TRIGONOMETRY

THE PYTHAGOREAN IDENTITIES

Question 1

Prove the validity of each of the following trigonometric identities.

a) $\dfrac{\cot^2 x}{1+\cot^2 x} \equiv \cos^2 x$

b) $\dfrac{1}{\sec x - \tan x} + \dfrac{1}{\sec x + \tan x} \equiv 2\sec x$

c) $\dfrac{\tan x \sec x}{1+\tan^2 x} \equiv \sin x$

d) $\dfrac{1}{\sec x - \tan x} - \dfrac{1}{\sec x + \tan x} \equiv 2\tan x$

e) $\dfrac{\cot x \operatorname{cosec} x}{1+\cot^2 x} \equiv \cos x$

(a) $\text{LHS} = \dfrac{\cot^2 x}{1+\cot^2 x} = \dfrac{\cot^2 x}{\operatorname{cosec}^2 x} = \cot^2 x \times \sin^2 x = \dfrac{\cos^2 x}{\cancel{\sin^2 x}} \times \cancel{\sin^2 x}$

$= \cos^2 x = \text{RHS}$

(b) $\text{LHS} = \dfrac{1}{\sec x - \tan x} + \dfrac{1}{\sec x + \tan x} = \dfrac{\sec x + \cancel{\tan x} + \sec x - \cancel{\tan x}}{(\sec x - \tan x)(\sec x + \tan x)}$

$= \dfrac{2\sec x}{\sec^2 x - \tan^2 x} = \dfrac{2\sec x}{(1+\cancel{\tan^2 x}) - \cancel{\tan^2 x}} = 2\sec x = \text{RHS}$

(c) $\text{LHS} = \dfrac{\tan x \sec x}{1+\tan^2 x} = \dfrac{\tan x \sec x}{\sec^2 x} = \dfrac{\tan x}{\sec x} = \tan x \cos x$

$= \dfrac{\sin x}{\cancel{\cos x}} \cancel{\cos x} = \sin x = \text{RHS}$

(d) $\text{LHS} = \dfrac{1}{\sec x - \tan x} - \dfrac{1}{\sec x + \tan x} = \dfrac{1(\sec x + \tan x) - 1(\sec x - \tan x)}{(\sec x - \tan x)(\sec x + \tan x)}$

$= \dfrac{2\tan x}{\sec^2 x - \tan^2 x} = \dfrac{2\tan x}{(1+\cancel{\tan^2 x}) - \cancel{\tan^2 x}} = 2\tan x = \text{RHS}$

(e) $\text{LHS} = \dfrac{\cot x \operatorname{cosec} x}{1+\cot^2 x} = \dfrac{\cot x \operatorname{cosec} x}{\operatorname{cosec}^2 x} = \dfrac{\cot x}{\operatorname{cosec} x} = \cot x \sin x$

$= \dfrac{\cos x}{\cancel{\sin x}} \cancel{\sin x} = \cos x = \text{RHS}$

Question 2

Prove the validity of each of the following trigonometric identities.

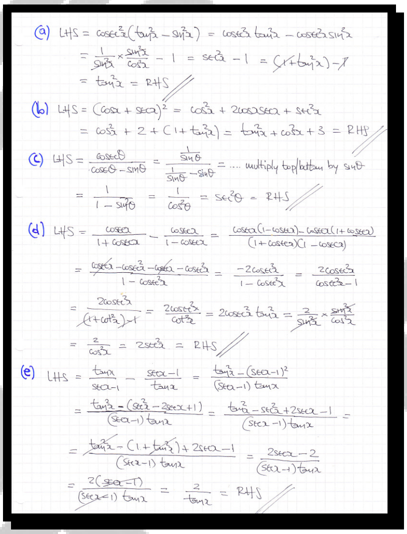
a) $\operatorname{cosec}^2 x\left(\tan^2 x - \sin^2 x\right) \equiv \tan^2 x$

b) $\left(\cos x + \sec x\right)^2 \equiv \tan^2 x + \cos^2 x + 3$

c) $\dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sin\theta} \equiv \sec^2\theta$

d) $\dfrac{\operatorname{cosec} x}{1+\operatorname{cosec} x} - \dfrac{\operatorname{cosec} x}{1-\operatorname{cosec} x} \equiv 2\sec^2 x$

e) $\dfrac{\tan x}{\sec x - 1} - \dfrac{\sec x - 1}{\tan x} \equiv 2\cot x$



(a) $\text{LHS} = \operatorname{cosec}^2 x(\tan^2 x - \sin^2 x) = \operatorname{cosec}^2 x \tan^2 x - \operatorname{cosec}^2 x \sin^2 x$

$= \frac{1}{\cancel{\sin^2 x}} \times \frac{\cancel{\sin^2 x}}{\cos^2 x} - 1 = \sec^2 x - 1 = (\cancel{1} + \tan^2 x) - \cancel{1}$

$= \tan^2 x = \text{RHS}$

(b) $\text{LHS} = (\cos x + \sec x)^2 = \cos^2 x + 2\cos x \sec x + \sec^2 x$

$= \cos^2 x + 2 + (1 + \tan^2 x) = \tan^2 x + \cos^2 x + 3 = \text{RHS}$

(c) $\text{LHS} = \frac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sin\theta} = \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \sin\theta} = \ldots$ multiply top/bottom by $\sin\theta$

$= \frac{1}{1 - \sin^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta = \text{RHS}$

(d) $\text{LHS} = \frac{\operatorname{cosec} x}{1 + \operatorname{cosec} x} - \frac{\operatorname{cosec} x}{1 - \operatorname{cosec} x} = \frac{\operatorname{cosec} x(1 - \operatorname{cosec} x) - \operatorname{cosec} x(1 + \operatorname{cosec} x)}{(1 + \operatorname{cosec} x)(1 - \operatorname{cosec} x)}$

$= \frac{\cancel{\operatorname{cosec} x} - \operatorname{cosec}^2 x - \cancel{\operatorname{cosec} x} - \operatorname{cosec}^2 x}{1 - \operatorname{cosec}^2 x} = \frac{-2\operatorname{cosec}^2 x}{1 - \operatorname{cosec}^2 x} = \frac{2\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x - 1}$

$= \frac{2\operatorname{cosec}^2 x}{(\cancel{1} + \cot^2 x) - \cancel{1}} = \frac{2\operatorname{cosec}^2 x}{\cot^2 x} = 2\operatorname{cosec}^2 x \tan^2 x = \frac{2}{\cancel{\sin^2 x}} \times \frac{\cancel{\sin^2 x}}{\cos^2 x}$

$= \frac{2}{\cos^2 x} = 2\sec^2 x = \text{RHS}$

(e) $\text{LHS} = \frac{\tan x}{\sec x - 1} - \frac{\sec x - 1}{\tan x} = \frac{\tan^2 x - (\sec x - 1)^2}{(\sec x - 1)\tan x}$

$= \frac{\tan^2 x - (\sec^2 x - 2\sec x + 1)}{(\sec x - 1)\tan x} = \frac{\tan^2 x - \sec^2 x + 2\sec x - 1}{(\sec x - 1)\tan x} =$

$= \frac{\cancel{\tan^2 x} - (1 + \cancel{\tan^2 x}) + 2\sec x - 1}{(\sec x - 1)\tan x} = \frac{2\sec x - 2}{(\sec x - 1)\tan x}$

$= \frac{2\cancel{(\sec x - 1)}}{\cancel{(\sec x - 1)}\tan x} = \frac{2}{\tan x} = \text{RHS}$

Question 3

Prove the validity of each of the following trigonometric identities.

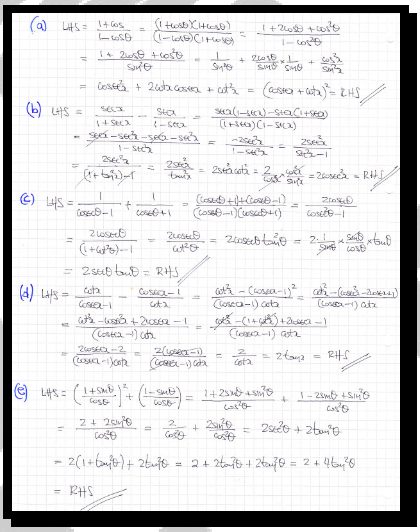
a) $\dfrac{1+\cos\theta}{1-\cos\theta} \equiv \left(\operatorname{cosec}\theta+\cot\theta\right)^2$

b) $\dfrac{\sec x}{1+\sec x}-\dfrac{\sec x}{1-\sec x} \equiv 2\operatorname{cosec}^2 x$

c) $\dfrac{1}{\operatorname{cosec}\theta-1}+\dfrac{1}{\operatorname{cosec}\theta+1} \equiv 2\sec\theta\tan\theta$

d) $\dfrac{\cot x}{\operatorname{cosec} x-1}-\dfrac{\operatorname{cosec} x-1}{\cot x} \equiv 2\tan x$

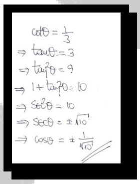
e) $\left(\dfrac{1+\sin\theta}{\cos\theta}\right)^2+\left(\dfrac{1-\sin\theta}{\cos\theta}\right)^2 \equiv 2+4\tan^2\theta$



Question 4

If $\cot\theta = \dfrac{1}{3}$, show clearly that $\cos\theta = \pm\dfrac{\sqrt{10}}{10}$.

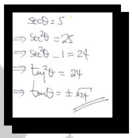
$\boxed{\text{proof}}$


$\cot\theta = \frac{1}{3}$
$\Rightarrow \tan\theta = 3$
$\Rightarrow \tan^2\theta = 9$
$\Rightarrow 1 + \tan^2\theta = 10$
$\Rightarrow \sec^2\theta = 10$
$\Rightarrow \sec\theta = \pm\sqrt{10}$
$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{10}}$

Question 5

If $\sec\theta = 5$, show clearly that $\tan\theta = \pm\sqrt{24}$.

$\boxed{\text{proof}}$


$\sec\theta = 5$
$\Rightarrow \sec^2\theta = 25$
$\Rightarrow \sec^2\theta - 1 = 24$
$\Rightarrow \tan^2\theta = 24$
$\Rightarrow \tan\theta = \pm\sqrt{24}$

Question 6

Solve each of the following equations.

a) $2\tan^2\theta = 11\sec\theta - 7$, $\quad 0 \le \theta < 360°$

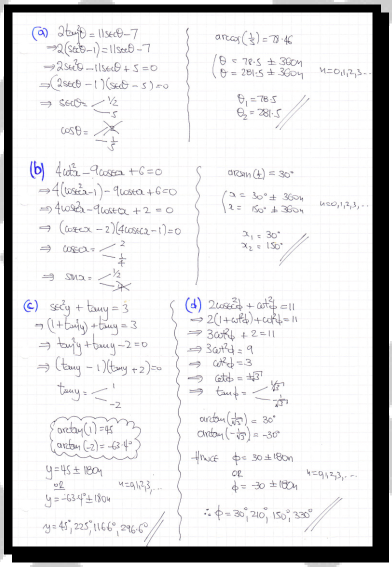
b) $4\cot^2 x - 9\operatorname{cosec} x + 6 = 0$, $\quad 0 \le x < 360°$

c) $\sec^2 y + \tan y = 3$, $\quad 0 \le y < 360°$

d) $2\operatorname{cosec}^2\varphi + \cot^2\varphi = 11$, $\quad 0 \le \varphi < 360°$

$\boxed{\theta = 78.5°,\ 281.5°}$, $\boxed{x = 30°,\ 150°}$, $\boxed{y = 45°,\ 225° \quad y \approx 116.6°,\ 296.6°}$,

$\boxed{\varphi = 30°,\ 150°,\ 210°,\ 330°}$



Question 7

Solve each of the following equations.

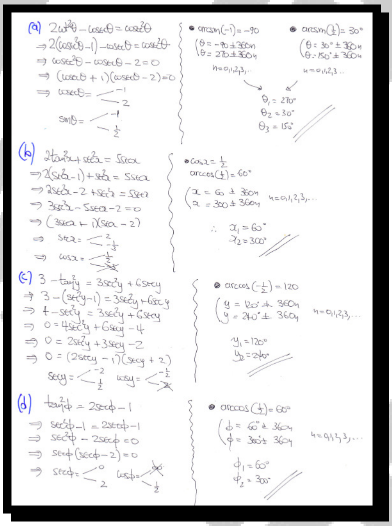
a) $2\cot^2\theta - \operatorname{cosec}\theta = \operatorname{cosec}^2\theta$, $\quad 0 \le \theta < 360°$

b) $2\tan^2 x + \sec^2 x = 5\sec x$, $\quad 0 \le x < 360°$

c) $3 - \tan^2 y = 3\sec^2 y + 6\sec y$, $\quad 0 \le y < 360°$

d) $\tan^2\varphi = 2\sec\varphi - 1$, $\quad 0 \le \varphi < 360°$

$\boxed{\theta = 30°,\ 150°,\ 270°}$, $\boxed{x = 60°,\ 300°}$, $\boxed{y = 120°,\ 240°}$, $\boxed{\varphi = 60°,\ 300°}$



(a) $2\cot^2\theta - \operatorname{cosec}\theta = \operatorname{cosec}^2\theta$
$\Rightarrow 2(\operatorname{cosec}^2\theta - 1) - \operatorname{cosec}\theta = \operatorname{cosec}^2\theta$
$\Rightarrow \operatorname{cosec}^2\theta - \operatorname{cosec}\theta - 2 = 0$
$\Rightarrow (\operatorname{cosec}\theta + 1)(\operatorname{cosec}\theta - 2) = 0$
$\Rightarrow \operatorname{cosec}\theta = -1,\ 2$
$\sin\theta = -1,\ \frac{1}{2}$

• $\arcsin(-1) = -90$
$\theta = -90 \pm 360n$
$\theta = 270 \pm 360n$
$n = 0,1,2,3,\dots$

• $\arcsin(\frac{1}{2}) = 30°$
$\theta = 30° \pm 360n$
$\theta = 150° \pm 360n$
$n = 0,1,2,3\dots$

$\theta_1 = 270°$
$\theta_2 = 30°$
$\theta_3 = 150°$

(b) $2\tan^2 x + \sec^2 x = 5\sec x$
$\Rightarrow 2(\sec^2 x - 1) + \sec^2 x = 5\sec x$
$\Rightarrow 2\sec^2 x - 2 + \sec^2 x = 5\sec x$
$\Rightarrow 3\sec^2 x - 5\sec x - 2 = 0$
$\Rightarrow (3\sec x + 1)(\sec x - 2)$
$\Rightarrow \sec x = 2,\ -\frac{1}{3}$
$\Rightarrow \cos x = \frac{1}{2},\ \cancel{-3}$

• $\cos x = \frac{1}{2}$
$\arccos(\frac{1}{2}) = 60°$
$x = 60 \pm 360n$
$x = 300 \pm 360n$
$n = 0,1,2,3,\dots$

$\therefore x_1 = 60°$
$x_2 = 300°$

(c) $3 - \tan^2 y = 3\sec^2 y + 6\sec y$
$\Rightarrow 3 - (\sec^2 y - 1) = 3\sec^2 y + 6\sec y$
$\Rightarrow 4 - \sec^2 y = 3\sec^2 y + 6\sec y$
$\Rightarrow 0 = 4\sec^2 y + 6\sec y - 4$
$\Rightarrow 0 = 2\sec^2 y + 3\sec y - 2$
$\Rightarrow 0 = (2\sec y - 1)(\sec y + 2)$
$\sec y = -2,\ \frac{1}{2}$ $\quad \cos y = -\frac{1}{2},\ \cancel{2}$

• $\arccos(-\frac{1}{2}) = 120$
$y = 120° \pm 360n$
$y = 240° \pm 360n$
$n = 0,1,2,3,\dots$

$y_1 = 120°$
$y_2 = 240°$

(d) $\tan^2\phi = 2\sec\phi - 1$
$\Rightarrow \sec^2\phi - 1 = 2\sec\phi - 1$
$\Rightarrow \sec^2\phi - 2\sec\phi = 0$
$\Rightarrow \sec\phi(\sec\phi - 2) = 0$
$\Rightarrow \sec\phi = 0,\ 2$ $\quad \cos\phi = \cancel{\infty},\ \frac{1}{2}$

• $\arccos(\frac{1}{2}) = 60°$
$\phi = 60° \pm 360n$
$\phi = 300° \pm 360n$
$n = 0,1,2,3,\dots$

$\phi_1 = 60°$
$\phi_2 = 300°$

Question 8

Solve each of the following equations.

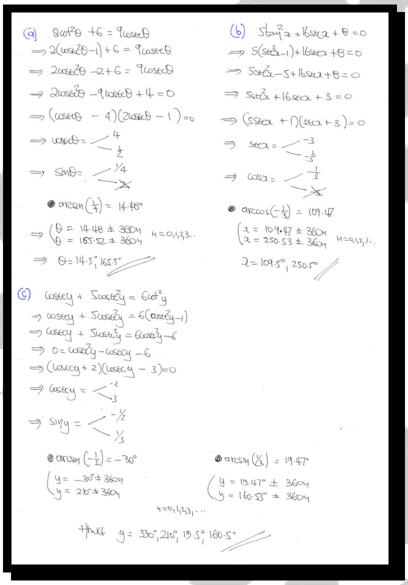
a) $2\cot^2\theta + 6 = 9\operatorname{cosec}\theta$, $\quad 0 \le \theta < 360°$

b) $5\tan^2 x + 16\sec x + 8 = 0$, $\quad 0 \le x < 360°$

c) $\operatorname{cosec} y + 5\operatorname{cosec}^2 y = 6\cot^2 y$, $\quad 0 \le y < 360°$

d) $2\tan^2\varphi = 15\sec\varphi - 9$, $\quad 0 \le \varphi < 360°$

$\boxed{\theta \approx 14.5°,\ 165.5°}$, $\boxed{x \approx 109.5°,\ 250.5°}$, $\boxed{y \approx 19.5°,\ 160.5° \quad y = 210°,\ 330°}$, $\boxed{\varphi \approx 81.8°,\ 278.2°}$



(a) $2\cot^2\theta + 6 = 9\operatorname{cosec}\theta$

$\Rightarrow 2(\operatorname{cosec}^2\theta - 1) + 6 = 9\operatorname{cosec}\theta$

$\Rightarrow 2\operatorname{cosec}^2\theta - 2 + 6 = 9\operatorname{cosec}\theta$

$\Rightarrow 2\operatorname{cosec}^2\theta - 9\operatorname{cosec}\theta + 4 = 0$

$\Rightarrow (\operatorname{cosec}\theta - 4)(2\operatorname{cosec}\theta - 1) = 0$

$\Rightarrow \operatorname{cosec}\theta = 4,\ \tfrac{1}{2}$

$\Rightarrow \sin\theta = 1/4,\ \cancel{2}$

• $\arcsin\left(\tfrac{1}{4}\right) = 14.48°$

$\Rightarrow \begin{cases} \theta = 14.48 \pm 360n \\ \theta = 165.52 \pm 360n \end{cases}$ $n = 0, 1, 2, 3 \ldots$

$\Rightarrow \theta = 14.5°, 165.5°$

(b) $5\tan^2 x + 16\sec x + 8 = 0$

$\Rightarrow 5(\sec^2 x - 1) + 16\sec x + 8 = 0$

$\Rightarrow 5\sec^2 x - 5 + 16\sec x + 8 = 0$

$\Rightarrow 5\sec^2 x + 16\sec x + 3 = 0$

$\Rightarrow (5\sec x + 1)(\sec x + 3) = 0$

$\Rightarrow \sec x = -3,\ -\tfrac{1}{5}$

$\Rightarrow \cos x = -\tfrac{1}{3},\ \cancel{-5}$

• $\arccos\left(-\tfrac{1}{3}\right) = 109.47$

$\begin{cases} x = 109.47 \pm 360n \\ x = 250.53 \pm 360n \end{cases}$ $n = 0, 1, 2, 3 \ldots$

$x = 109.5°, 250.5°$

(c) $\operatorname{cosec} y + 5\operatorname{cosec}^2 y = 6\cot^2 y$

$\Rightarrow \operatorname{cosec} y + 5\operatorname{cosec}^2 y = 6(\operatorname{cosec}^2 y - 1)$

$\Rightarrow \operatorname{cosec} y + 5\operatorname{cosec}^2 y = 6\operatorname{cosec}^2 y - 6$

$\Rightarrow 0 = \operatorname{cosec}^2 y - \operatorname{cosec} y - 6$

$\Rightarrow (\operatorname{cosec} y + 2)(\operatorname{cosec} y - 3) = 0$

$\Rightarrow \operatorname{cosec} y = -2,\ 3$

$\Rightarrow \sin y = -1/2,\ 1/3$

• $\arcsin\left(-\tfrac{1}{2}\right) = -30°$

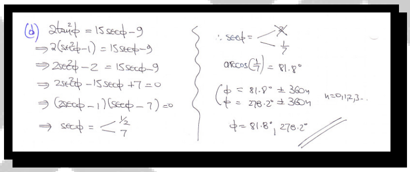
$\begin{cases} y = -30° \pm 360n \\ y = 210° \pm 360n \end{cases}$

• $\arcsin(1/3) = 19.47°$

$\begin{cases} y = 19.47° \pm 360n \\ y = 160.53° \pm 360n \end{cases}$

$n = 0, 1, 2, 3, \ldots$

thus $y = 330°, 210°, 19.5°, 160.5°$



(d) $2\tan^2\phi = 15\sec\phi - 9$

$\Rightarrow 2(\sec^2\phi - 1) = 15\sec\phi - 9$

$\Rightarrow 2\sec^2\phi - 2 = 15\sec\phi - 9$

$\Rightarrow 2\sec^2\phi - 15\sec\phi + 7 = 0$

$\Rightarrow (2\sec\phi - 1)(\sec\phi - 7) = 0$

$\Rightarrow \sec\phi = 1/2,\ 7$

$\therefore \sec\phi = \cancel{2},\ \tfrac{1}{7}$

$\arccos\left(\tfrac{1}{7}\right) = 81.8°$

$\begin{cases} \phi = 81.8° \pm 360n \\ \phi = 278.2° \pm 360n \end{cases}$ $n = 0, 1, 2, 3 \ldots$

$\phi = 81.8°, 278.2°$

Question 9

Solve each of the following equations.

a) $4\cot^2\theta = 1 + \operatorname{cosec}\theta$, $\quad 0 \le \theta < 360°$

b) $4\tan^2 x = 19\sec x + 1$, $\quad 0 \le x < 360°$

c) $4 - \cot^2 y = 8\operatorname{cosec} y + 3\operatorname{cosec}^2 y$, $\quad 0 \le y < 360°$

d) $\sec^2\varphi = 2\tan\varphi$, $\quad 0 \le \varphi < 360°$

$\boxed{\theta \approx 53.1°,\ 126.9° \quad \theta = 270°}$, $\boxed{x \approx 78.5°,\ 281.5°}$, $\boxed{y \approx 203.6°,\ 336.4°}$, $\boxed{\varphi = 45°,\ 225°}$

(a) $4\cot^2\theta = 1 + \operatorname{cosec}\theta$

$\Rightarrow 4(\operatorname{cosec}^2\theta - 1) = 1 + \operatorname{cosec}\theta$

$\Rightarrow 4\operatorname{cosec}^2\theta - 4 = 1 + \operatorname{cosec}\theta$

$\Rightarrow 4\operatorname{cosec}^2\theta - \operatorname{cosec}\theta - 5 = 0$

$\Rightarrow (4\operatorname{cosec}\theta - 5)(\operatorname{cosec}\theta + 1)$

$\Rightarrow \operatorname{cosec}\theta = \frac{5}{4}, -1$

$\Rightarrow \sin\theta = \frac{4}{5}, -1$

- $\arcsin\left(\frac{4}{5}\right) = 53.13$
- $\arcsin(-1) = -90°$

$\begin{cases}\theta = 53.1 \pm 360n \\ \theta = 126.9 \pm 360n\end{cases}$ $\quad \begin{cases}\theta = -90 \pm 360n \\ \theta = 270 \pm 360n\end{cases}$

$n = 0, 1, 2, 3, \ldots$

Hence $\theta_1 = 53.1°$, $\theta_2 = 126.9°$, $\theta_3 = 270°$

(b) $4\tan^2 x = 19\sec x + 1$

$\Rightarrow 4(\sec^2 x - 1) = 19\sec x + 1$

$\Rightarrow 4\sec^2 x - 4 = 19\sec x + 1$

$\Rightarrow 4\sec^2 x - 19\sec x - 5 = 0$

$\Rightarrow (4\sec x + 1)(\sec x - 5) = 0$

$\Rightarrow \sec x = -\frac{1}{4}, 5$

$\Rightarrow \cos x = \cancel{-4}, \frac{1}{5}$

- $\arccos\left(\frac{1}{5}\right) = 78.5°$

$\begin{cases}x = 78.5 \pm 360n \\ x = 281.5 \pm 360n\end{cases}$

$n = 0, 1, 2, 3, \ldots$

$x_1 = 78.5°$, $x_2 = 281.5°$

(c) $4 - \cot^2 y = 8\operatorname{cosec} y + 3\operatorname{cosec}^2 y$

$\Rightarrow 4 - (\operatorname{cosec}^2 y - 1) = 8\operatorname{cosec} y + 3\operatorname{cosec}^2 y$

$\Rightarrow 4 - \operatorname{cosec}^2 y + 1 = 8\operatorname{cosec} y + 3\operatorname{cosec}^2 y$

$\Rightarrow 0 = 4\operatorname{cosec}^2 y + 8\operatorname{cosec} y - 5$

$\Rightarrow 0 = (2\operatorname{cosec} y - 1)(2\operatorname{cosec} y + 5)$

$\operatorname{cosec} y = -\frac{5}{2}, \frac{1}{2}$

$\sin y = -\frac{2}{5}, \cancel{2}$

- $\arcsin\left(-\frac{2}{5}\right) = -23.58°$

$\begin{cases}y = -23.58 \pm 360n \\ y = 203.58 \pm 360n\end{cases}$ $\quad n = 0, 1, 2, 3, \ldots$

$\therefore y_1 = 203.6°$, $y_2 = 336.4°$

(d) $\sec^2\phi = 2\tan\phi$

$\Rightarrow 1 + \tan^2\phi = 2\tan\phi$

$\Rightarrow \tan^2\phi - 2\tan\phi + 1 = 0$

$\Rightarrow (\tan\phi - 1)^2 = 0$

$\tan\phi = 1$

Hence $\arctan(1) = 45°$

$(\phi = 45 \pm 180n \quad n = 0, 1, 2, 3, \ldots$

$\therefore \phi_1 = 45°$, $\phi_2 = 225°$

Question 10

Solve each of the following equations.

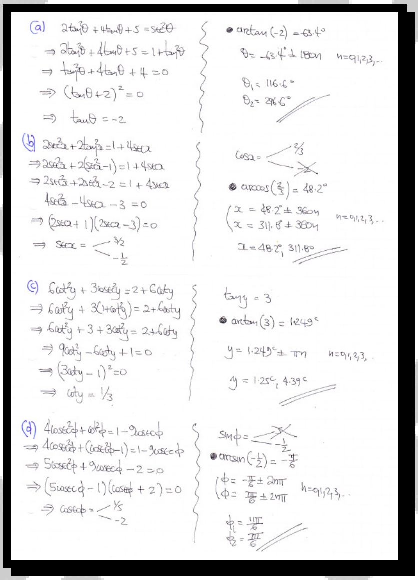
a) $2\tan^2\theta + 4\tan\theta + 5 = \sec^2\theta$, $\quad 0 \le \theta < 360°$

b) $2\sec^2 x + 2\tan^2 x = 1 + 4\sec x$, $\quad 0 \le x < 360°$

c) $6\cot^2 y + 3\text{cosec}^2 y = 2 + 6\cot y$, $\quad 0 \le y < 2\pi$

d) $4\text{cosec}^2\varphi + \cot^2\varphi = 1 - 9\,\text{cosec}\,\varphi$, $\quad 0 \le \varphi < 2\pi$

$\boxed{\theta \approx 116.6°,\ 296.6°}$, $\boxed{x \approx 48.2°,\ 311.8°}$, $\boxed{y \approx 1.25^c,\ 4.39^c}$, $\boxed{\varphi = \frac{7\pi}{6},\ \frac{11\pi}{6}}$



(a) $2\tan^2\theta + 4\tan\theta + 5 = \sec^2\theta$
$\Rightarrow 2\tan^2\theta + 4\tan\theta + 5 = 1 + \tan^2\theta$
$\Rightarrow \tan^2\theta + 4\tan\theta + 4 = 0$
$\Rightarrow (\tan\theta + 2)^2 = 0$
$\Rightarrow \tan\theta = -2$

• $\arctan(-2) = -63.4°$
$\theta = -63.4° \pm 180n \quad n = 0,1,2,3,\ldots$
$\theta_1 = 116.6°$
$\theta_2 = 296.6°$

(b) $2\sec^2 x + 2\tan^2 x = 1 + 4\sec x$
$\Rightarrow 2\sec^2 x + 2(\sec^2 x - 1) = 1 + 4\sec x$
$\Rightarrow 2\sec^2 x + 2\sec^2 x - 2 = 1 + 4\sec x$
$4\sec^2 x - 4\sec x - 3 = 0$
$\Rightarrow (2\sec x + 1)(2\sec x - 3) = 0$
$\Rightarrow \sec x = \frac{3}{2}, -\frac{1}{2}$

$\cos x = \frac{2}{3}, \cancel{-2}$
• $\arccos\left(\frac{2}{3}\right) = 48.2°$
$x = 48.2° \pm 360n$
$x = 311.8° \pm 360n \quad n = 0,1,2,3,\ldots$
$x = 48.2°, 311.8°$

(c) $6\cot^2 y + 3\text{cosec}^2 y = 2 + 6\cot y$
$\Rightarrow 6\cot^2 y + 3(1 + \cot^2 y) = 2 + 6\cot y$
$\Rightarrow 6\cot^2 y + 3 + 3\cot^2 y = 2 + 6\cot y$
$\Rightarrow 9\cot^2 y - 6\cot y + 1 = 0$
$\Rightarrow (3\cot y - 1)^2 = 0$
$\Rightarrow \cot y = \frac{1}{3}$

$\tan y = 3$
• $\arctan(3) = 1.249^c$
$y = 1.249^c \pm \pi n \quad n = 0,1,2,3,\ldots$
$y = 1.25^c, 4.39^c$

(d) $4\text{cosec}^2\phi + \cot^2\phi = 1 - 9\text{cosec}\phi$
$\Rightarrow 4\text{cosec}^2\phi + (\text{cosec}^2\phi - 1) = 1 - 9\text{cosec}\phi$
$\Rightarrow 5\text{cosec}^2\phi + 9\text{cosec}\phi - 2 = 0$
$\Rightarrow (5\text{cosec}\phi - 1)(\text{cosec}\phi + 2) = 0$
$\Rightarrow \text{cosec}\phi = \frac{1}{5}, -2$

$\sin\phi = \cancel{5}, -\frac{1}{2}$
• $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$
$\phi = -\frac{\pi}{6} \pm 2n\pi$
$\phi = \frac{7\pi}{6} \pm 2n\pi \quad n = 0,1,2,3,\ldots$
$\phi_1 = \frac{11\pi}{6}$
$\phi_2 = \frac{7\pi}{6}$

Question 11

Solve each of the following equations.

a) $10\sec^2\theta = 11\tan\theta + 16, \quad 0 \le \theta < 360°$

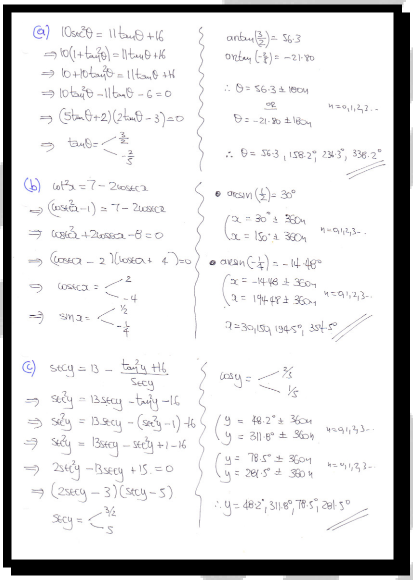
b) $\cot^2 x = 7 - 2\operatorname{cosec} x, \quad 0 \le x < 360°$

c) $\sec y = 13 - \dfrac{\tan^2 y + 16}{\sec y}, \quad 0 \le y < 360°$

d) $(\operatorname{cosec}\varphi + 1)^2 + 2(\cot\varphi - 1)^2 = 9 - 4\cot\varphi, \quad 0 \le \varphi < 360°$

$\boxed{\theta \approx 56.3°,\ 158.2°,\ 236.3°,\ 338.2°}$, $\boxed{x = 30°,\ 150° \quad x \approx 194.5°,\ 344.5°}$,

$\boxed{y \approx 48.2°,\ 78.5°,\ 281.5°,\ 311.8°}$, $\boxed{\varphi \approx 48.6°,\ 131.4° \quad \varphi = 210°,\ 330°}$



(a) $10\sec^2\theta = 11\tan\theta + 16$

$\Rightarrow 10(1+\tan^2\theta) = 11\tan\theta + 16$

$\Rightarrow 10 + 10\tan^2\theta = 11\tan\theta + 16$

$\Rightarrow 10\tan^2\theta - 11\tan\theta - 6 = 0$

$\Rightarrow (5\tan\theta + 2)(2\tan\theta - 3) = 0$

$\Rightarrow \tan\theta = \frac{3}{2}$ or $-\frac{2}{5}$

$\arctan\left(\frac{3}{2}\right) = 56.3$

$\arctan\left(-\frac{2}{5}\right) = -21.80$

$\therefore \theta = 56.3 \pm 180n$

OR

$\theta = -21.80 \pm 180n \quad n = 0,1,2,3\ldots$

$\therefore \theta = 56.3, 158.2°, 236.3°, 338.2°$

(b) $\cot^2 x = 7 - 2\operatorname{cosec} x$

$\Rightarrow (\operatorname{cosec}^2 x - 1) = 7 - 2\operatorname{cosec} x$

$\Rightarrow \operatorname{cosec}^2 x + 2\operatorname{cosec} x - 8 = 0$

$\Rightarrow (\operatorname{cosec} x - 2)(\operatorname{cosec} x + 4) = 0$

$\Rightarrow \operatorname{cosec} x = 2$ or -4

$\Rightarrow \sin x = \frac{1}{2}$ or $-\frac{1}{4}$

• $\arcsin\left(\frac{1}{2}\right) = 30°$

$x = 30° \pm 360n$, $x = 150° \pm 360n \quad n = 0,1,2,3\ldots$

• $\arcsin\left(-\frac{1}{4}\right) = -14.48°$

$x = -14.48 \pm 360n$, $x = 194.48 \pm 360n \quad n = 0,1,2,3\ldots$

$x = 30, 150, 194.5°, 354.5°$

(c) $\sec y = 13 - \frac{\tan^2 y + 16}{\sec y}$

$\Rightarrow \sec^2 y = 13\sec y - \tan^2 y - 16$

$\Rightarrow \sec^2 y = 13\sec y - (\sec^2 y - 1) - 16$

$\Rightarrow \sec^2 y = 13\sec y - \sec^2 y + 1 - 16$

$\Rightarrow 2\sec^2 y - 13\sec y + 15 = 0$

$\Rightarrow (2\sec y - 3)(\sec y - 5)$

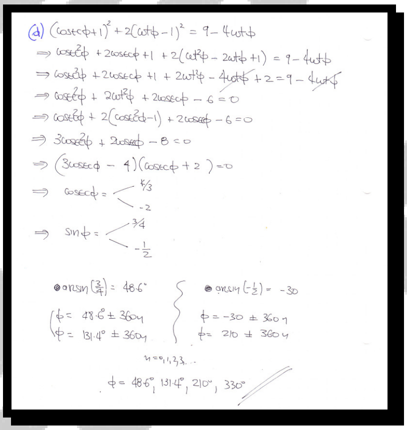
$\sec y = 3/2$ or 5

$\cos y = 2/3$ or $1/5$

$y = 48.2° \pm 360n$, $y = 311.8° \pm 360n \quad n = 0,1,2,3\ldots$

$y = 78.5° \pm 360n$, $y = 281.5° \pm 360n \quad n = 0,1,2,3\ldots$

$\therefore y = 48.2°, 311.8°, 78.5°, 281.5°$



(d) $(\operatorname{cosec}\phi + 1)^2 + 2(\cot\phi - 1)^2 = 9 - 4\cot\phi$

$\Rightarrow \operatorname{cosec}^2\phi + 2\operatorname{cosec}\phi + 1 + 2(\cot^2\phi - 2\cot\phi + 1) = 9 - 4\cot\phi$

$\Rightarrow \operatorname{cosec}^2\phi + 2\operatorname{cosec}\phi + 1 + 2\cot^2\phi - \cancel{4\cot\phi} + 2 = 9 - \cancel{4\cot\phi}$

$\Rightarrow \operatorname{cosec}^2\phi + 2\cot^2\phi + 2\operatorname{cosec}\phi - 6 = 0$

$\Rightarrow \operatorname{cosec}^2\phi + 2(\operatorname{cosec}^2\phi - 1) + 2\operatorname{cosec}\phi - 6 = 0$

$\Rightarrow 3\operatorname{cosec}^2\phi + 2\operatorname{cosec}\phi - 8 = 0$

$\Rightarrow (3\operatorname{cosec}\phi - 4)(\operatorname{cosec}\phi + 2) = 0$

$\Rightarrow \operatorname{cosec}\phi = 4/3$ or -2

$\Rightarrow \sin\phi = 3/4$ or $-\frac{1}{2}$

• $\arcsin\left(\frac{3}{4}\right) = 48.6°$

$\phi = 48.6° \pm 360n$, $\phi = 131.4° \pm 360n$

• $\arcsin\left(-\frac{1}{2}\right) = -30$

$\phi = -30 \pm 360n$, $\phi = 210 \pm 360n$

$n = 0,1,2,3\ldots$

$\phi = 48.6°, 131.4°, 210°, 330°$

Question 12

Solve each of the following equations.

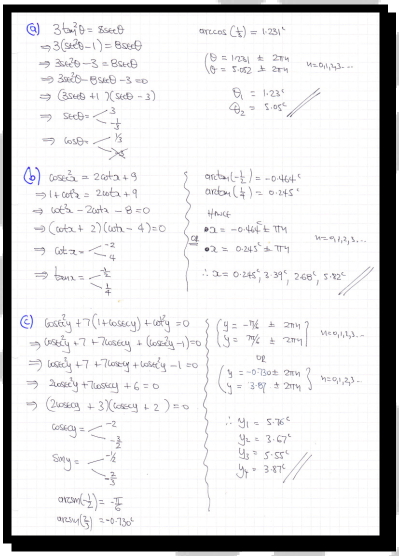
a) $3\tan^2\theta = 8\sec\theta$, $\quad 0 \le \theta < 2\pi$

b) $\text{cosec}^2 x = 2\cot x + 9$, $\quad 0 \le x < 2\pi$

c) $\text{cosec}^2 y + 7(1 + \text{cosec}\, y) + \cot^2 y = 0$, $\quad 0 \le y < 2\pi$

d) $6\tan\varphi = \dfrac{2 - 3\sec^2\varphi}{\tan\varphi - 1}$, $\quad 0 \le \varphi < 2\pi$

$\boxed{\theta \approx 1.23^{\text{c}},\ 5.05^{\text{c}}}$, $\boxed{x \approx 0.245^{\text{c}},\ 2.68^{\text{c}},\ 3.39^{\text{c}},\ 5.82^{\text{c}}}$, $\boxed{y \approx 3.67^{\text{c}},\ 3.87^{\text{c}},\ 5.55^{\text{c}},\ 5.76^{\text{c}}}$, $\boxed{\varphi \approx 0.322^{\text{c}},\ 3.46^{\text{c}}}$



(a) $3\tan^2\theta = 8\sec\theta$

$\Rightarrow 3(\sec^2\theta - 1) = 8\sec\theta$

$\Rightarrow 3\sec^2\theta - 3 = 8\sec\theta$

$\Rightarrow 3\sec^2\theta - 8\sec\theta - 3 = 0$

$\Rightarrow (3\sec\theta + 1)(\sec\theta - 3)$

$\Rightarrow \sec\theta = 3,\ -\frac{1}{3}$

$\Rightarrow \cos\theta = \frac{1}{3},\ \cancel{-3}$

$\arccos\left(\frac{1}{3}\right) = 1.231^{c}$

$\theta = 1.231 \pm 2\pi n$, $\theta = 5.052 \pm 2\pi n$, $n = 0,1,2,3,\ldots$

$\theta_1 = 1.23^{c}$, $\theta_2 = 5.05^{c}$

(b) $\text{cosec}^2 x = 2\cot x + 9$

$\Rightarrow 1 + \cot^2 x = 2\cot x + 9$

$\Rightarrow \cot^2 x - 2\cot x - 8 = 0$

$\Rightarrow (\cot x + 2)(\cot x - 4) = 0$

$\Rightarrow \cot x = -2,\ 4$

$\Rightarrow \tan x = -\frac{1}{2},\ \frac{1}{4}$

$\arctan\left(-\frac{1}{2}\right) = -0.464^{c}$

$\arctan\left(\frac{1}{4}\right) = 0.245^{c}$

Hence

- $x = -0.464^{c} \pm \pi n$

or

- $x = 0.245^{c} \pm \pi n$ $\quad n = 0,1,2,3,\ldots$

$\therefore x = 0.245^{c}, 3.39^{c}, 2.68^{c}, 5.82^{c}$

(c) $\text{cosec}^2 y + 7(1 + \text{cosec}\, y) + \cot^2 y = 0$

$\Rightarrow \text{cosec}^2 y + 7 + 7\text{cosec}\, y + (\text{cosec}^2 y - 1) = 0$

$\Rightarrow \text{cosec}^2 y + 7 + 7\text{cosec}\, y + \text{cosec}^2 y - 1 = 0$

$\Rightarrow 2\text{cosec}^2 y + 7\text{cosec}\, y + 6 = 0$

$\Rightarrow (2\text{cosec}\, y + 3)(\text{cosec}\, y + 2) = 0$

$\text{cosec}\, y = -2,\ -\frac{3}{2}$

$\sin y = -\frac{1}{2},\ -\frac{2}{3}$

$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

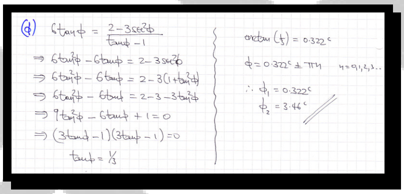
$\arcsin\left(\frac{2}{3}\right) = -0.730^{c}$

$y = -\pi/6 \pm 2\pi n$, $y = 7\pi/6 \pm 2\pi n$ $\quad n = 0,1,2,3,\ldots$

or

$y = -0.730 \pm 2\pi n$, $y = 3.87 \pm 2\pi n$ $\quad n = 0,1,2,3,\ldots$

$\therefore y_1 = 5.76^{c}$, $y_2 = 3.67^{c}$, $y_3 = 5.55^{c}$, $y_4 = 3.87^{c}$



(d) $6\tan\phi = \dfrac{2 - 3\sec^2\phi}{\tan\phi - 1}$

$\Rightarrow 6\tan^2\phi - 6\tan\phi = 2 - 3\sec^2\phi$

$\Rightarrow 6\tan^2\phi - 6\tan\phi = 2 - 3(1 + \tan^2\phi)$

$\Rightarrow 6\tan^2\phi - 6\tan\phi = 2 - 3 - 3\tan^2\phi$

$\Rightarrow 9\tan^2\phi - 6\tan\phi + 1 = 0$

$\Rightarrow (3\tan\phi - 1)(3\tan\phi - 1) = 0$

$\tan\phi = \frac{1}{3}$

$\arctan\left(\frac{1}{3}\right) = 0.322^{c}$

$\phi = 0.322^{c} \pm \pi n$ $\quad n = 0,1,2,3,\ldots$

$\therefore \phi_1 = 0.322^{c}$, $\phi_2 = 3.46^{c}$

Question 13

Solve each of the following equations.

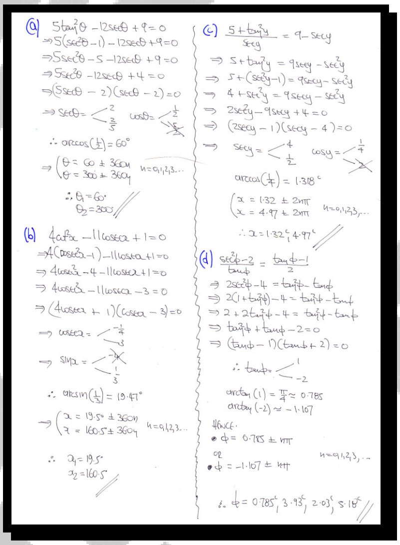
a) $5\tan^2\theta - 12\sec\theta + 9 = 0$, $\quad 0 \le \theta < 360°$

b) $4\cot^2 x - 11\operatorname{cosec} x + 1 = 0$, $\quad 0 \le x < 360°$

c) $\dfrac{5 + \tan^2 y}{\sec y} = 9 - \sec y$, $\quad 0 \le y < 2\pi$

d) $\dfrac{\sec^2\varphi - 2}{\tan\varphi} = \dfrac{\tan\varphi - 1}{2}$, $\quad 0 \le \varphi < 2\pi$

$\boxed{\theta = 60°,\ 300°}$, $\boxed{x \approx 19.5°,\ 160.5°}$, $\boxed{y \approx 1.32^{c},\ 4.97^{c}}$,
$\boxed{\varphi \approx 0.785^{c},\ 2.03^{c},\ 3.93^{c},\ 5.18^{c}}$



(a) $5\tan^2\theta - 12\sec\theta + 9 = 0$

$\Rightarrow 5(\sec^2\theta - 1) - 12\sec\theta + 9 = 0$

$\Rightarrow 5\sec^2\theta - 5 - 12\sec\theta + 9 = 0$

$\Rightarrow 5\sec^2\theta - 12\sec\theta + 4 = 0$

$\Rightarrow (5\sec\theta - 2)(\sec\theta - 2) = 0$

$\Rightarrow \sec\theta = 2,\ \frac{2}{5} \qquad \cos\theta = \frac{1}{2},\ \cancel{\frac{5}{2}}$

$\therefore \arccos\left(\frac{1}{2}\right) = 60°$

$\Rightarrow \begin{cases} \theta = 60 \pm 360n \\ \theta = 300 \pm 360n \end{cases} \quad n = 0,1,2,3\ldots$

$\therefore \theta_1 = 60°$, $\theta_2 = 300°$

(b) $4\cot^2 x - 11\operatorname{cosec} x + 1 = 0$

$\Rightarrow 4(\operatorname{cosec}^2 x - 1) - 11\operatorname{cosec} x + 1 = 0$

$\Rightarrow 4\operatorname{cosec}^2 x - 4 - 11\operatorname{cosec} x + 1 = 0$

$\Rightarrow 4\operatorname{cosec}^2 x - 11\operatorname{cosec} x - 3 = 0$

$\Rightarrow (4\operatorname{cosec} x + 1)(\operatorname{cosec} x - 3) = 0$

$\Rightarrow \operatorname{cosec} x = -\frac{1}{4},\ 3$

$\Rightarrow \sin x = \cancel{-4},\ \frac{1}{3}$

$\therefore \arcsin\left(\frac{1}{3}\right) = 19.47°$

$\Rightarrow \begin{cases} x = 19.5° \pm 360n \\ x = 160.5° \pm 360n \end{cases} \quad n = 0,1,2,3\ldots$

$\therefore x_1 = 19.5°$, $x_2 = 160.5°$

(c) $\dfrac{5 + \tan^2 y}{\sec y} = 9 - \sec y$

$\Rightarrow 5 + \tan^2 y = 9\sec y - \sec^2 y$

$\Rightarrow 5 + (\sec^2 y - 1) = 9\sec y - \sec^2 y$

$\Rightarrow 4 + \sec^2 y = 9\sec y - \sec^2 y$

$\Rightarrow 2\sec^2 y - 9\sec y + 4 = 0$

$\Rightarrow (2\sec y - 1)(\sec y - 4) = 0$

$\Rightarrow \sec y = 4,\ \frac{1}{2} \qquad \cos y = \frac{1}{4},\ \cancel{2}$

$\arccos\left(\frac{1}{4}\right) = 1.318^{c}$

$\begin{cases} x = 1.32 \pm 2n\pi \\ x = 4.97 \pm 2n\pi \end{cases} \quad n = 0,1,2,3,\ldots$

$\therefore x = 1.32^{c},\ 4.97^{c}$

(d) $\dfrac{\sec^2\phi - 2}{\tan\phi} = \dfrac{\tan\phi - 1}{2}$

$\Rightarrow 2\sec^2\phi - 4 = \tan^2\phi - \tan\phi$

$\Rightarrow 2(1 + \tan^2\phi) - 4 = \tan^2\phi - \tan\phi$

$\Rightarrow 2 + 2\tan^2\phi - 4 = \tan^2\phi - \tan\phi$

$\Rightarrow \tan^2\phi + \tan\phi - 2 = 0$

$\Rightarrow (\tan\phi - 1)(\tan\phi + 2) = 0$

$\therefore \tan\phi = 1,\ -2$

$\arctan(1) = \frac{\pi}{4} \approx 0.785$

$\arctan(-2) \approx -1.107$

Hence

- $\phi = 0.785 \pm n\pi$

or

- $\phi = -1.107 \pm n\pi \quad n = 0,1,2,3,\ldots$

$\therefore \phi = 0.785^{c},\ 3.93^{c},\ 2.03^{c},\ 5.18^{c}$

Question 14

Solve each of the following equations.

a) $\sec\theta = \dfrac{1-\tan^2\theta}{4\sec\theta - 9}, \qquad 0 \le \theta < 2\pi$

b) $\dfrac{\sec^2 x + 8}{4 - \tan x} = 3\tan x, \qquad 0 \le x < 2\pi$

c) $\dfrac{1 - 2\operatorname{cosec}^2 y}{2\cot y} - 2 = \cot y, \qquad 0 \le y < 2\pi$

d) $\dfrac{2\cot^2\varphi + 5}{\operatorname{cosec}\varphi} + 2\operatorname{cosec}\varphi = 13, \quad 0 \le \varphi < 2\pi$

$\boxed{\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}}$, $\boxed{x \approx 0.983^{c},\ 4.12^{c}}$, $\boxed{y \approx 2.03^{c},\ 5.18^{c}}$, $\boxed{\varphi \approx 0.340^{c},\ 2.80^{c}}$

(a) $\sec\theta = \dfrac{1-\tan^2\theta}{4\sec\theta-9}$

$\Rightarrow 4\sec^2\theta - 9\sec\theta = 1 - \tan^2\theta$

$\Rightarrow 4\sec^2\theta - 9\sec\theta = 1 - (\sec^2\theta - 1)$

$\Rightarrow 4\sec^2\theta - 9\sec\theta = 1 + 1 - \sec^2\theta$

$\Rightarrow 5\sec^2\theta - 9\sec\theta - 2 = 0$

$\Rightarrow (5\sec\theta + 1)(\sec\theta - 2)$

$\Rightarrow \sec\theta = \begin{cases} 2 \\ -\frac{1}{5} \end{cases}$ i.e. $\cos\theta = \begin{cases} \frac{1}{2} \\ \cancel{-5} \end{cases}$

$\cos\theta = \frac{1}{2}$, $\arccos(\frac{1}{2}) = \frac{\pi}{3}$

$\begin{cases} \theta = \frac{\pi}{3} \pm 2n\pi \\ \theta = \frac{5\pi}{3} \pm 2n\pi \end{cases}$ $n = 0, 1, 2, 3, \ldots$

$\theta_1 = \frac{\pi}{3}$

$\theta_2 = \frac{5\pi}{3}$ //

(b) $\dfrac{\sec^2 x + 8}{4 - \tan x} = 3\tan x$

$\Rightarrow \sec^2 x + 8 = 12\tan x - 3\tan^2 x$

$\Rightarrow (1 + \tan^2 x) + 8 = 12\tan x - 3\tan^2 x$

$\Rightarrow 4\tan^2 x - 12\tan x + 9 = 0$

$\Rightarrow (2\tan x - 3)^2 = 0$

$\tan x = 3/2$

$\arctan(3/2) = 0.983^{c}$

$\Rightarrow \{ x = 0.983^{c} \pm 2n\pi$ $n = 0, 1, 2, 3, \ldots$

$\therefore x_1 = 0.983^{c}$

$x_2 = 4.124^{c}$

(c) $\dfrac{1 - 2\operatorname{cosec}^2 y}{2\cot y} - 2 = \cot y$

$\Rightarrow \dfrac{1 - 2\operatorname{cosec}^2 y}{2\cot y} = 2 + \cot y$

$\Rightarrow 1 - 2\operatorname{cosec}^2 y = 4\cot y + 2\cot^2 y$

$\Rightarrow 1 - 2(1 + \cot^2 y) = 4\cot y + 2\cot^2 y$

$\Rightarrow 1 - 2 - 2\cot^2 y = 4\cot y + 2\cot^2 y$

$\Rightarrow 0 = 4\cot^2 y + 4\cot y + 1$

$\Rightarrow 0 = (2\cot y + 1)^2$

$\cot y = -\frac{1}{2}$

$\tan y = -2$

$\arctan(-2) = -1.107$

$\{ y = -1.107 \pm 2n\pi$ $n = 0, 1, 2, 3, \ldots$

$y_1 = 2.03^{c}$

$y_2 = 5.18^{c}$ //

(d) $\dfrac{2\cot^2\phi + 5}{\operatorname{cosec}\phi} + 2\operatorname{cosec}\phi = 13$

$\Rightarrow 2\cot^2\phi + 5 + 2\operatorname{cosec}^2\phi = 13\operatorname{cosec}\phi$

$\Rightarrow 2(\operatorname{cosec}^2\phi - 1) + 5 + 2\operatorname{cosec}^2\phi - 13\operatorname{cosec}\phi = 0$

$\Rightarrow 2\operatorname{cosec}^2\phi + 3 + 2\operatorname{cosec}^2\phi - 13\operatorname{cosec}\phi = 0$

$\Rightarrow 4\operatorname{cosec}^2\phi - 13\operatorname{cosec}\phi + 3 = 0$

$\Rightarrow (4\operatorname{cosec}\phi - 1)(\operatorname{cosec}\phi - 3) = 0$

$\operatorname{cosec}\phi = \begin{cases} 3 \\ \frac{1}{4} \end{cases}$ i.e. $\sin\phi = \begin{cases} \frac{1}{3} \\ \cancel{4} \end{cases}$

$\arcsin(\frac{1}{3}) = 0.340^{c}$

$\begin{cases} \phi = 0.340^{c} \pm 2n\pi \\ \phi = 2.802^{c} \pm 2n\pi \end{cases}$ $n = 0, 1, 2, 3, \ldots$

$\phi_1 = 0.340^{c}$

$\phi_2 = 2.81^{c}$ //