TRIGONOMETRY

R-TRANSFORMATIONS

Question 1

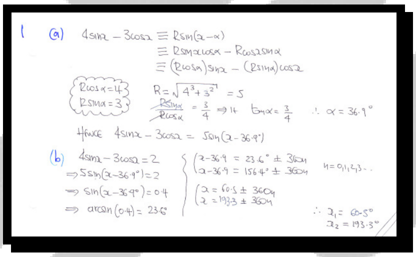
$$f(x) \equiv 4\sin x - 3\cos x .$$

a) Express $f(x)$ in the form $R\sin(x-\alpha)$, $R>0$, $0<\alpha<90°$.

b) Hence, solve the trigonometric equation

$$4\sin x - 3\cos x = 2, \quad 0 < x < 360° .$$

$\boxed{f(x) \equiv 4\sin x - 3\cos x \cong 5\sin(x-36.9°)}$, $\boxed{x \approx 60.5°,\ 193.3°}$



1 (a) $4\sin x - 3\cos x \equiv R\sin(x-\alpha)$
$\equiv R\sin x\cos\alpha - R\cos x\sin\alpha$
$\equiv (R\cos\alpha)\sin x - (R\sin\alpha)\cos x$

$R\cos\alpha = 4$
$R\sin\alpha = 3$

$R = \sqrt{4^2+3^2} = 5$

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{4} \Rightarrow \tan\alpha = \frac{3}{4}$ $\therefore \alpha = 36.9°$

Hence $4\sin x - 3\cos x = 5\sin(x-36.9°)$

(b) $4\sin x - 3\cos x = 2$
$\Rightarrow 5\sin(x-36.9°) = 2$
$\Rightarrow \sin(x-36.9°) = 0.4$
$\Rightarrow \arcsin(0.4) = 23.6°$

$x - 36.9 = 23.6° \pm 360n$
$x - 36.9 = 156.4° \pm 360n$ $n = 0,1,2,3\ldots$

$x = 60.5 \pm 360n$
$x = 193.3 \pm 360n$

$\therefore x_1 = 60.5°$
$x_2 = 193.3°$

Question 2

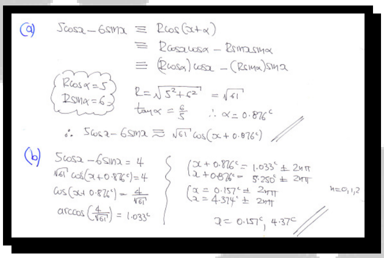
$$f(x) \equiv 5\cos x - 6\sin x, \quad x \in \mathbb{R} .$$

a) Express $f(x)$ in the form $R\cos(x+\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

b) Hence, solve the trigonometric equation

$$5\cos x - 6\sin x = 4, \quad 0 < x < 2\pi .$$

$\boxed{f(x) \equiv 5\cos x - 6\sin x \cong \sqrt{61}\cos\left(x+0.876^{c}\right)}$, $\boxed{x \approx 0.157^{c},\ 4.37^{c}}$



(a) $5\cos x - 6\sin x \equiv R\cos(x+\alpha)$
$\equiv R\cos x\cos\alpha - R\sin x\sin\alpha$
$\equiv (R\cos\alpha)\cos x - (R\sin\alpha)\sin x$

$R\cos\alpha = 5$
$R\sin\alpha = 6$

$R = \sqrt{5^2+6^2} = \sqrt{61}$

$\tan\alpha = \frac{6}{5}$ $\therefore \alpha = 0.876^{c}$

$\therefore 5\cos x - 6\sin x \equiv \sqrt{61}\cos(x+0.876^{c})$

(b) $5\cos x - 6\sin x = 4$
$\sqrt{61}\cos(x+0.876^{c}) = 4$
$\cos(x+0.876^{c}) = \frac{4}{\sqrt{61}}$
$\arccos\left(\frac{4}{\sqrt{61}}\right) = 1.033^{c}$

$x + 0.876^{c} = 1.033^{c} \pm 2n\pi$
$x + 0.876^{c} = 5.250^{c} \pm 2n\pi$ $n = 0,1,2$

$x = 0.157^{c} \pm 2n\pi$
$x = 4.374^{c} \pm 2n\pi$

$x = 0.157^{c}, 4.37^{c}$

Question 3

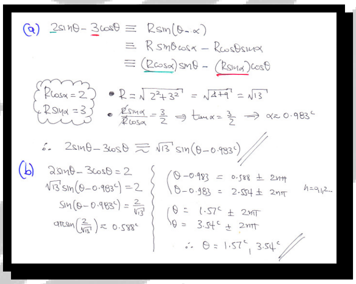
$$f(\theta) \equiv 2\sin\theta - 3\cos\theta\,,\ \theta \in \mathbb{R}\,.$$

a) Express $f(\theta)$ in the form $R\sin(\theta-\alpha)$, $R>0$, $0<\alpha<\dfrac{\pi}{2}$.

b) Hence, solve the trigonometric equation

$$2\sin\theta - 3\cos\theta = 2\,,\ 0<\theta<2\pi\,.$$

$\boxed{f(\theta) \equiv 2\sin\theta - 3\cos\theta \cong \sqrt{13}\sin\left(\theta - 0.983^{c}\right)}$, $\boxed{\theta \approx 1.57^{c},\ 3.54^{c}}$



(a) $2\sin\theta - 3\cos\theta \equiv R\sin(\theta-\alpha)$
$\equiv R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$
$\equiv (R\cos\alpha)\sin\theta - (R\sin\alpha)\cos\theta$

$R\cos\alpha = 2$, $R\sin\alpha = 3$

• $R = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$

• $\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{2} \Rightarrow \tan\alpha = \frac{3}{2} \Rightarrow \alpha \approx 0.983^{c}$

$\therefore\ 2\sin\theta - 3\cos\theta \equiv \sqrt{13}\sin(\theta - 0.983^{c})$

(b) $2\sin\theta - 3\cos\theta = 2$
$\sqrt{13}\sin(\theta-0.983^{c}) = 2$
$\sin(\theta - 0.983^{c}) = \frac{2}{\sqrt{13}}$
$\arcsin\left(\frac{2}{\sqrt{13}}\right) \approx 0.588^{c}$

$\theta - 0.983 = 0.588 \pm 2n\pi$
$\theta - 0.983 = 2.554 \pm 2n\pi$ $\quad n = 0,1,2\ldots$

$\theta = 1.57^{c} \pm 2n\pi$
$\theta = 3.54^{c} \pm 2n\pi$

$\therefore\ \theta = 1.57^{c},\ 3.54^{c}$

Question 4

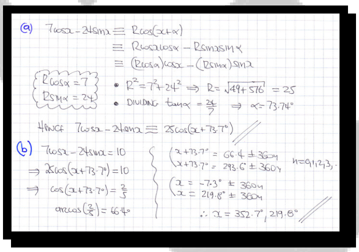
$$f(x) \equiv 7\cos x - 24\sin x\,,\ x \in \mathbb{R}\,.$$

a) Express $f(x)$ in the form $R\cos(x+\alpha)$, $R>0$, $0<\alpha<90°$.

b) Hence, solve the trigonometric equation

$$7\cos x - 24\sin x = 10\,,\ 0<x<360°\,.$$

$\boxed{f(x) \equiv 7\cos x - 24\sin x \cong 25\cos(x+73.7°)}$, $\boxed{x \approx 219.9°,\ 352.7°}$



(a) $7\cos x - 24\sin x \equiv R\cos(x+\alpha)$
$\equiv R\cos x\cos\alpha - R\sin x\sin\alpha$
$\equiv (R\cos\alpha)\cos x - (R\sin\alpha)\sin x$

$R\cos\alpha = 7$, $R\sin\alpha = 24$

• $R^2 = 7^2+24^2 \Rightarrow R = \sqrt{49+576} = 25$

• DIVIDING $\tan\alpha = \frac{24}{7} \Rightarrow \alpha \approx 73.74°$

HENCE $7\cos x - 24\sin x \equiv 25\cos(x+73.7°)$

(b) $7\cos x - 24\sin x = 10$
$\Rightarrow 25\cos(x+73.7°) = 10$
$\Rightarrow \cos(x+73.7°) = \frac{2}{5}$
$\arccos\left(\frac{2}{5}\right) = 66.4°$

$x + 73.7° = 66.4 \pm 360n$
$x + 73.7° = 293.6° \pm 360n$ $\quad n = 0,1,2,3,\ldots$

$x = -7.3° \pm 360n$
$x = 219.8° \pm 360n$

$\therefore\ x = 352.7°,\ 219.8°$

Question 5

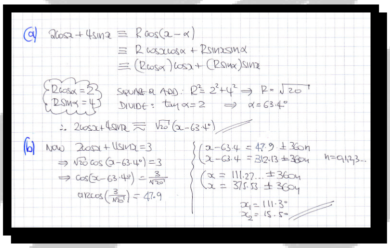
$$f(x) \equiv 2\cos x + 4\sin x, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\cos(x-\alpha)$, $R>0$, $0<\alpha<90°$.

b) Hence, solve the trigonometric equation

$$2\cos x + 4\sin x = 3, \quad 0 < x < 360°.$$

$\boxed{f(x) \equiv 2\cos x + 4\sin x \cong \sqrt{20}\cos(x-63.4°)}$, $\boxed{x \approx 15.6°, \ 111.3°}$



(a) $2\cos x + 4\sin x \equiv R\cos(x-\alpha)$

$\equiv R\cos x\cos\alpha + R\sin x\sin\alpha$

$\equiv (R\cos\alpha)\cos x + (R\sin\alpha)\sin x$

$R\cos\alpha = 2$, $R\sin\alpha = 4$

SQUARE & ADD: $R^2 = 2^2 + 4^2 \Rightarrow R = \sqrt{20}$

DIVIDE: $\tan\alpha = 2 \Rightarrow \alpha = 63.4°$

$\therefore 2\cos x + 4\sin x \cong \sqrt{20}(x - 63.4°)$

(b) NOW $2\cos x + 4\sin x = 3$

$\Rightarrow \sqrt{20}\cos(x-63.4°) = 3$

$\Rightarrow \cos(x-63.4°) = \frac{3}{\sqrt{20}}$

$\arccos\left(\frac{3}{\sqrt{20}}\right) = 47.9$

$x - 63.4 = 47.9 \pm 360n$

$x - 63.4 = 312.13 \pm 360n$ $n = 0,1,2,3\ldots$

$x = 111.27\ldots \pm 360n$

$x = 375.53 \pm 360n$

$x_1 = 111.3°$

$x_2 = 15.5°$

Question 6

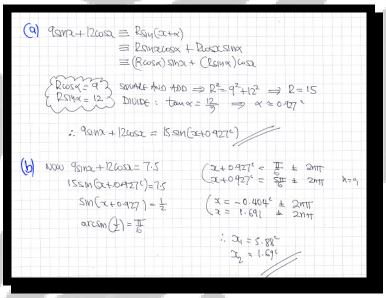
$$f(x) \equiv 9\sin x + 12\cos x, \quad x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\sin(x+\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

b) Hence, solve the trigonometric equation

$$9\sin x + 12\cos x = 7.5, \quad 0 < x < 2\pi.$$

$\boxed{f(x) \equiv 9\sin x + 12\cos x \cong 15\sin\left(x+0.927^{c}\right)}$, $\boxed{x \approx 1.69^{c},\ 5.88^{c}}$



(a) $9\sin x + 12\cos x \equiv R\sin(x+\alpha)$

$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

$\equiv (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$

$R\cos\alpha = 9$, $R\sin\alpha = 12$ SQUARE AND ADD $\Rightarrow R^2 = 9^2+12^2 \Rightarrow R = 15$

DIVIDE : $\tan\alpha = \frac{12}{9} \Rightarrow \alpha \approx 0.927^{c}$

$\therefore 9\sin x + 12\cos x = 15\sin(x+0.927^{c})$

(b) NOW $9\sin x + 12\cos x = 7.5$

$15\sin(x+0.927^{c}) = 7.5$

$\sin(x+0.927) = \frac{1}{2}$

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$x + 0.927^{c} = \frac{\pi}{6} \pm 2n\pi$

$x + 0.927^{c} = \frac{5\pi}{6} \pm 2n\pi$ $\quad n = 0, 1$

$x = -0.404^{c} \pm 2n\pi$

$x = 1.691 \pm 2n\pi$

$\therefore x_1 = 5.88^{c}$

$x_2 = 1.69^{c}$

Question 7

$$f(\theta) \equiv 9\cos 2\theta + 3\sin 2\theta, \ \theta \in \mathbb{R}.$$

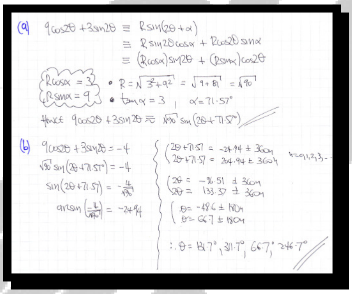
a) Express $f(\theta)$ in the form $R\sin(2\theta+\alpha)$, $R>0$, $0<\alpha<90°$.

b) Hence, solve the trigonometric equation

$$9\cos 2\theta + 3\sin 2\theta = -4, \quad 0<\theta<360°.$$

$\boxed{f(\theta) \equiv 9\cos 2\theta + 3\sin 2\theta \cong \sqrt{90}\sin(2\theta+71.6°)}$,

$\boxed{\theta \approx 66.7°, \ 131.7°, \ 246.7°, \ 311.7°}$



(a) $9\cos 2\theta + 3\sin 2\theta \equiv R\sin(2\theta+\alpha)$

$\equiv R\sin 2\theta\cos\alpha + R\cos 2\theta\sin\alpha$

$\equiv (R\cos\alpha)\sin 2\theta + (R\sin\alpha)\cos 2\theta$

$R\cos\alpha = 3$, $R\sin\alpha = 9$

• $R = \sqrt{3^2+9^2} = \sqrt{9+81} = \sqrt{90}$

• $\tan\alpha = 3$, $\alpha = 71.57°$

Hence $9\cos 2\theta + 3\sin 2\theta \approx \sqrt{90}\sin(2\theta + 71.57°)$

(b) $9\cos 2\theta + 3\sin 2\theta = -4$

$\sqrt{90}\sin(2\theta+71.57°) = -4$

$\sin(2\theta+71.57) = -\frac{4}{\sqrt{90}}$

$\arcsin\left(\frac{-4}{\sqrt{90}}\right) = -24.94$

$2\theta+71.57 = -24.94 \pm 360n$

$2\theta+71.57 = 204.94 \pm 360n$ $\quad n=0,1,2,3,\ldots$

$2\theta = -96.51 \pm 360n$

$2\theta = 133.37 \pm 360n$

$\theta = -48.6 \pm 180n$

$\theta = 66.7 \pm 180n$

$\therefore \theta = 131.7°, 311.7°, 66.7°, 246.7°$

Question 8

$$f(\theta) \equiv 9\sin\theta + 12\cos\theta,\ \theta \in \mathbb{R}.$$

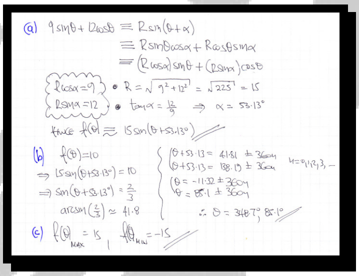
a) Express $f(\theta)$ the form $R\sin(\theta+\alpha)$, $R>0$, $0<\alpha<90°$.

b) Hence, solve the trigonometric equation

$$f(\theta)=10,\ \ 0<\theta<360°.$$

c) Write down the minimum and the maximum value of $f(\theta)$.

$\boxed{f(\theta) \cong 15\sin(\theta+53.1°)}$, $\boxed{\theta \approx 85.1°,\ 348.7°}$, $\boxed{f(\theta)_{\text{min}} = -15,\ \ f(\theta)_{\text{max}} = 15}$



(a) $9\sin\theta + 12\cos\theta \equiv R\sin(\theta+\alpha)$

$\equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$\equiv (R\cos\alpha)\sin\theta + (R\sin\alpha)\cos\theta$

$R\cos\alpha = 9$, $R\sin\alpha = 12$

• $R = \sqrt{9^2+12^2} = \sqrt{225} = 15$

• $\tan\alpha = \frac{12}{9} \Rightarrow \alpha = 53.13°$

Hence $f(\theta) \equiv 15\sin(\theta + 53.13°)$

(b) $f(\theta) = 10$

$\Rightarrow 15\sin(\theta+53.13°) = 10$

$\Rightarrow \sin(\theta+53.13°) = \frac{2}{3}$

$\arcsin\left(\frac{2}{3}\right) \approx 41.8$

$\theta + 53.13 = 41.81 \pm 360n$

$\theta + 53.13 = 138.19 \pm 360n$ $\quad n = 0, 1, 2, 3, \ldots$

$\theta = -11.32 \pm 360n$

$\theta = 85.1 \pm 360n$

$\therefore \theta = 348.7°, 85.1°$

(c) $f(\theta)_{\text{MAX}} = 15$, $f(\theta)_{\text{MIN}} = -15$

Question 9

$$f(\theta) \equiv 4\sin\theta + 3\cos\theta, \ \theta \in \mathbb{R}.$$

a) Write the above expression in the form $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

b) Write down the maximum value of $f(\theta)$.

c) Find the smallest positive value of θ for which this maximum value occurs.

$$\boxed{f(\theta) \equiv 4\sin\theta + 3\cos\theta \cong 5\sin(\theta + 36.9°)}, \ \boxed{f(\theta)_{\max} = 5}, \ \boxed{\theta \approx 53.1°}$$

(a) $4\sin\theta + 3\cos\theta \equiv R\sin(\theta+\alpha)$

$\equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$\equiv (R\cos\alpha)\sin\theta + (R\sin\alpha)\cos\theta$

$R\cos\alpha = 4$

$R\sin\alpha = 3$

$R = \sqrt{4^2+3^2} = 5$

$\tan\alpha = \frac{3}{4} \Rightarrow \alpha \simeq 36.87°$

$\therefore 4\sin\theta + 3\cos\theta \cong 5\sin(\theta + 36.9°)$

(b) $4\sin\theta + 3\cos\theta \cong 5\sin(\theta + 36.9°)$

MAX = 5

(c) FOR MAX OF 5 $\Rightarrow$ $5\sin(\theta + 36.9) = 5$

$\sin(\theta + 36.9) = 1$

$\theta + 36.9° = 90°$

$\theta = 53.1°$

Question 10

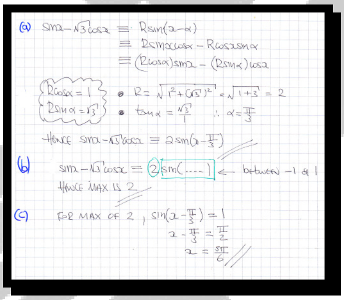
$$f(x) \equiv \sin x - \sqrt{3}\cos x, \quad x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\sin(x-\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

b) Write down the maximum value of $f(x)$.

c) Find the smallest positive value of x for which this maximum value occurs.

$$\boxed{f(x) \equiv \sin x - \sqrt{3}\cos x \equiv 2\sin\left(x-\frac{\pi}{3}\right)}, \quad \boxed{f(x)_{\text{max}} = 2}, \quad \boxed{x = \frac{5\pi}{6}}$$



(a) $\sin x - \sqrt{3}\cos x \equiv R\sin(x-\alpha)$

$\equiv R\sin x\cos\alpha - R\cos x\sin\alpha$

$\equiv (R\cos\alpha)\sin x - (R\sin\alpha)\cos x$

$R\cos\alpha = 1$, $R\sin\alpha = \sqrt{3}$

• $R = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$

• $\tan\alpha = \frac{\sqrt{3}}{1}$ $\therefore \alpha = \frac{\pi}{3}$

HENCE $\sin x - \sqrt{3}\cos x \equiv 2\sin\left(x-\frac{\pi}{3}\right)$

(b) $\sin x - \sqrt{3}\cos x \equiv 2\boxed{\sin(\ldots)}$ ← between −1 & 1

HENCE MAX IS 2

(c) FOR MAX OF 2, $\sin\left(x-\frac{\pi}{3}\right) = 1$

$x - \frac{\pi}{3} = \frac{\pi}{2}$

$x = \frac{5\pi}{6}$

Question 11

$$f(\theta) \equiv 10\sin\theta + 24\cos\theta, \ \theta \in \mathbb{R}.$$

a) Express $f(\theta)$ in the form $R\sin(\theta+\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

b) Write down the minimum value of $f(\theta)$.

c) Find the smallest positive value of θ for which this minimum value occurs.

$$\boxed{f(\theta) \cong 26\sin\left(\theta + 1.176^{\mathrm{c}}\right)}, \ \boxed{f(\theta)_{\text{min}} = -26}, \ \boxed{\theta \approx 3.54^{\mathrm{c}}}$$

(a) $f(\theta) = 10\sin\theta + 24\cos\theta \equiv R\sin(\theta+\alpha)$

$\equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$\equiv (R\cos\alpha)\sin\theta + (R\sin\alpha)\cos\theta$

$R\cos\alpha = 10$
$R\sin\alpha = 24$

• $R = \sqrt{10^2+24^2} = 26$

• DIVIDE EQUATIONS $\tan\alpha = \frac{24}{10} \Rightarrow \alpha = 1.176^{c}$

$\therefore \ f(\theta) = 26\sin(\theta + 1.176^{c})$

(b) $f(\theta)_{\text{MIN}} = -26$

(c) FOR MINIMUM $26\sin(\theta + 1.176^{c}) = -26$

$\sin(\theta + 1.176^{c}) = -1$

$\theta + 1.176^{c} = -\frac{\pi}{2}$

$\theta = -2.746\ldots$

$\downarrow +2\pi$

$\theta = 3.54^{c}$

Question 12

$$f(x) \equiv 3\sin x + 2\cos x\,, \quad x \in \mathbb{R}\,.$$

a) Express $f(x)$ in the form $R\sin(x+\alpha)$, $R>0$, $0<\alpha<\dfrac{\pi}{2}$.

b) Hence solve the trigonometric equation

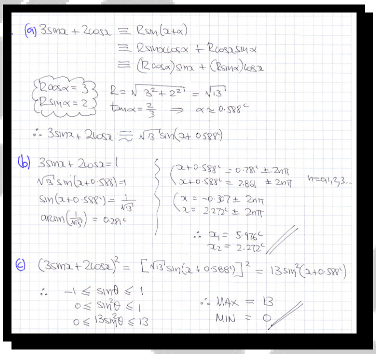
$$3\sin x + 2\cos x = 1\,, \quad 0 < x < 2\pi\,.$$

c) Write down the minimum value and the maximum value of

$$(3\sin x + 2\cos x)^2\,.$$

$\boxed{f(x) \equiv 3\sin x + 2\cos x \cong \sqrt{13}\sin\left(x + 0.588^{c}\right)}$, $\boxed{x \approx 2.27^{c}, 5.98^{c}}$,

$\boxed{\min = 0,\ \max = 13}$



(a) $3\sin x + 2\cos x \equiv R\sin(x+\alpha)$

$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

$\equiv (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$

$R\cos\alpha = 3$, $R\sin\alpha = 2$ 　 $R = \sqrt{3^2+2^2} = \sqrt{13}$, $\tan\alpha = \frac{2}{3} \Rightarrow \alpha \simeq 0.588^c$

$\therefore 3\sin x + 2\cos x \cong \sqrt{13}\sin(x + 0.588^c)$

(b) $3\sin x + 2\cos x = 1$

$\sqrt{13}\sin(x+0.588) = 1$

$\sin(x+0.588^c) = \frac{1}{\sqrt{13}}$

$\arcsin\left(\frac{1}{\sqrt{13}}\right) = 0.281^c$

$x + 0.588^c = 0.281^c \pm 2n\pi$
$x + 0.588^c = 2.861 \pm 2n\pi$ 　 $n = 0,1,2,3\ldots$

$x = -0.307 \pm 2n\pi$
$x = 2.272^c \pm 2n\pi$

$\therefore x_1 = 5.976^c$
$x_2 = 2.272^c$

(c) $(3\sin x + 2\cos x)^2 = \left[\sqrt{13}\sin(x+0.588^c)\right]^2 = 13\sin^2(x+0.588^c)$

$\therefore -1 \leqslant \sin\theta \leqslant 1$
$0 \leqslant \sin^2\theta \leqslant 1$
$0 \leqslant 13\sin^2\theta \leqslant 13$

$\therefore$ MAX $= 13$
MIN $= 0$

Question 13

$$y \equiv 5\sin x - 2\cos x, \ 0 < x < 2\pi .$$

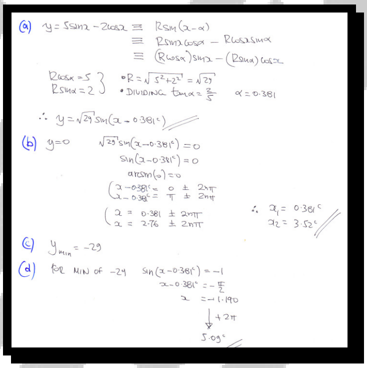
a) Express y in the form $R\sin(x-\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

b) Find the coordinates where the graph of y crosses the x axis.

c) Write down the minimum value of y.

d) Find the smallest positive value of x for which this minimum value of y occurs.

$\boxed{y \cong \sqrt{29}\sin\left(x - 0.381^{\text{c}}\right)}$, $\boxed{x \approx 0.381^{\text{c}}, 3.52^{\text{c}}}$, $\boxed{y_{\min} = -\sqrt{29}}$, $\boxed{x \approx 5.09^{\text{c}}}$


(a) $y = 5\sin x - 2\cos x \equiv R\sin(x-\alpha)$
$\equiv R\sin x\cos\alpha - R\cos x\sin\alpha$
$\equiv (R\cos\alpha)\sin x - (R\sin\alpha)\cos x$

$\left.\begin{matrix} R\cos\alpha = 5 \\ R\sin\alpha = 2 \end{matrix}\right\}$ • $R = \sqrt{5^2+2^2} = \sqrt{29}$
• DIVIDING $\tan\alpha = \frac{2}{5}$ $\alpha = 0.381$

$\therefore y = \sqrt{29}\sin(x - 0.381^{c})$

(b) $y=0$ $\sqrt{29}\sin(x-0.381^{c}) = 0$
$\sin(x-0.381^{c}) = 0$
$\arcsin(0) = 0$
$\left(\begin{matrix} x - 0.381^{c} = 0 \pm 2n\pi \\ x - 0.381^{c} = \pi \pm 2n\pi \end{matrix} \right.$
$\left(\begin{matrix} x = 0.381 \pm 2n\pi \\ x = 2.76 \pm 2n\pi \end{matrix} \right.$ $\therefore x_1 = 0.381^{c}$, $x_2 = 3.52^{c}$

(c) $y_{\min} = -29$

(d) FOR MIN OF -29 $\sin(x - 0.381^{c}) = -1$
$x - 0.381^{c} = -\frac{\pi}{2}$
$x = -1.190$
$\downarrow +2\pi$
5.09^{c}

Question 14

$$f(x) \equiv 2\sqrt{2}\cos x + 2\sqrt{2}\sin x, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\sin(x+\alpha)$, $R>0$, $0<\alpha<\dfrac{\pi}{2}$.

b) Solve the trigonometric equation

$$f(x) = 2 \text{ for } 0 < x < 2\pi.$$

c) Write down the maximum value of $f(x)$.

d) Find the smallest positive value of x for which this maximum value occurs.

$$\boxed{f(x) \equiv 4\sin\left(x+\frac{\pi}{4}\right)}, \ \boxed{x = \frac{7\pi}{12}, \frac{23\pi}{12}}, \ \boxed{f(x)_{\max} = 4}, \ \boxed{x = \frac{\pi}{4}}$$

(a) $2\sqrt{2}\cos x + 2\sqrt{2}\sin x \equiv R\sin(x+\alpha)$

$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

$\equiv (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$

$R\cos\alpha = 2\sqrt{2}$, $R\sin\alpha = 2\sqrt{2}$

- $R = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{8+8} = \sqrt{16} = 4$
- $\tan\alpha = 1 \quad \alpha = \frac{\pi}{4}$

$\therefore 2\sqrt{2}\cos x + 2\sqrt{2}\sin x \equiv 4\sin\left(x+\frac{\pi}{4}\right)$ //

(b) $2\sqrt{2}\cos x + 2\sqrt{2}\sin x = 2$

$4\sin\left(x+\frac{\pi}{4}\right) = 2$

$\sin\left(x+\frac{\pi}{4}\right) = \frac{1}{2}$

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$x + \frac{\pi}{4} = \frac{\pi}{6} \pm 2n\pi$

$x + \frac{\pi}{4} = \frac{5\pi}{6} \pm 2n\pi$ $\quad n = 0,1,2,3,\ldots$

$x = -\frac{\pi}{12} \pm 2n\pi$

$x = \frac{7\pi}{12} \pm 2n\pi$

$\therefore x_1 = \frac{23\pi}{12}$

$x_2 = \frac{7\pi}{12}$ //

(c) MAX IS 4 //

(d) TO GET THIS MAX $\Rightarrow$ $\sin\left(x+\frac{\pi}{4}\right) = 1$

$x + \frac{\pi}{4} = \frac{\pi}{2}$

$x = \frac{\pi}{4}$ //

Question 15

$$f(x) \equiv 3\sin x + \cos x, \ x \in \mathbb{R}$$

a) Express $f(x)$ in the form $R\cos(x-\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

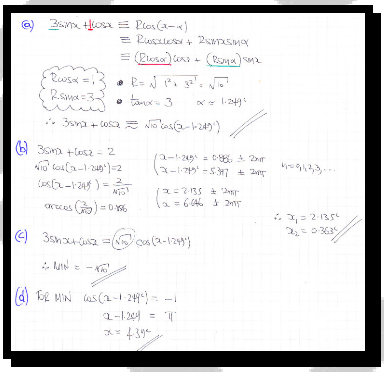
b) Solve the equation

$$f(x) = 2 \ \text{for} \ \ 0 < x < 2\pi .$$

c) Write down the minimum value of $f(x)$.

d) Find the smallest positive value of x for which this minimum value occurs.

$\boxed{f(x) \cong \sqrt{10}\cos\left(x - 1.249^{c}\right)}$, $\boxed{x = 0.363^{c}, 2.135^{c}}$, $\boxed{f(x)_{\text{min}} = -\sqrt{10}}$, $\boxed{x = 4.391^{c}}$



(a) $3\sin x + 1\cos x \equiv R\cos(x-\alpha)$

$\equiv R\cos x\cos\alpha + R\sin x\sin\alpha$

$\equiv (R\cos\alpha)\cos x + (R\sin\alpha)\sin x$

$R\cos\alpha = 1$
$R\sin\alpha = 3$

- $R = \sqrt{1^2 + 3^2} = \sqrt{10}$
- $\tan\alpha = 3 \quad \alpha \simeq 1.249^{c}$

$\therefore 3\sin x + \cos x \cong \sqrt{10}\cos(x - 1.249^{c})$

(b) $3\sin x + \cos x = 2$

$\sqrt{10}\cos(x - 1.249^{c}) = 2$

$\cos(x - 1.249^{c}) = \frac{2}{\sqrt{10}}$

$\arccos\left(\frac{2}{\sqrt{10}}\right) = 0.886$

$\left(\begin{array}{l} x - 1.249^{c} = 0.886 \pm 2n\pi \\ x - 1.249^{c} = 5.397 \pm 2n\pi \end{array}\right.$ $\quad n = 0, 1, 2, 3, \ldots$

$\left(\begin{array}{l} x = 2.135 \pm 2n\pi \\ x = 6.646 \pm 2n\pi \end{array}\right.$

$\therefore x_1 = 2.135^{c}$
$x_2 = 0.363^{c}$

(c) $3\sin x + \cos x = \sqrt{10}\cos(x - 1.249^{c})$

$\therefore \text{MIN} = -\sqrt{10}$

(d) FOR MIN $\cos(x - 1.249^{c}) = -1$

$x - 1.249 = \pi$

$x = 4.39^{c}$

Question 16

$$y \equiv \sqrt{2}\cos\theta - \sqrt{6}\sin\theta\,,\ 0 < \theta < 360°\,.$$

a) Express y in the form $R\cos(\theta+\alpha)$, $R > 0$, $0 < \alpha < 90°$.

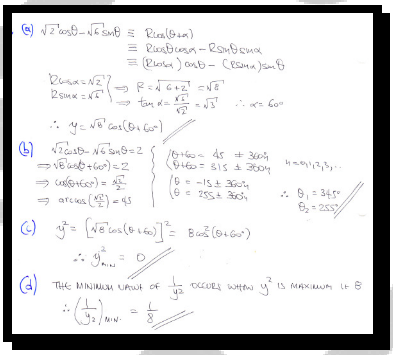
b) Solve the equation $y = 2$.

c) Write down the minimum values of …

i. … y^2.

ii. … $\dfrac{1}{y^2}$.

$$\boxed{y \equiv \sqrt{8}\cos(\theta+60°)}\,,\ \boxed{\theta = 255°, 345°}\,,\ \boxed{\min = 0}\,,\ \boxed{\min = \frac{1}{8}}$$


(a) $\sqrt{2}\cos\theta - \sqrt{6}\sin\theta \equiv R\cos(\theta+\alpha)$
$\equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$
$\equiv (R\cos\alpha)\cos\theta - (R\sin\alpha)\sin\theta$

$\left.\begin{array}{l} R\cos\alpha = \sqrt{2} \\ R\sin\alpha = \sqrt{6} \end{array}\right\} \Rightarrow R = \sqrt{6+2} = \sqrt{8}$
$\Rightarrow \tan\alpha = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \quad \therefore \alpha = 60°$

$\therefore y = \sqrt{8}\cos(\theta + 60°)$

(b) $\sqrt{2}\cos\theta - \sqrt{6}\sin\theta = 2$
$\Rightarrow \sqrt{8}\cos(\theta+60°) = 2$
$\Rightarrow \cos(\theta+60°) = \frac{\sqrt{2}}{2}$
$\Rightarrow \arccos\left(\frac{\sqrt{2}}{2}\right) = 45$

$\theta + 60 = 45 \pm 360n$
$\theta + 60 = 315 \pm 360n \quad n = 0, 1, 2, 3, \ldots$
$\theta = -15 \pm 360n$
$\theta = 255 \pm 360n$

$\therefore \theta_1 = 345°$
$\theta_2 = 255°$

(c) $y^2 = \left[\sqrt{8}\cos(\theta+60)\right]^2 = 8\cos^2(\theta+60°)$

$\therefore y^2_{MIN} = 0$

(d) THE MINIMUM VALUE OF $\frac{1}{y^2}$ OCCURS WHEN y^2 IS MAXIMUM IE 8

$\therefore \left(\frac{1}{y^2}\right)_{MIN} = \frac{1}{8}$

Question 17

$$f(x) \equiv 2\sin x + 2\cos x, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\sin(x+\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

b) State the minimum and the maximum value of …

i. … $y = f\left(x - \frac{\pi}{2}\right)$.

ii. … $y = 2f(x)+1$.

iii. … $y = \left[f(x)\right]^2$.

iv. … $y = \dfrac{10}{f(x)+3\sqrt{2}}$.

$\boxed{f(x) \equiv \sqrt{8}\sin\left(x+\frac{\pi}{4}\right)}$, $\boxed{\left[-\sqrt{8}, \sqrt{8}\right]}$, $\boxed{\left[-2\sqrt{8}+1, 2\sqrt{8}+1\right]}$, $\boxed{[0,8]}$, $\boxed{\left[\sqrt{2}, 5\sqrt{2}\right]}$

(a) $f(x) \equiv 2\sin x + 2\cos x \equiv R\sin(x+\alpha)$
$\equiv R\sin x\cos\alpha + R\cos x\sin\alpha$
$\equiv (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$

$\left.\begin{array}{l} R\cos\alpha = 2 \\ R\sin\alpha = 2 \end{array}\right\} \Rightarrow \begin{array}{l} R = \sqrt{2^2+2^2} = \sqrt{8} \\ \tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4} \end{array}$

$\therefore f(x) = \sqrt{8}\sin\left(x+\frac{\pi}{4}\right)$

(b) (I) MIN $= -\sqrt{8}$ MAX $= \sqrt{8}$ • (TRANSLATION RIGHT DOES NOT AFFECT y CO-ORDINATE)

(II) MIN $= -2\sqrt{8}+1$ MAX $= 2\sqrt{8}+1$ • (STRETCH IN y BY FACTOR OF 2 THEN TRANSLATION "UP" BY 1 UNIT)

(III) MIN $= 0$ MAX $= 8$

(IV) MIN $= \sqrt{2}$ MAX $= 5\sqrt{2}$

$\frac{10}{\sqrt{8}+3\sqrt{2}}$ $\frac{10}{-\sqrt{8}+3\sqrt{2}}$