frac{2}{3}n\pi\\ \arcsin\frac{1}{x}=-\frac{\pi}{2}\pm 2n\pi\end{cases}$

- BUT THE ARCSINE FUNCTION IS BOUNDED, IE $-\frac{\pi}{2}\leqslant\arcsin A\leqslant\frac{\pi}{2}$

HENCE WE OBTAIN

- $\arcsin\frac{1}{x}=\frac{\pi}{6}$, $\frac{1}{x}=\sin\frac{\pi}{6}$, $\frac{1}{x}=\frac{1}{2}$, $x=2$

OR

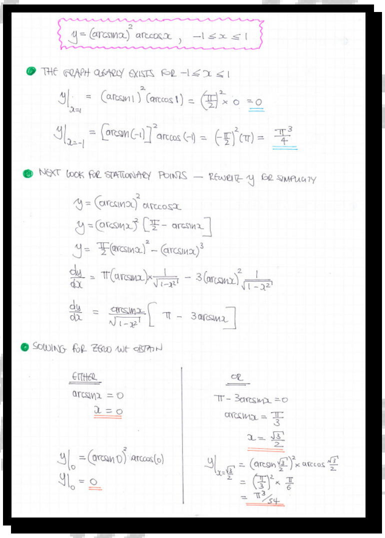
- $\arcsin\frac{1}{x}=-\frac{\pi}{2}$, $\frac{1}{x}=\sin\left(-\frac{\pi}{2}\right)$, $\frac{1}{x}=-1$, $x=-1$

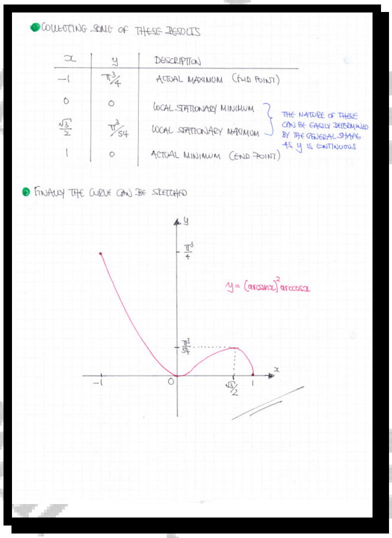
Question 129 (*****)

Sketch the graph of

$$y = (\arcsin x)^2 \arccos x \,, \quad -1 \le x \le 1$$

[ ] , [graph]





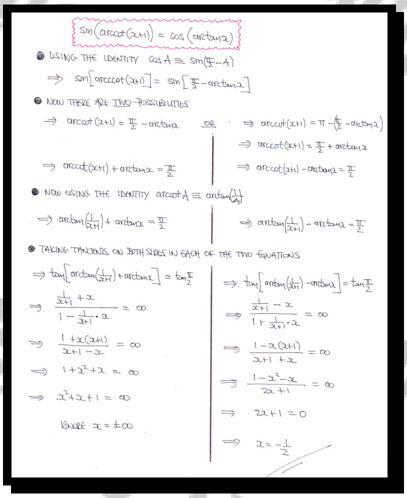
Question 130 (*****)

Solve the following trigonometric equation

$$\sin\left[\operatorname{arccot}(x+1)\right] = \cos(\arctan x).$$

You may assume that $y = \operatorname{arccot} x$ is the inverse function for $y = \cot x$, $0 \le x \le \pi$.

$\boxed{\phantom{x = -\frac{1}{2}}}$, $\boxed{x = -\frac{1}{2}}$



$\sin(\operatorname{arccot}(x+1)) = \cos(\arctan x)$

- USING THE IDENTITY $\cos A \equiv \sin\left(\frac{\pi}{2} - A\right)$

$\Rightarrow \sin[\operatorname{arccot}(x+1)] = \sin\left[\frac{\pi}{2} - \arctan x\right]$

- NOW THERE ARE TWO POSSIBILITIES

$\Rightarrow \operatorname{arccot}(x+1) = \frac{\pi}{2} - \arctan x$ OR $\Rightarrow \operatorname{arccot}(x+1) = \pi - \left(\frac{\pi}{2} - \arctan x\right)$

$\Rightarrow \operatorname{arccot}(x+1) = \frac{\pi}{2} + \arctan x$

$\Rightarrow \operatorname{arccot}(x+1) + \arctan x = \frac{\pi}{2}$ | $\Rightarrow \operatorname{arccot}(x+1) - \arctan x = \frac{\pi}{2}$

- NOW USING THE IDENTITY $\operatorname{arccot} A \equiv \arctan\left(\frac{1}{A}\right)$

$\Rightarrow \arctan\left(\frac{1}{x+1}\right) + \arctan x = \frac{\pi}{2}$ | $\Rightarrow \arctan\left(\frac{1}{x+1}\right) - \arctan x = \frac{\pi}{2}$

- TAKING TANGENTS ON BOTH SIDES IN EACH OF THE TWO EQUATIONS

$\Rightarrow \tan\left[\arctan\left(\frac{1}{x+1}\right) + \arctan x\right] = \tan\frac{\pi}{2}$

$\Rightarrow \dfrac{\frac{1}{x+1} + x}{1 - \frac{1}{x+1} \cdot x} = \infty$

$\Rightarrow \dfrac{1 + x(x+1)}{x+1-x} = \infty$

$\Rightarrow 1 + x^2 + x = \infty$

$\Rightarrow x^2 + x + 1 = \infty$

IGNORE $x = \pm\infty$

$\Rightarrow \tan\left[\arctan\left(\frac{1}{x+1}\right) - \arctan x\right] = \tan\frac{\pi}{2}$

$\Rightarrow \dfrac{\frac{1}{x+1} - x}{1 + \frac{1}{x+1} \cdot x} = \infty$

$\Rightarrow \dfrac{1 - x(x+1)}{x+1+x} = \infty$

$\Rightarrow \dfrac{1 - x^2 - x}{2x+1} = \infty$

$\Rightarrow 2x + 1 = 0$

$\Rightarrow x = -\frac{1}{2}$

Question 131 (*****)

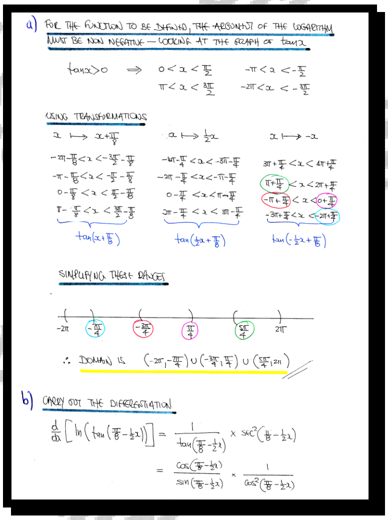
The function f is defined in the largest possible real domain, contained in the interval $(-2\pi, 2\pi)$, and its equation is

$$f(x) \equiv \ln\left[\tan\left(\tfrac{1}{8}\pi - \tfrac{1}{2}x\right)\right].$$

a) Find the domain of f.

b) Show that $f'(x) \equiv \dfrac{k}{\sqrt{1-\sin 2x}}$, for some constant k.

SPX, $\left(-2\pi, -\tfrac{7}{4}\pi\right) \cup \left(-\tfrac{3}{4}\pi, \tfrac{1}{4}\pi\right) \cup \left(\tfrac{5}{4}\pi, 2\pi\right)$



a) FOR THE FUNCTION TO BE DEFINED, THE ARGUMENT OF THE LOGARITHM MUST BE NON NEGATIVE — LOOKING AT THE GRAPH OF $\tan x$

$\tan x > 0 \Rightarrow 0 < x < \frac{\pi}{2}$, $-\pi < x < -\frac{\pi}{2}$, $\pi < x < \frac{3\pi}{2}$, $-2\pi < x < -\frac{3\pi}{2}$

USING TRANSFORMATIONS

$x \mapsto x + \frac{\pi}{8}$: $-2\pi - \frac{\pi}{8} < x < -\frac{3\pi}{2} - \frac{\pi}{8}$, $-\pi - \frac{\pi}{8} < x < -\frac{\pi}{2} - \frac{\pi}{8}$, $0 - \frac{\pi}{8} < x < \frac{\pi}{2} - \frac{\pi}{8}$, $\pi - \frac{\pi}{8} < x < \frac{3\pi}{2} - \frac{\pi}{8}$ — $\tan\left(x + \frac{\pi}{8}\right)$

$x \mapsto \frac{1}{2}x$: $-4\pi - \frac{\pi}{4} < x < -3\pi - \frac{\pi}{4}$, $-2\pi - \frac{\pi}{4} < x < -\pi - \frac{\pi}{4}$, $0 - \frac{\pi}{4} < x < \pi - \frac{\pi}{4}$, $2\pi - \frac{\pi}{4} < x < 3\pi - \frac{\pi}{4}$ — $\tan\left(\frac{1}{2}x + \frac{\pi}{8}\right)$

$x \mapsto -x$: $3\pi + \frac{\pi}{4} < x < 4\pi + \frac{\pi}{4}$, $\pi + \frac{\pi}{4} < x < 2\pi + \frac{\pi}{4}$, $-\pi + \frac{\pi}{4} < x < 0 + \frac{\pi}{4}$, $-3\pi + \frac{\pi}{4} < x < -2\pi + \frac{\pi}{4}$ — $\tan\left(-\frac{1}{2}x + \frac{\pi}{8}\right)$

SIMPLIFYING THESE RANGES

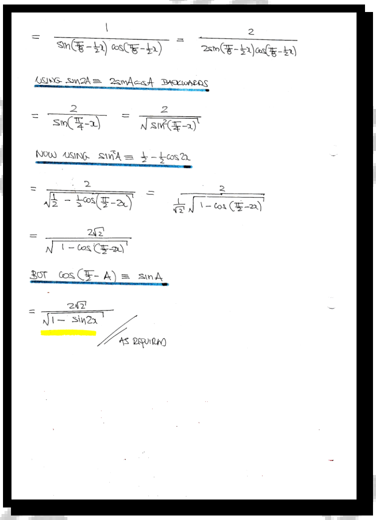
-2π, $-\frac{7\pi}{4}$, $-\frac{3\pi}{4}$, $\frac{\pi}{4}$, $\frac{5\pi}{4}$, 2π

$\therefore$ DOMAIN IS $\left(-2\pi, -\frac{7\pi}{4}\right) \cup \left(-\frac{3\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$

b) CARRY OUT THE DIFFERENTIATION

$\frac{d}{dx}\left[\ln\left(\tan\left(\frac{\pi}{8} - \frac{1}{2}x\right)\right)\right] = \frac{1}{\tan\left(\frac{\pi}{8} - \frac{1}{2}x\right)} \times \sec^2\left(\frac{\pi}{8} - \frac{1}{2}x\right)$

$= \frac{\cos\left(\frac{\pi}{8} - \frac{1}{2}x\right)}{\sin\left(\frac{\pi}{8} - \frac{1}{2}x\right)} \times \frac{1}{\cos^2\left(\frac{\pi}{8} - \frac{1}{2}x\right)}$



$= \frac{1}{\sin\left(\frac{\pi}{8} - \frac{1}{2}x\right)\cos\left(\frac{\pi}{8} - \frac{1}{2}x\right)} = \frac{2}{2\sin\left(\frac{\pi}{8} - \frac{1}{2}x\right)\cos\left(\frac{\pi}{8} - \frac{1}{2}x\right)}$

USING $\sin 2A \equiv 2\sin A\cos A$ BACKWARDS

$= \frac{2}{\sin\left(\frac{\pi}{4} - x\right)} = \frac{2}{\sqrt{\sin^2\left(\frac{\pi}{4} - x\right)}}$

NOW USING $\sin^2 A \equiv \frac{1}{2} - \frac{1}{2}\cos 2x$

$= \frac{2}{\sqrt{\frac{1}{2} - \frac{1}{2}\cos\left(\frac{\pi}{2} - 2x\right)}} = \frac{2}{\frac{1}{\sqrt{2}}\sqrt{1 - \cos\left(\frac{\pi}{2} - 2x\right)}}$

$= \frac{2\sqrt{2}}{\sqrt{1 - \cos\left(\frac{\pi}{2} - 2x\right)}}$

BUT $\cos\left(\frac{\pi}{2} - A\right) \equiv \sin A$

$= \frac{2\sqrt{2}}{\sqrt{1 - \sin 2x}}$ AS REQUIRED

Question 132 (*****)

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

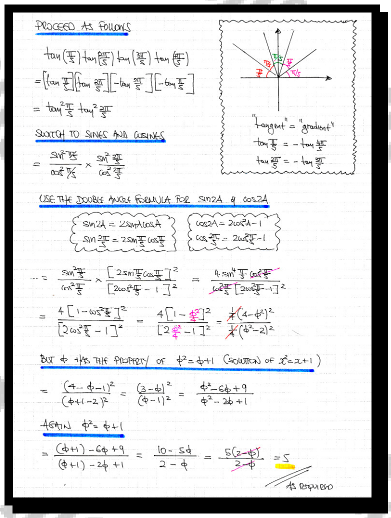
It can be shown, and you **may assume in this question**, that $\cos\left(\frac{1}{5}\pi\right) = \frac{1}{2}\phi$.

Use trigonometric identities to show that

$$\tan\left(\tfrac{1}{5}\pi\right)\tan\left(\tfrac{2}{5}\pi\right)\tan\left(\tfrac{3}{5}\pi\right)\tan\left(\tfrac{4}{5}\pi\right) = 5.$$

You may not use complex numbers in this question.

proof



Question 133 (*****)

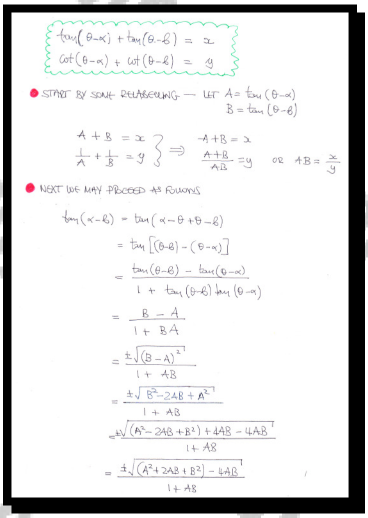
It is given that θ, α and β are distinct real numbers which satisfy.

$$\tan(\theta-\alpha)+\tan(\theta-b)=x$$

$$\cot(\theta-\alpha)+\cot(\theta-b)=y\,.$$

Find, in exact simplified form, an expression for $\tan(\alpha-\beta)$, in terms of x and y.

$\boxed{}$, $\boxed{\tan(\alpha-\beta)=\pm\dfrac{\sqrt{x^2y^2-4xy}}{x+y}}$



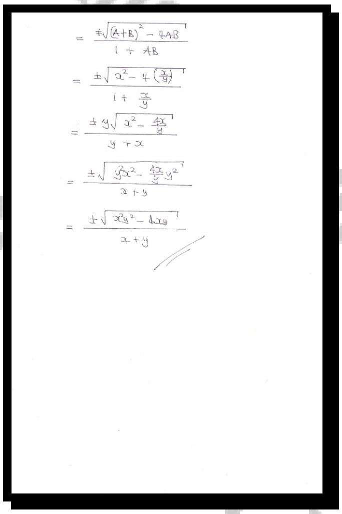
$\tan(\theta-\alpha)+\tan(\theta-\beta)=x$
$\cot(\theta-\alpha)+\cot(\theta-\beta)=y$

- START BY SOME RELABELLING — LET $A=\tan(\theta-\alpha)$, $B=\tan(\theta-\beta)$

$A+B=x$, $\frac{1}{A}+\frac{1}{B}=y \Rightarrow A+B=x$, $\frac{A+B}{AB}=y$ OR $AB=\frac{x}{y}$

- NEXT WE MAY PROCEED AS FOLLOWS

$\tan(\alpha-\beta)=\tan(\alpha-\theta+\theta-\beta)$
$=\tan[(\theta-\beta)-(\theta-\alpha)]$
$=\frac{\tan(\theta-\beta)-\tan(\theta-\alpha)}{1+\tan(\theta-\beta)\tan(\theta-\alpha)}$
$=\frac{B-A}{1+BA}$
$=\frac{\pm\sqrt{(B-A)^2}}{1+AB}$
$=\frac{\pm\sqrt{B^2-2AB+A^2}}{1+AB}$
$=\frac{\pm\sqrt{(A^2-2AB+B^2)+4AB-4AB}}{1+AB}$
$=\frac{\pm\sqrt{(A^2+2AB+B^2)-4AB}}{1+AB}$



$=\frac{\pm\sqrt{(A+B)^2-4AB}}{1+AB}$
$=\frac{\pm\sqrt{x^2-4\left(\frac{x}{y}\right)}}{1+\frac{x}{y}}$
$=\frac{\pm y\sqrt{x^2-\frac{4x}{y}}}{y+x}$
$=\frac{\pm\sqrt{y^2x^2-\frac{4x}{y}y^2}}{x+y}$
$=\frac{\pm\sqrt{x^2y^2-4xy}}{x+y}$

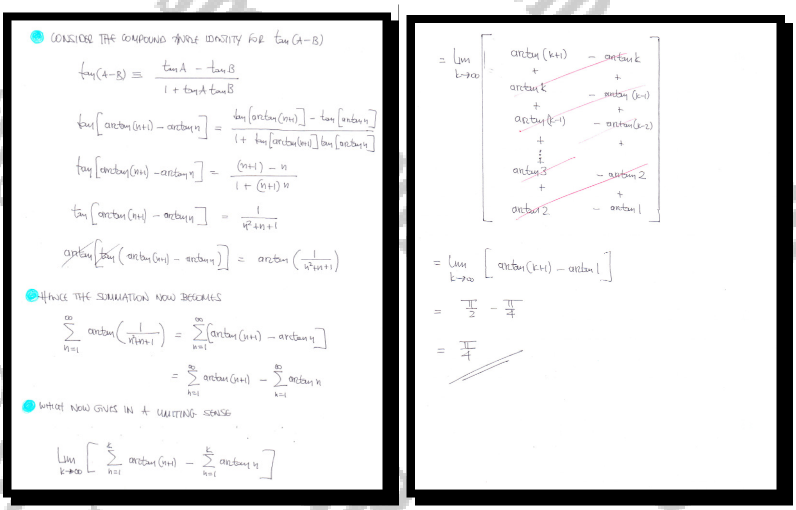
Question 134 (*****)

By considering the trigonometric identity for $\tan(A-B)$, with $A = \arctan(n+1)$ and $B = \arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right).$$

You may assume the series converges.

$\boxed{}$, $\boxed{\dfrac{\pi}{4}}$



Question 135 (*****)

It is given that

$$\left(\arcsin x\right)^3 + \left(\arccos x\right)^3 = k\pi^3, \quad |x| \le 1,$$

for some constant k.

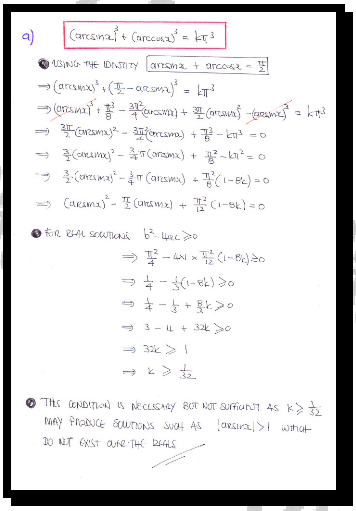
a) Show that a necessary but not sufficient condition for the above equation to have solutions is that

$$k \ge \frac{1}{32}.$$

b) Solve the equation given that it only has one solution.

c) Given instead that that $k = \frac{7}{96}$, find the two solutions of the equation, giving the answers in the form $x = \sin(a\pi)$, where $a \in \mathbb{Q}$.

$\boxed{}$, $\boxed{x = \frac{\sqrt{2}}{2}}$, $\boxed{x = \sin\left(\frac{\pi}{12}\right), \quad x = \sin\left(\frac{5\pi}{12}\right)}$



a) $\boxed{(\arcsin x)^3 + (\arccos x)^3 = k\pi^3}$

• USING THE IDENTITY $\boxed{\arcsin x + \arccos x = \frac{\pi}{2}}$

$\Rightarrow (\arcsin x)^3 + (\frac{\pi}{2} - \arcsin x)^3 = k\pi^3$

$\Rightarrow (\arcsin x)^3 + \frac{\pi^3}{8} - \frac{3\pi^2}{4}(\arcsin x) + \frac{3\pi}{2}(\arcsin x)^2 - (\arcsin x)^3 = k\pi^3$

$\Rightarrow \frac{3\pi}{2}(\arcsin x)^2 - \frac{3\pi^2}{4}(\arcsin x) + \frac{\pi^3}{8} - k\pi^3 = 0$

$\Rightarrow \frac{3}{2}(\arcsin x)^2 - \frac{3}{4}\pi(\arcsin x) + \frac{\pi^2}{8} - k\pi^2 = 0$

$\Rightarrow \frac{3}{2}(\arcsin x)^2 - \frac{3}{4}\pi(\arcsin x) + \frac{\pi^2}{8}(1-8k) = 0$

$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{\pi^2}{12}(1-8k) = 0$

• FOR REAL SOLUTIONS $b^2 - 4ac \ge 0$

$\Rightarrow \frac{\pi^2}{4} - 4 \times 1 \times \frac{\pi^2}{12}(1-8k) \ge 0$

$\Rightarrow \frac{1}{4} - \frac{1}{3}(1-8k) \ge 0$

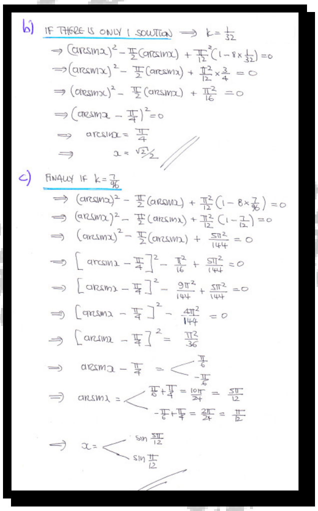
$\Rightarrow \frac{1}{4} - \frac{1}{3} + \frac{8}{3}k \ge 0$

$\Rightarrow 3 - 4 + 32k \ge 0$

$\Rightarrow 32k \ge 1$

$\Rightarrow k \ge \frac{1}{32}$

• THIS CONDITION IS NECESSARY BUT NOT SUFFICIENT AS $k \ge \frac{1}{32}$ MAY PRODUCE SOLUTIONS SUCH AS $|\arcsin x| > 1$ WHICH DO NOT EXIST OVER THE REALS



b) IF THERE IS ONLY 1 SOLUTION $\Rightarrow k = \frac{1}{32}$

$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{\pi^2}{12}(1 - 8 \times \frac{1}{32}) = 0$

$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{\pi^2}{12} \times \frac{3}{4} = 0$

$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{\pi^2}{16} = 0$

$\Rightarrow (\arcsin x - \frac{\pi}{4})^2 = 0$

$\Rightarrow \arcsin x = \frac{\pi}{4}$

$\Rightarrow x = \sqrt{2}/2$

c) FINALLY IF $k = \frac{7}{96}$

$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{\pi^2}{12}(1 - 8 \times \frac{7}{96}) = 0$

$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{\pi^2}{12}(1 - \frac{7}{12}) = 0$

$\Rightarrow (\arcsin x)^2 - \frac{\pi}{2}(\arcsin x) + \frac{5\pi^2}{144} = 0$

$\Rightarrow [\arcsin x - \frac{\pi}{4}]^2 - \frac{\pi^2}{16} + \frac{5\pi^2}{144} = 0$

$\Rightarrow [\arcsin x - \frac{\pi}{4}]^2 - \frac{9\pi^2}{144} + \frac{5\pi^2}{144} = 0$

$\Rightarrow [\arcsin x - \frac{\pi}{4}]^2 - \frac{4\pi^2}{144} = 0$

$\Rightarrow [\arcsin x - \frac{\pi}{4}]^2 = \frac{\pi^2}{36}$

$\Rightarrow \arcsin x - \frac{\pi}{4} = \frac{\pi}{6}$ or $-\frac{\pi}{6}$

$\Rightarrow \arcsin x = \frac{\pi}{6} + \frac{\pi}{4} = \frac{10\pi}{24} = \frac{5\pi}{12}$ or $-\frac{\pi}{6} + \frac{\pi}{4} = \frac{2\pi}{24} = \frac{\pi}{12}$

$\Rightarrow x = \sin\frac{5\pi}{12}$ or $\sin\frac{\pi}{12}$

Question 136 (*****)

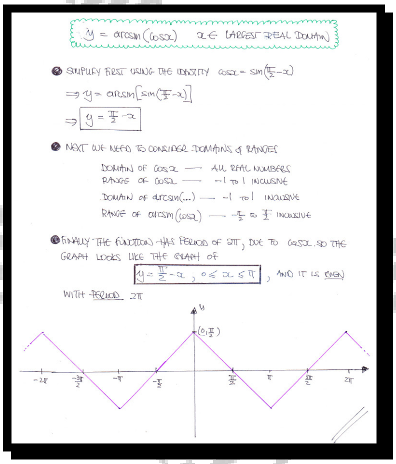
Sketch the graph of

$$f(x) = \arcsin(\cos x),$$

in the largest domain that the function is defined.

Indicate the coordinates of any intersections with the axes, and the coordinates of the cusps of the curve.

[ ], [graph]



Question 137 (*****)

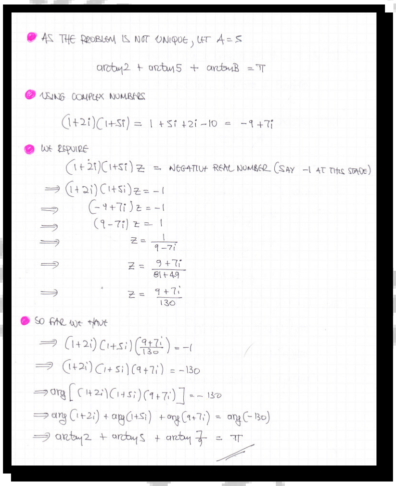
It is given that

$$\arctan 2 + \arctan A + \arctan B = \pi .$$

It is further given that A and B are distinct positive real numbers other than unity.

Determine a pair of possible values for A and B.

$\boxed{}$, $\boxed{5 \ \& \ \frac{7}{9}}$


AS THE PROBLEM IS NOT UNIQUE, LET $A=5$

$$\arctan 2 + \arctan 5 + \arctan B = \pi$$

USING COMPLEX NUMBERS

$$(1+2i)(1+5i) = 1+5i+2i-10 = -9+7i$$

WE REQUIRE

$$(1+2i)(1+5i)z = \text{NEGATIVE REAL NUMBER (SAY } -1 \text{ AT THIS STAGE)}$$

$\Rightarrow (1+2i)(1+5i)z = -1$

$\Rightarrow (-9+7i)z = -1$

$\Rightarrow (9-7i)z = 1$

$\Rightarrow z = \frac{1}{9-7i}$

$\Rightarrow z = \frac{9+7i}{81+49}$

$\Rightarrow z = \frac{9+7i}{130}$

SO FAR WE HAVE

$\Rightarrow (1+2i)(1+5i)\left(\frac{9+7i}{130}\right) = -1$

$\Rightarrow (1+2i)(1+5i)(9+7i) = -130$

$\Rightarrow \arg\left[(1+2i)(1+5i)(9+7i)\right] = -130$

$\Rightarrow \arg(1+2i) + \arg(1+5i) + \arg(9+7i) = \arg(-130)$

$\Rightarrow \arctan 2 + \arctan 5 + \arctan \frac{7}{9} = \pi$

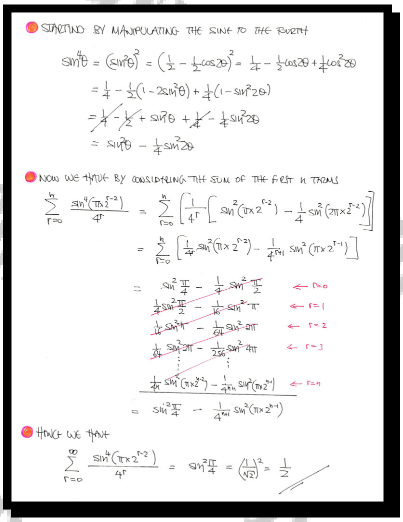
Question 138 (*****)

Find the value of

$$\sum_{r=0}^{\infty}\left[\frac{\sin^4\left(\pi\times 2^{r-2}\right)}{4^r}\right].$$

Hint: Express $\sin^4\theta$ in terms of $\sin^2\theta$ and $\sin^2 2\theta$ only.

$\boxed{}$, $\boxed{\frac{1}{2}}$



STARTING BY MANIPULATING THE SINE TO THE FOURTH

$\sin^4\theta = (\sin^2\theta)^2 = \left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)^2 = \frac{1}{4}-\frac{1}{2}\cos 2\theta+\frac{1}{4}\cos^2 2\theta$

$= \frac{1}{4}-\frac{1}{2}(1-2\sin^2\theta)+\frac{1}{4}(1-\sin^2 2\theta)$

$= \frac{1}{4}-\frac{1}{2}+\sin^2\theta+\frac{1}{4}-\frac{1}{4}\sin^2 2\theta$

$= \sin^2\theta-\frac{1}{4}\sin^2 2\theta$

NOW WE PROVE BY CONSIDERING THE SUM OF THE FIRST n TERMS

$\sum_{r=0}^{n}\frac{\sin^4(\pi\times 2^{r-2})}{4^r} = \sum_{r=0}^{n}\left[\frac{1}{4^r}\left[\sin^2(\pi\times 2^{r-2})-\frac{1}{4}\sin^2(2\pi\times 2^{r-2})\right]\right]$

$= \sum_{r=0}^{n}\left[\frac{1}{4^r}\sin^2(\pi\times 2^{r-2})-\frac{1}{4^{r+1}}\sin^2(\pi\times 2^{r-1})\right]$

$= \sin^2\frac{\pi}{4}-\frac{1}{4}\sin^2\frac{\pi}{2}$ ← r=0

$\frac{1}{4}\sin^2\frac{\pi}{2}-\frac{1}{16}\sin^2\pi$ ← r=1

$\frac{1}{16}\sin^2\pi-\frac{1}{64}\sin^2 2\pi$ ← r=2

$\frac{1}{64}\sin^2 2\pi-\frac{1}{256}\sin^2 4\pi$ ← r=3

⋮

$\frac{1}{4^n}\sin^2(\pi\times 2^{n-2})-\frac{1}{4^{n+1}}\sin^2(\pi\times 2^{n-1})$ ← r=n

$= \sin^2\frac{\pi}{4}-\frac{1}{4^{n+1}}\sin^2(\pi\times 2^{n-1})$

HENCE WE HAVE

$\sum_{r=0}^{\infty}\frac{\sin^4(\pi\times 2^{r-2})}{4^r} = \sin^2\frac{\pi}{4} = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

Question 139 (*****)

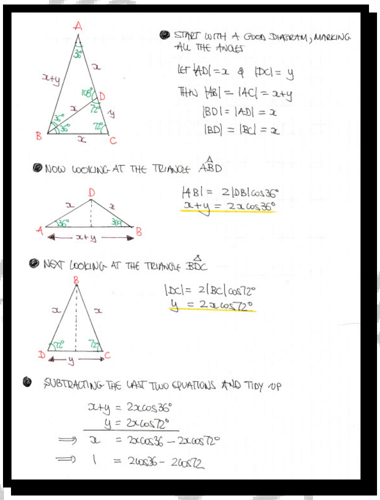
The triangle ABC is isosceles with $|AB| = |AC|$ and $\measuredangle BAC = 36°$.

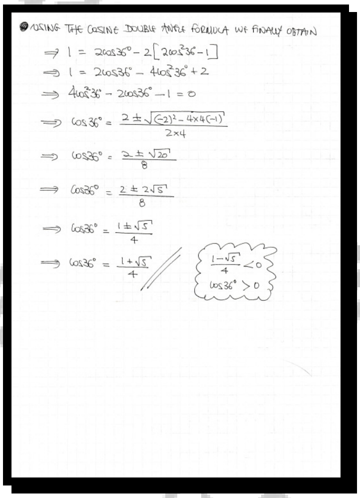
The angle bisector of $\measuredangle ABC$ meets AC at the point D.

By using trigonometry in the above construction, or otherwise, show that

$$\cos 36° = \tfrac{1}{2}\left(1+\sqrt{5}\right).$$

[ ], [proof]





Question 140 (*****)

$$f(x) = \frac{\cos 2x}{\sqrt{1+\sin 2x}}, \quad x \in \mathbb{R}, \quad \sin 2x \neq -1.$$

a) Express $f(x)$ in the form

$$f(x) = \frac{g(x)\,g(-x)}{|g(x)|},$$

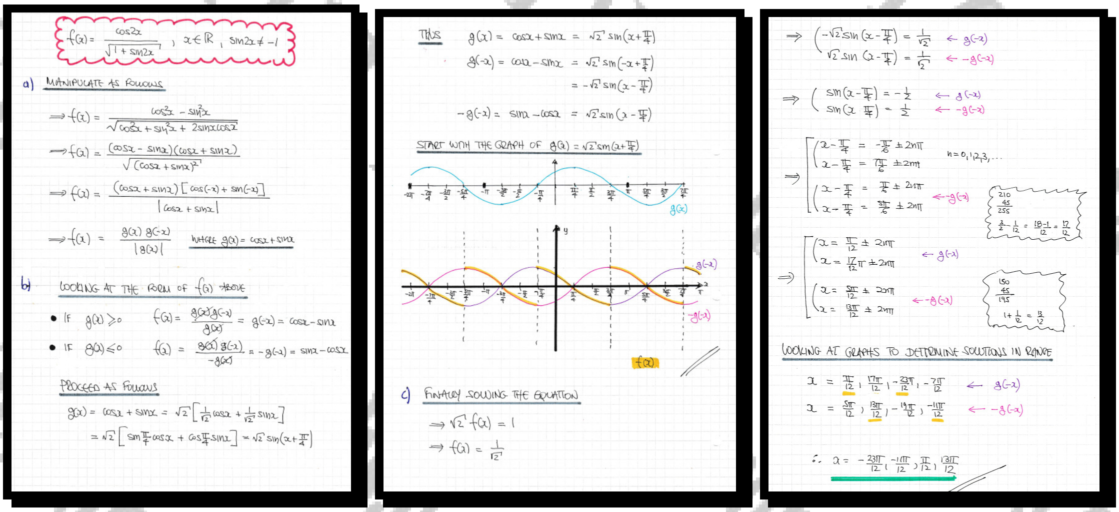
where $g(x)$ is a function to be found.

b) Sketch the graph of $f(x)$ for $-2\pi \le x \le 2\pi$.

c) Hence solve the trigonometric equation

$$\sqrt{2}\,f(x) = 1, \quad -2\pi \le x \le 2\pi.$$

$\boxed{}$, $\boxed{g(x) = \cos x + \sin x}$, $\boxed{x = -\frac{23}{12}\pi, \ -\frac{11}{12}\pi, \ \frac{1}{12}\pi, \ \frac{13}{12}\pi}$

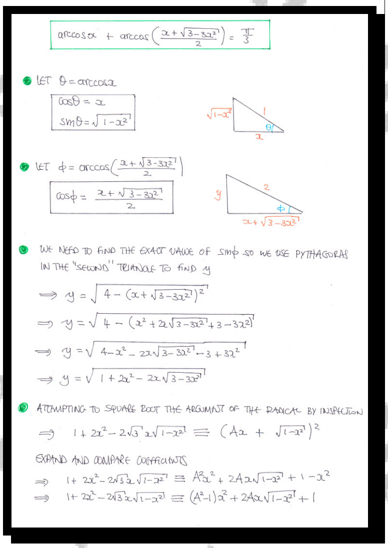


Question 141 (*****)

Prove that for all x such that $-1 \le x \le 1$

$$\arccos x + \arccos\left[\frac{1}{2}\left(x + \sqrt{3-3x^2}\right)\right] = \frac{\pi}{3}.$$

proof



$$\arccos x + \arccos\left(\frac{x+\sqrt{3-3x^2}}{2}\right) = \frac{\pi}{3}$$

- LET $\theta = \arccos x$

$\cos\theta = x$

$\sin\theta = \sqrt{1-x^2}$

- LET $\phi = \arccos\left(\frac{x+\sqrt{3-3x^2}}{2}\right)$

$\cos\phi = \frac{x+\sqrt{3-3x^2}}{2}$

- WE NEED TO FIND THE EXACT VALUE OF $\sin\phi$ SO WE USE PYTHAGORAS IN THE "SECOND" TRIANGLE TO FIND y

$\Rightarrow y = \sqrt{4-\left(x+\sqrt{3-3x^2}\right)^2}$

$\Rightarrow y = \sqrt{4-\left(x^2+2x\sqrt{3-3x^2}+3-3x^2\right)}$

$\Rightarrow y = \sqrt{4-x^2-2x\sqrt{3-3x^2}-3+3x^2}$

$\Rightarrow y = \sqrt{1+2x^2-2x\sqrt{3-3x^2}}$

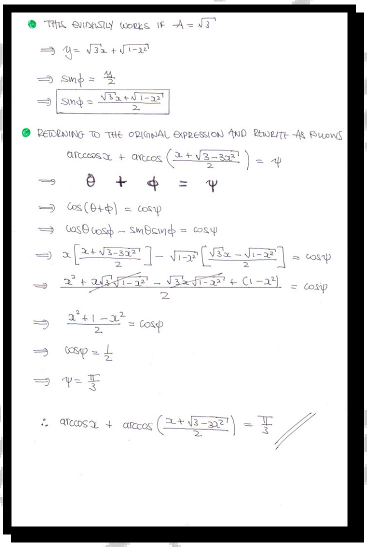
- ATTEMPTING TO SQUARE ROOT THE ARGUMENT OF THE RADICAL BY INSPECTION

$\Rightarrow 1+2x^2-2\sqrt{3}x\sqrt{1-x^2} \equiv \left(Ax+\sqrt{1-x^2}\right)^2$

EXPAND AND COMPARE COEFFICIENTS

$\Rightarrow 1+2x^2-2\sqrt{3}x\sqrt{1-x^2} \equiv A^2x^2+2Ax\sqrt{1-x^2}+1-x^2$

$\Rightarrow 1+2x^2-2\sqrt{3}x\sqrt{1-x^2} \equiv (A^2-1)x^2+2Ax\sqrt{1-x^2}+1$



- THIS EVIDENTLY WORKS IF $A=\sqrt{3}$

$\Rightarrow y = \sqrt{3}x+\sqrt{1-x^2}$

$\Rightarrow \sin\phi = \frac{y}{2}$

$\Rightarrow \sin\phi = \frac{\sqrt{3}x+\sqrt{1-x^2}}{2}$

- RETURNING TO THE ORIGINAL EXPRESSION AND REWRITE AS FOLLOWS

$\arccos x + \arccos\left(\frac{x+\sqrt{3-3x^2}}{2}\right) = \psi$

$\Rightarrow \theta + \phi = \psi$

$\Rightarrow \cos(\theta+\phi) = \cos\psi$

$\Rightarrow \cos\theta\cos\phi - \sin\theta\sin\phi = \cos\psi$

$\Rightarrow x\left[\frac{x+\sqrt{3-3x^2}}{2}\right] - \sqrt{1-x^2}\left[\frac{\sqrt{3}x-\sqrt{1-x^2}}{2}\right] = \cos\psi$

$\Rightarrow \frac{x^2+x\sqrt{3}\sqrt{1-x^2}-\sqrt{3}x\sqrt{1-x^2}+(1-x^2)}{2} = \cos\psi$

$\Rightarrow \frac{x^2+1-x^2}{2} = \cos\psi$

$\Rightarrow \cos\psi = \frac{1}{2}$

$\Rightarrow \psi = \frac{\pi}{3}$

$\therefore \arccos x + \arccos\left(\frac{x+\sqrt{3-3x^2}}{2}\right) = \frac{\pi}{3}$

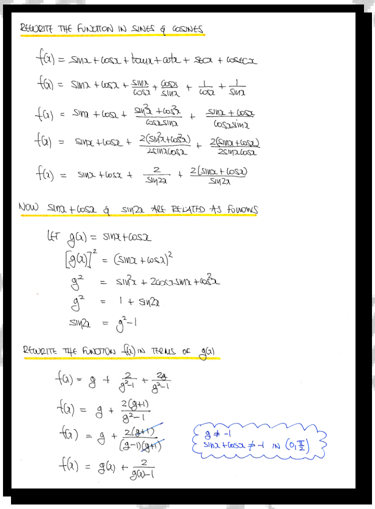
Question 142 (*****)

The function f is defined as

$$f(x) \equiv \sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x, \qquad x \in \left(0, \tfrac{1}{2}\pi\right).$$

Determine with full justification the range of f.

$\boxed{}$, $\boxed{f(x) \in \left[2 + 3\sqrt{2}, \infty\right)}$



REWRITE THE FUNCTION IN SINES & COSINES

$f(x) = \sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x$

$f(x) = \sin x + \cos x + \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x}$

$f(x) = \sin x + \cos x + \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} + \frac{\sin x + \cos x}{\cos x \sin x}$

$f(x) = \sin x + \cos x + \frac{2(\sin^2 x + \cos^2 x)}{2\sin x \cos x} + \frac{2(\sin x + \cos x)}{2 \sin x \cos x}$

$f(x) = \sin x + \cos x + \frac{2}{\sin 2x} + \frac{2(\sin x + \cos x)}{\sin 2x}$

NOW $\sin x + \cos x$ & $\sin 2x$ ARE RELATED AS FOLLOWS

LET $g(x) = \sin x + \cos x$

$[g(x)]^2 = (\sin x + \cos x)^2$

$g^2 = \sin^2 x + 2\cos x \sin x + \cos^2 x$

$g^2 = 1 + \sin 2x$

$\sin 2x = g^2 - 1$

REWRITE THE FUNCTION $f(x)$ IN TERMS OF $g(x)$

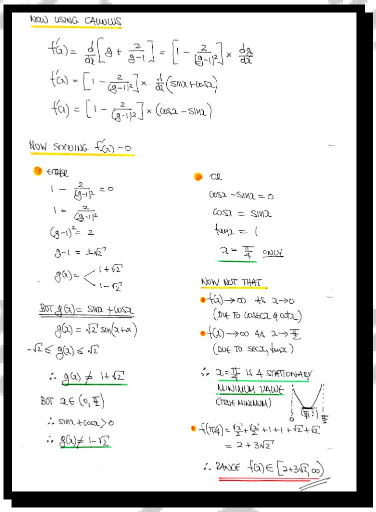
$f(x) = g + \frac{2}{g^2-1} + \frac{2g}{g^2-1}$

$f(x) = g + \frac{2(g+1)}{g^2-1}$

$f(x) = g + \frac{2(g+1)}{(g-1)(g+1)}$

$g \neq -1$
$\sin x + \cos x \neq -1$ IN $\left(0, \frac{\pi}{2}\right)$

$f(x) = g(x) + \frac{2}{g(x)-1}$



NOW USING CALCULUS

$f'(x) = \frac{d}{dx}\left[g + \frac{2}{g-1}\right] = \left[1 - \frac{2}{(g-1)^2}\right] \times \frac{dg}{dx}$

$f'(x) = \left[1 - \frac{2}{(g-1)^2}\right] \times \frac{d}{dx}(\sin x + \cos x)$

$f'(x) = \left[1 - \frac{2}{(g-1)^2}\right] \times (\cos x - \sin x)$

NOW SOLVING $f'(x) = 0$

- EITHER

$1 - \frac{2}{(g-1)^2} = 0$

$1 = \frac{2}{(g-1)^2}$

$(g-1)^2 = 2$

$g - 1 = \pm\sqrt{2}$

$g(x) = 1+\sqrt{2}$ or $1-\sqrt{2}$

BUT $g(x) = \sin x + \cos x$

$g(x) = \sqrt{2}\sin(x + \alpha)$

$-\sqrt{2} \leqslant g(x) \leqslant \sqrt{2}$

$\therefore g(x) \neq 1 + \sqrt{2}$

BUT $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \sin x + \cos x > 0$

$\therefore g(x) \neq 1 - \sqrt{2}$

- OR

$\cos x - \sin x = 0$

$\cos x = \sin x$

$\tan x = 1$

$x = \frac{\pi}{4}$ ONLY

NOW NOT THAT

- $f(x) \to \infty$ AS $x \to 0$ (DUE TO $\operatorname{cosec} x$ & $\cot x$)
- $f(x) \to \infty$ AS $x \to \frac{\pi}{2}$ (DUE TO $\sec x$, $\tan x$)

$\therefore x = \frac{\pi}{4}$ IS A STATIONARY MINIMUM VALUE (TRUE MINIMUM)

- $f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 1 + 1 + \sqrt{2} + \sqrt{2}$
$= 2 + 3\sqrt{2}$

$\therefore$ RANGE $f(x) \in \left[2 + 3\sqrt{2}, \infty\right)$

Question 143 **(*****)**

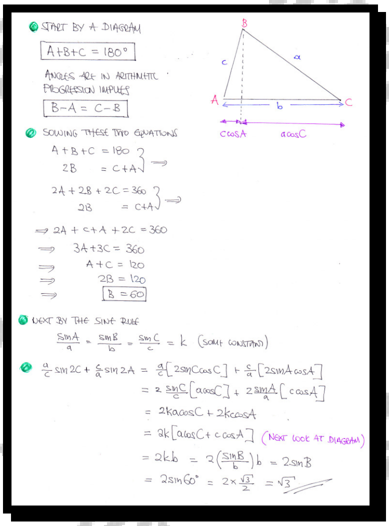
It is given that

- the angles A, B and C are the three angles of a triangle ABC.
- the angles A, B and C are in an increasing arithmetic progression, in that order.
- The lengths of the triangle ABC, opposite each of the angles A, B and C are denoted by a, b and c.

Show that

$$\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A = \sqrt{3}.$$

SP-V, proof



START BY A DIAGRAM

$A+B+C=180°$

ANGLES ARE IN ARITHMETIC PROGRESSION IMPLIES

$B-A=C-B$

SOLVING THESE TWO EQUATIONS

$$\left.\begin{array}{l} A+B+C=180 \\ 2B=C+A \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2A+2B+2C=360 \\ 2B=C+A \end{array}\right\} \Rightarrow$$

$\Rightarrow 2A+C+A+2C=360$

$\Rightarrow 3A+3C=360$

$\Rightarrow A+C=120$

$\Rightarrow 2B=120$

$\Rightarrow B=60$

NEXT BY THE SINE RULE

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k \quad \text{(SOME CONSTANT)}$$

$$\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A = \frac{a}{c}[2\sin C\cos C]+\frac{c}{a}[2\sin A\cos A]$$
$$= 2\frac{\sin C}{c}[a\cos C]+2\frac{\sin A}{a}[c\cos A]$$
$$= 2ka\cos C+2kc\cos A$$
$$= 2k[a\cos C+c\cos A] \quad \text{(NEXT LOOK AT DIAGRAM)}$$
$$= 2kb = 2\left(\frac{\sin B}{b}\right)b = 2\sin B$$
$$= 2\sin 60° = 2\times\frac{\sqrt{3}}{2} = \sqrt{3}$$

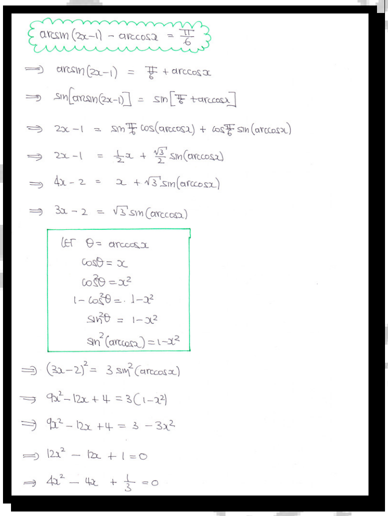
Question 144 (*****)

Find, in exact surd form, the only real solution of the following trigonometric equation

$$\arcsin(2x-1) - \arccos x = \frac{\pi}{6}.$$

The rejection of any additional solutions must be fully justified.

$\boxed{}$, $\boxed{x = \frac{1}{2} - \frac{1}{6}\sqrt{6}}$



$\arcsin(2x-1) - \arccos x = \frac{\pi}{6}$

$\Rightarrow \arcsin(2x-1) = \frac{\pi}{6} + \arccos x$

$\Rightarrow \sin[\arcsin(2x-1)] = \sin[\frac{\pi}{6} + \arccos x]$

$\Rightarrow 2x-1 = \sin\frac{\pi}{6}\cos(\arccos x) + \cos\frac{\pi}{6}\sin(\arccos x)$

$\Rightarrow 2x-1 = \frac{1}{2}x + \frac{\sqrt{3}}{2}\sin(\arccos x)$

$\Rightarrow 4x-2 = x + \sqrt{3}\sin(\arccos x)$

$\Rightarrow 3x-2 = \sqrt{3}\sin(\arccos x)$

LET $\theta = \arccos x$
$\cos\theta = x$
$\cos^2\theta = x^2$
$1-\cos^2\theta = 1-x^2$
$\sin^2\theta = 1-x^2$
$\sin^2(\arccos x) = 1-x^2$

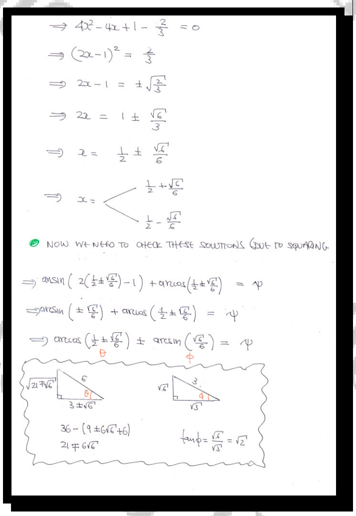
$\Rightarrow (3x-2)^2 = 3\sin^2(\arccos x)$

$\Rightarrow 9x^2-12x+4 = 3(1-x^2)$

$\Rightarrow 9x^2-12x+4 = 3-3x^2$

$\Rightarrow 12x^2-12x+1=0$

$\Rightarrow 4x^2-4x+\frac{1}{3}=0$



$\Rightarrow 4x^2-4x+1-\frac{2}{3}=0$

$\Rightarrow (2x-1)^2 = \frac{2}{3}$

$\Rightarrow 2x-1 = \pm\sqrt{\frac{2}{3}}$

$\Rightarrow 2x = 1 \pm \frac{\sqrt{6}}{3}$

$\Rightarrow x = \frac{1}{2} \pm \frac{\sqrt{6}}{6}$

$\Rightarrow x = \frac{1}{2}+\frac{\sqrt{6}}{6}$ or $\frac{1}{2}-\frac{\sqrt{6}}{6}$

NOW WE NEED TO CHECK THESE SOLUTIONS (DUE TO SQUARING

$\Rightarrow \arcsin\left(2\left(\frac{1}{2}\pm\frac{\sqrt{6}}{6}\right)-1\right) + \arccos\left(\frac{1}{2}\pm\frac{\sqrt{6}}{6}\right) = \psi$

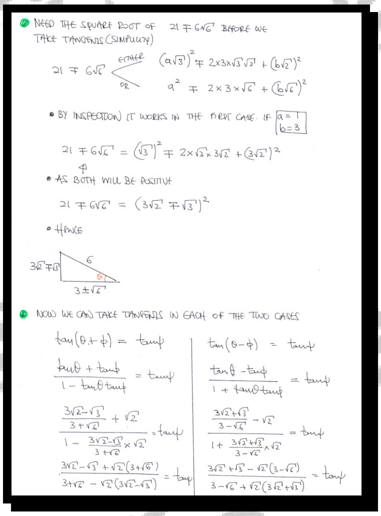
$\Rightarrow \arcsin\left(\pm\frac{\sqrt{6}}{6}\right) + \arccos\left(\frac{1}{2}\pm\frac{\sqrt{6}}{6}\right) = \psi$

$\Rightarrow \arccos\left(\frac{1}{2}\pm\frac{\sqrt{6}}{6}\right) \pm \arcsin\left(\frac{\sqrt{6}}{6}\right) = \psi$

θ, φ

$36-(9\pm6\sqrt{6}+6)$
$21\mp6\sqrt{6}$

$\tan\phi = \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$



NEED THE SQUARE ROOT OF $21\mp6\sqrt{6}$ BEFORE WE TAKE TANGENTS (SIMPLICITY)

$21\mp6\sqrt{6}$ EITHER $(a\sqrt{3})^2 \mp 2\times3\times\sqrt{3}\sqrt{2} + (b\sqrt{2})^2$ OR $a^2 \mp 2\times3\times\sqrt{6} + (b\sqrt{6})^2$

- BY INSPECTION IT WORKS IN THE FIRST CASE. IF $a=1$, $b=3$

$21\mp6\sqrt{6} = (\sqrt{3})^2 \mp 2\times\sqrt{3}\times3\sqrt{2} + (3\sqrt{2})^2$

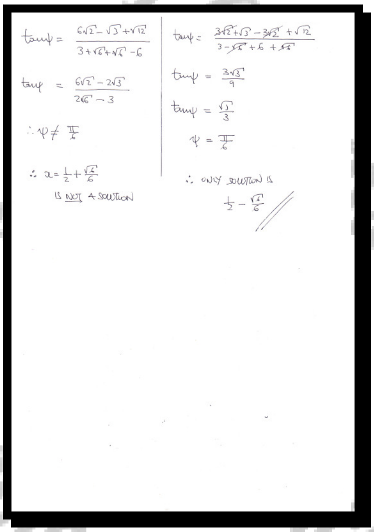
- AS BOTH WILL BE POSITIVE

$21\mp6\sqrt{6} = (3\sqrt{2}\mp\sqrt{3})^2$

- HENCE

NOW WE CAN TAKE TANGENTS IN EACH OF THE TWO CASES

$\tan(\theta+\phi) = \tan\psi$	$\tan(\theta-\phi) = \tan\psi$
$\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} = \tan\psi$	$\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi} = \tan\psi$
$\frac{\frac{3\sqrt{2}-\sqrt{3}}{3+\sqrt{6}}+\sqrt{2}}{1-\frac{3\sqrt{2}-\sqrt{3}}{3+\sqrt{6}}\times\sqrt{2}} = \tan\psi$	$\frac{\frac{3\sqrt{2}+\sqrt{3}}{3-\sqrt{6}}-\sqrt{2}}{1+\frac{3\sqrt{2}+\sqrt{3}}{3-\sqrt{6}}\times\sqrt{2}} = \tan\psi$
$\frac{3\sqrt{2}-\sqrt{3}+\sqrt{2}(3+\sqrt{6})}{3+\sqrt{6}-\sqrt{2}(3\sqrt{2}-\sqrt{3})} = \tan\psi$	$\frac{3\sqrt{2}+\sqrt{3}-\sqrt{2}(3-\sqrt{6})}{3-\sqrt{6}+\sqrt{2}(3\sqrt{2}+\sqrt{3})} = \tan\psi$



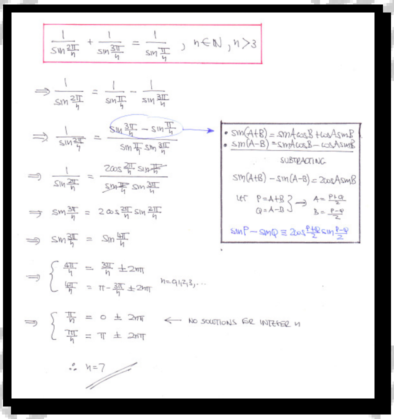
$\tan\psi = \frac{6\sqrt{2}-\sqrt{3}+\sqrt{12}}{3+\sqrt{6}+\sqrt{6}-6}$	$\tan\psi = \frac{3\sqrt{2}+\sqrt{3}-3\sqrt{2}+\sqrt{12}}{3-\sqrt{6}+6+\sqrt{6}}$
$\tan\psi = \frac{6\sqrt{2}-2\sqrt{3}}{2\sqrt{6}-3}$	$\tan\psi = \frac{3\sqrt{3}}{9}$
$\therefore \psi \neq \frac{\pi}{6}$	$\tan\psi = \frac{\sqrt{3}}{3}$
	$\psi = \frac{\pi}{6}$
$\therefore x = \frac{1}{2}+\frac{\sqrt{6}}{6}$ IS NOT A SOLUTION	$\therefore$ ONLY SOLUTION IS $\frac{1}{2}-\frac{\sqrt{6}}{6}$

Question 145 (*****)

Given that n is an integer such that $n > 3$, use a detailed method to solve the following trigonometric equation.

$$\frac{1}{\sin\left[\frac{2\pi}{n}\right]} + \frac{1}{\sin\left[\frac{3\pi}{n}\right]} = \frac{1}{\sin\left[\frac{\pi}{n}\right]}.$$

$\boxed{}$, $\boxed{n = 7}$



$$\frac{1}{\sin\frac{2\pi}{n}} + \frac{1}{\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{\pi}{n}}, \quad n \in \mathbb{N}, n > 3$$

$$\Rightarrow \frac{1}{\sin\frac{2\pi}{n}} = \frac{1}{\sin\frac{\pi}{n}} - \frac{1}{\sin\frac{3\pi}{n}}$$

$$\Rightarrow \frac{1}{\sin\frac{2\pi}{n}} = \frac{\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}}$$

- $\sin(A+B) = \sin A\cos B + \cos A\sin B$
- $\sin(A-B) = \sin A\cos B - \cos A\sin B$

SUBTRACTING

$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

LET $P = A+B$, $Q = A-B$ $\Rightarrow$ $A = \frac{P+Q}{2}$, $B = \frac{P-Q}{2}$

$\sin P - \sin Q \equiv 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$$\Rightarrow \frac{1}{\sin\frac{2\pi}{n}} = \frac{2\cos\frac{2\pi}{n}\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}}$$

$$\Rightarrow \sin\frac{3\pi}{n} = 2\cos\frac{2\pi}{n}\sin\frac{2\pi}{n}$$

$$\Rightarrow \sin\frac{3\pi}{n} = \sin\frac{4\pi}{n}$$

$$\Rightarrow \begin{cases} \frac{4\pi}{n} = \frac{3\pi}{n} \pm 2m\pi \\ \frac{4\pi}{n} = \pi - \frac{3\pi}{n} \pm 2m\pi \end{cases} \quad m = 0,1,2,3,\ldots$$

$$\Rightarrow \begin{cases} \frac{\pi}{n} = 0 \pm 2m\pi \\ \frac{7\pi}{n} = \pi \pm 2m\pi \end{cases}$$ ← NO SOLUTIONS FOR INTEGER n

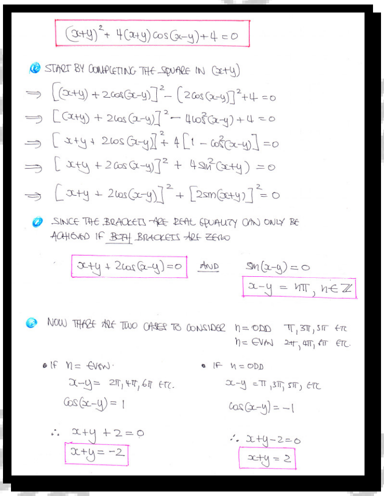
$\therefore n = 7$

Question 146 (*****)

Find, in terms of π, the general solution of the equation

$$(x+y)^2+4(x+y)\cos(x-y)+4=0.$$

$\boxed{}$, $\boxed{(x,y)=\left(-1+\frac{k\pi}{2},-1-\frac{k\pi}{2}\right), k=\text{even}}$, $\boxed{(x,y)=\left(1+\frac{k\pi}{2},1-\frac{k\pi}{2}\right), k=\text{odd}}$



$(x+y)^2+4(x+y)\cos(x-y)+4=0$

START BY COMPLETING THE SQUARE IN $(x+y)$

$\Rightarrow [(x+y)+2\cos(x-y)]^2-[2\cos(x-y)]^2+4=0$

$\Rightarrow [(x+y)+2\cos(x-y)]^2-4\cos^2(x-y)+4=0$

$\Rightarrow [x+y+2\cos(x-y)]^2+4[1-\cos^2(x-y)]=0$

$\Rightarrow [x+y+2\cos(x-y)]^2+4\sin^2(x+y)=0$

$\Rightarrow [x+y+2\cos(x-y)]^2+[2\sin(x+y)]^2=0$

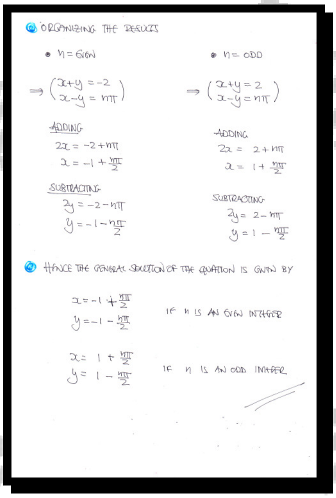
SINCE THE BRACKETS ARE REAL EQUALITY CAN ONLY BE ACHIEVED IF BOTH BRACKETS ARE ZERO

$x+y+2\cos(x-y)=0$ AND $\sin(x-y)=0$

$x-y=n\pi, n\in\mathbb{Z}$

NOW THERE ARE TWO CASES TO CONSIDER $n=\text{ODD}$ $\pi, 3\pi, 5\pi$ ETC; $n=\text{EVEN}$ $2\pi, 4\pi, 6\pi$ ETC.

- IF $n=\text{EVEN}$:
  $x-y=2\pi, 4\pi, 6\pi$ ETC.
  $\cos(x-y)=1$
  $\therefore x+y+2=0$
  $x+y=-2$
- IF $n=\text{ODD}$
  $x-y=\pi, 3\pi, 5\pi$, ETC
  $\cos(x-y)=-1$
  $\therefore x+y-2=0$
  $x+y=2$



ORGANIZING THE RESULTS

- $n=\text{EVEN}$
  $\Rightarrow \begin{pmatrix} x+y=-2 \\ x-y=n\pi \end{pmatrix}$
  ADDING
  $2x=-2+n\pi$
  $x=-1+\frac{n\pi}{2}$
  SUBTRACTING
  $2y=-2-n\pi$
  $y=-1-\frac{n\pi}{2}$
- $n=\text{ODD}$
  $\Rightarrow \begin{pmatrix} x+y=2 \\ x-y=n\pi \end{pmatrix}$
  ADDING
  $2x=2+n\pi$
  $x=1+\frac{n\pi}{2}$
  SUBTRACTING
  $2y=2-n\pi$
  $y=1-\frac{n\pi}{2}$

HENCE THE GENERAL SOLUTION OF THE EQUATION IS GIVEN BY

$x=-1+\frac{n\pi}{2}$, $y=-1-\frac{n\pi}{2}$ IF n IS AN EVEN INTEGER

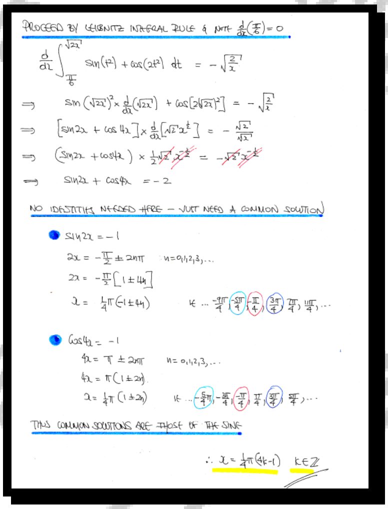
$x=1+\frac{n\pi}{2}$, $y=1-\frac{n\pi}{2}$ IF n IS AN ODD INTEGER

Question 147 (*****)

Find the general solution of the following equation

$$\frac{d}{dx}\left[\int_{\frac{1}{6}\pi}^{\sqrt{2x}} \sin\left(t^2\right)+\cos\left(2t^2\right)\ dx\right] = -\sqrt{\frac{2}{x}}, \quad x\in\mathbb{R}.$$

$\boxed{}$, $\boxed{x=\frac{1}{4}\pi(4k-1) \quad k\in\mathbb{Z}}$



PROCEED BY LEIBNITZ INTEGRAL RULE & NOTE $\frac{d}{dx}\left(\frac{\pi}{6}\right)=0$

$$\frac{d}{dx}\int_{\frac{\pi}{6}}^{\sqrt{2x}} \sin(t^2)+\cos(2t^2)\ dt = -\sqrt{\frac{2}{x}}$$

$$\Rightarrow \sin\left(\sqrt{2x}\right)^2\times\frac{d}{dx}\left(\sqrt{2x}\right)+\cos\left[2\left(\sqrt{2x}\right)^2\right] = -\sqrt{\frac{2}{x}}$$

$$\Rightarrow \left[\sin 2x+\cos 4x\right]\times\frac{d}{dx}\left[\sqrt{2}x^{\frac{1}{2}}\right] = -\frac{\sqrt{2}}{\sqrt{x}}$$

$$\Rightarrow \left(\sin 2x+\cos 4x\right)\times\frac{1}{2}\sqrt{2}x^{-\frac{1}{2}} = -\sqrt{2}x^{-\frac{1}{2}}$$

$$\Rightarrow \sin 2x+\cos 4x = -2$$

NO IDENTITIES NEEDED HERE – JUST NEED A COMMON SOLUTION

- $\sin 2x = -1$

$2x = -\frac{\pi}{2}\pm 2n\pi \quad n=0,1,2,3,\ldots$

$2x = -\frac{\pi}{2}\left[1\pm 4n\right]$

$x = \frac{1}{4}\pi(-1\pm 4n) \quad$ i.e. $\ldots -\frac{9\pi}{4}, -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \ldots$

- $\cos 4x = -1$

$4x = \pi\pm 2n\pi \quad n=0,1,2,3,\ldots$

$4x = \pi(1\pm 2n)$

$x = \frac{1}{4}\pi(1\pm 2n) \quad$ i.e. $\ldots -\frac{5\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$

THUS COMMON SOLUTIONS ARE THOSE OF THE SINE

$\therefore x = \frac{1}{4}\pi(4k-1) \quad k\in\mathbb{Z}$