DIFFERENTIATION

Question 1 (**)

Differentiate each of the following expressions with respect to x, simplifying the final answers as far as possible

a) $y = \left(x^2 - 4\right)^3$

b) $y = x\cos 2x$

c) $y = \dfrac{\sin x}{x}$

$\boxed{}$, $\boxed{\dfrac{dy}{dx} = 6x\left(x^2 - 4\right)^2}$, $\boxed{\dfrac{dy}{dx} = \cos 2x - 2x\sin 2x}$, $\boxed{\dfrac{dy}{dx} = \dfrac{x\cos x - \sin x}{x^2}}$

(a) $y = (x^2-4)^3$

$\frac{dy}{dx} = 3(x^2-4)^2 \times 2x$

$\frac{dy}{dx} = 6x(x^2-4)^2$

(b) $y = x\cos 2x$

$\frac{dy}{dx} = 1 \times \cos 2x + x(-2\sin 2x)$

$\frac{dy}{dx} = \cos 2x - 2x\sin 2x$

(c) $y = \frac{\sin x}{x} \Rightarrow \frac{dy}{dx} = \frac{x(\cos x) - \sin x \times 1}{x^2} = \frac{x\cos x - \sin x}{x^2}$

Question 2 (**)

Differentiate each of the following expressions with respect to x, simplifying the final answers as far as possible.

a) $y = \left(1 - x^2\right)^6$

b) $y = x^3 \sin 3x$

c) $y = \dfrac{5x}{x^3 + 2}$

$\boxed{}$, $\boxed{\dfrac{dy}{dx} = -12x\left(1 - x^2\right)^5}$, $\boxed{\dfrac{dy}{dx} = 3x^2\left(\sin 3x + x\cos 3x\right)}$, $\boxed{\dfrac{dy}{dx} = \dfrac{10\left(1 - x^3\right)}{\left(x^3 + 2\right)^2}}$

(a) $y = (1-x^2)^6$

$\frac{dy}{dx} = 6(1-x^2)^5 \times (-2x)$

$\frac{dy}{dx} = -12x(1-x^2)^5$

(b) $y = x^3\sin 3x$

$\frac{dy}{dx} = 3x^2\sin 3x + x^3(3\cos 3x)$

$\frac{dy}{dx} = 3x^2\sin 3x + 3x^3\cos 3x$

$\frac{dy}{dx} = 3x^2(\sin 3x + x\cos 3x)$

(c) $y = \frac{5x}{x^3+2}$

$\frac{dy}{dx} = \frac{(x^3+2)\times 5 - 5x(3x^2)}{(x^3+2)^2}$

$\frac{dy}{dx} = \frac{5x^3+10-15x^3}{(x^3+2)^2}$

$\frac{dy}{dx} = \frac{10-10x^3}{(x^3+2)^2}$

Question 3 (***)

Differentiate each of the following expressions with respect to x, writing the final answers as simplified fractions.

a) $y = \dfrac{\ln x}{1+\ln x}$.

b) $y = \ln\left(\dfrac{1}{x^2+9}\right)$.

$\boxed{}$, $\boxed{\dfrac{dy}{dx} = \dfrac{1}{x(1+\ln x)^2}}$, $\boxed{\dfrac{dy}{dx} = -\dfrac{2x}{x^2+9}}$

(a) $\dfrac{d}{dx}\left(\dfrac{\ln x}{1+\ln x}\right) = \dfrac{(1+\ln x)\times\frac{1}{x} - \ln x\left(\frac{1}{x}\right)}{(1+\ln x)^2} = \dfrac{\frac{1}{x} + \cancel{\frac{1}{x}\ln x} - \cancel{\frac{1}{x}\ln x}}{(1+\ln x)^2}$

$= \dfrac{1}{x(1+\ln x)^2}$

(b) $\dfrac{d}{dx}\left(\ln\left(\dfrac{1}{x^2+9}\right)\right) = \dfrac{d}{dx}\left[-\ln(x^2+9)\right] = -\dfrac{1}{x^2+9}\times 2x = -\dfrac{2x}{x^2+9}$

Question 4 (***)

Differentiate each of the following expressions with respect to x, simplifying the final answers as far as possible

a) $y = \dfrac{4}{(2x-1)^2}$.

b) $y = x^3 e^{-2x}$.

c) $y = \dfrac{2x^2+1}{3x^2+1}$.

$\boxed{}$, $\boxed{\dfrac{dy}{dx} = -\dfrac{16}{(2x-1)^3}}$, $\boxed{\dfrac{dy}{dx} = x^2(3-2x)e^{-2x}}$, $\boxed{\dfrac{dy}{dx} = -\dfrac{2x}{\left(3x^2+1\right)^2}}$

(a) $y = 4(2x-1)^{-2}$

$\dfrac{dy}{dx} = -8(2x-1)^{-3}\times 2$

$\dfrac{dy}{dx} = -16(2x-1)^{-3}$

$\dfrac{dy}{dx} = -\dfrac{16}{(2x-1)^3}$

(b) $y = x^3 e^{-2x}$

$\dfrac{dy}{dx} = 3x^2 e^{-2x} + x^3 e^{-2x}(-2)$

$\dfrac{dy}{dx} = 3x^2 e^{-2x} - 2x^3 e^{-2x}$

$\dfrac{dy}{dx} = x^2 e^{-2x}(3-2x)$

(c) $y = \dfrac{2x^2+1}{3x^2+1}$

$\dfrac{dy}{dx} = \dfrac{(3x^2+1)(4x) - (2x^2+1)(6x)}{(3x^2+1)^2}$

$\dfrac{dy}{dx} = \dfrac{4x + 12x^3 - 12x^3 - 6x}{(3x^2+1)^2} = -\dfrac{2x}{(3x^2+1)^2}$

Question 5 (***)

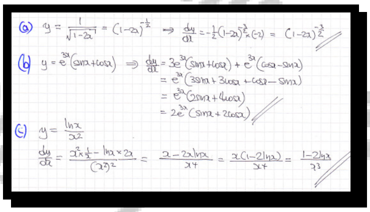
Differentiate each of the following expressions with respect to x, simplifying the final answers where possible.

a) $y = \dfrac{1}{\sqrt{1-2x}}$.

b) $y = \mathrm{e}^{3x}\left(\sin x + \cos x\right)$.

c) $y = \dfrac{\ln x}{x^2}$.

$\boxed{}$, $\boxed{\dfrac{dy}{dx} = (1-2x)^{-\frac{3}{2}}}$, $\boxed{\dfrac{dy}{dx} = 2\mathrm{e}^{3x}\left(\sin x + 2\cos x\right)}$, $\boxed{\dfrac{dy}{dx} = \dfrac{1-2\ln x}{x^3}}$



(a) $y = \dfrac{1}{\sqrt{1-2x}} = (1-2x)^{-\frac{1}{2}} \Rightarrow \dfrac{dy}{dx} = -\frac{1}{2}(1-2x)^{-\frac{3}{2}} \times (-2) = (1-2x)^{-\frac{3}{2}}$

(b) $y = e^{3x}(\sin x + \cos x) \Rightarrow \dfrac{dy}{dx} = 3e^{3x}(\sin x + \cos x) + e^{3x}(\cos x - \sin x)$

$= e^{3x}(3\sin x + 3\cos x + \cos x - \sin x)$

$= e^{3x}(2\sin x + 4\cos x)$

$= 2e^{3x}(\sin x + 2\cos x)$

(c) $y = \dfrac{\ln x}{x^2}$

$\dfrac{dy}{dx} = \dfrac{x^2 \times \frac{1}{x} - \ln x \times 2x}{(x^2)^2} = \dfrac{x - 2x\ln x}{x^4} = \dfrac{x(1-2\ln x)}{x^4} = \dfrac{1-2\ln x}{x^3}$

Question 6 (***)

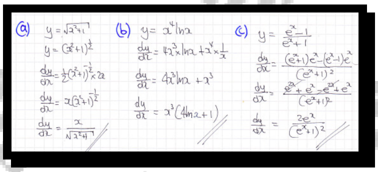
Differentiate each of the following expressions with respect to x, simplifying the final answers where possible.

a) $y=\sqrt{x^2-1}$

b) $y=x^4\ln x$

c) $y=\dfrac{e^x-1}{e^x+1}$

$\boxed{}$, $\boxed{\dfrac{dy}{dx}=x\left(x^2-1\right)^{-\frac{1}{2}}}$, $\boxed{\dfrac{dy}{dx}=4x^3\ln x+x^3}$, $\boxed{\dfrac{dy}{dx}=\dfrac{2e^x}{\left(e^x+1\right)^2}}$



(a) $y=\sqrt{x^2+1}$

$y=(x^2+1)^{\frac{1}{2}}$

$\frac{dy}{dx}=\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\times 2x$

$\frac{dy}{dx}=x(x^2+1)^{-\frac{1}{2}}$

$\frac{dy}{dx}=\frac{x}{\sqrt{x^2+1}}$

(b) $y=x^4\ln x$

$\frac{dy}{dx}=4x^3\times\ln x+x^4\times\frac{1}{x}$

$\frac{dy}{dx}=4x^3\ln x+x^3$

$\frac{dy}{dx}=x^3(4\ln x+1)$

(c) $y=\frac{e^x-1}{e^x+1}$

$\frac{dy}{dx}=\frac{(e^x+1)e^x-(e^x-1)e^x}{(e^x+1)^2}$

$\frac{dy}{dx}=\frac{\cancel{e^{2x}}+e^x-\cancel{e^{2x}}+e^x}{(e^x+1)^2}$

$\frac{dy}{dx}=\frac{2e^x}{(e^x+1)^2}$

Question 7 **(***+)**

Show clearly that …

i. … $\dfrac{d}{dx}\left(x^{\frac{3}{2}}\,\mathrm{e}^{2x}\right)=\dfrac{1}{2}(4x+3)\,x^{\frac{1}{2}}\,\mathrm{e}^{2x}$.

ii. … $\dfrac{d}{dx}\left(\dfrac{4x+1}{1-2x}\right)=\dfrac{6}{(1-2x)^2}$.

iii. … $\dfrac{d}{dx}\left(\ln(\sec x+\tan x)\right)=\sec x$.

$\boxed{\text{proof}}$

(I) $\dfrac{d}{dx}\left(x^{\frac{3}{2}}e^{2x}\right) = \dfrac{3}{2}x^{\frac{1}{2}}e^{2x} + 2x^{\frac{3}{2}}e^{2x} = \dfrac{1}{2}x^{\frac{1}{2}}e^{2x}(3+4x)$

$= \dfrac{1}{2}x^{\frac{1}{2}}(4x+3)e^{2x}$

(II) $\dfrac{d}{dx}\left(\dfrac{4x+1}{1-2x}\right) = \dfrac{(1-2x)4-(4x+1)(-2)}{(1-2x)^2} = \dfrac{4-\cancel{8x}+\cancel{8x}+2}{(1-2x)^2} = \dfrac{6}{(1-2x)^2}$

(III) $\dfrac{d}{dx}\left[\ln(\sec x+\tan x)\right] = \dfrac{1}{\sec x+\tan x}\times(\sec x\tan x+\sec^2 x)$

$= \dfrac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}$

$= \dfrac{\sec x\,\cancel{(\tan x+\sec x)}}{\cancel{\sec x+\tan x}}$

$= \sec x$

Question 8 **(***+)**

Show clearly that …

i. … $\dfrac{d}{dx}\left[2x^3(2x+3)^5\right] = 2x^2(16x+9)(2x+3)^4$.

ii. … $\dfrac{d}{dx}\left[\dfrac{2x^2+1}{3x^2+1}\right] = -\dfrac{2x}{(3x^2+1)^2}$.

iii. … $\dfrac{d}{dx}\left[\ln(\sec x+\tan x)\right] = \sec x$.

$\boxed{\text{proof}}$

(a) $\dfrac{d}{dx}\left[2x^3(2x+3)^5\right] = 6x^2(2x+3)^5 + 2x^3 \times 10(2x+3)^4$
$= 6x^2(2x+3)^5 + 20x^3(2x+3)^4$
$= 2x^2(2x+3)^4\left[3(2x+3)+10x\right]$
$= 2x^2(2x+3)^4(16x+9)$

(b) $\dfrac{d}{dx}\left[\dfrac{2x^2+1}{3x^2+1}\right] = \dfrac{(3x^2+1)\times 4x - (2x^2+1)(6x)}{(3x^2+1)^2} = \dfrac{\cancel{12x^3} + 4x - \cancel{12x^3} - 6x}{(3x^2+1)^2} = -\dfrac{2x}{(3x^2+1)^2}$

(c) $\dfrac{d}{dx}\left[\ln(\sec x+\tan x)\right] = \dfrac{1}{\sec x+\tan x} \times (\sec x\tan x + \sec^2 x)$
$= \dfrac{\sec x\tan x + \sec^2 x}{\sec x+\tan x} = \dfrac{\sec x\,\cancel{(\tan x+\sec x)}}{\cancel{\sec x+\tan x}} = \sec x$

Question 9 (***+)

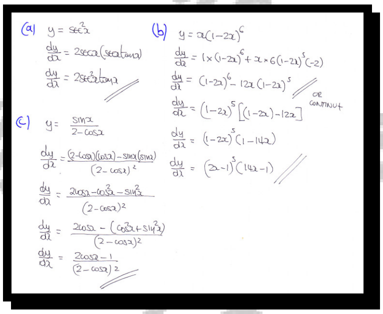
Differentiate each of the following expressions with respect to x, simplifying the final answer as far as possible.

a) $y = \sec^2 x$.

b) $y = x(1-2x)^6$.

c) $y = \dfrac{\sin x}{2-\cos x}$.

$\boxed{\dfrac{dy}{dx} = 2\sec^2 x \tan x}$, $\boxed{\dfrac{dy}{dx} = (14x-1)(2x-1)^5 = (1-14x)(1-2x)^5}$, $\boxed{\dfrac{dy}{dx} = \dfrac{2\cos x - 1}{(2-\cos x)^2}}$



(a) $y = \sec^2 x$

$\frac{dy}{dx} = 2\sec x(\sec x \tan x)$

$\frac{dy}{dx} = 2\sec^2 x \tan x$

(b) $y = x(1-2x)^6$

$\frac{dy}{dx} = 1 \times (1-2x)^6 + x \times 6(1-2x)^5(-2)$

$\frac{dy}{dx} = (1-2x)^6 - 12x(1-2x)^5$

OR CONTINUE

$\frac{dy}{dx} = (1-2x)^5[(1-2x)-12x]$

$\frac{dy}{dx} = (1-2x)^5(1-14x)$

$\frac{dy}{dx} = (2x-1)^5(14x-1)$

(c) $y = \frac{\sin x}{2-\cos x}$

$\frac{dy}{dx} = \frac{(2-\cos x)(\cos x) - \sin x(\sin x)}{(2-\cos x)^2}$

$\frac{dy}{dx} = \frac{2\cos x - \cos^2 x - \sin^2 x}{(2-\cos x)^2}$

$\frac{dy}{dx} = \frac{2\cos x - (\cos^2 x + \sin^2 x)}{(2-\cos x)^2}$

$\frac{dy}{dx} = \frac{2\cos x - 1}{(2-\cos x)^2}$

Question 10 (****)

Differentiate each the following expressions with respect to x, simplifying the final answers as far as possible.

(Fractional answers must not involve double fractions)

a) $y = \sin^3 2x$.

b) $y = x \tan 4x$.

c) $y = \ln\left(\dfrac{x+1}{x}\right)$.

$\boxed{\dfrac{dy}{dx} = 6\sin^2 2x \cos 2x}$, $\boxed{\dfrac{dy}{dx} = \tan 4x + 4x\sec^2 4x}$, $\boxed{\dfrac{dy}{dx} = -\dfrac{1}{x^2+x}$}

(a) $y = \sin^3 2x$

$y = (\sin 2x)^3$

$\dfrac{dy}{dx} = 3(\sin 2x)^2 \times 2\cos 2x$

$\dfrac{dy}{dx} = 6\sin^2 2x \cos 2x$

(b) $y = x\tan 4x$

$\dfrac{dy}{dx} = (1 \times \tan 4x) + (x \times 4\sec^2 4x)$

$\dfrac{dy}{dx} = \tan 4x + 4x\sec^2 4x$

(c) $y = \ln\left(\dfrac{x+1}{x}\right) = \ln(x+1) - \ln x$

$\dfrac{dy}{dx} = \dfrac{1}{x+1} - \dfrac{1}{x}$

$\left(= \dfrac{x-(x+1)}{x(x+1)} = -\dfrac{1}{x(x+1)} \right)$

Question 11 (****)

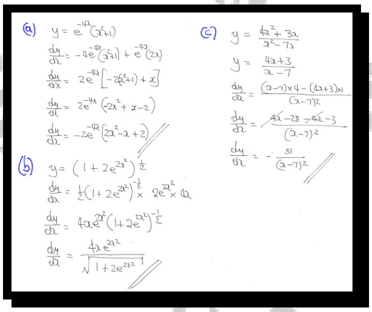
Differentiate each of the following expressions with respect to x, simplifying the answers as far as possible.

a) $y = \mathrm{e}^{-4x}\left(x^2+1\right)$.

b) $y = \sqrt{1+2\mathrm{e}^{2x^2}}$.

c) $y = \dfrac{4x^2+3x}{x^2-7x}$.

$$\boxed{\frac{dy}{dx} = -2\mathrm{e}^{-4x}\left(2x^2-x+2\right)}, \quad \boxed{\frac{dy}{dx} = \frac{4x\mathrm{e}^{2x^2}}{\sqrt{1+2\mathrm{e}^{2x^2}}}}, \quad \boxed{\frac{dy}{dx} = -\frac{31}{(x-7)^2}}$$



(a) $y = e^{-4x}(x^2+1)$

$\frac{dy}{dx} = -4e^{-4x}(x^2+1) + e^{-4x}(2x)$

$\frac{dy}{dx} = 2e^{-4x}\left[-2(x^2+1) + x\right]$

$\frac{dy}{dx} = 2e^{-4x}\left(-2x^2+x-2\right)$

$\frac{dy}{dx} = -2e^{-4x}\left(2x^2-x+2\right)$

(b) $y = \left(1+2e^{2x^2}\right)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}\left(1+2e^{2x^2}\right)^{-\frac{1}{2}} \times 2e^{2x^2} \times 4x$

$\frac{dy}{dx} = 4xe^{2x^2}\left(1+2e^{2x^2}\right)^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{4xe^{2x^2}}{\sqrt{1+2e^{2x^2}}}$

(c) $y = \frac{4x^2+3x}{x^2-7x}$

$y = \frac{4x+3}{x-7}$

$\frac{dy}{dx} = \frac{(x-7)\times 4 - (4x+3)\times 1}{(x-7)^2}$

$\frac{dy}{dx} = \frac{\cancel{4x}-28-\cancel{4x}-3}{(x-7)^2}$

$\frac{dy}{dx} = -\frac{31}{(x-7)^2}$

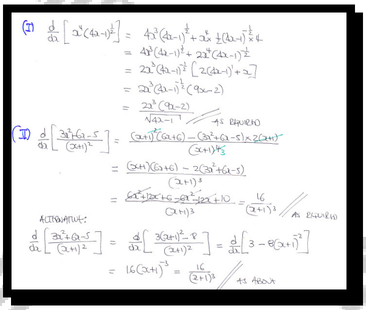
Question 12 (****)

Prove that …

i. … $\dfrac{d}{dx}\left(x^4\sqrt{4x-1}\right)=\dfrac{2x^3(9x-2)}{\sqrt{4x-1}}$.

ii. … $\dfrac{d}{dx}\left(\dfrac{3x^2+6x-5}{(x+1)^2}\right)=\dfrac{16}{(x+1)^3}$.

$\boxed{\text{proof}}$



(I) $\frac{d}{dx}\left[x^4(4x-1)^{\frac{1}{2}}\right] = 4x^3(4x-1)^{\frac{1}{2}} + x^4 \times \frac{1}{2}(4x-1)^{-\frac{1}{2}} \times 4$

$= 4x^3(4x-1)^{\frac{1}{2}} + 2x^4(4x-1)^{-\frac{1}{2}}$

$= 2x^3(4x-1)^{-\frac{1}{2}}\left[2(4x-1)^1 + x\right]$

$= 2x^3(4x-1)^{-\frac{1}{2}}(9x-2)$

$= \frac{2x^3(9x-2)}{\sqrt{4x-1}}$ // AS REQUIRED

(II) $\frac{d}{dx}\left[\frac{3x^2+6x-5}{(x+1)^2}\right] = \frac{(x+1)^{\cancel{2}}(6x+6)-(3x^2+6x-5)\times 2\cancel{(x+1)}}{(x+1)^{\cancel{4}\,3}}$

$= \frac{(x+1)(6x+6)-2(3x^2+6x-5)}{(x+1)^3}$

$= \frac{\cancel{6x^2}+\cancel{12x}+6-\cancel{6x^2}-\cancel{12x}+10}{(x+1)^3} = \frac{16}{(x+1)^3}$ // AS REQUIRED

ALTERNATIVE:

$\frac{d}{dx}\left[\frac{3x^2+6x-5}{(x+1)^2}\right] = \frac{d}{dx}\left[\frac{3(x+1)^2-8}{(x+1)^2}\right] = \frac{d}{dx}\left[3-8(x+1)^{-2}\right]$

$= 16(x+1)^{-3} = \frac{16}{(x+1)^3}$ // AS ABOVE

Question 13 (****)

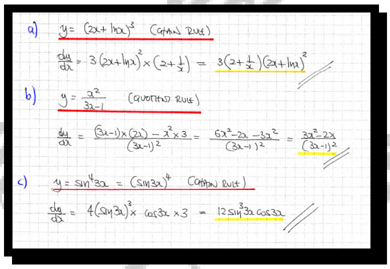
Differentiate each of the following expressions with respect to x.

a) $y = (2x + \ln x)^3$.

b) $y = \dfrac{x^2}{3x-1}$.

c) $y = \sin^4 3x$.

$\boxed{\dfrac{dy}{dx} = 3(2x+\ln x)^2\left(2+\dfrac{1}{x}\right)}$, $\boxed{\dfrac{dy}{dx} = \dfrac{3x^2-2x}{(3x-1)^2}}$, $\boxed{\dfrac{dy}{dx} = 12\sin^3 3x\cos 3x}$



a) $y = (2x+\ln x)^3$ (CHAIN RULE)

$\frac{dy}{dx} = 3(2x+\ln x)^2 \times \left(2+\frac{1}{x}\right) = 3\left(2+\frac{1}{x}\right)(2x+\ln x)^2$

b) $y = \frac{x^2}{3x-1}$ (QUOTIENT RULE)

$\frac{dy}{dx} = \frac{(3x-1)\times(2x) - x^2\times 3}{(3x-1)^2} = \frac{6x^2-2x-3x^2}{(3x-1)^2} = \frac{3x^2-2x}{(3x-1)^2}$

c) $y = \sin^4 3x = (\sin 3x)^4$ (CHAIN RULE)

$\frac{dy}{dx} = 4(\sin 3x)^3 \times \cos 3x \times 3 = 12\sin^3 3x\cos 3x$

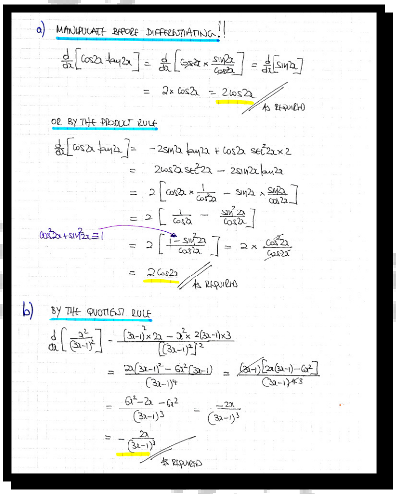
Question 14 (****)

Show, with detailed workings, that

a) $\frac{d}{dx}\left(\cos 2x \tan 2x\right) = 2\cos 2x$.

b) $\frac{d}{dx}\left(\frac{x^2}{(3x-1)^2}\right) = -\frac{2x}{(3x-1)^3}$.

$\boxed{}$, $\boxed{\text{proof}}$



a) MANIPULATE BEFORE DIFFERENTIATING!!

$$\frac{d}{dx}\left[\cos 2x \tan 2x\right] = \frac{d}{dx}\left[\cancel{\cos 2x} \times \frac{\sin 2x}{\cancel{\cos 2x}}\right] = \frac{d}{dx}\left[\sin 2x\right]$$

$$= 2 \times \cos 2x = 2\cos 2x$$

AS REQUIRED

OR BY THE PRODUCT RULE

$$\frac{d}{dx}\left[\cos 2x \tan 2x\right] = -2\sin 2x \tan 2x + \cos 2x \sec^2 2x \times 2$$

$$= 2\cos 2x \sec^2 2x - 2\sin 2x \tan 2x$$

$$= 2\left[\cos 2x \times \frac{1}{\cos^2 2x} - \sin 2x \times \frac{\sin 2x}{\cos 2x}\right]$$

$$= 2\left[\frac{1}{\cos 2x} - \frac{\sin^2 2x}{\cos 2x}\right]$$

$\cos^2 2x + \sin^2 2x \equiv 1$

$$= 2\left[\frac{1 - \sin^2 2x}{\cos 2x}\right] = 2 \times \frac{\cos^{\cancel{2}} 2x}{\cancel{\cos 2x}}$$

$$= 2\cos 2x$$

AS REQUIRED

b) BY THE QUOTIENT RULE

$$\frac{d}{dx}\left[\frac{x^2}{(3x-1)^2}\right] = \frac{(3x-1)^2 \times 2x - x^2 \times 2(3x-1)\times 3}{\left[(3x-1)^2\right]^2}$$

$$= \frac{2x(3x-1)^2 - 6x^2(3x-1)}{(3x-1)^4} = \frac{\cancel{(3x-1)}\left[2x(3x-1) - 6x^2\right]}{(3x-1)^{\cancel{4}\,3}}$$

$$= \frac{6x^2 - 2x - 6x^2}{(3x-1)^3} = \frac{-2x}{(3x-1)^3}$$

$$= -\frac{2x}{(3x-1)^3}$$

AS REQUIRED

Question 15 (*****)

Show clearly that …

i. … $\dfrac{d}{dx}\left(\dfrac{x-4}{\sqrt{x}+2}\right)=\dfrac{1}{2\sqrt{x}}$

ii. … $\dfrac{d}{dx}\left(\dfrac{4x-8\sqrt{x}+3}{\left(\sqrt{x}-1\right)^{2}}\right)=\dfrac{1}{\sqrt{x}\left(\sqrt{x}-1\right)^{3}}$.

$\boxed{}$, $\boxed{\text{proof}}$