EXPONENTIALS
&
LOGARITHMS
EXAM QUESTIONS

Question 1 **(**)**

Solve the equation

$$5 + e^{2x-4} = 7$$

giving the answer in the form $k + \ln\sqrt{k}$, where k is an integer.

$\boxed{k = 2}$

$5 + e^{2x-4} = 7$
$\Rightarrow e^{2x-4} = 2$
$\Rightarrow 2x - 4 = \ln 2$
$\Rightarrow 2x = 4 + \ln 2$
$\Rightarrow x = 2 + \frac{1}{2}\ln 2$
$\Rightarrow x = 2 + \ln 2^{\frac{1}{2}}$
$\Rightarrow x = 2 + \ln\sqrt{2}$ //
$k = 2$

Question 2 **(**)**

A new antibiotic is tested by spraying it on a lab dish covered in bacteria.

Initially 12000 bacteria were placed on the dish and 24 hours later this number has fallen to 2000.

The number of bacteria N on this lab dish reduces according to the equation

$$N = A e^{-kt}, \quad t \geq 0,$$

where t is the time in hours since the bacteria were first placed on the dish and, A and k are positive constants.

a) Show that $k = 0.07466$, correct to 4 significant figures.

b) Find the value of t when the bacteria will reach 1000.

$\boxed{\quad}$, $\boxed{t \approx 33.3}$

$\boxed{N = Ae^{-kt}}$

N = bacteria
t = time (hours)
$t = 0 \quad N = 12000$
$t = 24 \quad N = 2000$

(a)
- $12000 = Ae^{0}$
  $A = 12000$
  $\therefore \boxed{N = 12000e^{-kt}}$
- $2000 = 12000e^{-24k}$
  $\Rightarrow \frac{1}{6} = e^{-24k}$
  $\Rightarrow 6 = e^{24k}$
  $\Rightarrow \ln 6 = 24k$
  $\Rightarrow k = \frac{1}{24}\ln 6 \simeq 0.07466$ //

(b)
$\boxed{N = 12000e^{-0.07466t}}$
$\Rightarrow 1000 = 12000e^{-0.07466t}$
$\Rightarrow \frac{1}{12} = e^{-0.07466t}$
$\Rightarrow 12 = e^{0.07466t}$
$\Rightarrow \ln 12 = 0.07466t$
$\Rightarrow t = \frac{\ln 12}{0.07466} \simeq 33.2844\ldots$
$\Rightarrow t \simeq 33.3$ // (3sf)

Question 3 (**)

Find, in exact form where appropriate, the solution of each of the following equations.

a) $e^{2x} = 9$

b) $\ln(4-y) = 2$

c) $\ln t + \ln 3 = \ln 12$

$\boxed{}$, $\boxed{x = \ln 3}$, $\boxed{y = 4 - e^2 \approx -3.39}$, $\boxed{t = 4}$

(a) $e^{2x} = 9$
$\Rightarrow 2x = \ln 9$
$\Rightarrow 2x = 2\ln 3$
$\Rightarrow x = \ln 3$

(b) $\ln(4-y) = 2$
$\Rightarrow e^{\ln(4-y)} = e^2$
$\Rightarrow 4 - y = e^2$
$\Rightarrow 4 - e^2 = y$
$\Rightarrow y = 4 - e^2$

(c) $\ln t + \ln 3 = \ln 12$
$\ln(3t) = \ln 12$
$3t = 12$
$t = 4$

Question 4 (**)

A preservation programme, for elephants in Africa, was introduced 8 years ago. The elephants were then released to the wild. Let t be the number of years since the start of the programme.

The population of elephants P, is given by

$$P = 400e^{\frac{1}{12}(t-8)}, \quad t \geq 0.$$

Assuming that P can be treated as a continuous variable, find …

a) … the number of elephants when the programme started.

b) … the number of elephants released to the wild.

c) … the value of t when the number of elephants will reach 1000.

$\boxed{}$, $\boxed{205}$, $\boxed{400}$, $\boxed{t = 19}$

(a) $t=0$ $\quad P = 400e^{\frac{1}{12}(0-8)} = 205.366\ldots \approx 205$ ELEPHANTS

(b) $t=8$ $\quad P = 400e^{\frac{1}{12}(8-8)} = 400e^0 = 400$ ELEPHANTS

(c) $1000 = 400e^{\frac{1}{12}(t-8)}$
$\Rightarrow \frac{5}{2} = e^{\frac{1}{12}(t-8)}$
$\Rightarrow \ln\frac{5}{2} = \frac{1}{12}(t-8)$
$\Rightarrow 12\ln\frac{5}{2} = t-8$
$\Rightarrow t = 8 + 12\ln\frac{5}{2} \approx 19$

Question 5 **(**)**

The value £ V of a certain model of car, t years after it was purchased, is given by

$$V = B\mathrm{e}^{-kt}, \ t \geq 0$$

where B and k are positive constants.

The value of the car when new was £21000 and after five years it dropped to £5000 .

Find the value of B and the value of k .

$\boxed{B = 21000}$, $\boxed{k \approx 0.2870}$

$\boxed{V = Be^{-kt}}$

- $t=0 \quad V=21000$

$21000 = Be^{0}$

$B = 21000$ //

$\boxed{V = 21000e^{-kt}}$

- when $t=5$, $V=5000$

$5000 = 21000e^{-5k}$

$\frac{5}{21} = e^{-5k}$

$\frac{21}{5} = e^{5k}$

$\ln\left(\frac{21}{5}\right) = 5k$

$k = \frac{1}{5}\ln\left(\frac{21}{5}\right) \approx 0.2870$ //

Question 6 (**)

A curve has equation

$$y = Ae^{kx},$$

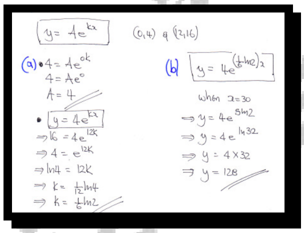
where A and k are non zero constants.

The curve passes through the points $(0,4)$ and $(12,16)$.

a) Find the value of A and the exact value of k.

b) Determine the value of y when $x = 30$

$\boxed{A=4}$, $\boxed{k=\frac{1}{6}\ln 2}$, $\boxed{y=128}$



$\boxed{y = Ae^{kx}}$ $(0,4)$ & $(12,16)$

(a) • $4 = Ae^{0k}$
$4 = Ae^{0}$
$A = 4$
• $\boxed{y = 4e^{kx}}$
$\Rightarrow 16 = 4e^{12k}$
$\Rightarrow 4 = e^{12k}$
$\Rightarrow \ln 4 = 12k$
$\Rightarrow k = \frac{1}{12}\ln 4$
$\Rightarrow k = \frac{1}{6}\ln 2$

(b) $\boxed{y = 4e^{(\frac{1}{6}\ln 2)x}}$
when $x = 30$
$\Rightarrow y = 4e^{5\ln 2}$
$\Rightarrow y = 4e^{\ln 32}$
$\Rightarrow y = 4 \times 32$
$\Rightarrow y = 128$

Question 7 (**)

A cup of coffee is cooling down in a room.

The temperature T °C of the coffee, t minutes after it was made is modelled by

$$T = 20 + 50\mathrm{e}^{-\frac{t}{15}}, \ t \geq 0.$$

a) State the temperature of the coffee when it was first made.

b) Find the temperature of the coffee, after 30 minutes.

c) Calculate, to the nearest minute, the value of t when the temperature of the coffee has reached $35\,°\mathrm{C}$.

$\boxed{}$, $\boxed{T = 70}$, $\boxed{26.8\,°\mathrm{C}}$, $\boxed{t = 18}$

(a) $T = 20 + 50e^{-\frac{1}{15}t}$

• when $t = 0$

$T = 20 + 50e^{0}$

$T = 20 + 50$

$T = 70°C$

(b) • when $t = 30$

$T = 20 + 50e^{-2}$

$T \approx 26.8°C$

(c) when $T = 35$

$35 = 20 + 50e^{-\frac{1}{15}t}$

$\Rightarrow 15 = 50e^{-\frac{1}{15}t}$

$\Rightarrow \frac{3}{10} = e^{-\frac{1}{15}t}$

$\Rightarrow \frac{10}{3} = e^{\frac{1}{15}t}$

$\Rightarrow \ln\left(\frac{10}{3}\right) = \frac{1}{15}t$

$\Rightarrow t = 15\ln\left(\frac{10}{3}\right)$

$\Rightarrow t \simeq 18$

Question 8 (**+)

Find, in exact form where appropriate, the solution of each of the following equations.

a) $4 - 3\mathrm{e}^{2x} = 3$

b) $\ln(2w+1) = 1 + \ln(w-1)$

$\boxed{}$, $\boxed{x = -\frac{1}{2}\ln 3 \approx -0.549}$, $\boxed{w = \frac{\mathrm{e}+1}{\mathrm{e}-2} \approx 5.18}$

(a) $4 - 3e^{2x} = 3$

$\Rightarrow 1 = 3e^{2x}$

$\Rightarrow \frac{1}{3} = e^{2x}$

$\Rightarrow \ln\frac{1}{3} = 2x$

$\Rightarrow x = \frac{1}{2}\ln\frac{1}{3}$

$\Rightarrow x = -\frac{1}{2}\ln 3 \approx -0.549$

(b) $\ln(2w+1) = 1 + \ln(w-1)$

$\Rightarrow \ln(2w+1) - \ln(w-1) = 1$

$\Rightarrow \ln\left(\frac{2w+1}{w-1}\right) = 1$

$\Rightarrow \frac{2w+1}{w-1} = e^{1}$

$\Rightarrow 2w+1 = ew - e$

$\Rightarrow e+1 = ew - 2w$

$\Rightarrow e+1 = w(e-2)$

$\Rightarrow w = \frac{e+1}{e-2} \approx 5.18$

Question 9 **(**+)**

A microbiologist models the population of bacteria in culture by the equation

$$P = 1000 - 950\mathrm{e}^{-\frac{1}{2}t}, \quad t \geq 0$$

where P is the number of bacteria in time t hours.

a) Find the initial number of bacteria in the culture.

b) Show mathematically that the limiting value for P is 1000.

c) Find the value of t when $P = 500$.

$\boxed{P_0 = 50}$, $\boxed{t \approx 1.28}$

(a) $P = 1000 - 950e^{-\frac{1}{2}t}$

• $t=0$ $P = 1000 - 950e^{0}$

$P = 50$

(b) As $t \to \infty$ $e^{-\frac{1}{2}t} \to 0$

$950e^{-\frac{1}{2}t} \to 0$

$P \to 1000$

(c) • $P = 500$

$500 = 1000 - 950e^{-\frac{1}{2}t}$

$\Rightarrow 950e^{-\frac{1}{2}t} = 500$

$\Rightarrow e^{-\frac{1}{2}t} = \frac{10}{19}$

$\Rightarrow e^{\frac{1}{2}t} = \frac{19}{10}$

$\Rightarrow \frac{1}{2}t = \ln\left(\frac{19}{10}\right)$

$\Rightarrow t = 2\ln\left(\frac{19}{10}\right) \approx 1.28$

Question 10 **(**+)**

It is given that

$$\frac{7}{4}\ln 16 - \frac{2}{3}\ln 8 \equiv k \ln 2 .$$

Determine the value of k.

$\boxed{k = 5}$

$\frac{7}{4}\ln 16 - \frac{2}{3}\ln 8 = \frac{7}{4} \times \ln 2^4 - \frac{2}{3} \times \ln 2^3 = \frac{7}{4} \times 4\ln 2 - \frac{2}{3} \times 3\ln 2$

$= 7\ln 2 - 2\ln 2 = 5\ln 2$

Question 11 (**+)

The function f is given by

$$f(x) = 4 - \ln(2x-1), \ x \in \mathbb{R}, \ x > \tfrac{1}{2}.$$

a) Find an expression for $f^{-1}(x)$, in its simplest form.

b) Find the exact value of $ff(1)$.

c) Hence, or otherwise, solve the equation $f(x) = ff(1)$.

$\boxed{f^{-1}(x) = \frac{1}{2}\left(1 + e^{4-x}\right)}$, $\boxed{ff(1) = 4 - \ln 7}$, $\boxed{x = 4}$

(a) $y = 4 - \ln(2x-1)$

$\Rightarrow \ln(2x-1) = 4 - y$

$\Rightarrow 2x - 1 = e^{4-y}$

$\Rightarrow 2x = 1 + e^{4-y}$

$\Rightarrow x = \frac{1}{2}(1 + e^{4-y})$

$\therefore f^{-1}(x) = \frac{1}{2}(1 + e^{4-x})$

(b) $f(f(1)) = f(4 - \cancel{\ln 1})$

$= f(4)$

$= 4 - \ln 7$

(c) $\cancel{4} - \ln(2x-1) = \cancel{4} - \ln 7$

$\Rightarrow \ln(2x-1) = \ln 7$

$\Rightarrow 2x - 1 = 7$

$\Rightarrow x = 4$

Question 12 (***)

Find, in exact form where appropriate, the solution of each of the following equations.

a) $e^{2x+1} = 4$

b) $\ln(4y+1) = 2$

c) $2\ln t + \ln 3 = \ln(5t+2)$

$\boxed{x = \frac{1}{2}(-1 + \ln 4)}$, $\boxed{y = \frac{1}{4}\left(e^2 - 1\right)}$, $\boxed{t = 2}$

(a) $e^{2x+1} = 4$

$\Rightarrow 2x + 1 = \ln 4$

$\Rightarrow 2x = -1 + \ln 4$

$\Rightarrow x = \frac{1}{2}(-1 + \ln 4)$

(b) $\ln(4y+1) = 2$

$\Rightarrow 4y + 1 = e^2$

$\Rightarrow 4y = e^2 - 1$

$\Rightarrow y = \frac{1}{4}(e^2 - 1)$

(c) $2\ln t + \ln 3 = \ln(5t+2)$

$\Rightarrow \ln t^2 + \ln 3 = \ln(5t+2)$

$\Rightarrow \ln(3t^2) = \ln(5t+2)$

$\Rightarrow 3t^2 = 5t + 2$

$\Rightarrow 3t^2 - 5t - 2 = 0$

$\Rightarrow (3t+1)(t-2) = 0$

$t = 2, \ \cancel{-\frac{1}{3}}$

Question 13 (***)

Find, without the use of a calculating aid, the solution of the equation

$$x\ln 9 + \ln 28 = \ln 12 + x\ln 49$$

giving the answer as an exact fraction.

$$\boxed{x = \frac{1}{2}}$$

$$x\ln 9 + \ln 28 = \ln 12 + x\ln 49$$
$$\Rightarrow x\ln 9 - x\ln 49 = \ln 12 - \ln 28$$
$$\Rightarrow x(\ln 9 - \ln 49) = \ln 12 - \ln 28$$
$$\Rightarrow x = \frac{\ln 12 - \ln 28}{\ln 9 - \ln 49}$$
$$\Rightarrow x = \frac{\ln \frac{12}{28}}{\ln \frac{9}{49}} = \frac{\ln \frac{3}{7}}{\ln\left(\frac{3}{7}\right)^2}$$
$$\Rightarrow x = \frac{\ln\left(\frac{3}{7}\right)}{2\ln\left(\frac{3}{7}\right)}$$
$$\Rightarrow x = \frac{1}{2}$$

Question 14 (***)

Rearrange each of the following equations for x.

a) $y = \ln(x-2)+3, \quad x \in \mathbb{R},\ x > 2$

b) $y = \frac{1}{2}\left(e^{x-4}+3\right)$

$\boxed{x = e^{y-3}+2}$, $\boxed{x = 4+\ln(2y-3)}$

(a) $y = \ln(x-2)+3$
$$\Rightarrow y-3 = \ln(x-2)$$
$$\Rightarrow e^{y-3} = x-2$$
$$\Rightarrow e^{y-3}+2 = x$$
$$\Rightarrow x = 2+e^{y-3}$$

(b) $y = \frac{1}{2}\left(e^{x-4}+3\right)$
$$\Rightarrow 2y = e^{x-4}+3$$
$$\Rightarrow 2y-3 = e^{x-4}$$
$$\Rightarrow \ln(2y-3) = x-4$$
$$\Rightarrow 4+\ln(2y-3) = x$$
$$\Rightarrow x = 4+\ln(2y-3)$$

Question 15 (***)

A liquid is cooling down and its temperature θ °C satisfies

$$\theta = 20 + 30\mathrm{e}^{-\frac{t}{20}}, \ t \geq 0$$

where t is the time in minutes after a given instant.

Find the value of t when the temperature of the liquid has reduced to half its initial temperature.

$\boxed{}$, $\boxed{t = 35.8}$

$\theta = 20 + 30e^{-\frac{1}{20}t}$

WHEN $t=0$ (INITIALLY)

$\theta = 20 + 30e^{0}$

$\theta = 20 + 30 \times 1$

$\theta = 50$

INITIAL TEMPERATURE IS 50°C IE HALF THE INITIAL TEMPERATURE IS 25°C

$\Rightarrow 25 = 20 + 30e^{-\frac{1}{20}t}$

$\Rightarrow 5 = 30e^{-\frac{1}{20}t}$

$\Rightarrow \frac{1}{6} = e^{-\frac{1}{20}t}$

$\Rightarrow 6 = e^{\frac{1}{20}t}$

$\Rightarrow \ln 6 = \frac{1}{20}t$

$\Rightarrow t = 20\ln 6$

or 35.8

Question 16 (***)

A car tyre develops a puncture.

The tyre pressure P, measured in suitable units known as p.s.i., t minutes after the tyre got punctured is given by the expression

$$P = 8 + 32e^{-kt}, \ t \geq 0,$$

where k is a positive constant.

a) State the tyre pressure when the tyre got punctured.

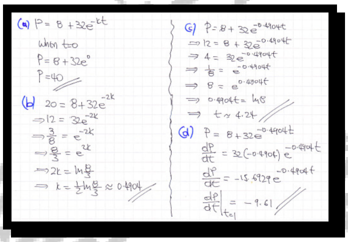
The tyre pressure halves 2 minutes after the puncture occurred.

b) Show that $k = 0.4904$, correct to 4 significant figures.

c) Calculate the time it takes for the tyre pressure to drop to 12 p.s.i.

d) Find the rate at which the pressure of the tyre is changing one minute after the puncture occurred.

$\boxed{}$, $\boxed{P = 40}$, $\boxed{t \approx 4.24}$, $\boxed{-9.61 \text{ p.s.i/min}}$



Question 17 (***)

The population P, in thousands, of a colony of rabbits in time t years after a certain instant, is given by

$$P = 5 + a\mathrm{e}^{-bt}, \ t \geq 0$$

where a and b are positive constants.

It is given that the initial population is 8 thousands rabbits, and one year later this population has reduced by 2 thousands.

a) Find the value of a and the value of b.

b) Explain mathematically, why the population can never reach 1000, according to this model.

$\boxed{a = 3}$, $\boxed{b = \ln 3}$

(a) $P = 5 + a e^{-bt}$

- $t=0 \quad P=8 \Rightarrow 8 = 5 + a e^{0}$
  $8 = 5 + a$
  $a = 3$
- $t=1 \quad P=6 \Rightarrow 6 = 5 + 3e^{-b \times 1}$
  $\Rightarrow 1 = 3e^{-b}$
  $\Rightarrow \frac{1}{3} = e^{-b}$
  $\Rightarrow 3 = e^{b}$
  $\Rightarrow b = \ln 3$

(b) As $t \to \infty$

$e^{-bt} \to 0$

$\therefore P \to 5$

IE POPULATION CAN ONLY REDUCE TO 5000

$\therefore$ CAN NEVER DROP TO 1000

Question 18 (***)

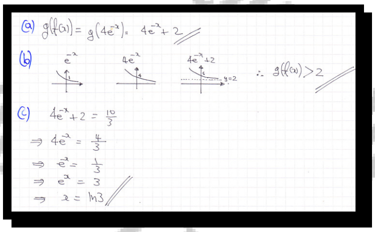
$$f(x) = 4e^{-x}, \ x \in \mathbb{R}$$
$$g(x) = x + 2, \ x \in \mathbb{R}$$

a) Find an expression for $gf(x)$.

b) State the range of $gf(x)$.

c) Solve the equation $gf(x) = \dfrac{10}{3}$.

$\boxed{gf(x) = 4e^{-x} + 2}$, $\boxed{gf(x) > 2}$, $\boxed{x = \ln 3}$



(a) $g(f(x)) = g(4e^{-x}) = 4e^{-x} + 2$

(b) e^{-x} $\quad 4e^{-x}$ $\quad 4e^{-x}+2$ $\quad y=2$ $\quad \therefore g(f(x)) > 2$

(c) $4e^{-x} + 2 = \frac{10}{3}$

$\Rightarrow 4e^{-x} = \frac{4}{3}$

$\Rightarrow e^{-x} = \frac{1}{3}$

$\Rightarrow e^{x} = 3$

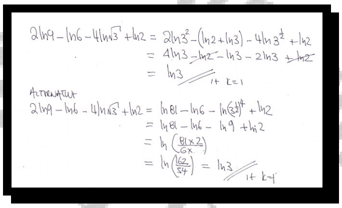
$\Rightarrow x = \ln 3$

Question 19 (***)

Simplify the following expression, giving the final answer in the form $k\ln 3$, where k is an integer.

$$2\ln 9 - \ln 6 - 4\ln\sqrt{3} + \ln 2$$

$\boxed{k = 1}$



$2\ln 9 - \ln 6 - 4\ln\sqrt{3} + \ln 2 = 2\ln 3^2 - (\ln 2 + \ln 3) - 4\ln 3^{\frac{1}{2}} + \ln 2$

$= 4\ln 3 \cancel{- \ln 2} - \ln 3 - 2\ln 3 \cancel{+ \ln 2}$

$= \ln 3$ if $k=1$

ALTERNATIVE

$2\ln 9 - \ln 6 - 4\ln\sqrt{3} + \ln 2 = \ln 81 - \ln 6 - \ln(3^{\frac{1}{2}})^4 + \ln 2$

$= \ln 81 - \ln 6 - \ln 9 + \ln 2$

$= \ln\left(\frac{81 \times 2}{6 \times 9}\right)$

$= \ln\left(\frac{162}{54}\right) = \ln 3$ if $k=1$

Question 20 (***)

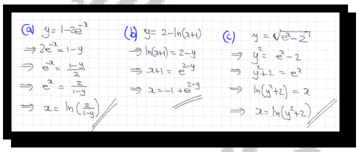
Rearrange each of the following equations for x.

a) $y = 1 - 2e^{-x}$

b) $y = 2 - \ln(x+1), \quad x > -1$

c) $y = \sqrt{e^x - 2}, \quad x \geq \ln 2$

$$\boxed{x = -\ln\left(\frac{1-y}{2}\right) = \ln\left(\frac{2}{1-y}\right)}, \quad \boxed{x = e^{2-y} - 1}, \quad \boxed{x = \ln\left(y^2 + 2\right)}$$



(a) $y = 1 - 2e^{-x}$
$\Rightarrow 2e^{-x} = 1 - y$
$\Rightarrow e^{-x} = \frac{1-y}{2}$
$\Rightarrow e^{x} = \frac{2}{1-y}$
$\Rightarrow x = \ln\left(\frac{2}{1-y}\right)$

(b) $y = 2 - \ln(x+1)$
$\Rightarrow \ln(x+1) = 2 - y$
$\Rightarrow x+1 = e^{2-y}$
$\Rightarrow x = -1 + e^{2-y}$

(c) $y = \sqrt{e^x - 2}$
$\Rightarrow y^2 = e^x - 2$
$\Rightarrow y^2 + 2 = e^x$
$\Rightarrow \ln(y^2+2) = x$
$\Rightarrow x = \ln(y^2+2)$

Question 21 (***)

A hot drink is cooling down in a room.

The temperature T °C of the drink, t minutes after it was made is modelled by

$$T = 22 + 50\mathrm{e}^{-\frac{1}{8}t}, \quad t \geq 0 .$$

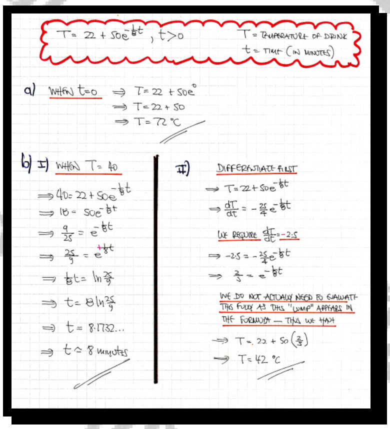
a) State the temperature of the drink when it was first made.

b) Assuming the drink is not consumed ...

i. ... calculate, to the nearest minute, the value of t when the temperature of the drink has reached $40°\text{C}$.

ii. ... determine the value of T, when the drink is cooling at the rate of $2.5°\text{C}$ per minute.

$\boxed{}$, $\boxed{T = 72}$, $\boxed{t \approx 8 \text{ min}}$, $\boxed{T = 42}$



$T = 22 + 50e^{-\frac{1}{8}t}$, $t>0$ T = TEMPERATURE OF DRINK, t = TIME (IN MINUTES)

a) WHEN $t=0$ $\Rightarrow T = 22 + 50e^0$
$\Rightarrow T = 22 + 50$
$\Rightarrow T = 72\,°\text{C}$

b) I) WHEN $T = 40$
$\Rightarrow 40 = 22 + 50e^{-\frac{1}{8}t}$
$\Rightarrow 18 = 50e^{-\frac{1}{8}t}$
$\Rightarrow \frac{9}{25} = e^{-\frac{1}{8}t}$
$\Rightarrow \frac{25}{9} = e^{+\frac{1}{8}t}$
$\Rightarrow \frac{1}{8}t = \ln\frac{25}{9}$
$\Rightarrow t = 8\ln\frac{25}{9}$
$\Rightarrow t = 8.1732\ldots$
$\Rightarrow t \simeq 8$ minutes

II) DIFFERENTIATE FIRST
$\Rightarrow T = 22 + 50e^{-\frac{1}{8}t}$
$\Rightarrow \frac{dT}{dt} = -\frac{25}{4}e^{-\frac{1}{8}t}$
WE REQUIRE $\frac{dT}{dt} = -2.5$
$\Rightarrow -2.5 = -\frac{25}{4}e^{-\frac{1}{8}t}$
$\Rightarrow \frac{2}{5} = e^{-\frac{1}{8}t}$
WE DO NOT ACTUALLY NEED TO EVALUATE THIS FULLY AS THIS "LUMP" APPEARS IN THE FORMULA — THUS WE HAVE
$\Rightarrow T = 22 + 50\left(\frac{2}{5}\right)$
$\Rightarrow T = 42\,°\text{C}$

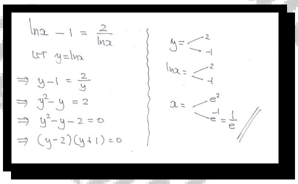
Question 22 **(***+)**

Solve the equation

$$\ln x = \frac{2}{\ln x} + 1, \quad x > 0,$$

giving the answers in exact form.

$$\boxed{x = \frac{1}{e}, e^2}$$



Question 23 **(***+)**

A population P is decreasing according to the formula

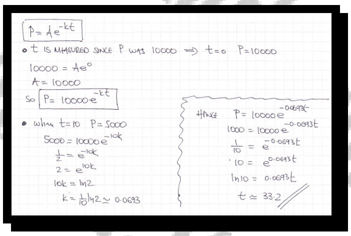
$$P = A e^{-kt}, \quad t \geq 0$$

where A and k are positive constants and t is the time in years since the population was 10000.

Ten years later the population has reduced to 5000.

Find the value of t when the population reaches 1000.

$$\boxed{t \approx 33.22}$$



Question 24 (***+)

Find in exact form the solution of the equation

$$\ln\left(x^2-2x-8\right)=1+\ln\left(x^2-6x+8\right).$$

$$\boxed{x=\frac{2\mathrm{e}+2}{\mathrm{e}-1}}$$

$$\ln(x^2-2x-8)=1+\ln(x^2-6x+8)$$
$$\Rightarrow \ln(x^2-2x-8)-\ln(x^2-6x+8)=1$$
$$\Rightarrow \ln\left(\frac{x^2-2x-8}{x^2-6x+8}\right)=1$$
$$\Rightarrow \frac{x^2-2x-8}{x^2-6x+8}=e^1$$
$$\Rightarrow \frac{(x+2)\cancel{(x-4)}}{(x-2)\cancel{(x-4)}}=e$$
$$\Rightarrow x+2=ex-2e$$
$$\Rightarrow 2+2e=ex-x$$
$$\Rightarrow 2e+2=x(e-1)$$
$$\Rightarrow x=\frac{2e+2}{e-1}$$

Question 25 (***+)

The volume of water in a tank V m^3, t hours after midnight, is given by the equation

$$V=10+8\mathrm{e}^{-\frac{1}{12}t},\ t\geq 0.$$

a) State the volume of water in the tank at midnight.

b) Find the time, using 24 hour clock notation, when the volume of the water in the tank is 14 m^3.

c) Determine the rate at which the volume of the water is changing at midday, explaining the significance of its sign.

d) State the limiting value of V.

$\boxed{}$, $\boxed{V=18}$, $\boxed{08:19}$, $\boxed{-\frac{2}{3\mathrm{e}}\approx -0.245,\ \text{decrease}}$, $\boxed{t\to\infty,\ V\to 10}$

(a) $V=10+8e^{-\frac{1}{12}t}$

$t=0 \quad V=10+8e^0=18$

(b) $14=10+8e^{-\frac{1}{12}t}$

$4=8e^{-\frac{1}{12}t}$

$\frac{1}{2}=e^{-\frac{1}{12}t}$

$2=e^{\frac{1}{12}t}$

$\ln 2=\frac{1}{12}t$

$t=12\ln 2$

$t\simeq 8.317\ldots$ $\quad \therefore 0.317\times 60\approx 19$ $\quad \therefore 08:19$

(c) $V=10+8e^{-\frac{1}{12}t}$

$\frac{dV}{dt}=-\frac{2}{3}e^{-\frac{1}{12}t}$

$\left.\frac{dV}{dt}\right|_{t=12}=-\frac{2}{3}e^{-1}\approx -0.245$

(minus = decrease)

(d) As $t\to\infty$ $\quad e^{-\frac{1}{12}t}\to 0$

$8e^{-\frac{1}{12}t}\to 0$

$V\to 10$

Question 26 (***+)

Find the exact solution of the simultaneous equations

$$e^x + e^y = 8$$

$$2e^x + e^{2y} = 16\,.$$

$\boxed{x = \ln 6}$, $\boxed{y = \ln 2}$

$$\left.\begin{array}{l} e^x + e^y = 8 \\ 2e^x + e^{2y} = 16 \end{array}\right\} \quad \left.\begin{array}{l} e^x + e^y = 8 \\ 2e^x + (e^y)^2 = 16 \end{array}\right\} \Rightarrow \begin{array}{l} X + Y = 8 \\ 2X + Y^2 = 16 \end{array} \quad \text{with } \begin{array}{l} X = e^x \\ Y = e^y \end{array}$$

HENCE $X = 8 - Y \Rightarrow$ SUB INTO THE OTHER

$$2(8-Y) + Y^2 = 16$$
$$16 - 2Y + Y^2 = 16$$
$$Y^2 - 2Y = 0$$
$$Y(Y-2) = 0$$
$$Y = \begin{cases} 0 \\ 2 \end{cases}$$
$$e^y = \begin{cases} 0 \\ 2 \end{cases}$$
$$y = \ln 2$$

$$X = 8 - Y$$
$$e^x = 8 - e^y$$
$$e^x = 8 - 2$$
$$e^x = 6$$
$$x = \ln 6$$

$\therefore (\ln 6, \ln 2)$ //

Question 27 (***+)

A curve has equation

$$y = A\ln\left(Bx\right),\ x > 0$$

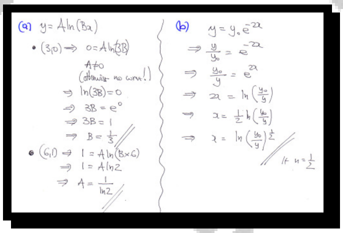
where A and B are positive constants.

a) Given the curve passes through the points with coordinates $(3,0)$ and $(6,1)$ find the exact values of A and B.

A different curve has equation $y = y_0\,\mathrm{e}^{-2x}$, where y_0 is a constant.

b) Show clearly that $x = \ln\left(\dfrac{y_0}{y}\right)^n$, where n is a constant to be found.

$$\boxed{A = \frac{1}{\ln 2}},\ \boxed{B = \frac{1}{3}},\ \boxed{n = \frac{1}{2}}$$



(a) $y = A\ln(Bx)$

- $(3,0) \Rightarrow 0 = A\ln(3B)$

$A \neq 0$ (otherwise no curve!)

$\Rightarrow \ln(3B) = 0$

$\Rightarrow 3B = e^0$

$\Rightarrow 3B = 1$

$\Rightarrow B = \frac{1}{3}$

- $(6,1) \Rightarrow 1 = A\ln(B\times 6)$

$\Rightarrow 1 = A\ln 2$

$\Rightarrow A = \frac{1}{\ln 2}$

(b) $y = y_0 e^{-2x}$

$\Rightarrow \frac{y}{y_0} = e^{-2x}$

$\Rightarrow \frac{y_0}{y} = e^{2x}$

$\Rightarrow 2x = \ln\left(\frac{y_0}{y}\right)$

$\Rightarrow x = \frac{1}{2}\ln\left(\frac{y_0}{y}\right)$

$\Rightarrow x = \ln\left(\frac{y_0}{y}\right)^{\frac{1}{2}}$

i.e. $n = \frac{1}{2}$

Question 28 **(***+)**

Make x the subject in each of the following expressions.

a) $y = y_0 \, e^{0.2(x-1)}$.

b) $y = \ln\sqrt{\dfrac{4}{x-1}}$, $x > 1$.

$$\boxed{x = 1 + 5\ln\left(\frac{y}{y_0}\right)}, \quad \boxed{x = 1 + 4e^{-2y} = \frac{e^{2y}+4}{e^{2y}}}$$

(a) $y = y_0 e^{0.2(x-1)}$
$\Rightarrow \frac{y}{y_0} = e^{\frac{1}{5}(x-1)}$
$\Rightarrow \ln\left(\frac{y}{y_0}\right) = \frac{1}{5}(x-1)$
$\Rightarrow 5\ln\left(\frac{y}{y_0}\right) = x-1$
$\Rightarrow 5\ln\left(\frac{y}{y_0}\right) + 1 = x$
$\therefore x = 1 + 5\ln\left(\frac{y}{y_0}\right)$ //

(b) $y = \ln\sqrt{\frac{4}{x-1}}$
$\Rightarrow y = \ln\left(\frac{4}{x-1}\right)^{\frac{1}{2}}$
$\Rightarrow e^y = \left(\frac{4}{x-1}\right)^{\frac{1}{2}}$
$\Rightarrow e^{2y} = \frac{4}{x-1}$
$\Rightarrow xe^{2y} - e^{2y} = 4$
$\Rightarrow xe^{2y} = 4 + e^{2y}$
$\Rightarrow x = \frac{4+e^{2y}}{e^{2y}}$ //

Question 29 **(***+)**

Determine, in exact simplified form where appropriate, the solution of each of the following equations.

a) $2\ln 54 - x\ln 3 = \ln 12$

b) $e^{-3y} + 5 = 32$

c) $\ln(1-2w) = 1 + \ln w$

$$\boxed{x = 5}, \quad \boxed{y = -\ln 3}, \quad \boxed{w = \frac{1}{2+e} \approx 0.212}$$

(a) $2\ln 54 - x\ln 3 = \ln 12$
$\Rightarrow 2\ln 54 - \ln 12 = x\ln 3$
$\Rightarrow \ln 2916 - \ln 12 = x\ln 3$
$\Rightarrow \ln 243 = x\ln 3$
$\Rightarrow x = \frac{\ln 243}{\ln 3}$ BUT $3^5 = 243$
$\Rightarrow x = \frac{5\cancel{\ln 3}}{\cancel{\ln 3}} = 5$ //

(b) $e^{-3y} + 5 = 32$
$\Rightarrow e^{-3y} = 27$
$\Rightarrow -3y = \ln 27$
$\Rightarrow -\cancel{3}y = \cancel{3}\ln 3$
$\Rightarrow y = -\ln 3$ //

(c) $\ln(1-2w) = 1 + \ln w$
$\Rightarrow \ln(1-2w) - \ln w = 1$
$\Rightarrow \ln\left(\frac{1-2w}{w}\right) = 1$
$\Rightarrow \frac{1-2w}{w} = e^1$
$\Rightarrow 1 - 2w = ew$
$\Rightarrow 1 = ew + 2w$
$\Rightarrow 1 = w(e+2)$
$\Rightarrow w = \frac{1}{e+2}$ //

Question 30 **(***+)**

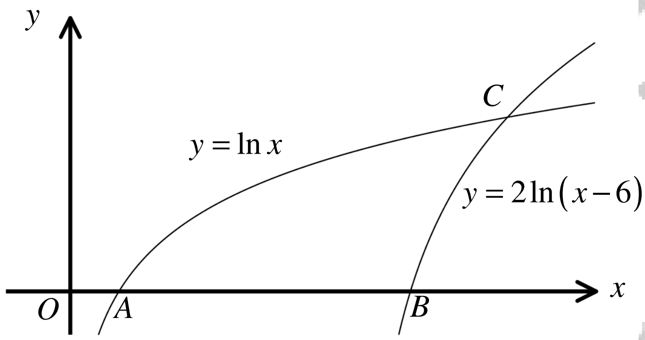
Show that $x = -\ln 2$ is the only real solution of the equation

$$4\mathrm{e}^{-2x} - \mathrm{e}^{-4x} = 0 .$$

proof

$$4e^{-2x} - e^{-4x} = 0$$
$$\Rightarrow 4e^{-2x} = e^{-4x}$$
$$\Rightarrow \frac{4}{e^{2x}} = \frac{1}{e^{4x}}$$
$$\Rightarrow 4e^{4x} = e^{2x}$$
$$\Rightarrow 4\frac{e^{4x}}{e^{2x}} = 1 \quad (e^{2x} \neq 0)$$
$$\Rightarrow 4e^{2x} = 1$$
$$\Rightarrow e^{2x} = \frac{1}{4}$$
$$\Rightarrow 2x = \ln\frac{1}{4}$$
$$\Rightarrow x = \frac{1}{2}\ln\frac{1}{4}$$
$$\Rightarrow x = -\frac{1}{2}\ln 4$$
$$\Rightarrow x = -\frac{1}{2}\ln 2^2$$
$$\Rightarrow x = -\ln 2$$

Question 31 (***+)



The figure above shows the curves with equations

$$y = \ln x\,,\ x > 0 \quad \text{and} \quad y = 2\ln(x-6)\,,\ x > 6\,.$$

The curves cross the x axis at the points A and B respectively, and intersect each other at the point C.

a) Write down the coordinates of A and B.

b) Determine the exact coordinates of C.

$\boxed{A(1,0)}$, $\boxed{B(7,0)}$, $\boxed{B(9,2\ln 3)}$

(a) • $y = 2\ln(x-6)$
$0 = \cancel{2}\ln(x-6)$
$e^0 = x-6$
$x = 7$
$B(7,0)$ //

• $y = \ln x$
$0 = \ln x$
$e^0 = x$
$x = 1$
$A(1,0)$ //

(b) $\left.\begin{array}{l} y = \ln x \\ y = 2\ln(x-6) \end{array}\right\} \Rightarrow \ln x = 2\ln(x-6)$
$\Rightarrow \ln x = \ln(x-6)^2$
$\Rightarrow x = (x-6)^2$
$\Rightarrow x = x^2 - 12x + 36$
$\Rightarrow 0 = x^2 - 13x + 36$
$\Rightarrow 0 = (x-9)(x-4)$
$\Rightarrow x = \begin{cases} 9 \\ \cancel{4} \end{cases}$

thus $y = \ln 9 = 2\ln 3$
$\therefore C(9, 2\ln 3)$ //

Question 32 (***+)

Find the exact solution of the following equation

$$e^x - e^{-x} = \frac{3}{2}.$$

$\boxed{\ }$, $\boxed{x = \ln 2}$

$$e^x - e^{-x} = \frac{3}{2}$$
$$\Rightarrow 2e^x - 2e^{-x} = 3$$
$$\Rightarrow 2e^x - \frac{2}{e^x} = 3$$
$$\Rightarrow 2y - \frac{2}{y} = 3 \quad (y = e^x)$$
$$\Rightarrow 2y^2 - 2 = 3y$$
$$\Rightarrow 2y^2 - 3y - 2 = 0$$
$$\Rightarrow (2y+1)(y-2)$$
$$\Rightarrow y = \begin{cases} 2 \\ -\frac{1}{2} \end{cases}$$
$$\Rightarrow e^x = \begin{cases} 2 \\ \cancel{-\frac{1}{2}} \end{cases}$$
$$\Rightarrow e^x = 2$$
$$\Rightarrow x = \ln 2$$

Question 33 (***+)

Show clearly that

$$5\ln\left(\frac{5}{6}\right) + 2\ln\left(\frac{4}{3}\right) - \left[5\ln\left(\frac{2}{3}\right) + 2\ln\left(\frac{5}{3}\right)\right] \equiv 3(\ln a - \ln b)$$

where a and b are integers to be found.

$\boxed{a = 5}$, $\boxed{b = 4}$

$$5\ln\left(\frac{5}{6}\right) + 2\ln\left(\frac{4}{3}\right) - \left[5\ln\left(\frac{2}{3}\right) + 2\ln\left(\frac{5}{3}\right)\right] = 5\ln\left(\frac{5}{6}\right) - 5\ln\left(\frac{2}{3}\right) + 2\ln\left(\frac{4}{3}\right) - 2\ln\left(\frac{5}{3}\right)$$
$$= 5\left[\ln\left(\frac{5}{6}\right) - \ln\left(\frac{2}{3}\right)\right] + 2\left[\ln\left(\frac{4}{3}\right) - \ln\left(\frac{5}{3}\right)\right] = 5\ln\left(\frac{5/6}{2/3}\right) + 2\ln\left(\frac{4/3}{5/3}\right)$$
$$= 5\ln\left(\frac{5}{4}\right) + 2\ln\left(\frac{4}{5}\right) = 5\ln\left(\frac{5}{4}\right) - 2\ln\left(\frac{5}{4}\right) = 3\ln\left(\frac{5}{4}\right) = 3(\ln 5 - \ln 4)$$

$a = 5$
$b = 4$

Question 34 (***+)

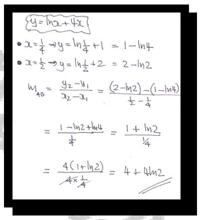
A curve has equation

$$y = \ln x + 4x, \quad x > 0.$$

The points A and B lie on this curve, where $x = \frac{1}{4}$ and $x = \frac{1}{2}$, respectively.

Show that the gradient of the straight line segment AB is $4 + 4\ln 2$.

$\boxed{\text{proof}}$



$\boxed{y = \ln x + 4x}$

- $x = \frac{1}{4} \Rightarrow y = \ln\frac{1}{4} + 1 = 1 - \ln 4$
- $x = \frac{1}{2} \Rightarrow y = \ln\frac{1}{2} + 2 = 2 - \ln 2$

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(2-\ln 2) - (1 - \ln 4)}{\frac{1}{2} - \frac{1}{4}}$$

$$= \frac{1 - \ln 2 + \ln 4}{\frac{1}{4}} = \frac{1 + \ln 2}{\frac{1}{4}}$$

$$= \frac{4(1 + \ln 2)}{\cancel{4 \times \frac{1}{4}}} = 4 + 4\ln 2$$

Question 35 (***+)

The functions $f(x)$ and $g(x)$ are defined

$$f(x) = 4 + \ln x, \ \ x \in \mathbb{R}, \ \ x > 0$$

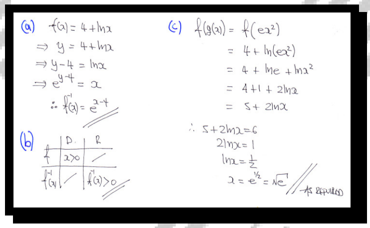
$$g(x) = \mathrm{e}\, x^2, \ \ x \in \mathbb{R}.$$

a) Find an expression for $f^{-1}(x)$.

b) State the range of $f^{-1}(x)$.

c) Show that $x = \sqrt{\mathrm{e}}$ is the solution of the equation $fg(x) = 6$.

$\boxed{f^{-1}(x) = \mathrm{e}^{x-4}}$, $\boxed{f^{-1}(x) > 0}$


(a) $f(x) = 4 + \ln x$

$\Rightarrow y = 4 + \ln x$

$\Rightarrow y - 4 = \ln x$

$\Rightarrow e^{y-4} = x$

$\therefore f^{-1}(x) = e^{x-4}$

(b)

	D.	R
f	$x>0$	/
$f^{-1}(x)$	/	$f^{-1}(x) > 0$

(c) $f(g(x)) = f(ex^2)$

$= 4 + \ln(ex^2)$

$= 4 + \ln e + \ln x^2$

$= 4 + 1 + 2\ln x$

$= 5 + 2\ln x$

$\therefore 5 + 2\ln x = 6$

$2\ln x = 1$

$\ln x = \frac{1}{2}$

$x = e^{1/2} = \sqrt{e}$ — AS REQUIRED

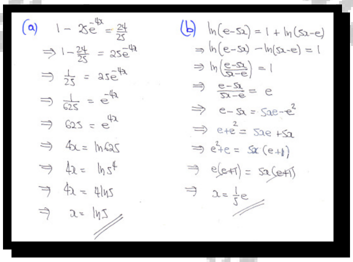
Question 36 (***+)

Find, in exact simplified form, the solution of each of the following equations.

a) $1-25e^{-4x}=\dfrac{24}{25}$.

b) $\ln(e-5x)=1+\ln(5x-e)$.

$\boxed{x=\ln 5}$, $\boxed{x=\frac{1}{5}e}$



(a) $1-25e^{-4x}=\frac{24}{25}$

$\Rightarrow 1-\frac{24}{25}=25e^{-4x}$

$\Rightarrow \frac{1}{25}=25e^{-4x}$

$\Rightarrow \frac{1}{625}=e^{-4x}$

$\Rightarrow 625=e^{4x}$

$\Rightarrow 4x=\ln 625$

$\Rightarrow 4x=\ln 5^4$

$\Rightarrow 4x=4\ln 5$

$\Rightarrow x=\ln 5$

(b) $\ln(e-5x)=1+\ln(5x-e)$

$\Rightarrow \ln(e-5x)-\ln(5x-e)=1$

$\Rightarrow \ln\left(\frac{e-5x}{5x-e}\right)=1$

$\Rightarrow \frac{e-5x}{5x-e}=e$

$\Rightarrow e-5x=5xe-e^2$

$\Rightarrow e+e^2=5xe+5x$

$\Rightarrow e^2+e=5x(e+1)$

$\Rightarrow e\cancel{(e+1)}=5x\cancel{(e+1)}$

$\Rightarrow x=\frac{1}{5}e$

Question 37 (***+)

The value, £V, of a painting is modelled by the equation

$$V = 600 + 200\ln(2t+1), \ t \geq 0$$

where t is the number of years since the painting was purchased.

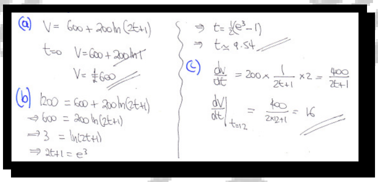
Determine…

a) … the value of the painting when it was first purchased.

b) … the value of t when the value of the painting doubles.

c) … the rate at which the value of the painting is increasing twelve years after it was purchased.

$\boxed{V = 600}$, $\boxed{t \approx 9.54}$, $\boxed{\text{£}16 \text{ per year}}$

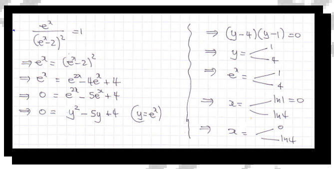


Question 38 (***+)

Find, in exact form where appropriate, the solutions of the equation

$$\frac{e^x}{(e^x - 2)^2} = 1.$$

$\boxed{x = 0, \ x = \ln 4}$



Question 39 (***+)

The function $f(x)$ is given by

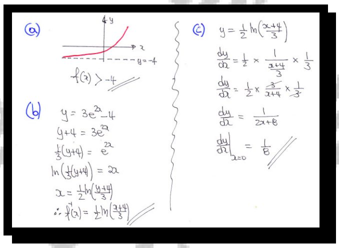
$$f(x) = 3e^{2x} - 4, \quad x \in \mathbb{R}.$$

a) State the range of $f(x)$.

b) Find an expression for $f^{-1}(x)$.

c) Find the value of the gradient on $f^{-1}(x)$ at the point where $x = 0$.

$\boxed{f(x) > -4}$, $\boxed{f^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+4}{3}\right)}$, $\boxed{\frac{1}{8}}$



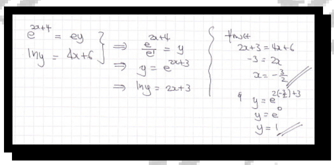
Question 40 (***+)

Find the solution of the following simultaneous equations

$$e^{2x+4} = e\,y$$

$$\ln y = 4x + 6.$$

$\boxed{}$, $\boxed{x = -\frac{3}{2}, \ y = 1}$



Question 41 (***+)

The functions $f(x)$ and $g(x)$ are defined

$$f(x) = x^2 - 10x, \quad x \in \mathbb{R}$$

$$g(x) = \mathrm{e}^x + 5, \quad x \in \mathbb{R}.$$

a) Find, showing all the steps of the calculation, the value of $g(3\ln 2)$.

b) Find, in its simplest form, an expression for $fg(x)$.

c) Show that $g(2x) - fg(x) = k$, stating the value of the constant k.

d) Solve the equation $gf(x) = 6$.

$\boxed{g(3\ln 2) = 13}$, $\boxed{fg(x) = \mathrm{e}^{2x} - 25}$, $\boxed{k = 30}$, $\boxed{x = 0,10}$

(a) $g(3\ln 2) = e^{3\ln 2} + 5 = e^{\ln 8} + 5 = 8 + 5 = 13$ //

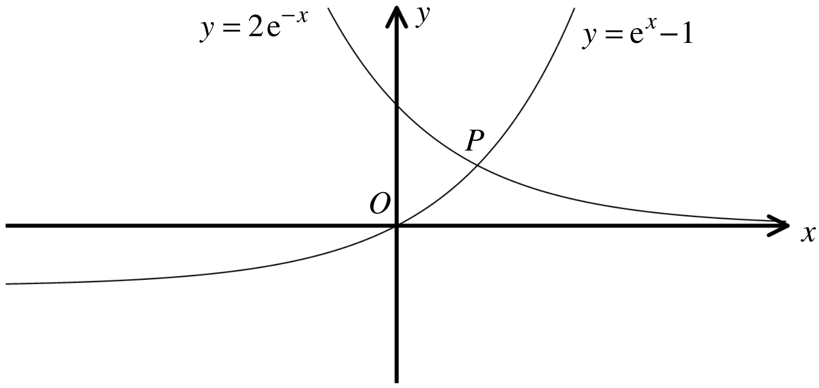
(b) $f(g(x)) = f(e^x + 5) = (e^x+5)^2 - 10(e^x+5) = e^{2x} + \cancel{10e^x} + 25 - \cancel{10e^x} - 50$
$= e^{2x} - 25$

(c) $g(2x) - fg(x) = (e^{2x} + 5) - (e^{2x} - 25) = 30$ // if $k = 30$

(d) $g(f(x)) = 6$
$\Rightarrow g(x^2 - 10x) = 6$
$\Rightarrow e^{x^2-10x} + 5 = 6$
$\Rightarrow e^{x^2-10x} = 1$

$\therefore x^2 - 10x = 0$
$x(x-10) = 0$
$x = 0, 10$ //

Question 42 (***+)



The figure above shows the graphs of the curves with equations

$$y = 2e^{-x} \quad \text{and} \quad y = e^{x} - 1.$$

The two graphs intersect at the point P.

Determine the exact coordinates of P.

$\boxed{}$, $\boxed{P(\ln 2, 1)}$

SOLVING THE EQUATIONS SIMULTANEOUSLY

$$\left.\begin{array}{l} y = 2e^{-x} \\ y = e^{x} - 1 \end{array}\right\} \implies e^{x} - 1 = 2e^{-x}$$

$$\implies e^{x} - 1 = \frac{2}{e^{x}}$$

$$\implies A - 1 = \frac{2}{A} \qquad (A = e^{x})$$

$$\implies A^2 - A = 2$$

$$\implies A^2 - A - 2 = 0$$

$$\implies (A+1)(A-2) = 0$$

$$\implies A = \begin{cases} -1 \\ 2 \end{cases}$$

$$\implies e^{x} = \begin{cases} \cancel{-1} \\ 2 \end{cases}$$

$$\implies x = \ln 2$$

USING $e^{x} = 2$ INTO THE SECOND EQUATION YIELDS $y = 1$

$\therefore P(\ln 2, 1)$

ALTERNATIVE

- $y = 2e^{-x}$
$\implies \frac{y}{2} = e^{-x}$
$\implies \frac{2}{y} = e^{x}$

- $y = e^{x} - 1$
$\implies e^{x} = y + 1$
$\implies \frac{2}{y} = y + 1$
$\implies 2 = y^2 + y$
$\implies y^2 + y - 2 = 0$
$\implies (y-1)(y+2) = 0$
$\implies y = \begin{cases} 1 \\ -2 \end{cases}$ $\quad e^{x} = y + 1$

$e^{x} = \begin{cases} 2 \\ \cancel{-1} \end{cases}$

$x = \ln 2$

HENCE $P(\ln 2, 1)$ AS BEFORE

Question 43 (***+)

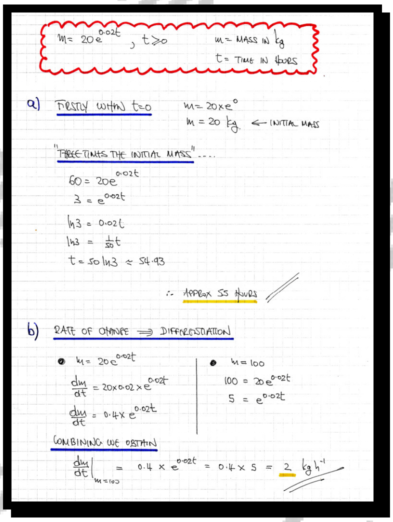
During a chemical process the mass of a substance, m kg, at time t hours grows exponentially according to the formula

$$m = 20e^{0.02t}, \ t \geq 0.$$

a) Find the time taken for the substance to increase to three times its initial mass.

b) Calculate the rate of change of m, when $m = 100$.

$\boxed{}$, $\boxed{t = 50\ln 3 \approx 54.93 \text{ hours}}$, $\boxed{\left.\dfrac{dm}{dt}\right|_{m=100} = 2}$



$m = 20e^{0.02t}$, $t \geqslant 0$ m = MASS IN kg, t = TIME IN HOURS

a) FIRSTLY WHEN $t=0$ $m = 20 \times e^0$, $m = 20$ kg ← INITIAL MASS

"THREE TIMES THE INITIAL MASS"....

$60 = 20e^{0.02t}$

$3 = e^{0.02t}$

$\ln 3 = 0.02t$

$\ln 3 = \frac{1}{50}t$

$t = 50\ln 3 \approx 54.93$

$\therefore$ APPROX 55 HOURS

b) RATE OF CHANGE $\Longrightarrow$ DIFFERENTIATION

• $m = 20e^{0.02t}$

$\frac{dm}{dt} = 20 \times 0.02 \times e^{0.02t}$

$\frac{dm}{dt} = 0.4 \times e^{0.02t}$

• $m = 100$

$100 = 20e^{0.02t}$

$5 = e^{0.02t}$

COMBINING WE OBTAIN

$\left.\frac{dm}{dt}\right|_{m=100} = 0.4 \times e^{0.02t} = 0.4 \times 5 = 2 \text{ kg h}^{-1}$

Question 44 (***+)

The half life of a radioactive isotope is the time it takes for a given mass to reduce to half the size of that given mass.

A radioactive substance reduces from 12 g to exactly 10.24 g in 30 days.

Assuming that the isotope decays exponentially, determine the half life of the isotope, correct to the nearest day.

$\boxed{}$, $\boxed{\approx 131 \text{ days}}$

STARTING WITH A STANDARD EXPONENTIALLY DECAYING MODEL

$\Rightarrow M = m_0 e^{-kt} \quad k>0 \quad (m_0 = \text{INITIAL MASS})$

$\Rightarrow 10.24 = 12 e^{-k \times 30}$

$\Rightarrow e^{-30k} = \frac{64}{75}$

$\Rightarrow e^{30k} = \frac{75}{64}$

$\Rightarrow 30k = \ln \frac{75}{64}$

$\Rightarrow \boxed{k = \frac{1}{30} \ln \frac{75}{64}}$

NOW WHEN $t = T$, $M = \frac{1}{2} m_0 = 6$

$\Rightarrow M = 12 e^{-\left(\frac{1}{30} \ln \frac{75}{64}\right)t}$

$\Rightarrow 6 = 12 e^{-\left(\frac{1}{30} \ln \frac{75}{64}\right)T}$

$\Rightarrow \frac{1}{2} = e^{-\left(\frac{1}{30} \ln \frac{75}{64}\right)T}$

$\Rightarrow 2 = e^{\left(\frac{1}{30} \ln \frac{75}{64}\right)T}$

$\Rightarrow \ln 2 = \left(\frac{1}{30} \ln \frac{75}{64}\right)T$

$\Rightarrow T = \frac{30 \ln 2}{\ln 75/64}$

$\Rightarrow T = 131.108187...$ $\quad \therefore$ HALF LIFE $\simeq 131$ days

Question 45 (****)

A curve has equation

$$y = \ln(Ax + B), \ x > 0,$$

where A and B are non zero constants.

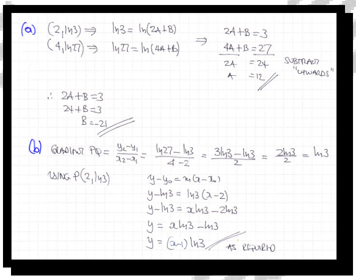
The curve passes through the points $P(2, \ln 3)$ and $Q(4, \ln 27)$.

a) Find the value of A and the value of B.

b) Show that the equation of the straight line through P and Q can be written as

$$y = (x-1)\ln 3.$$

$\boxed{A = 12}$, $\boxed{B = -21}$


(a) $(2, \ln 3) \Rightarrow \ln 3 = \ln(2A+B)$
$(4, \ln 27) \Rightarrow \ln 27 = \ln(4A+B)$ $\Rightarrow$ $2A+B=3$, $4A+B=27$
$2A = 24$
$A = 12$
SUBTRACT "UPWARDS"

$\therefore 2A+B=3$
$24+B=3$
$B=-21$

(b) GRADIENT $PQ = \dfrac{y_2-y_1}{x_2-x_1} = \dfrac{\ln 27 - \ln 3}{4-2} = \dfrac{3\ln 3 - \ln 3}{2} = \dfrac{2\ln 3}{2} = \ln 3$

USING $P(2, \ln 3)$
$y - y_0 = m(x - x_0)$
$y - \ln 3 = \ln 3(x-2)$
$y - \ln 3 = x\ln 3 - 2\ln 3$
$y = x\ln 3 - \ln 3$
$y = (x-1)\ln 3$ AS REQUIRED

Question 46 (****)

Find, in exact simplified form, the solution of each of the following equations.

a) $e^{2x-3} = 2e$.

b) $\ln(2y-1) = 1 + \ln(e - y)$.

$$\boxed{x = \frac{1}{2}(4 + \ln 2)}, \quad \boxed{y = \frac{e^2+1}{e+2}}$$

(a) $e^{2x-3} = 2e$
$\Rightarrow \frac{e^{2x-3}}{e} = 2$
$\Rightarrow e^{2x-4} = 2$
$\Rightarrow 2x - 4 = \ln 2$
$\Rightarrow 2x = 4 + \ln 2$
$\Rightarrow x = \frac{1}{2}(4 + \ln 2)$

(b) $\ln(2y-1) = 1 + \ln(e-y)$
$\Rightarrow \ln(2y-1) - \ln(e-y) = 1$
$\Rightarrow \ln\left(\frac{2y-1}{e-y}\right) = 1$
$\Rightarrow \frac{2y-1}{e-y} = e^1$
$\Rightarrow 2y - 1 = e^2 - ey$
$\Rightarrow 2y + ey = e^2 + 1$
$\Rightarrow y(2+e) = e^2 + 1$
$\Rightarrow y = \frac{e^2+1}{e+2}$

Question 47 (****)

Find, in exact form if appropriate, the solution of the following simultaneous equations

$$x + e^y = 5$$

$$\ln(x+1)^2 = 2y.$$

$$\boxed{}, \quad \boxed{x = 2}, \quad \boxed{y = \ln 3}$$

$\left.\begin{array}{l} x + e^y = 5 \\ 2y = \ln(x+1)^2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} e^y = 5 - x \\ 2y = 2\ln(x+1) \end{array}\right\} \Rightarrow \begin{array}{l} y = \ln(5-x) \\ y = \ln(x+1) \end{array}$

$\therefore \ln(5-x) = \ln(x+1)$
$5 - x = x + 1$
$4 = 2x$
$x = 2$ $\quad y = \ln 3$

Question 48 (****)

The function f is defined as

$$f(x) = a\ln(bx), \ x \in \mathbb{R}, \ x > 0$$

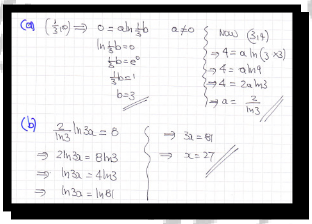
where a and b are positive constants.

a) Given that the graph of $f(x)$ passes through the points $\left(\frac{1}{3}, 0\right)$ and $(3, 4)$, find the exact value of a and the value of b.

b) Solve the equation

$$f(x) = 8.$$

$\boxed{a = \frac{2}{\ln 3}}$, $\boxed{b = 3}$, $\boxed{x = 27}$



(a) $\left(\frac{1}{3}, 0\right) \Rightarrow 0 = a\ln\frac{1}{3}b$ $a \neq 0$

$\ln\frac{1}{3}b = 0$

$\frac{1}{3}b = e^0$

$\frac{1}{3}b = 1$

$b = 3$

NOW $(3, 4)$

$\Rightarrow 4 = a\ln(3 \times 3)$

$\Rightarrow 4 = a\ln 9$

$\Rightarrow 4 = 2a\ln 3$

$\Rightarrow a = \frac{2}{\ln 3}$

(b) $\frac{2}{\ln 3}\ln 3x = 8$

$\Rightarrow 2\ln 3x = 8\ln 3$

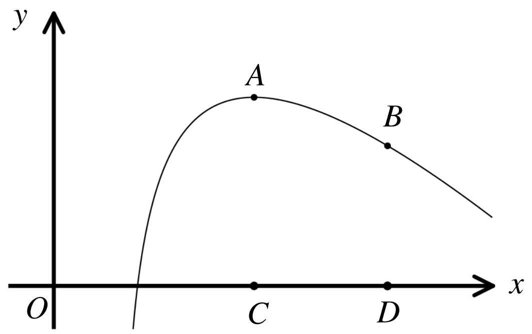
$\Rightarrow \ln 3x = 4\ln 3$

$\Rightarrow \ln 3x = \ln 81$

$\Rightarrow 3x = 81$

$\Rightarrow x = 27$

Question 49 (****)



The figure above shows the graph of the curve with equation

$$y = 4 - x + 2\ln(x-1), \quad x > 1.$$

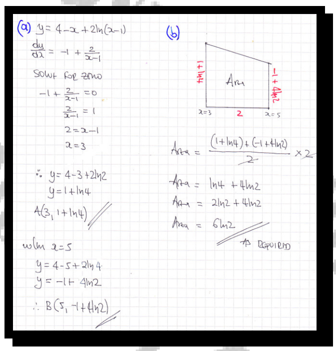
The point A is the maximum point of the curve and the point B lies on the curve where $x = 5$.

a) Find the coordinates of A and B.

The points C and D lie on the x axis directly below the points A and B, respectively.

b) Show that the area of the trapezium $ABDC$ is $6\ln 2$ square units.

$\boxed{A(3, 1+2\ln 2), B(5, -1+4\ln 2)}$



Question 50 (****)

An area is to be replanted with eucalyptus trees after a large fire.

The height, H m, of one such tree is given by the formula

$$H = 25 - 24e^{-0.1t}, \quad t \geq 0,$$

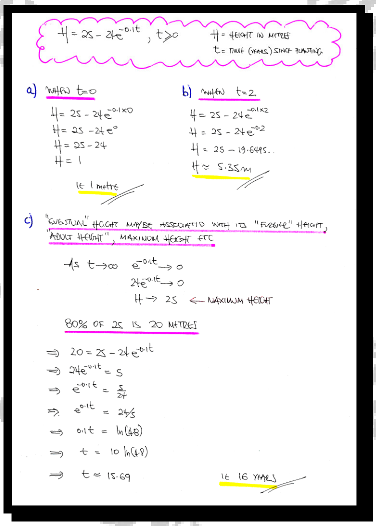
where t is the time in years since the tree was planted.

a) State the height of a newly planted tree.

b) Find the height of a tree, after 2 years.

c) Calculate, to the nearest integer, the value of t when the height of a tree has reached 80% of its eventual height.

$\boxed{}$, $\boxed{H_0 = 1 \text{ m}}$, $\boxed{H \approx 5.35 \text{ m}}$, $\boxed{t \approx 16}$



$H = 25 - 24e^{-0.1t}$, $t \geq 0$ H = HEIGHT IN METRES
t = TIME (YEARS) SINCE PLANTING

a) WHEN $t=0$

$H = 25 - 24e^{-0.1 \times 0}$
$H = 25 - 24e^{0}$
$H = 25 - 24$
$H = 1$

IE 1 metre

b) WHEN $t=2$

$H = 25 - 24e^{-0.1 \times 2}$
$H = 25 - 24e^{-0.2}$
$H = 25 - 19.6495..$
$H \approx 5.35$ m

c) "EVENTUAL" HEIGHT MAYBE ASSOCIATED WITH ITS "FOREVER" HEIGHT, "ADULT HEIGHT", MAXIMUM HEIGHT ETC

As $t \to \infty$ $e^{-0.1t} \to 0$
$24e^{-0.1t} \to 0$
$H \to 25$ ← MAXIMUM HEIGHT

80% OF 25 IS 20 METRES

$\Rightarrow 20 = 25 - 24e^{-0.1t}$
$\Rightarrow 24e^{-0.1t} = 5$
$\Rightarrow e^{-0.1t} = \frac{5}{24}$
$\Rightarrow e^{0.1t} = 24/5$
$\Rightarrow 0.1t = \ln(4.8)$
$\Rightarrow t = 10\ln(4.8)$
$\Rightarrow t \approx 15.69$

IE 16 YEARS

Question 51 (****)

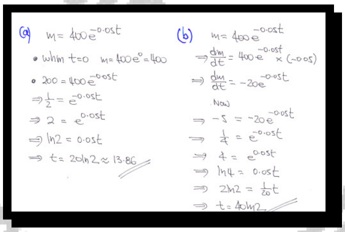
A radioactive substance decays so that its mass, m kg, at time t years from now, satisfies the exponential equation

$$m = 400\mathrm{e}^{-0.05t}, \; t \geq 0 .$$

a) Find the time it takes for the substance to halve its mass.

b) Determine the **exact** value of t when the radioactive substance is decaying at the rate of 5 kg per year, giving the answer in terms of $\ln 2$.

$\boxed{t = 20\ln 2 \approx 13.86 \text{ years}}$, $\boxed{t = 40\ln 2}$



(a) $m = 400e^{-0.05t}$

• when $t=0$ $m = 400e^{0} = 400$

• $200 = 400e^{-0.05t}$

$\Rightarrow \frac{1}{2} = e^{-0.05t}$

$\Rightarrow 2 = e^{0.05t}$

$\Rightarrow \ln 2 = 0.05t$

$\Rightarrow t = 20\ln 2 \approx 13.86$

(b) $m = 400e^{-0.05t}$

$\Rightarrow \frac{dm}{dt} = 400e^{-0.05t} \times (-0.05)$

$\Rightarrow \frac{dm}{dt} = -20e^{-0.05t}$

Now

$\Rightarrow -5 = -20e^{-0.05t}$

$\Rightarrow \frac{1}{4} = e^{-0.05t}$

$\Rightarrow 4 = e^{0.05t}$

$\Rightarrow \ln 4 = 0.05t$

$\Rightarrow 2\ln 2 = \frac{1}{20}t$

$\Rightarrow t = 40\ln 2$

Question 52 (****)

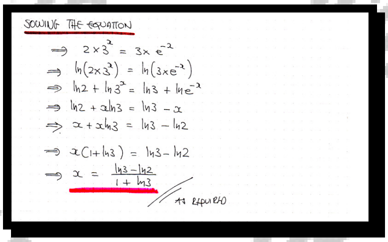
It is given that

$$x = \frac{\ln 3 - \ln 2}{1 + \ln 3} .$$

Show clearly that x is the solution of the equation

$$2 \times 3^{x} = 3 \times \mathrm{e}^{-x} .$$

$\boxed{}$, $\boxed{\text{proof}}$



SOLVING THE EQUATION

$\Rightarrow 2 \times 3^{x} = 3 \times e^{-x}$

$\Rightarrow \ln(2 \times 3^{x}) = \ln(3 \times e^{-x})$

$\Rightarrow \ln 2 + \ln 3^{x} = \ln 3 + \ln e^{-x}$

$\Rightarrow \ln 2 + x\ln 3 = \ln 3 - x$

$\Rightarrow x + x\ln 3 = \ln 3 - \ln 2$

$\Rightarrow x(1 + \ln 3) = \ln 3 - \ln 2$

$\Rightarrow x = \frac{\ln 3 - \ln 2}{1 + \ln 3}$

AS REQUIRED

Question 53 (****)

Find, in exact form where appropriate, the solutions of each of the following equations.

a) $2\ln 56 - \left[\ln 168 - \ln\left(\frac{3}{7}\right)\right] = x \ln 2$.

b) $e^{y} 3^{e} = 3$.

c) $e^{\cos(\ln w)} = 1$, $1 \le w < 5$.

$\boxed{}$, $\boxed{x = 3}$, $\boxed{y = (1-e)\ln 3}$, $\boxed{w = e^{\frac{\pi}{2}}}$

(a) $2\ln 56 - \left[\ln 168 - \ln\frac{3}{7}\right] = x\ln 2$
$\Rightarrow \ln 3136 - \ln 168 + \ln\frac{3}{7} = x\ln 2$
$\Rightarrow \ln\left(\frac{56}{3}\right) + \ln\left(\frac{3}{7}\right) = x\ln 2$
$\Rightarrow \ln\left(\frac{56}{3}\times\frac{3}{7}\right) = x\ln 2$
$\Rightarrow \ln 8 = x\ln 2$
$\Rightarrow 3\cancel{\ln 2} = x\cancel{\ln 2}$
$\Rightarrow x = 3$

(b) $e^{y} 3^{e} = 3$
$\Rightarrow e^{y} = \frac{3}{3^{e}}$
$\Rightarrow e^{y} = 3^{1-e}$
$\Rightarrow y = \ln 3^{1-e}$
$\Rightarrow y = (1-e)\ln 3$

(c) $e^{\cos(\ln w)} = 1$
$\Rightarrow \cos(\ln w) = 0$
$\Rightarrow \ln w = \frac{\pi}{2} + 2\pi n$, $\frac{3\pi}{2} + 2\pi n$ $\quad n = 0,1,2,3,\ldots$
$\Rightarrow w = e^{\frac{\pi}{2}}$

Question 54 (****)

$$f(x) = \ln(4x),\ x \in \mathbb{R},\ x > 0.$$

Find, in exact simplified form, the solution of the equation

$$f(x) + f\left(x^2\right) + f\left(x^3\right) = 6.$$

$\boxed{}$, $\boxed{x = \frac{1}{2}e}$

$f(x) = \ln 4x \quad x > 0$

TIDYING UP THE EQUATION

$\Rightarrow f(x) + f(x^2) + f(x^3) = 6$
$\Rightarrow \ln 4x + \ln(4x^2) + \ln(4x^3) = 6$
$\Rightarrow \ln\left[4x \times 4x^2 \times 4x^3\right] = 6$
$\Rightarrow \ln(64x^6) = 6$
$\Rightarrow 64x^6 = e^6$
$\Rightarrow x^6 = \frac{e^6}{64}$
$\Rightarrow x^6 = \frac{e^6}{2^6}$
$\Rightarrow x^6 = \left(\frac{1}{2}e\right)^6$
$\Rightarrow x = +\frac{1}{2}e \quad x > 0$

Question 55 (****)

Water is heated in a kettle which is kept in a kitchen. The kitchen is kept at a constant temperature, T_0.

The temperature, T °C, of the water in the kettle satisfies

$$T = 95 - 75e^{-t}, \ t \geq 0,$$

where t is the time in minutes since the kettle was switched on.

a) Find the time it takes for the water in the kettle to reach a temperature of 85 °C.

b) Determine the initial rate of the temperature rise of the water in the kettle.

Once the water has reached a temperature of 85 °C the kettle is switched off and is allowed to cool. Its temperature is now given by

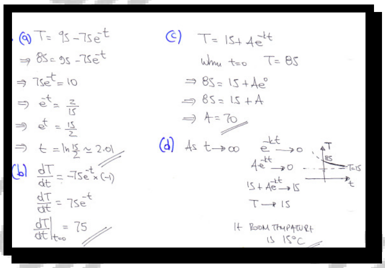
$$T = 15 + Ae^{-kt}, \ t \geq 0,$$

where A and k are positive constants, and t now represents the time in minutes since the kettle was switched off.

c) Find the value of A.

d) State, with a reason, the constant temperature of the kitchen, T_0.

$\boxed{}$, $\boxed{t \approx 2.01 \text{ minutes}}$, $\boxed{75\,°\text{C/min}}$, $\boxed{A = 70}$, $\boxed{T_0 = 15}$


(a) $T = 95 - 75e^{-t}$

$\Rightarrow 85 = 95 - 75e^{-t}$

$\Rightarrow 75e^{-t} = 10$

$\Rightarrow e^{-t} = \frac{2}{15}$

$\Rightarrow e^{t} = \frac{15}{2}$

$\Rightarrow t = \ln\frac{15}{2} \simeq 2.01$

(b) $\frac{dT}{dt} = -75e^{-t} \times (-1)$

$\frac{dT}{dt} = 75e^{-t}$

$\left.\frac{dT}{dt}\right|_{t=0} = 75$

(c) $T = 15 + Ae^{-kt}$

when $t=0$ $T = 85$

$\Rightarrow 85 = 15 + Ae^{0}$

$\Rightarrow 85 = 15 + A$

$\Rightarrow A = 70$

(d) As $t \to \infty$ $e^{-kt} \to 0$

$Ae^{-kt} \to 0$

$15 + Ae^{-kt} \to 15$

$T \to 15$

i.e. ROOM TEMPERATURE IS 15°C

Question 56 (****)

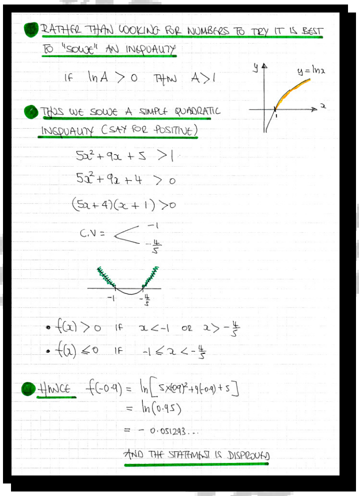
$$f(x) = \ln\left(5x^2 + 9x + 5\right), \ x \in \mathbb{R}.$$

Show that the statement

$$\text{“} f(x) \textit{ is positive for all real values of } x \text{”}$$

is in fact false.

$\boxed{}$, $\boxed{\text{proof}}$



RATHER THAN LOOKING FOR NUMBERS TO TRY IT IS BEST TO "SOLVE" AN INEQUALITY

IF $\ln A > 0$ THEN $A > 1$

THUS WE SOLVE A SIMPLE QUADRATIC INEQUALITY (SAY FOR POSITIVE)

$$5x^2 + 9x + 5 > 1$$

$$5x^2 + 9x + 4 > 0$$

$$(5x+4)(x+1) > 0$$

$$\text{C.V} = \begin{cases} -1 \\ -\frac{4}{5} \end{cases}$$

- $f(x) > 0$ IF $x < -1$ OR $x > -\frac{4}{5}$
- $f(x) \leqslant 0$ IF $-1 \leqslant x < -\frac{4}{5}$

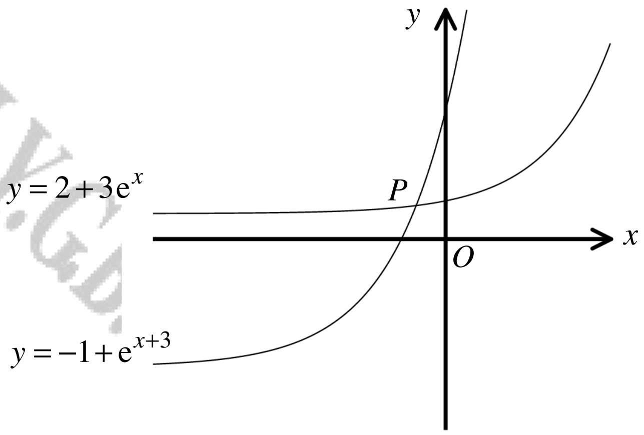
HENCE $f(-0.9) = \ln\left[5 \times (-0.9)^2 + 9(-0.9) + 5\right]$

$= \ln(0.95)$

$= -0.051293\ldots$

AND THE STATEMENT IS DISPROVED

Question 57 (****)



The figure above shows the graphs of

$$y = 2 + 3e^x \quad \text{and} \quad y = -1 + e^{x+3}.$$

The graphs meet at the point P.

Show that the y coordinate of P is

$$\frac{2e^3 + 3}{e^3 - 3}.$$

$\boxed{}$, $\boxed{\text{proof}}$

$y = 2 + 3e^x$
$y = -1 + e^{x+3}$

$2 + 3e^x = -1 + e^{x+3}$
$3 = e^{x+3} - 3e^x$
$3 = e^x e^3 - 3e^x$
$3 = e^x(e^3 - 3)$
$e^x = \frac{3}{e^3 - 3}$
! NO NEED TO LOG TO GET x!

$\therefore\ y = 2 + 3e^x$
$y = 2 + 3\left(\frac{3}{e^3-3}\right)$
$y = 2 + \frac{9}{e^3-3}$
$y = \frac{2e^3 - 6 + 9}{e^3 - 3}$
$y = \frac{2e^3+3}{e^3-3}$ // AS REQUIRED

ALTERNATIVE

$y = 2 + 3e^x$
$y = -1 + e^{x+3}$

$3e^x = y - 2$
$e^{x+3} = y + 1$
DIVIDE EQUATIONS

$\Rightarrow \frac{3e^x}{e^{x+3}} = \frac{y-2}{y+1}$
$\Rightarrow 3e^{-3} = \frac{y-2}{y+1}$
$\Rightarrow \frac{3}{e^3} = \frac{y-2}{y+1}$
$\Rightarrow 3y + 3 = ye^3 - 2e^3$
$\Rightarrow 3 + 2e^3 = ye^3 - 3y$
$\Rightarrow 3 + 2e^3 = y(e^3 - 3)$
$\Rightarrow y = \frac{2e^3+3}{e^3-3}$ // AS REQUIRED

Question 58 (****)

Find, in exact form where appropriate, the solution for each of the following equations.

a) $2\ln x = \ln(4x+12)$.

b) $3e^{y} + 2e^{-y} = 7$.

c) $\dfrac{e^{t}}{2^{t}} = e$.

$\boxed{x = 6,\ x \neq -2}$, $\boxed{y = \ln 2,\ -\ln 3}$, $\boxed{t = \dfrac{1}{1-\ln 2}}$

(a) $2\ln x = \ln(4x+12)$

$\Rightarrow \ln x^2 = \ln(4x+12)$

$\Rightarrow x^2 = 4x+12$

$\Rightarrow x^2 - 4x - 12 = 0$

$\Rightarrow (x+2)(x-6) = 0$

$\Rightarrow x = \begin{cases} 6 \\ \cancel{-2} \end{cases}$

(b) $3e^{y} + 2e^{-y} = 7$

$\Rightarrow 3e^{y} + \dfrac{2}{e^{y}} = 7$

$\Rightarrow 3a + \dfrac{2}{a} = 7 \quad (a = e^{y})$

$\Rightarrow 3a^2 + 2 = 7a$

$\Rightarrow 3a^2 - 7a + 2 = 0$

$\Rightarrow (3a-1)(a-2) = 0$

$\Rightarrow a = \begin{cases} 2 \\ \frac{1}{3} \end{cases}$ i.e. $e^{y} = \begin{cases} 2 \\ \frac{1}{3} \end{cases}$

$\Rightarrow y = \begin{cases} \ln 2 \\ \ln\frac{1}{3} = -\ln 3 \end{cases}$ //

(c) $\dfrac{e^{t}}{2^{t}} = e$

$\Rightarrow e^{t} = e \times 2^{t}$

$\Rightarrow \dfrac{e^{t}}{e} = 2^{t}$

$\Rightarrow e^{t-1} = 2^{t}$

$\Rightarrow t-1 = \ln 2^{t}$

$\Rightarrow t-1 = t\ln 2$

$\Rightarrow t - t\ln 2 = 1$

$\Rightarrow t(1-\ln 2) = 1$

$\Rightarrow t = \dfrac{1}{1-\ln 2}$ //

Question 59 (****)

When hot cooking oil cools down, its temperature, T °C, is related to the time, t minutes, for which it has been cooling, by the formula

$$T = 20 + 160e^{-0.1t}, \quad t \geq 0.$$

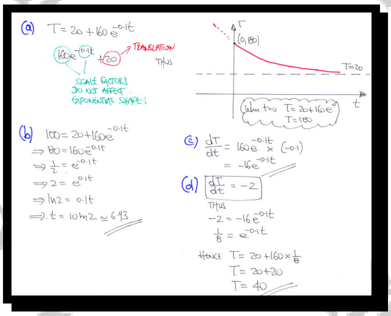
a) Sketch the graph of T against t, clearly marking its asymptote and the coordinates of the starting point of the curve.

b) Determine the value of t, when $T = 100$.

c) Find an expression for $\dfrac{dT}{dt}$.

d) Calculate the value of T, at the instant when the oil is cooling down at the rate of 2°C per minute.

$\boxed{}$, $\boxed{t = 10\ln 2 \approx 6.93}$, $\boxed{\dfrac{dT}{dt} = -16e^{-0.1t}}$, $\boxed{T = 40}$



Question 60 **(****)**

The point $P(\ln 2, 5b-3a)$ lies on the curve with equation

$$y = a + b\mathrm{e}^{x},$$

where a and b are non zero constants.

The gradient at P is 8.

a) Find the value of a and the value of b.

b) Determine the exact coordinates of the point where the normal to the curve at P crosses the x axis.

$\boxed{a=3,\ b=4}$, $\boxed{(88+\ln 2, 0)}$

(a) $P(\ln 2, 5b-3a)$

- $\boxed{y = a + b\mathrm{e}^{x}}$

$5b-3a = a + b\mathrm{e}^{\ln 2}$

$5b-3a = a + b \times 2$

$5b-3a = a + 2b$

$\boxed{3b = 4a}$

- $\dfrac{dy}{dx} = b\mathrm{e}^{x}$ & $\left.\dfrac{dy}{dx}\right|_{x=\ln 2} = 8$

$8 = b\mathrm{e}^{\ln 2}$

$8 = 2b$

$b = 4$

$a = 3$

(b) NORMAL GRADIENT IS $-\frac{1}{8}$

EQUATION OF NORMAL

$y - y_0 = m(x - x_0)$

$y - (5b-3a) = -\frac{1}{8}(x - \ln 2)$

$y - (20-9) = -\frac{1}{8}(x - \ln 2)$

$y - 11 = -\frac{1}{8}(x - \ln 2)$

when $y = 0$

$-11 = -\frac{1}{8}(x - \ln 2)$

$-88 = -x + \ln 2$

$x = 88 + \ln 2$

i.e. $(88+\ln 2, 0)$

Question 61 **(****)**

Find the exact solutions of the equation

$$3\mathrm{e}^{3x} - 4\mathrm{e}^{2x} - 5\mathrm{e}^{x} + 2 = 0.$$

$\boxed{x = \ln 2, -\ln 3}$

$3\mathrm{e}^{3x} - 4\mathrm{e}^{2x} - 5\mathrm{e}^{x} + 2 = 0$

$\Rightarrow 3(\mathrm{e}^{x})^3 - 4(\mathrm{e}^{x})^2 - 5(\mathrm{e}^{x}) + 2 = 0$

$\Rightarrow 3y^3 - 4y^2 - 5y + 2 = 0 \quad (y = \mathrm{e}^{x})$

$\Rightarrow (y+1)(3y^2 - 7y + 2)$

-7

$2y$

$\Rightarrow (y+1)(3y-1)(y-2) = 0$

$y = -1, 2, \frac{1}{3}$

$\mathrm{e}^{x} = \cancel{-1}, 2, \frac{1}{3}$

$x = \ln 2, \ \ln\frac{1}{3} = -\ln 3$

Question 62 (****)

A curve has equation

$$y = \frac{1}{4}x^3 - 6\ln x + 1, \ x > 0.$$

The points C and D lie on this curve, where $x = 4$ and $x = 8$, respectively.

Show that the gradient of the straight line segment CD is $28 - \frac{1}{2}\ln 2$.

$\boxed{\text{proof}}$

$y = \frac{1}{4}x^3 - 6\ln x + 1$

- when $x = 4$ $y = 16 - 6\ln 4 + 1 = 17 - 6\ln 4$ ∴ $C(4, 17 - 6\ln 4)$
- when $x = 8$ $y = 128 - 6\ln 8 + 1 = 129 - 6\ln 8$ $D(8, 129 - 6\ln 8)$

$\text{GRAD } CD = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(129 - 6\ln 8) - (17 - 6\ln 4)}{8 - 4} = \frac{112 - 6\ln 8 + 6\ln 4}{4}$

$= \frac{112 - 6(\ln 8 - \ln 4)}{4} = \frac{112 - 6\ln 2}{4} = 28 - \frac{3}{2}\ln 2$

Question 63 (****)

Find in exact simplified form the solution to the equation

$$2^x e^{3x+1} = 10.$$

$$\boxed{x = \frac{-1 + \ln 10}{3 + \ln 2}}$$

$2^x e^{3x+1} = 10$

$\Rightarrow \ln\left[2^x e^{3x+1}\right] = \ln 10$

$\Rightarrow \ln 2^x + \ln e^{3x+1} = \ln 10$

$\Rightarrow x\ln 2 + 3x + 1 = \ln 10$

$\Rightarrow x\ln 2 + 3x = -1 + \ln 10$

$\Rightarrow x(\ln 2 + 3) = -1 + \ln 10$

$\Rightarrow x = \frac{-1 + \ln 10}{3 + \ln 2}$

Question 64 (****)

Find the exact solution of the following simultaneous equations

$$x + e^{y} = 1$$

$$\ln(x+1)^{2} - 2y = 2 \,.$$

$\boxed{\quad}$, $\boxed{x = \dfrac{e-1}{e+1}}$, $\boxed{y = \ln\left(\dfrac{2}{e+1}\right)}$

PROCEED AS FOLLOWS

$\Rightarrow \ln(x+1)^2 - 2y = 2$ AND $x + e^{y} = 1$

$\Rightarrow 2\ln(x+1) - 2y = 2$ $\Rightarrow e^{y} = 1 - x$

$\Rightarrow \ln(x+1) - y = 1$ $\Rightarrow y = \ln(1-x)$

$\ln(x+1) - \ln(1-x) = 1$

$\Rightarrow \ln\left(\frac{x+1}{1-x}\right) = 1$

$\Rightarrow \frac{x+1}{1-x} = e^{1}$

$\Rightarrow \frac{x+1}{1-x} = e$

$\Rightarrow x + 1 = e - ex$

$\Rightarrow x + ex = e - 1$

$\Rightarrow x(1+e) = e - 1$

$\Rightarrow x = \frac{e-1}{e+1}$ //

AND $y = \ln(1-x)$

$y = \ln\left(1 - \frac{e-1}{e+1}\right)$

$y = \ln\left(\frac{\cancel{e}+1-\cancel{e}+1}{e+1}\right)$

$y = \ln\left(\frac{2}{e+1}\right)$

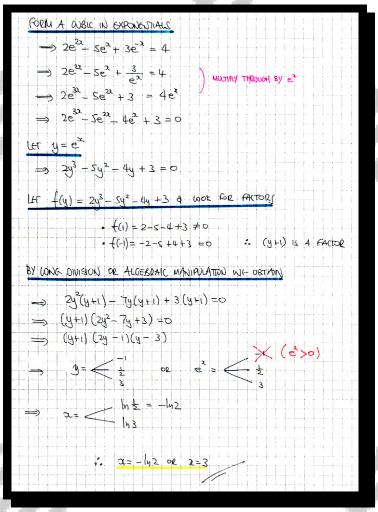
$\therefore (x,y) = \left[\frac{e-1}{e+1}, \ln\left(\frac{2}{e+1}\right)\right]$

Question 65 (****)

Find the exact solutions of the equation

$$2e^{2x} - 5e^{x} + 3e^{-x} = 4 .$$

$\boxed{}$, $\boxed{x = \ln 3, -\ln 2}$



FORM A CUBIC IN EXPONENTIALS

$\Rightarrow 2e^{2x} - 5e^{x} + 3e^{-x} = 4$

$\Rightarrow 2e^{2x} - 5e^{x} + \frac{3}{e^{x}} = 4$ MULTIPLY THROUGH BY e^{x}

$\Rightarrow 2e^{3x} - 5e^{2x} + 3 = 4e^{x}$

$\Rightarrow 2e^{3x} - 5e^{2x} - 4e^{x} + 3 = 0$

LET $y = e^{x}$

$\Rightarrow 2y^{3} - 5y^{2} - 4y + 3 = 0$

LET $f(y) = 2y^{3} - 5y^{2} - 4y + 3$ & LOOK FOR FACTORS

- $f(1) = 2 - 5 - 4 + 3 \neq 0$
- $f(-1) = -2 - 5 + 4 + 3 = 0$ $\therefore$ $(y+1)$ IS A FACTOR

BY LONG DIVISION OR ALGEBRAIC MANIPULATION WE OBTAIN

$\Rightarrow 2y^{2}(y+1) - 7y(y+1) + 3(y+1) = 0$

$\Rightarrow (y+1)(2y^{2} - 7y + 3) = 0$

$\Rightarrow (y+1)(2y-1)(y-3)$

$\Rightarrow y = \begin{cases} -1 \\ \frac{1}{2} \\ 3 \end{cases}$ OR $e^{x} = \begin{cases} \cancel{-1} \ (e^{x} > 0) \\ \frac{1}{2} \\ 3 \end{cases}$

$\Rightarrow x = \begin{cases} \ln\frac{1}{2} = -\ln 2 \\ \ln 3 \end{cases}$

$\therefore$ $x = -\ln 2$ OR $x = 3$

Question 66 (****)

Find, in exact form where appropriate, the solutions for each of the following equations.

a) $e^{1-x} = 3e$.

b) $e^{w} - 3 = \dfrac{8}{e^{w}-1}$.

$\boxed{}$, $\boxed{x = -\ln 3}$, $\boxed{w = \ln 5}$

a) TWO SENSIBLE METHODS

- $e^{1-x} = 3e$
$\ln(e^{1-x}) = \ln(3e)$
$1-x = \ln 3 + \ln e$
$1-x = \ln 3 + 1$
$-x = \ln 3$
$x = -\ln 3$

- $e^{1-x} = 3e$
$\frac{e^{1-x}}{e^{1}} = 3$
$e^{1-x-1} = 3$
$e^{-x} = 3$
$-x = \ln 3$
$x = \ln 3$ AS BEFORE

b) THIS IS A "HIDDEN QUADRATIC"

$e^{w} - 3 = \frac{8}{e^{w}-1} \implies (e^{w}-3)(e^{w}-1) = 8$
$\implies e^{2w} - e^{w} - 3e^{w} + 3 = 8$
$\implies e^{2w} - 4e^{w} - 5 = 0$
$\implies (e^{w}+1)(e^{w}-5) = 0$
$\implies e^{w} = \cancel{-1}, 5$
$\implies w = \ln 5$

Question 67 **(****)**

Show that the following simultaneous equations

$$e^{2y} + 4 = x$$

$$\ln(x+1) = 2y - 1$$

are satisfied by the solution pair

$$x = \frac{4+e}{1-e}, \quad y = \frac{1}{2}\ln\left(\frac{5e}{1-e}\right)$$

and hence explain why the equations have no real solutions.

SYN-M , proof

BY ANY SENSIBLE SUBSTITUTION

$e^{2y} = x - 4$ & $2y = 1 + \ln(x+1)$

$2y = \ln(x-4)$

$\Longrightarrow \ln(x-4) = 1 + \ln(x+1)$

$\Longrightarrow \ln(x-4) - \ln(x+1) = 1$

$\Longrightarrow \ln\left(\frac{x-4}{x+1}\right) = 1$

$\Longrightarrow \frac{x-4}{x+1} = e$

$\Longrightarrow x+4 = ex + e$

$\Longrightarrow x - ex = e + 4$

$\Longrightarrow x(1-e) = e+4$

$\Longrightarrow x = \frac{e+4}{1-e}$

or $x = \frac{4+e}{1-e}$

SUBSTITUTING INTO $2y = \ln(x-4)$

$2y = \ln\left(\frac{4+e}{1-e} - 4\right) = \ln\left(\frac{4+e-4(1-e)}{1-e}\right) = \ln\left(\frac{\cancel{4}+e-\cancel{4}+4e}{1-e}\right)$

$2y = \ln\left(\frac{5}{1-e}\right)$

$y = \frac{1}{2}\ln\left(\frac{5}{1-e}\right)$

NO REAL SOLUTION AS EVIDENTLY THE ARGUMENT OF THE LOGARITHM IN $y = \frac{1}{2}\ln\left(\frac{5}{1-e}\right)$ IS NEGATIVE

ALTERNATIVE

$x = e^{2y} + 4$ $\qquad$ $\ln(x+1) = 2y - 1$

$\Longrightarrow \ln(e^{2y} + 4 + 1) = 2y - 1$

$\Longrightarrow \ln(e^{2y} + 5) = 2y - 1$

$\Longrightarrow e^{2y} + 5 = e^{2y-1}$

$\Longrightarrow 5 = e^{2y-1} - e^{2y}$

$\Longrightarrow 5 = e^{2y}(e^{-1} - 1)$

$\Longrightarrow e^{2y} = \frac{5}{e^{-1} - 1}$

$\Longrightarrow e^{2y} = \frac{5e^{1}}{e^{-1}e^{1} - 1e^{1}}$

$\Longrightarrow e^{2y} = \frac{5e}{1-e}$

$\Longrightarrow 2y = \ln\left(\frac{5e}{1-e}\right)$

$\Longrightarrow y = \frac{1}{2}\ln\left(\frac{5e}{1-e}\right)$

AND SINCE $e^{2y} = \frac{5e}{1-e}$

$x = \frac{5e}{1-e} + 4 = \frac{5e + 4(1-e)}{1-e} = \frac{5e + 4 - 4e}{1-e} = \frac{e+4}{1-e}$

AS BEFORE

Question 68 (****)

Solve the following logarithmic equation

$$\ln x^2 + \frac{3}{\ln x} = 7 .$$

$\boxed{}$, $\boxed{x = e^3, \sqrt{e}}$

MANIPULATE AS FOLLOWS

$$\Rightarrow \ln x^2 + \frac{3}{\ln x} = 7$$

$$\Rightarrow 2\ln x + \frac{3}{\ln x} = 7$$

$$\Rightarrow 2(\ln x)^2 + 3 = 7\ln x$$

$$\Rightarrow 2(\ln x)^2 - 7\ln x + 3 = 0$$

FACTORIZE THE QUADRATIC

$$\Rightarrow (2\ln x - 1)(\ln x - 3)$$

$$\Rightarrow \ln x = \begin{cases} \frac{1}{2} \\ 3 \end{cases}$$

$$\Rightarrow x = \begin{cases} e^{\frac{1}{2}} = \sqrt{e} \\ e^3 \end{cases}$$

Question 69 (****)

The value V, in £, of a computer system t years after it was purchased is modelled by the following expression

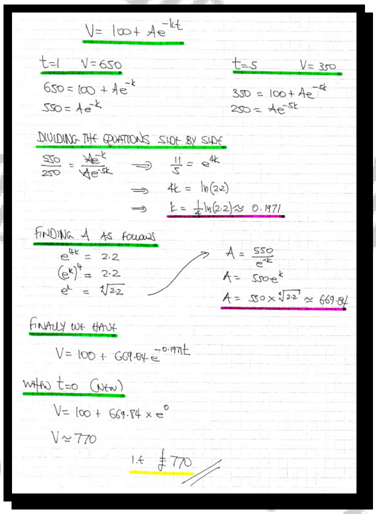
$$V = 100 + A\mathrm{e}^{-kt}, \ t \geq 0,$$

where A and k are positive constants.

Its value after one year was £650 and after a further period of four years £350.

Find, correct to the nearest £, the value of the system when new.

$\boxed{}$, $\boxed{£770}$



$V = 100 + A\mathrm{e}^{-kt}$

$t=1 \quad V=650$

$650 = 100 + A\mathrm{e}^{-k}$

$550 = A\mathrm{e}^{-k}$

$t=5 \quad V=350$

$350 = 100 + A\mathrm{e}^{-5k}$

$250 = A\mathrm{e}^{-5k}$

DIVIDING THE EQUATIONS SIDE BY SIDE

$\frac{550}{250} = \frac{\cancel{A}\mathrm{e}^{-k}}{\cancel{A}\mathrm{e}^{-5k}} \Rightarrow \frac{11}{5} = \mathrm{e}^{4k}$

$\Rightarrow 4k = \ln(2.2)$

$\Rightarrow k = \frac{1}{4}\ln(2.2) \approx 0.1971$

FINDING A AS FOLLOWS

$\mathrm{e}^{4k} = 2.2$

$(\mathrm{e}^{k})^4 = 2.2$

$\mathrm{e}^{k} = \sqrt[4]{2.2}$

$A = \frac{550}{\mathrm{e}^{-k}}$

$A = 550\mathrm{e}^{k}$

$A = 550 \times \sqrt[4]{2.2} \approx 669.84$

FINALLY WE HAVE

$V = 100 + 669.84\mathrm{e}^{-0.1971t}$

WHEN $t=0$ (NEW)

$V = 100 + 669.84 \times \mathrm{e}^{0}$

$V \simeq 770$

i.e. £770

Question 70 (****)

The curve C has equation

$$y = e^{2x} - 4e^{x} - 16x\,.$$

a) Show that the x coordinates of the stationary points of C satisfy the equation

$$e^{2x} - 2e^{x} - 8 = 0\,.$$

b) Hence find the exact coordinates of the stationary point of C, giving the answer in terms of $\ln 2$.

$\boxed{C3P}$, $\boxed{(2\ln 2, -32\ln 2)}$

(a) $y = e^{2x} - 4e^{x} - 16x$

$\Rightarrow \frac{dy}{dx} = 2e^{2x} - 4e^{x} - 16$

SET TO ZERO

$\Rightarrow 2e^{2x} - 4e^{x} - 16 = 0$

$\Rightarrow e^{2x} - 2e^{x} - 8 = 0$ AS REQUIRED

(b) $e^{2x} - 2e^{x} - 8 = 0$

$\Rightarrow (e^{x})^2 - 2(e^{x}) - 8 = 0$

$\Rightarrow a^2 - 2a - 8 = 0 \quad (a = e^{x})$

$\Rightarrow (a-4)(a+2) = 0$

$\Rightarrow a = 4, -2 \qquad e^{x} = 4, \cancel{-2}$

$\boxed{x = \ln 4 = 2\ln 2}$

$\therefore y = e^{2\ln 4} - 4e^{\ln 4} - 16\ln 4$

$y = \cancel{16} - \cancel{16} - 16\ln 4 = -32\ln 2$

$\therefore (2\ln 2, -32\ln 2)$

Question 71 (****)

Find, in exact form where appropriate, the solutions of the following equation

$$\frac{d}{dx}\left(\frac{4}{3 - e^{x}}\right) = 1\,.$$

$\boxed{x = 0,\ \ln 9}$

$\frac{d}{dx}\left(\frac{4}{3-e^{x}}\right) = 1$

$\Rightarrow \frac{d}{dx}\left[4(3-e^{x})^{-1}\right] = 1$

$\Rightarrow -4(3-e^{x})^{-2} \times (-e^{x}) = 1$

$\Rightarrow \frac{4e^{x}}{(3-e^{x})^2} = 1$

$\Rightarrow 4e^{x} = (3-e^{x})^2$

$\Rightarrow 4e^{x} = 9 - 6e^{x} + e^{2x}$

$\Rightarrow 0 = e^{2x} - 10e^{x} + 9$

$\Rightarrow (e^{x})^2 - 10(e^{x}) + 9 = 0$

$\Rightarrow a^2 - 10a + 9 = 0$

$\Rightarrow (a-1)(a-9) = 0$

$\Rightarrow a = 1, 9$

$\Rightarrow e^{x} = 1, 9$

$x = \ln 1 = 0, \ \ln 9$

Question 72 **(****)**

The temperature of an oven is modelled by

$$\theta = 225 - 200\mathrm{e}^{-0.1t}, \ t \geq 0$$

where θ °C is the temperature of the oven, t minutes after it was switched on.

a) State the highest temperature of the oven according to this model.

b) Determine the value of t when the oven temperature reaches $125°\mathrm{C}$.

c) Show clearly that

$$\frac{d\theta}{dt} = \frac{1}{10}(225 - \theta),$$

and hence find the rate at which the temperature of the oven is increasing when its temperature has reached $125°\mathrm{C}$.

$\boxed{}$, $\boxed{\theta_{\text{max}} = 225}$, $\boxed{t \approx 6.93}$, $\boxed{10°\mathrm{C}/\min}$

a) WHEN t BECOMES VERY LARGE, IE $t \to \infty$

$e^{-0.1t} \to 0$

$200e^{-0.1t} \to 0$

$\therefore \ \theta \to 225$

$\therefore$ MAX TEMPERATURE IS $225°\mathrm{C}$

b) WHEN $\theta = 125$

$\Rightarrow 125 = 225 - 200e^{-0.1t}$

$\Rightarrow 200e^{-0.1t} = 100$

$\Rightarrow e^{-0.1t} = \frac{1}{2}$

$\Rightarrow e^{0.1t} = 2$

$\Rightarrow 0.1t = \ln 2$

$\Rightarrow t = 10\ln 2 \simeq 6.93147... \simeq 7 \text{ min}$

c) WORK AS FOLLOWS

$\theta = 225 - 200e^{-0.1t}$

$\frac{d\theta}{dt} = 0 + 20e^{-0.1t}$

$\frac{d\theta}{dt} = 20e^{-0.1t}$

$10\frac{d\theta}{dt} = 200e^{-0.1t}$

$10\frac{d\theta}{dt} = 225 - \theta$

$\frac{d\theta}{dt} = \frac{1}{10}(225 - \theta)$ // AS REQUIRED

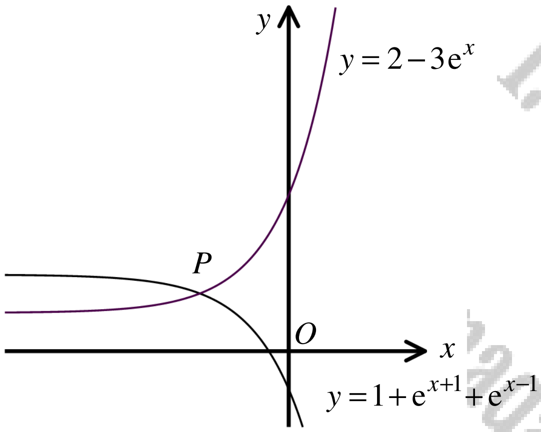
FROM THE ORIGINAL EQUATION

$\theta = 225 - 200e^{-0.1t}$

$200e^{-0.1t} = 225 - \theta$

$\left.\frac{d\theta}{dt}\right|_{\theta = 125} = 10 \ °\mathrm{C}/\text{min}$ //

Question 73 (****)



The figure above shows the graphs of

$$y = 2 - 3e^{x} \qquad \text{and} \qquad y = 1 + e^{x+1} + e^{x-1}.$$

The graphs meet at the point P.

Show that the y coordinate of P is

$$\frac{2e^{2} + 3e + 2}{e^{2} + 3e + 1}.$$

$\boxed{}$, $\boxed{\text{proof}}$

PROCEED AS FOLLOWS AS THE x CO-ORDINATE IS NOT REQUIRED

- $y = 2 - 3e^{x}$

  $3e^{x} = 2 - y$

- $y = 1 + e^{x+1} + e^{x-1}$

  $y = 1 + e^{x} \times e + e^{x} \times e^{-1}$

  $3y = 3e^{x} \times e + 3e^{x} \times \frac{1}{e} + 3$

  $3y = (2-y)e + (2-y) \times \frac{1}{e} + 3$

  $3ey = e^{2}(2-y) + 2 - y + 3e$

  $3ey = 2e^{2} - e^{2}y + 2 - y + 3e$

  $3ey + e^{2}y + y = 2e^{2} + 3e + 2$

  $y(e^{2} + 3e + 1) = 2e^{2} + 3e + 2$

  $y = \frac{2e^{2} + 3e + 2}{e^{2} + 3e + 1}$

Question 74 (****)

The functions $f(x)$ and $g(x)$ are defined by

$$f(x) = e^x - 3, \ x \in \mathbb{R}$$

$$g(x) = x + 1, \quad x \in \mathbb{R}.$$

a) Find an expression for $f^{-1}(x)$, the inverse of $f(x)$.

b) State the domain and range of $f^{-1}(x)$.

c) Solve the equation

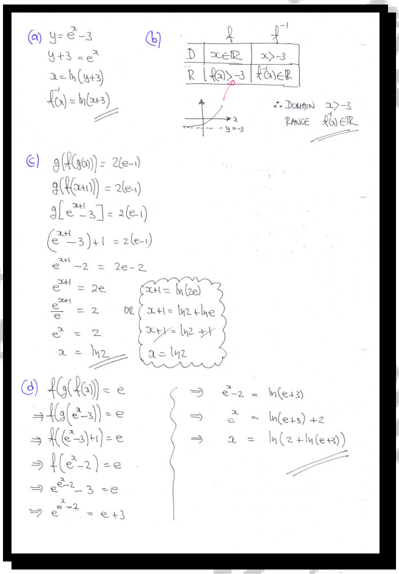
$$gfg(x) = 2(e-1),$$

giving the final answer in terms of logarithms in its simplest form.

d) Solve the equation

$$fgf(x) = e.$$

$\boxed{f^{-1}(x) = \ln(x+3)}$, $\boxed{x \in \mathbb{R}, \ x > -3}$, $\boxed{f^{-1}(x) \in \mathbb{R}}$, $\boxed{x = \ln 2}$, $\boxed{x = \ln\left[2 + \ln(3+e)\right]}$



(a) $y = e^x - 3$

$y + 3 = e^x$

$x = \ln(y+3)$

$f^{-1}(x) = \ln(x+3)$

(b)

	f	f^{-1}
D	$x \in \mathbb{R}$	$x > -3$
R	$f(x) > -3$	$f^{-1}(x) \in \mathbb{R}$

x $y=-3$

$\therefore$ DOMAIN $x > -3$

RANGE $f^{-1}(x) \in \mathbb{R}$

(c) $g(f(g(x))) = 2(e-1)$

$g(f(x+1)) = 2(e-1)$

$g[e^{x+1} - 3] = 2(e-1)$

$(e^{x+1} - 3) + 1 = 2(e-1)$

$e^{x+1} - 2 = 2e - 2$

$e^{x+1} = 2e$

$\frac{e^{x+1}}{e} = 2$

$e^x = 2$

$x = \ln 2$

OR

$x+1 = \ln(2e)$

$x+1 = \ln 2 + \ln e$

$x + \cancel{1} = \ln 2 + \cancel{1}$

$x = \ln 2$

(d) $f(g(f(x))) = e$

$\Rightarrow f(g(e^x - 3)) = e$

$\Rightarrow f((e^x - 3) + 1) = e$

$\Rightarrow f(e^x - 2) = e$

$\Rightarrow e^{e^x - 2} - 3 = e$

$\Rightarrow e^{e^x - 2} = e + 3$

$\Rightarrow e^x - 2 = \ln(e+3)$

$\Rightarrow e^x = \ln(e+3) + 2$

$\Rightarrow x = \ln(2 + \ln(e+3))$

Question 75 (****)

The number of bacteria N present in a culture at time t hours, is modelled by the equation

$$N = A\mathrm{e}^{kt}, \ t \geq 0.$$

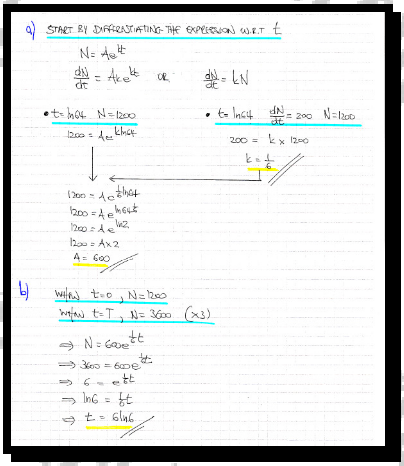
At the instant when $t = \ln 64$, there are 1200 bacteria present in the culture and bacteria are increasing at the rate of 200 bacteria per hour.

a) Find the value of A and the value of k.

The time T it takes for the bacteria to triple in number is constant.

b) Find the exact value of T.

$\boxed{A = 600}$, $\boxed{k = \frac{1}{6}}$, $\boxed{T = 6\ln 3}$


a) START BY DIFFERENTIATING THE EXPRESSION W.R.T t

$N = Ae^{kt}$

$\frac{dN}{dt} = Ake^{kt}$ OR $\frac{dN}{dt} = kN$

• $t = \ln 64$ $N = 1200$

$1200 = Ae^{k\ln 64}$

• $t = \ln 64$ $\frac{dN}{dt} = 200$ $N = 1200$

$200 = k \times 1200$

$k = \frac{1}{6}$

$1200 = Ae^{\frac{1}{6}\ln 64}$

$1200 = Ae^{\ln 64^{\frac{1}{6}}}$

$1200 = Ae^{\ln 2}$

$1200 = A \times 2$

$A = 600$

b) WHEN $t = 0$, $N = 1200$

WHEN $t = T$, $N = 3600$ $(\times 3)$

$\Rightarrow N = 600e^{\frac{1}{6}t}$

$\Rightarrow 3600 = 600e^{\frac{1}{6}t}$

$\Rightarrow 6 = e^{\frac{1}{6}t}$

$\Rightarrow \ln 6 = \frac{1}{6}t$

$\Rightarrow t = 6\ln 6$

Question 76 **(****)**

Simplify the following logarithmic expression

$$\ln\left(2\sqrt{e}\right)-\frac{1}{3}\ln\left(\frac{8}{e^2}\right)-\ln\left(\frac{1}{3}e\right),$$

giving the answer in the form $\frac{1}{a}+\ln b$, where a and b are positive integers.

$\boxed{}$, $\boxed{\frac{1}{2}+\ln 3}$

METHOD A

$$\ln(2\sqrt{e})-\tfrac{1}{3}\ln\left(\tfrac{8}{e^2}\right)-\ln\left(\tfrac{1}{3}e\right)=\tfrac{1}{3}\left[3\ln(2e^{\frac{1}{2}})-\ln\left(\tfrac{8}{e^2}\right)-3\ln\left(\tfrac{e}{3}\right)\right]$$
$$=\tfrac{1}{3}\left[\ln(8e^{\frac{3}{2}})-\ln\left(\tfrac{8}{e^2}\right)-\ln\left(\tfrac{e^3}{27}\right)\right]$$
$$=\tfrac{1}{3}\left[\ln(8e^{\frac{3}{2}})+\ln\left(\tfrac{e^2}{8}\right)+\ln\left(\tfrac{27}{e^3}\right)\right]$$
$$=\tfrac{1}{3}\ln\left[\cancel{8}e^{\frac{3}{2}}\times\tfrac{e^2}{\cancel{8}}\times\tfrac{27}{e^3}\right]$$
$$=\tfrac{1}{3}\ln\left[27e^{\frac{1}{2}}\right]$$
$$=\tfrac{1}{3}\left[\ln 27+\ln e^{\frac{1}{2}}\right]$$
$$=\tfrac{1}{3}\left[\ln 3^3+\tfrac{1}{2}\ln e\right]$$
$$=\tfrac{1}{3}\left[3\ln 3+\tfrac{1}{2}\times 1\right]$$
$$=\ln 3+\tfrac{1}{6}$$

$a=6 \quad b=3$

METHOD B

$$\ln(2\sqrt{e})-\tfrac{1}{3}\ln\left(\tfrac{8}{e^2}\right)-\ln\left(\tfrac{1}{3}e\right)=\ln 2+\ln\sqrt{e}-\tfrac{1}{3}\left[\ln 8-\ln e^2\right]-\left[\ln\tfrac{1}{3}+\ln e\right]$$
$$=\ln 2+\ln e^{\frac{1}{2}}-\tfrac{1}{3}\left[\ln 2^3-2\ln e\right]-\left[\ln\tfrac{1}{3}+1\right]$$
$$=\ln 2+\tfrac{1}{2}\ln e-\tfrac{1}{3}\left[3\ln 2-2\times 1\right]-\left[\ln\tfrac{1}{3}+1\right]$$
$$=\ln 2+\tfrac{1}{2}\times 1-\tfrac{1}{3}\left[3\ln 2-2\right]-\left[\ln\tfrac{1}{3}+1\right]$$
$$=\cancel{\ln 2}+\tfrac{1}{2}-\cancel{\ln 2}+\tfrac{2}{3}-\ln\tfrac{1}{3}-1$$
$$=\tfrac{1}{6}-\ln\tfrac{1}{3}$$
$$=\tfrac{1}{6}+\ln 3$$

AS BEFORE

Question 77 (****)

Find, in exact form where appropriate, the solutions for each of the following equations.

a) $\ln(2x) - \ln(x+2) = 1$.

b) $(3\mathrm{e})^y = \dfrac{1}{9}$.

c) $\mathrm{e}^{2t} + 15 = 8\mathrm{e}^t$.

$\boxed{x = \dfrac{2\mathrm{e}}{2-\mathrm{e}} \approx -7.57}$, $\boxed{y = -\dfrac{2\ln 3}{1+\ln 3}}$, $\boxed{t = \ln 3, \ln 5}$

(a) $\ln(2x) - \ln(x+2) = 1$

$\Rightarrow \ln\left(\frac{2x}{x+2}\right) = 1$

$\Rightarrow \frac{2x}{x+2} = e$

$\Rightarrow 2x = xe + 2e$

$\Rightarrow 2x - xe = 2e$

$\Rightarrow x(2-e) = 2e$

$\Rightarrow x = \frac{2e}{2-e}$

(b) $(3e)^y = \frac{1}{9}$

$\Rightarrow 3^y e^y = 3^{-2}$

$\Rightarrow e^y = \frac{3^{-2}}{3^y}$

$\Rightarrow e^y = 3^{-2-y}$

$\Rightarrow y = \ln 3$

$\Rightarrow y = (-2-y)\ln 3$

$\Rightarrow y = -2\ln 3 - y\ln 3$

$\Rightarrow y + y\ln 3 = -2\ln 3$

$\Rightarrow y(1+\ln 3) = -2\ln 3$

$\Rightarrow y = \frac{-2\ln 3}{1+\ln 3}$

ALT: $(3e)^y = \frac{1}{9}$

$\Rightarrow \ln(3e)^y = \ln\left(\frac{1}{9}\right)$

$\Rightarrow y\ln(3e) = \ln(1/9)$

$\Rightarrow y = \frac{\ln(1/9)}{\ln(3e)}$

$\Rightarrow y = \frac{\ln 3^{-2}}{\ln 3 + \ln e}$

$\Rightarrow y = \frac{-2\ln 3}{1+\ln 3}$

(c) $e^{2t} + 15 = 8e^t$

$\Rightarrow e^{2t} - 8e^t + 15 = 0$

$\Rightarrow (e^t)^2 - 8(e^t) + 15 = 0$

$\Rightarrow (e^t - 3)(e^t - 5) = 0$

$\therefore e^t = 5, 3$ $\quad\therefore t = \ln 5, \ln 3$

Question 78 **(****)**

Solve the equation

$$\ln\left(\mathrm{e}\,x^2\right)\ln x = 1,\ \ x > 0,$$

giving the answers in exact form.

$\boxed{}$, $\boxed{x = \frac{1}{\mathrm{e}}, \sqrt{\mathrm{e}}}$

$\ln(\mathrm{e}x^2)\ln x = 1$

$\Rightarrow \ln(\mathrm{e}x^2) = \frac{1}{\ln x}$

$\Rightarrow \ln \mathrm{e} + \ln x^2 = \frac{1}{\ln x}$

$\Rightarrow 1 + 2\ln x = \frac{1}{\ln x}$

$\Rightarrow 1 + 2y = \frac{1}{y}$ $(y = \ln x)$

$\Rightarrow y + 2y^2 = 1$

$\Rightarrow 2y^2 + y - 1 = 0$

$\Rightarrow (2y-1)(y+1) = 0$

$\Rightarrow y = \begin{cases} -1 \\ \frac{1}{2} \end{cases}$

$\Rightarrow \ln x = \begin{cases} -1 \\ \frac{1}{2} \end{cases}$

$\Rightarrow x = \begin{cases} \mathrm{e}^{-1} \\ \mathrm{e}^{\frac{1}{2}} \end{cases}$

$\Rightarrow x = \begin{cases} \frac{1}{\mathrm{e}} \\ \sqrt{\mathrm{e}} \end{cases}$ //

Question 79 (****)

A curve has equation

$$f(x) = \mathrm{e}^{x} + 10\mathrm{e}^{-x} - 7, \quad x \in \mathbb{R}.$$

a) Solve the equation $f(x) = 0$.

b) Hence, or otherwise, solve the equation

$$\mathrm{e}^{2x-2} - 7\mathrm{e}^{x-1} + 10 = 0.$$

$\boxed{\text{[illegible]}}$, $\boxed{x = \ln 2,\ x = \ln 5}$, $\boxed{x = \ln(2\mathrm{e}),\ x = \ln(5\mathrm{e})}$

a) PROCEED AS FOLLOWS TO SOLVE $f(x)=0$

$\Rightarrow e^{x} + 10e^{-x} - 7 = 0$

$\Rightarrow e^{x} + \frac{10}{e^{x}} - 7 = 0$

$\Rightarrow E + \frac{10}{E} - 7 = 0$, WHERE $E = e^{x}$

$\Rightarrow E^{2} + 10 - 7E = 0$

$\Rightarrow E^{2} - 7E + 10 = 0$

$\Rightarrow (E-2)(E-5) = 0$

$\Rightarrow E = e^{x} = \left\langle \begin{matrix} 2 \\ 5 \end{matrix} \right.$

$\therefore x = \ln 2$ OR $x = \ln 5$

b) REWRITE THE EQUATION AS FOLLOWS

$\Rightarrow e^{2x-2} - 7e^{x-1} + 10 = 0$

$\Rightarrow e^{2(x-1)} - 7e^{(x-1)} + 10 = 0$

COMPARE THIS EQUATION WITH

$e^{2x} - 7e^{x} + 10 = 0 \quad [E^{2} - 7E + 10 = 0]$

$\Rightarrow x - 1 = \left\langle \begin{matrix} \ln 2 \\ \ln 5 \end{matrix} \right.$

$\Rightarrow x = \left\langle \begin{matrix} 1 + \ln 2 \\ 1 + \ln 5 \end{matrix} \right.$

Question 80 (****)

A hot metal rod is cooled down by dipping it into a large pool of water which is maintained at constant temperature.

The temperature of the metal rod, T °C, is given by

$$T = 20 + 480\mathrm{e}^{-0.1t}, \ t \geq 0,$$

where t is time in minutes since the rod was dipped in the water.

a) State the temperature …

- **i.** … of the rod before it enters the water.
- **ii.** … of the water.

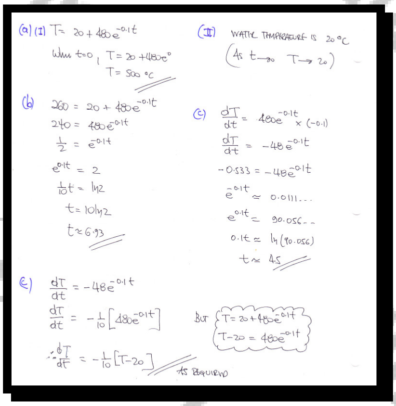
b) Determine the value of t when the rod reaches a temperature of 260 °C.

c) Find the value of t when the rod is cooling at the rate of 0.533 °C per minute.

d) Show clearly that

$$\frac{dT}{dt} = -\frac{1}{10}(T - 20).$$

$\boxed{}$, $\boxed{T = 500}$, $\boxed{T_{\text{water}} = 20}$, $\boxed{t \approx 6.93}$, $\boxed{t \approx 45}$



(a)(I) $T = 20 + 480e^{-0.1t}$

When $t = 0$, $T = 20 + 480e^{0}$

$T = 500$ °C

(II) WATER TEMPERATURE IS 20 °C

(As $t \to \infty$ $T \to 20$)

(b) $260 = 20 + 480e^{-0.1t}$

$240 = 480e^{-0.1t}$

$\frac{1}{2} = e^{-0.1t}$

$e^{0.1t} = 2$

$\frac{1}{10}t = \ln 2$

$t = 10\ln 2$

$t \approx 6.93$

(c) $\frac{dT}{dt} = 480e^{-0.1t} \times (-0.1)$

$\frac{dT}{dt} = -48e^{-0.1t}$

$-0.533 = -48e^{-0.1t}$

$e^{-0.1t} \simeq 0.0111...$

$e^{0.1t} = 90.056...$

$0.1t \simeq \ln(90.056)$

$t \simeq 45$

(d) $\frac{dT}{dt} = -48e^{-0.1t}$

$\frac{dT}{dt} = -\frac{1}{10}\left[480e^{-0.1t}\right]$

BUT $T = 20 + 480e^{-0.1t}$

$T - 20 = 480e^{-0.1t}$

$\therefore \frac{dT}{dt} = -\frac{1}{10}[T - 20]$

AS REQUIRED

Question 81 (****)

The mass, M grams, of a leaf t days after it was picked from a tree is given by

$$M = A\mathrm{e}^{-kt}\,,\ t \geq 0$$

where A and k are positive constants.

When the leaf is picked its mass is 10 grams and 5 days later its mass is 5 grams.

a) Show clearly that

$$k = \frac{1}{5}\ln 2\,.$$

b) Find the value of t that satisfies the equation

$$\frac{dM}{dt} = \ln\left(\frac{\sqrt{2}}{2}\right).$$

$\boxed{}$, $\boxed{t = 10}$

(a) $\boxed{M = Ae^{-kt}}$

$t=0 \quad M=10 \Rightarrow A=10$

$\boxed{M = 10e^{-kt}}$

$t=5 \quad M=5$

$5 = 10e^{-5k}$

$\frac{1}{2} = e^{-5k}$

$e^{5k} = 2$

$5k = \ln 2$

$k = \frac{1}{5}\ln 2$

AS REQUIRED

(b) $M = Ae^{-kt}$

$\frac{dM}{dt} = -Ake^{-kt}$

$\ln\frac{\sqrt{2}}{2} = -10 \times \frac{1}{5}\ln 2 \times e^{-\left(\frac{1}{5}\ln 2\right)t}$

$\ln\frac{\sqrt{2}}{2} = -2\ln 2 \times e^{-\frac{1}{5}t\ln 2}$

$\ln\sqrt{2} - \ln 2 = -2\ln 2 \times e^{-\frac{1}{5}t\ln 2}$

$\frac{1}{2}\ln 2 - \ln 2 = -2\ln 2 \times e^{-\frac{1}{5}t\ln 2}$

$-\frac{1}{2}\cancel{\ln 2} = -2\cancel{\ln 2} \times e^{-\frac{1}{5}t\ln 2}$

$\frac{1}{4} = e^{-\frac{1}{5}t\ln 2}$

$4 = e^{\frac{1}{5}t\ln 2}$

$\ln 4 = \frac{1}{5}t\ln 2$

$\frac{5\ln 4}{\ln 2} = t$

$t = \frac{10\cancel{\ln 2}}{\cancel{\ln 2}} \qquad t = 10$

Question 82 (****+)

A scientist investigating the growth of a certain species of mushroom observes that this mushroom grows to a height of 41 mm in 5 hours.

He decides to model the height, H mm, t hours after the mushroom started forming, by the equation

$$H = k\left(1 - e^{-\frac{1}{12}t}\right), \quad t \geq 0$$

where k is a positive constant.

a) Show that $k = 120$, correct to three significant figures.

The equation

$$H = k\left(1 - e^{-\frac{1}{12}t}\right), \quad t \geq 0,$$

is to be used for the rest of this question.

b) Determine the value of t when $H = 90$, giving the answer in the form $a \ln 2$, where a is an integer.

c) Show clearly that

$$\frac{dH}{dt} = 10 - \frac{1}{12}H.$$

d) Hence find the value of H when the height is the mushroom is growing at the rate of 7.5 mm per hour.

e) State the maximum height of the mushroom according to this model.

$\boxed{}$, $\boxed{a = 24}$, $\boxed{H = 30}$, $\boxed{H_{\text{max}} = 120}$

(a) $H = k\left(1 - e^{-\frac{1}{12}t}\right)$

when $t=5$ $H=41$

$\Rightarrow 41 = k\left(1 - e^{-\frac{5}{12}}\right)$

$\Rightarrow 41 \simeq k(0.3407...)$

$\Rightarrow k \simeq 120.3195$

$\Rightarrow k \simeq 120$

(b) $H = 120\left(1 - e^{-\frac{1}{12}t}\right)$

$\Rightarrow 90 = 120\left(1 - e^{-\frac{1}{12}t}\right)$

$\Rightarrow \frac{3}{4} = 1 - e^{-\frac{1}{12}t}$

$\Rightarrow e^{-\frac{1}{12}t} = \frac{1}{4}$

$\Rightarrow e^{\frac{1}{12}t} = 4$

$\Rightarrow \frac{1}{12}t = \ln 4$

$\Rightarrow t = 12\ln 4$

$\Rightarrow t = 24\ln 2$

$a = 24$

(c) $H = 120\left(1 - e^{-\frac{1}{12}t}\right)$

$\Rightarrow \frac{dH}{dt} = 120\left(\frac{1}{12}e^{-\frac{1}{12}t}\right)$

$\Rightarrow \frac{dH}{dt} = 10e^{-\frac{1}{12}t}$

$\Rightarrow 12\frac{dH}{dt} = 120e^{-\frac{1}{12}t}$

BUT $H = 120 - 120e^{-\frac{1}{12}t}$

$120e^{-\frac{1}{12}t} = 120 - H$

$\Rightarrow 12\frac{dH}{dt} = 120 - H$

$\Rightarrow \frac{dH}{dt} = 10 - \frac{1}{12}H$

(d) $\frac{dH}{dt} = 10 - \frac{1}{12}H$

$\Rightarrow 7.5 = 10 - \frac{1}{12}H$

$\Rightarrow \frac{1}{12}H = 2.5$

$\Rightarrow H = 30$

(e) As $t \to \infty$ $e^{-\frac{1}{12}t} \to 0$

$\therefore H \to 120(1-0)$

$\therefore H_{\text{MAX}} = 120$

Question 83 (****+)

Determine, in exact simplified form where appropriate, the solutions for each of the following equations.

a) $e^{2x} + 2 = 3e^{x}$.

b) $e^{2y-2} + 2 = 3e^{y-1}$.

c) $e^{t} = 3^{\frac{3}{\ln 3}}$.

SY[illegible]P , $\boxed{x = 0, \quad x = \ln 2}$, $\boxed{y = 1, \quad y = 1 + \ln 2}$, $\boxed{t = 3}$

a) THIS IS A QUADRATIC IN e^{x}

$\Rightarrow e^{2x} + 2 = 3e^{x}$

$\Rightarrow e^{2x} - 3e^{x} + 2 = 0$

$\Rightarrow (e^{x})^2 - 3(e^{x}) + 2 = 0$

$\Rightarrow (e^{x} - 1)(e^{x} - 2) = 0$

$\Rightarrow e^{x} = \begin{cases} 1 \\ 2 \end{cases}$ $\quad \therefore x = \begin{cases} 0 \\ \ln 2 \end{cases}$

b) REWRITE THE EQUATION AS FOLLOWS

$\Rightarrow e^{2y-2} + 2 = 3e^{y-1}$

$\Rightarrow e^{2y-2} - 3e^{y-1} + 2 = 0$

$\Rightarrow e^{2(y-1)} - 3e^{y-1} + 2 = 0$

$\Rightarrow e^{2x} - 3e^{x} + 2 = 0$ WHERE $x = y - 1$

USING PART (a)

$x = y - 1 = \begin{cases} 0 \\ \ln 2 \end{cases}$

$\therefore y = \begin{cases} 1 \\ 1 + \ln 2 \end{cases}$

c) USING EITHER OF THE APPROACHES SHOWN BELOW

- $e^{t} = 3^{\frac{3}{\ln 3}}$

$\Rightarrow \ln(e^{t}) = \ln\left(3^{\frac{3}{\ln 3}}\right)$

$\Rightarrow t = \frac{3}{\cancel{\ln 3}} \times \cancel{\ln 3}$

$\Rightarrow t = 3$

- $e^{t} = 3^{\frac{3}{\ln 3}}$

$\Rightarrow (e^{t})^{\ln 3} = \left(3^{\frac{3}{\ln 3}}\right)^{\ln 3}$

$\Rightarrow (e^{\ln 3})^{t} = 3^{3}$

$\Rightarrow 3^{t} = 3^{3}$

$\Rightarrow t = 3$

Question 84 (****+)

Find the solution of the following logarithmic equation

$$2\ln 108 - x\ln 48 = x\ln 3 - 8\ln 2 .$$

$\boxed{x=3}$

$2\ln 108 - x\ln 48 = x\ln 3 - 8\ln 2$

$\Rightarrow 2\ln 108 + 8\ln 2 = x\ln 3 + x\ln 48$

$\Rightarrow x(\ln 3 + \ln 48) = 2\ln 108 + 8\ln 2$

$\Rightarrow x = \dfrac{2\ln 108 + 8\ln 2}{\ln 3 + \ln 48}$

$\Rightarrow x = \dfrac{2\ln(27\times 4) + 8\ln 2}{\ln 3 + \ln(16\times 3)}$

$\Rightarrow x = \dfrac{2\ln 27 + 2\ln 4 + 8\ln 2}{\ln 3 + \ln 16 + \ln 3}$

$\Rightarrow x = \dfrac{2\ln 3^3 + 4\ln 2 + 8\ln 2}{2\ln 3 + \ln 2^4}$

$\Rightarrow x = \dfrac{6\ln 3 + 12\ln 2}{2\ln 3 + 4\ln 2}$

$\Rightarrow x = \dfrac{6\cancel{(\ln 3 + 2\ln 2)}}{2\cancel{(\ln 3 + 2\ln 2)}}$

$\therefore x = 3$

OR

$x = \dfrac{\ln 108^2 + \ln 2^8}{\ln(3\times 48)} = \dfrac{\ln(108^2\times 2^8)}{\ln(144)} = \dfrac{\ln(2985984)}{\ln(144)}$

$= \dfrac{\ln 144^3}{\ln 144} = \dfrac{3\cancel{\ln 144}}{\cancel{\ln 144}} = 3$

Question 85 (****+)

The function f is defined as

$$f(x) = 3 - \ln 4x, \ \ x \in \mathbb{R}, \ x > 0$$

a) Determine, in exact form, the coordinates of the point where the graph of f crosses the x axis.

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\ln x \xrightarrow{T_1} \ln 4x \xrightarrow{T_2} -\ln 4x \xrightarrow{T_3} 3 - \ln 4x$$

b) Describe geometrically each of the transformations T_1, T_2 and T_3, and hence sketch the graph of $f(x)$.
Indicate clearly any intersections with the coordinate axes.

The function g is defined by

$$g(x) = e^{5-x}, \ \ x \in \mathbb{R}.$$

c) Show that

$$fg(x) = x - k - k\ln k \ ,$$

where k is a positive integer.

$\boxed{}$, $\boxed{\left(\frac{1}{4}e^3, 0\right)}$, $\boxed{T_1 = \text{stretch in } x\text{, scale factor } \frac{1}{4}}$, $\boxed{T_2 = \text{reflection in the } x\text{-axis}}$, $\boxed{T_3 = \text{translation, "up", 4 units}}$, $\boxed{k = 2}$

a) SOLVING $y=0$ YIELDS

$\Rightarrow 0 = 3 - \ln 4x$

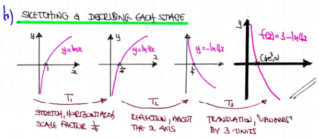
$\Rightarrow \ln 4x = 3$

$\Rightarrow 4x = e^3$

$\Rightarrow x = \frac{1}{4}e^3$

$\therefore \left(\frac{1}{4}e^3, 0\right)$

b) SKETCHING & DESCRIBING EACH STAGE



c) FINALLY THE COMPOSITION

$f(g(x)) = f(e^{5-x})$

$= 3 - \ln(4e^{5-x})$

$= 3 - [\ln 4 + \ln e^{5-x}]$

$= 3 - [\ln 4 + (5-x)]$

$= 3 - \ln 4 - 5 + x$

$= x - 2 - \ln 4$

$= x - 2 - 2\ln 2$ I.E $k=2$

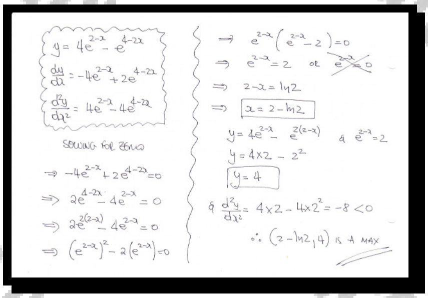
Question 86 (****+)

A curve has equation

$$y = 4e^{2-x} - e^{4-2x}, \ x \in \mathbb{R}.$$

Use differentiation to find the exact coordinates of the stationary point of the curve, and determine its nature.

$\boxed{}$, $\boxed{\max(2 - \ln 2, 4)}$



$y = 4e^{2-x} - e^{4-2x}$

$\frac{dy}{dx} = -4e^{2-x} + 2e^{4-2x}$

$\frac{d^2y}{dx^2} = 4e^{2-x} - 4e^{4-2x}$

SOLVING FOR ZERO

$\Rightarrow -4e^{2-x} + 2e^{4-2x} = 0$

$\Rightarrow 2e^{4-2x} - 4e^{2-x} = 0$

$\Rightarrow 2e^{2(2-x)} - 4e^{2-x} = 0$

$\Rightarrow (e^{2-x})^2 - 2(e^{2-x}) = 0$

$\Rightarrow e^{2-x}(e^{2-x} - 2) = 0$

$\Rightarrow e^{2-x} = 2$ or $\cancel{e^{2-x} = 0}$

$\Rightarrow 2 - x = \ln 2$

$\Rightarrow \boxed{x = 2 - \ln 2}$

$y = 4e^{2-x} - e^{2(2-x)}$ & $e^{2-x} = 2$

$y = 4 \times 2 - 2^2$

$\boxed{y = 4}$

& $\frac{d^2y}{dx^2} = 4 \times 2 - 4 \times 2^2 = -8 < 0$

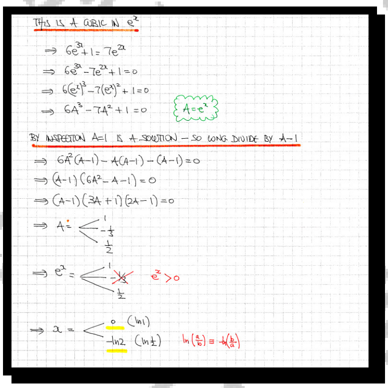
$\therefore (2 - \ln 2, 4)$ IS A MAX

Question 87 (****+)

Find, in exact form where appropriate, the solutions of the following equation

$$6e^{3x} + 1 = 7e^{2x}.$$

$\boxed{}$, $\boxed{x = 0, -\ln 2}$



THIS IS A CUBIC IN e^x

$\Rightarrow 6e^{3x} + 1 = 7e^{2x}$

$\Rightarrow 6e^{3x} - 7e^{2x} + 1 = 0$

$\Rightarrow 6(e^x)^3 - 7(e^x)^2 + 1 = 0$

$\Rightarrow 6A^3 - 7A^2 + 1 = 0$ $A = e^x$

BY INSPECTION $A=1$ IS A SOLUTION – SO LONG DIVIDE BY $A-1$

$\Rightarrow 6A^2(A-1) - A(A-1) - (A-1) = 0$

$\Rightarrow (A-1)(6A^2 - A - 1) = 0$

$\Rightarrow (A-1)(3A+1)(2A-1) = 0$

$\Rightarrow A = 1, -\frac{1}{3}, \frac{1}{2}$

$\Rightarrow e^x = 1, \cancel{-\frac{1}{3}}, \frac{1}{2}$ $e^x > 0$

$\Rightarrow x = 0 \ (\ln 1), \ -\ln 2 \ (\ln \frac{1}{2})$ $\ln\left(\frac{a}{b}\right) \equiv -\ln\left(\frac{b}{a}\right)$

Question 88 (****+)

When a tree of a certain species was planted it was 2 metres in height and after 2 years its height was measured at 3.81 metres.

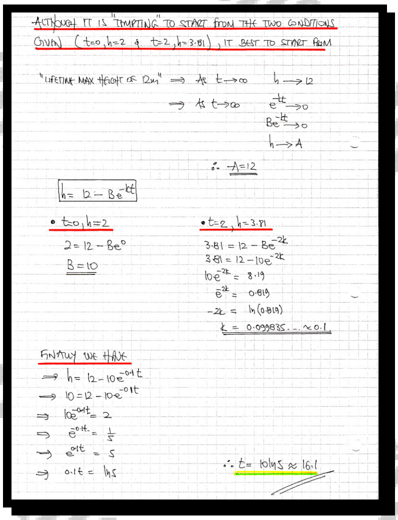
The height, h metres, of this tree, t years after it was planted, is modelled by the equation

$$h = A - B\mathrm{e}^{-kt},$$

where A, B and k are positive constants.

Given that this species of tree will reach in its lifetime a maximum height of 12 metres, find the value of t when $h = 10$.

$\boxed{\text{MP1}}$, $\boxed{t \approx 16}$



ALTHOUGH IT IS "TEMPTING" TO START FROM THE TWO CONDITIONS GIVEN ($t=0, h=2$ & $t=2, h=3.81$), IT BEST TO START FROM

"LIFETIME MAX HEIGHT OF 12m" $\Rightarrow$ AS $t \to \infty$ $\quad h \to 12$

$\Rightarrow$ AS $t \to \infty$ $\quad e^{-kt} \to 0$

$Be^{-kt} \to 0$

$h \to A$

$\therefore A = 12$

$\boxed{h = 12 - Be^{-kt}}$

- $t=0, h=2$

$2 = 12 - Be^{0}$

$B = 10$

- $t=2, h=3.81$

$3.81 = 12 - Be^{-2k}$

$3.81 = 12 - 10e^{-2k}$

$10e^{-2k} = 8.19$

$e^{-2k} = 0.819$

$-2k = \ln(0.819)$

$k = 0.099835\ldots \approx 0.1$

FINALLY WE HAVE

$\Rightarrow h = 12 - 10e^{-0.1t}$

$\Rightarrow 10 = 12 - 10e^{-0.1t}$

$\Rightarrow 10e^{-0.1t} = 2$

$\Rightarrow e^{-0.1t} = \frac{1}{5}$

$\Rightarrow e^{0.1t} = 5$

$\Rightarrow 0.1t = \ln 5$

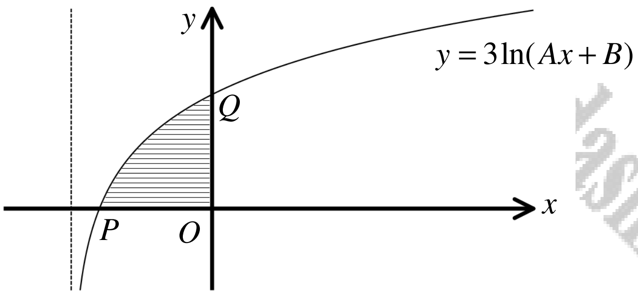
$\therefore t = 10\ln 5 \approx 16.1$

Question 89 (****+)

The figure below shows the curve with equation

$$y = 3\ln(Ax + B), \ Ax + B > 0,$$

where A and B are positive constants.



The curve passes through the points $P(-1,0)$ and $Q(0,\ln 27)$.

a) Find the values of A and B.

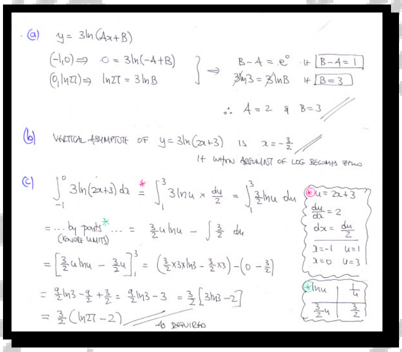
b) State the equation of the vertical asymptote to the curve.

The shaded region R is bounded by the curve and the coordinate axes.

c) Show that the area of R is

$$\frac{3}{2}(\ln 27 - 2).$$

$\boxed{A = 2}$, $\boxed{B = 3}$, $\boxed{x = -\frac{3}{2}}$



(a) $y = 3\ln(Ax+B)$

$(-1,0) \Rightarrow 0 = 3\ln(-A+B)$
$(0,\ln 27) \Rightarrow \ln 27 = 3\ln B$
$\Rightarrow$ $B - A = e^0$ ∴ $\boxed{B-A=1}$
$\cancel{3}\ln 3 = \cancel{3}\ln B$ ∴ $\boxed{B=3}$

∴ $A = 2$ & $B = 3$

(b) VERTICAL ASYMPTOTE OF $y = 3\ln(2x+3)$ IS $x = -\frac{3}{2}$
IT WHEN ARGUMENT OF LOG BECOMES ZERO

(c) $\int_{-1}^{0} 3\ln(2x+3)\,dx \overset{*}{=} \int_{1}^{3} 3\ln u \times \frac{du}{2} = \int_{1}^{3} \frac{3}{2}\ln u \, du$

$= \ldots$ by parts* $\ldots = \frac{3}{2}u\ln u - \int \frac{3}{2}\, du$
(IGNORE LIMITS)

$= \left[\frac{3}{2}u\ln u - \frac{3}{2}u\right]_1^3 = \left(\frac{3}{2}\times 3\times\ln 3 - \frac{3}{2}\times 3\right) - \left(0 - \frac{3}{2}\right)$

$= \frac{9}{2}\ln 3 - \frac{9}{2} + \frac{3}{2} = \frac{9}{2}\ln 3 - 3 = \frac{3}{2}[3\ln 3 - 2]$

$= \frac{3}{2}(\ln 27 - 2)$ AS REQUIRED

$*$ $u = 2x+3$
$\frac{du}{dx} = 2$
$dx = \frac{du}{2}$
$x=-1$ $u=1$
$x=0$ $u=3$

$\ln u$	$\frac{1}{u}$
$\frac{3}{2}u$	$\frac{3}{2}$

Question 90 (****+)

Find, in exact form where appropriate, the solutions for each of the following equations.

a) $4e^{2x+1} = e^{3x-4}$.

b) $\dfrac{2-2\sqrt{2}+1}{\ln x} = \ln x$.

$\boxed{\quad}$, $\boxed{x = 5 + 2\ln 2}$, $\boxed{y = e^{\pm\left(\sqrt{2}-1\right)}}$

(a) $4e^{2x+1} = e^{3x-4}$

$\Rightarrow 4 = \dfrac{e^{3x-4}}{e^{2x+1}}$

$\Rightarrow 4 = e^{x-5}$

$\Rightarrow \ln 4 = x - 5$

$\Rightarrow 2\ln 2 + 5 = x$

$\Rightarrow x = 5 + 2\ln 2$ //

(b) $\dfrac{2-2\sqrt{2}+1}{\ln x} = \ln x$

$\Rightarrow 2 - 2\sqrt{2} + 1 = (\ln x)^2$

$\Rightarrow \sqrt{2}^2 - 2\times 1\times\sqrt{2} + 1^2 = (\ln x)^2$

$\Rightarrow (\sqrt{2}-1)^2 = (\ln x)^2$

$\Rightarrow \pm(\sqrt{2}-1) = \ln x$

$\Rightarrow x = e^{\pm(\sqrt{2}-1)}$ //

Question 91 (****+)

The function f is defined as

$$f : x \mapsto 6 - \ln(x+3), \ x \in \mathbb{R}, \ x \geq -2 .$$

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\ln x \xrightarrow{T_1} \ln(x+3) \xrightarrow{T_2} -\ln(x+3) \xrightarrow{T_3} -\ln(x+3)+6 .$$

a) Describe geometrically T_1, T_2 and T_3, and hence sketch the graph of $f(x)$. Indicate clearly any intersections with the axes and the graph's starting point.

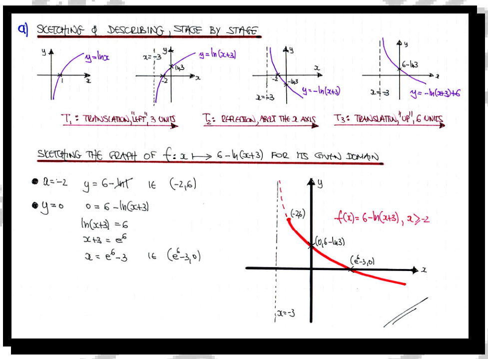
b) Find, in its simplest form, an expression for $f^{-1}(x)$, stating further the domain and range of $f^{-1}(x)$.

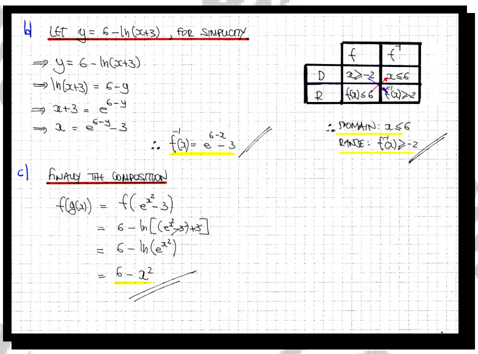
The function g satisfies

$$g : x \to e^{x^2} - 3, \quad x \in \mathbb{R} .$$

c) Find, in its simplest form, an expression for the composition $fg(x)$.

$\boxed{}$, $\boxed{T_1 = \text{translation, "left", 3 units}}$, $\boxed{T_2 = \text{reflection about } x\text{-axis}}$, $\boxed{T_3 = \text{translation, "up", 6 units}}$, $\boxed{(0, 6-\ln 3)}$, $\boxed{(-3+e^6, 0)}$, $\boxed{(-2, 6)}$, $\boxed{f^{-1} : x \mapsto -3 + e^{6-x}}$, $\boxed{x \leq 6, \quad f^{-1}(x) \geq -2}$, $\boxed{fg : x \mapsto 6 - x^2}$





Question 92 (****+)

Show that the value of x in the following expression

$$\ln x = \frac{3\ln 2}{2\ln 2 - 1}$$

satisfies the logarithmic equation

$$2 + \log_2 x = 2\ln\left(\frac{x}{\sqrt{e}}\right), \quad x > 0.$$

$\boxed{}$, $\boxed{\text{proof}}$

CHANGE THE LOG BASE 2 INTO NATURAL LOGS

$\log_a x = \frac{\log_b x}{\log_b a}$ HERE $\log_2 x = \frac{\log_e x}{\log_e 2} = \frac{\ln x}{\ln 2}$

$\Rightarrow 2 + \log_2 x = 2\ln\left(\frac{x}{\sqrt{e}}\right)$

$\Rightarrow 2 + \frac{\ln x}{\ln 2} = 2\ln x - 2\ln\sqrt{e}$

$\Rightarrow 2 + \frac{\ln x}{\ln 2} = 2\ln x - 2\ln e^{\frac{1}{2}}$

$\Rightarrow 2 + \frac{\ln x}{\ln 2} = 2\ln x - \ln e$

$\Rightarrow 2 + \frac{\ln x}{\ln 2} = 2\ln x - 1$

$\Rightarrow 3 + \frac{\ln x}{\ln 2} = 2\ln x$

GETTING RID OF THE FRACTIONS AND TIDY

$\Rightarrow 3\ln 2 + \ln x = 2\ln x\ln 2$

$\Rightarrow 3\ln 2 = 2\ln x\ln 2 - \ln x$

$\Rightarrow 3\ln 2 = \ln x\left[2\ln 2 - 1\right]$

$\Rightarrow \ln x = \frac{3\ln 2}{2\ln 2 - 1}$

AS REQUIRED

Question 93 **(****+)** **non calculator**

An implicit relationship between x and y is given below, in terms of a constant A.

$$y\sec^2 x = A + 2\ln(\sec x),\ 0 \le x < \frac{\pi}{2}.$$

Given that $y = 2$ at $x = \frac{\pi}{3}$, **show clearly** that when $x = \frac{1}{6}\pi$

$$y = \frac{3}{4}(8 - \ln 3).$$

$\boxed{}$, $\boxed{\text{proof}}$

START BY FINDING THE VALUE OF A

$$\left(\tfrac{\pi}{3}, 2\right) \Rightarrow 2 \times \sec^2\tfrac{\pi}{3} = A + 2\ln\left(\sec\tfrac{\pi}{3}\right)$$

$$\Rightarrow \frac{2}{(\cos\frac{\pi}{3})^2} = A + 2\ln\left(\frac{1}{\cos\frac{\pi}{3}}\right)$$

$$\Rightarrow \frac{2}{\frac{1}{4}} = A + 2\ln\left(\frac{1}{\frac{1}{2}}\right)$$

$$\Rightarrow 8 = A + 2\ln 2$$

$$\Rightarrow A = 8 - 2\ln 2$$

REWRITE WITH THE VALUE OF A FOUND & $x = \pi/6$

$$\Rightarrow \frac{y}{\cos^2 x} = 8 - 2\ln 2 + 2\ln\left(\frac{1}{\cos x}\right)$$

$$\Rightarrow \frac{y}{(\cos\frac{\pi}{6})^2} = 8 - 2\ln 2 + 2\ln\left(\frac{1}{\cos\frac{\pi}{6}}\right)$$

$$\Rightarrow \frac{y}{3/4} = 8 - 2\ln 2 + 2\ln\left(\frac{1}{\sqrt{3}/2}\right)$$

$$\Rightarrow \frac{4}{3}y = 8 - 2\ln 2 + 2\ln\frac{2}{\sqrt{3}}$$

$$\Rightarrow \frac{4}{3}y = 8 - 2\ln 2 + 2[\ln 2 - \ln\sqrt{3}]$$

$$\Rightarrow \frac{4}{3}y = 8 - \cancel{2\ln 2} + \cancel{2\ln 2} - 2\ln\sqrt{3}$$

$$\Rightarrow \frac{4}{3}y = 8 - 2\ln 3^{\frac{1}{2}}$$

$$\Rightarrow \frac{4}{3}y = 8 - \ln 3$$

$$\Rightarrow y = \frac{3}{4}(8 - \ln 3)$$

AS REQUIRED

Question 94 (****+)

Find, in its simplest form, the solution of the following equation

$$e^{4x} = 16^{\frac{1}{\ln 2}}.$$

The answer must be supported by detailed workings.

$\boxed{}$, $\boxed{x=1}$

TAKING NATURAL LOGARITHMS BOTH SIDES

$$e^{4x} = 16^{\frac{1}{\ln 2}}$$
$$\Rightarrow 4x = \ln 16^{\frac{1}{\ln 2}}$$
$$\Rightarrow 4x = \frac{1}{\ln 2} \times \ln 16$$
$$\Rightarrow 4x = \frac{\ln 16}{\ln 2}$$
$$\Rightarrow 4x = \frac{4\cancel{\ln 2}}{\cancel{\ln 2}}$$
$$\Rightarrow x = 1$$

ALTERNATIVE FOR FUN

$\log_a b \equiv \frac{1}{\log_b a}$

$$e^{4x} = 16^{\frac{1}{\ln 2}} = (2^4)^{\frac{1}{\ln 2}} = \dots \left[\frac{1}{\ln 2} = \frac{1}{\log_e 2} = \log_2 e\right]$$
$$\dots = (2^4)^{\log_2 e}$$
$$= 2^{4\log_2 e}$$
$$= 2^{\log_2 e^4}$$
$$= e^4$$

$a^{\log_a x} \equiv x$

$$\therefore e^{4x} = e^4$$
$$\Rightarrow x = 1$$

Question 95 (****+)

A population of bacteria P is growing exponentially with time t and the table below shows some of these values.

t	12	36	60
P	576	2304	a

Show clearly that $a = 9216$.

$\boxed{}$, $\boxed{\text{proof}}$

IF THE POPULATION IS INCREASING EXPONENTIALLY WITH TIME, THEN

$$P = Ae^{kt}, \quad A, k \text{ POSITIVE CONSTANTS}$$

USING THE FIRST TWO PAIRS OF VALUES FROM THE TABLE

$$\left.\begin{array}{l} t=12, P=576 \\ t=36, P=2304 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 576 = Ae^{12k} \\ 2304 = Ae^{36k} \end{array}\right\} \text{ DIVIDING "UPWARDS"}$$

$$\Rightarrow e^{24k} = \frac{2304}{576} = \frac{1152}{288} = \frac{576}{144} = \frac{4 \times 576}{576}$$

$$\Rightarrow e^{24k} = 4$$

$$\Rightarrow \underline{\underline{e^{12k} = +2}}$$

THUS $576 = Ae^{12k}$

$576 = A \times 2$

$\underline{\underline{A = 288}}$

FINALLY TO FIND P WHEN $t = 60$

$P = Ae^{kt}$

$P = 288 \times e^{60k}$

$P = 288 \times (e^{12k})^5$

$P = 288 \times 2^5$

$P = 288 \times 32$

$P = 9416$

$$\begin{array}{r} 288 \\ 32 \\ \hline 576 \\ 884 \\ \hline 9416 \end{array}$$

Question 96 (****+)

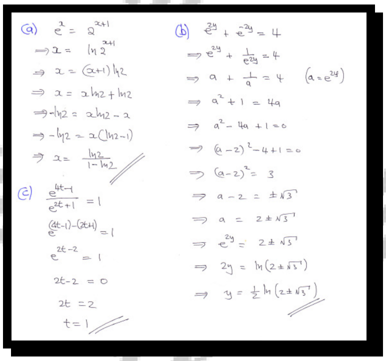
Find, in exact form where appropriate, the solutions for each of the following equations.

a) $e^x = 2^{x+1}$.

b) $e^{2y} + e^{-2y} = 4$.

c) $\dfrac{e^{4t-1}}{e^{2t+1}} = 1$.

$\boxed{x = \dfrac{\ln 2}{1-\ln 2}}$, $\boxed{y = \dfrac{1}{2}\ln\left(2 \pm \sqrt{3}\right) = \pm\dfrac{1}{2}\ln\left(2+\sqrt{3}\right)}$, $\boxed{t=1}$



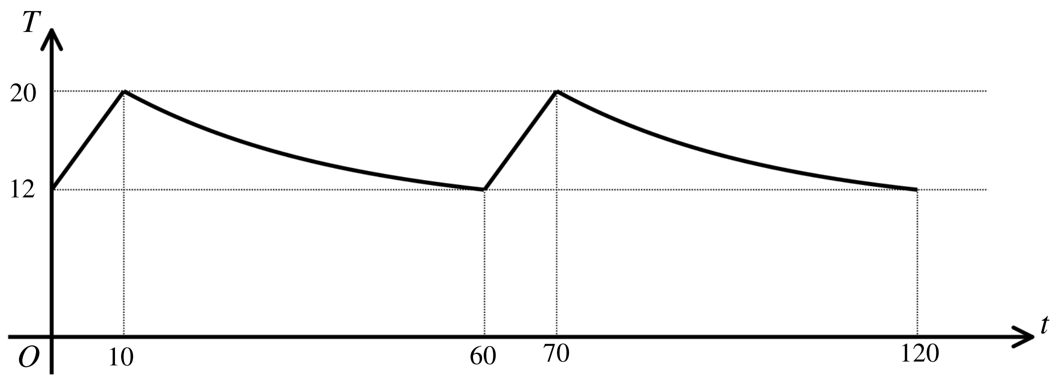
Question 97 (****+)

The temperature T °C in a warehouse is regulated by a thermostat so that when the temperature drops to 12 °C the heating system is switched on.

It takes 10 minutes for the temperature to rise to 20 °C and at that point the thermostat switches the heating system off. It takes 50 minutes for the temperature to drop back down to 12 °C.

The cycle then repeats.

This is shown in the graph below, where the time t is in minutes.



The section of the graph for which $10 \le t \le 60$ is modelled by the equation

$$T = 10 + Ae^{-kt}, \quad 10 \le t \le 60$$

where A and k are positive constants.

a) Find, correct to 5 significant figures, the value of A and the value of k.

[continues overleaf]

[continued from overleaf]

The section of the graph for which $70 \le t \le 120$ is a horizontal translation of the section of the graph for which $10 \le t \le 60$.

This section is modelled by the equation

$$T = 10 + B\mathrm{e}^{-kt}, \quad 70 \le t \le 120$$

where B is a positive constant.

b) Find, to 4 significant figures, the value of B.

$\boxed{A = 13.797}$, $\boxed{k = 0.032189}$, $\boxed{B = 95.18}$

(a) $T = 10 + Ae^{-kt}$

when $t=10$ $T=20$ $\Rightarrow$ $20 = 10 + Ae^{-10k}$
$t=60$ $T=12$ $\Rightarrow$ $12 = 10 + Ae^{-60k}$ $\Rightarrow$

$Ae^{-10k} = 10$
$Ae^{-60k} = 2$ DIVIDE $e^{50k} = 5$

$50k = \ln 5$

$k = \frac{1}{50}\ln 5$

$k = 0.032189$ (5 sf)

USING $Ae^{-10k} = 10$.

$A = \dfrac{10}{e^{-10(0.032189...)}} \approx 13.797$ (5 sf)

(b) TRANSLATE GRAPH 60 UNITS TO THE "RIGHT"

$f(t-60) = 10 + Ae^{-k(t-60)}$
$= 10 + Ae^{-kt+60k}$
$= 10 + Ae^{-kt} \times e^{60k}$
$= 10 + \underbrace{\left(Ae^{60k}\right)}_{B}e^{-kt}$

$\therefore B = 13.797... \times e^{60(0.032189...)}$

$B \simeq 95.18$ (4 sf)

Question 98 (**+)**

Find, in exact form if appropriate, the solutions of the following equation.

$$e^{\frac{3}{2}x} = e^{3x} - 2 .$$

$\boxed{}$, $\boxed{x = \frac{2}{3}\ln 2}$

REWRITE AS A QUADRATIC

$\Rightarrow e^{\frac{3}{2}x} = e^{3x} - 2$

$\Rightarrow 0 = e^{3x} - e^{\frac{3}{2}x} - 2$

$\Rightarrow \underbrace{\left(e^{\frac{3}{2}x}\right)^2}_{e^{3x}} - e^{3x} + 2 = 0$

LET $A = e^{\frac{3}{2}x}$ FOR SIMPLICITY

$\Rightarrow A^2 - A + 2 = 0$

$\Rightarrow (A+1)(A-2) = 0$

$\Rightarrow A = \begin{cases} -1 \\ 2 \end{cases}$

$\Rightarrow e^{\frac{3}{2}x} = \begin{cases} \cancel{-1} \\ 2 \end{cases}$

$\Rightarrow \frac{3}{2}x = \ln 2$

$\Rightarrow x = \frac{2}{3}\ln 2$

Question 99 (****+)

A scientist is investigating the population growth of farm mice.

The number of farm mice N, t months since the start of the investigation, is modelled by the equation

$$N = \frac{600}{1+5e^{-0.25t}}, \quad t \geq 0.$$

a) State the number of farm mice at the start of the investigation.

b) Calculate the number of months that it will take the population of farm mice to reach 455.

c) Show clearly that

$$\frac{dN}{dt} = \frac{1}{4}N - \frac{1}{2400}N^2.$$

d) Find the value of t when the rate of growth of the population of these farm mice is largest.

$\boxed{\phantom{\text{proof}}}$, $\boxed{100}$, $\boxed{t \approx 11}$, $\boxed{t = 4\ln 5 \approx 8}$

a) $N = \dfrac{600}{1+5e^{-0.25t}}$

$t=0, \quad N = \dfrac{600}{1+5e^{0}} = \dfrac{600}{6} = 100$

b) $N=455, \quad 455 = \dfrac{600}{1+5e^{-0.25t}}$

$1+5e^{-0.25t} = \dfrac{120}{91}$

$5e^{-0.25t} = \dfrac{29}{91}$

$e^{-0.25t} = \dfrac{29}{455}$

$e^{0.25t} = \dfrac{455}{29}$

$0.25t = \ln\left(\dfrac{455}{29}\right)$

$t = 4\ln\left(\dfrac{455}{29}\right)$

$t \simeq 11$ months

c) $N = 600\left(1+5e^{-0.25t}\right)^{-1}$

$\dfrac{dN}{dt} = -600\left(1+5e^{-0.25t}\right)^{-2} \times \left(-\dfrac{5}{4}e^{-0.25t}\right) = \dfrac{750e^{-0.25t}}{\left(1+5e^{-0.25t}\right)^2}$

REARRANGING $1+5e^{-0.25t} = \dfrac{600}{N}$

$e^{-0.25t} = \dfrac{120}{N} - \dfrac{1}{5}$

$\therefore \dfrac{dN}{dt} = \dfrac{750\left(\frac{120}{N} - \frac{1}{5}\right)}{\left(\frac{600}{N}\right)^2} = \dfrac{750N^2\left(\frac{120}{N} - \frac{1}{5}\right)}{360000}$

$\therefore \dfrac{dN}{dt} = \dfrac{N^2}{480}\left(\dfrac{120}{N} - \dfrac{1}{5}\right) = \dfrac{1}{4}N - \dfrac{1}{2400}N^2$

AS REQUIRED

d) LET $f(N) = \dfrac{dN}{dt} = \dfrac{1}{4}N - \dfrac{1}{2400}N^2$

$f'(N) = \dfrac{1}{4} - \dfrac{1}{1200}N$

• SOLVE FOR ZERO

$\dfrac{1}{4} - \dfrac{1}{1200}N = 0$

$\dfrac{1}{4} = \dfrac{1}{1200}N$

$N = 300$

$\therefore 300 = \dfrac{600}{1+5e^{-0.25t}}$

$\Rightarrow 1+5e^{-0.25t} = 2$

$\Rightarrow 5e^{-0.25t} = 1$

$\Rightarrow e^{-0.25t} = \dfrac{1}{5}$

$\Rightarrow e^{0.25t} = 5$

$\Rightarrow 0.25t = \ln 5$

$\Rightarrow t = 4\ln 5$

$\Rightarrow t \simeq 8$ months

Question 100 (****+)

The populations P_1 and P_2 of two bacterial cultures, t hours after a certain instant, are modelled by the following equations

$$P_1 = 1600e^{\frac{1}{4}t}, \quad P_2 = 100e^{\frac{1}{2}t}, \quad t \in \mathbb{R}, \ t \geq 0.$$

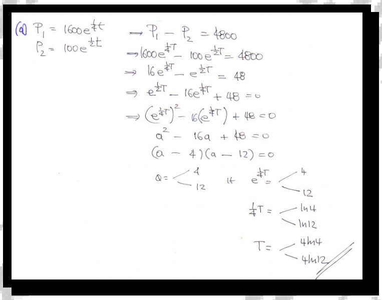
When $t = T$, P_1 contains 4800 more bacteria than P_2.

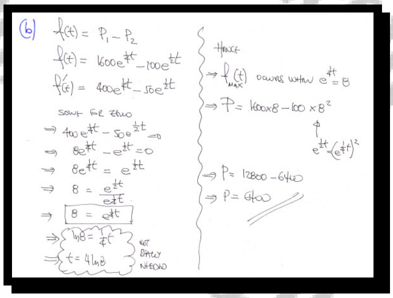
a) Find, in terms of natural logarithms, the possible values of T.

At a certain time there are P extra bacteria in P_1 compared with P_2.

b) Determine the greatest value of P.

$\boxed{T = 8\ln 2 \text{ or } 4\ln 12}$, $\boxed{P = 6400}$





Question 101 (****+)

The function f is defined as

$$f(x) = \ln(4-2x), \; x \in \mathbb{R}, \; x < 2.$$

a) Find in exact form the coordinates of the points where the graph of $f(x)$ crosses the coordinate axes.

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\ln x \xrightarrow{T_1} \ln(x+4) \xrightarrow{T_2} \ln(2x+4) \xrightarrow{T_3} \ln(-2x+4)$$

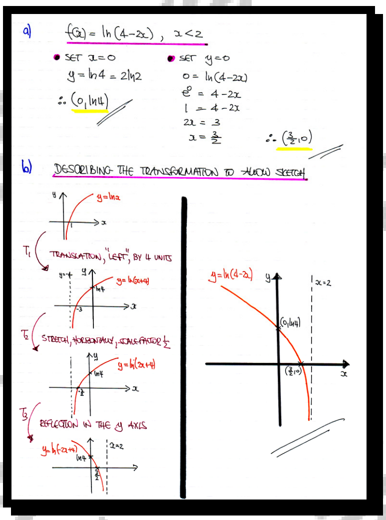
b) Describe geometrically the transformations T_1, T_2 and T_3, and hence sketch the graph of $f(x)$.

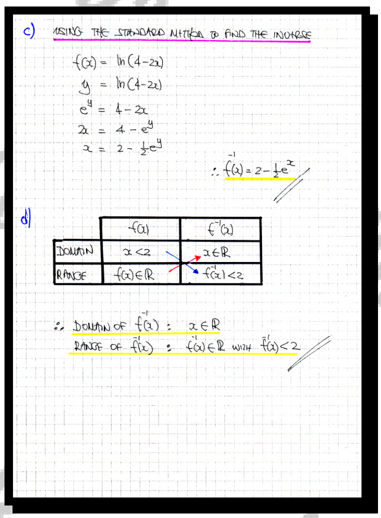
Indicate clearly any asymptotes and coordinates of intersections with the axes.

c) Find, an expression for $f^{-1}(x)$, the inverse function of $f(x)$.

d) State the domain and range of $f^{-1}(x)$.

$\boxed{\text{M.P}}$, $\boxed{\left(\frac{3}{2},0\right), \; (0,\ln 4)}$, $\boxed{T_1 = \text{translation, "left", 4 units}}$, $\boxed{T_2 = \text{stretch in } x\text{, scale factor } \frac{1}{2}}$, $\boxed{T_3 = \text{reflection in the } y\text{-axis}}$, $\boxed{\text{asymptote } x = 2}$, $\boxed{f^{-1}(x) = 2 - \frac{1}{2}e^x}$, $\boxed{x \in \mathbb{R}}$, $\boxed{f^{-1}(x) < 2}$





Question 102 (****+)

The amount X milligrams, of an anaesthetic drug in the bloodstream of a patient, is given by

$$X = D\mathrm{e}^{-0.2t}, \ t \geq 0$$

where D is the dose, in milligrams, of the anaesthetic administered and t is the time in hours since the dose was administered.

A patient undergoing an operation is given an initial dose of 20 milligrams.

This patient will remain asleep if there are more than 12 milligrams of anaesthetic in his bloodstream.

a) Show that one hour later $X = 16.37$, correct to two decimal places.

b) Show, by calculation, that two hours after the initial dose was administered, the patient should still be asleep.

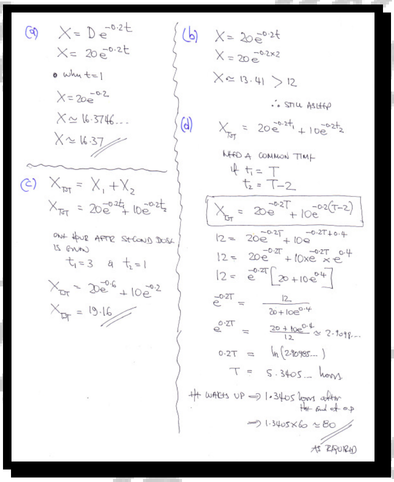
Two hours after the initial dose was administered a further dose of **10** milligrams is given to the patient.

c) Find the amount of the anaesthetic in the patient's bloodstream one hour after the second dose is given.

No more anaesthetic is given and the operation lasts for 4 hours.

d) Show by solving a relevant equation that the patient should "wake up" approximately 80 minutes after the end of his operation.

$\boxed{}$, $\boxed{X \approx 19.16}$



(a) $X = De^{-0.2t}$

$X = 20e^{-0.2t}$

• when $t=1$

$X = 20e^{-0.2}$

$X \simeq 16.3746...$

$X \simeq 16.37$

(c) $X_{TOT} = X_1 + X_2$

$X_{TOT} = 20e^{-0.2t_1} + 10e^{-0.2t_2}$

ONE HOUR AFTER SECOND DOSE IS GIVEN

$t_1 = 3$ & $t_2 = 1$

$X_{TOT} = 20e^{-0.6} + 10e^{-0.2}$

$X_{TOT} = 19.16$

(b) $X = 20e^{-0.2t}$

$X = 20e^{-0.2\times 2}$

$X \simeq 13.41 > 12$

$\therefore$ STILL ASLEEP

(d) $X_{TOT} = 20e^{-0.2t_1} + 10e^{-0.2t_2}$

NEED A COMMON TIME

if $t_1 = T$

$t_2 = T-2$

$\boxed{X_{TOT} = 20e^{-0.2T} + 10e^{-0.2(T-2)}}$

$12 = 20e^{-0.2T} + 10e^{-0.2T+0.4}$

$12 = 20e^{-0.2T} + 10\times e^{-0.2T}\times e^{0.4}$

$12 = e^{-0.2T}\left[20 + 10e^{0.4}\right]$

$e^{-0.2T} = \frac{12}{20+10e^{0.4}}$

$e^{0.2T} = \frac{20+10e^{0.4}}{12} \simeq 2.9098...$

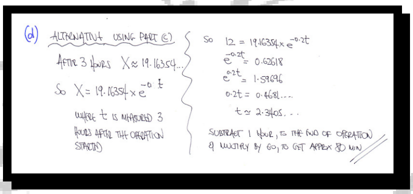
$0.2T = \ln(2.90985...)$

$T = 5.3405...$ hours

He WAKES UP $\Rightarrow$ 1.3405 hours after the end of op

$\Rightarrow 1.3405\times 60 \simeq 80$

AS REQUIRED



(d) ALTERNATIVE USING PART (c)

AFTER 3 HOURS $X \approx 19.16354...$

So $X = 19.16354\times e^{-0.2t}$

WHERE t IS MEASURED 3 HOURS AFTER THE OPERATION STARTED

So $12 = 19.16354\times e^{-0.2t}$

$e^{-0.2t} = 0.62618$

$e^{0.2t} = 1.59696$

$0.2t = 0.4681...$

$t \simeq 2.3405...$

SUBTRACT 1 HOUR, TO THE END OF OPERATION & MULTIPLY BY 60, TO GET APPROX 80 MIN

Question 103 (****+)

The temperature, θ °C, of an oven t minutes after it was switched on is given by

$$\theta = 300 - 280\mathrm{e}^{-0.05t}, \ t \geq 0.$$

a) State the initial temperature of the oven.

b) Find the value of t when the temperature of the oven …

i. … reaches 160 °C.

ii. … is increasing at the rate of 4 °C per minute.

c) Determine, with justification, the maximum temperature this oven can reach.

The temperature θ °C of a **different** oven t minutes after it was switched on is modelled by a similar equation

$$\theta = 250 - 230\mathrm{e}^{-0.1t}, \ t \geq 0.$$

d) Assuming that both ovens are switched on at the same time find the time when both ovens will have the same temperature since they were switched on.

$\boxed{}$, $\boxed{20°\text{C}}$, $\boxed{20\ln 2 \approx 13.86}$, $\boxed{20\ln\left(\frac{7}{2}\right) \approx 25.06}$, $\boxed{300°\text{C}}$, $\boxed{20\ln\left(\frac{23}{5}\right) \approx 30.52}$

(a) $t=0 \quad \theta = 300 - 280e^{0} = 20°C$

(b)(I) $\theta = 300 - 280e^{-0.05t}$

$\Rightarrow 160 = 300 - 280e^{-0.05t}$

$\Rightarrow 280e^{-0.05t} = 140$

$\Rightarrow e^{-0.05t} = \frac{1}{2}$

$\Rightarrow e^{0.05t} = 2$

$\Rightarrow 0.05t = \ln 2$

$\Rightarrow t = 20\ln 2 \approx 13.86$

(II) $\frac{d\theta}{dt} = 14e^{-0.05t}$

$4 = 14e^{-0.05t}$

$\frac{2}{7} = e^{-0.05t}$

$\frac{7}{2} = e^{0.05t}$

$\ln\frac{7}{2} = 0.05t$

$t = 20\ln\frac{7}{2} \approx 25.06$

(c) As $t \to \infty$, $e^{-0.05t} \to 0$

$280e^{-0.05t} \to 0$

$\theta \to 300$

$\therefore$ MAX IS 300°C

(d) $300 - 280e^{-0.05t} = 250 - 230e^{-0.1t}$

$50 = 280e^{-0.05t} - 230e^{-0.1t}$

$50 = \frac{280}{e^{0.05t}} - \frac{230}{e^{0.1t}}$

$50 = \frac{280}{e^{0.05t}} - \frac{230}{(e^{0.05t})^2}$

$50 = \frac{280}{x} - \frac{230}{x^2}$

$50x^2 = 280x - 230$

$50x^2 - 280x + 230 = 0$

$5x^2 - 28x + 23 = 0$

$(5x - 23)(x - 1) = 0$

$x = 1, \ \frac{23}{5}$

Hence

$e^{0.05t} = 1, \ \frac{23}{5}$

$0.05t = \cancel{0}, \ \ln\frac{23}{5}$

$t = 20\ln\frac{23}{5} \approx 30.52$

Question 104 (*****)

The population P of seals on an island obeys the equation

$$P = \frac{800k\,\mathrm{e}^{0.25t}}{1 + k\,\mathrm{e}^{0.25t}}, \quad t \geq 0,$$

where k is a positive constant and t is the time, in years, measured from a certain instant.

Initially there were 175 seals on the island.

a) Find, showing a detailed method, …

i. … the value of t when the number of seals reaches 560.

ii. … the long term prospects of the population of these seals.

b) Show further that

$$\frac{dP}{dt} = \frac{P(800 - P)}{3200}.$$

$\boxed{}$, $\boxed{t \approx 8.48}$, $\boxed{P_{\max} = 800}$

a) I) START BY OBTAINING k, USING $t=0$ $P=175$

$$\Rightarrow P = \frac{800ke^{0.25t}}{1+ke^{0.25t}}$$

$$\Rightarrow 175 = \frac{800k \times 1}{1+k\times 1}$$

$$\Rightarrow 175 = \frac{800k}{1+k}$$

$$\Rightarrow 175k + 175 = 800k$$

$$\Rightarrow 175 = 625k$$

$$\Rightarrow k = \frac{7}{25}$$

NOW USING THE EQUATION WITH THE ABOVE VALUE OF k

$$\Rightarrow P = \frac{800 \times \frac{7}{25}e^{0.25t}}{1 + \frac{7}{25}e^{0.25t}}$$

$$\Rightarrow P = \frac{224e^{0.25t}}{1 + \frac{7}{25}e^{0.25t}}$$

MULTIPLY NUMERATOR & DENOMINATOR OF THE FRACTION BY 25

$$\Rightarrow P = \frac{5600e^{0.25t}}{25 + 7e^{0.25t}}$$

$$\Rightarrow 560 = \frac{5600e^{0.25t}}{25 + 7e^{0.25t}}$$

$\div 560$

$$\Rightarrow 1 = \frac{10e^{0.25t}}{25 + 7e^{0.25t}}$$

$$\Rightarrow 25 + 7e^{0.25t} = 10e^{0.25t}$$

$$\Rightarrow 25 = 3e^{0.25t}$$

$$\Rightarrow \frac{25}{3} = e^{\frac{1}{4}t}$$

$$\Rightarrow \ln\frac{25}{3} = \frac{1}{4}t$$

$$\Rightarrow t = 4\ln\frac{25}{3} \approx 8.48$$

II) REWRITING THE FORMULA FOR SIMPLICITY

$$\Rightarrow P = \frac{5600e^{0.25t}}{25 + 7e^{0.25t}}$$

$$\Rightarrow P = \frac{5600e^{0.25t}e^{-0.25t}}{25e^{-0.25t} + 7e^{0.25t}e^{-0.25t}}$$

$$\Rightarrow \boxed{P = \frac{5600}{25e^{-0.25t} + 7}}$$

NOW AS $t \to \infty$ $e^{-0.25t} \to 0 \Rightarrow P \to \frac{5600}{7}$

$$\Rightarrow P \to 800$$

$\therefore$ THE POPULATION TENDS TO 800

b) USING THE EXPRESSION IN THE "RED BOX", ABOVE

$$P = \frac{5600}{25e^{-0.25t}+7} = 5600\left(7 + 25e^{-0.25t}\right)^{-1}$$

$$\Rightarrow \frac{dP}{dt} = -5600\left(7+25e^{-0.25t}\right)^{-2} \times 25e^{-0.25t} \times \left(-\frac{1}{4}\right)$$

$$\Rightarrow \frac{dP}{dt} = 35000e^{-0.25t}\left(7 + 25e^{-0.25t}\right)^{-2}$$

NOW REARRANGING THE FORMULA FOR P, FROM EARLIER

$$25e^{-0.25t} + 7 = \frac{5600}{P}$$

$$25e^{-0.25t} = \frac{5600}{P} - 7$$

$$\Rightarrow \frac{dP}{dt} = 1400 \times 25e^{-0.25t} \times \left(7 + 25e^{-0.25t}\right)^{-2}$$

$$\Rightarrow \frac{dP}{dt} = 1400 \times \left(\frac{5600}{P} - 7\right)\left(\frac{5600}{P}\right)^{-2}$$

$$\Rightarrow \frac{dP}{dt} = 1400 \times \left(\frac{5600}{P} - 7\right)\left(\frac{P^2}{5600^2}\right)$$

$$\Rightarrow \frac{dP}{dt} = \frac{P^2}{22400}\left(\frac{5600}{P} - 7\right)$$

$$\Rightarrow \frac{dP}{dt} = \frac{P}{4} - \frac{P^2}{3200}$$

$$\Rightarrow \frac{dP}{dt} = \frac{800P - P^2}{3200}$$

$$\Rightarrow \frac{dP}{dt} = \frac{P(800 - P)}{3200}$$

AS REQUIRED

Question 105 (*****)

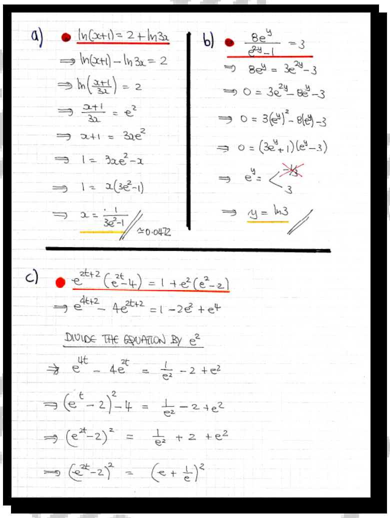
Find, in exact form where appropriate, the solutions for each of the following equations.

a) $\ln(x+1) = 2 + \ln(3x)$.

b) $\dfrac{8\mathrm{e}^{y}}{\mathrm{e}^{2y}-1} = 3$.

c) $\mathrm{e}^{2t+2}\left(\mathrm{e}^{2t}-4\right) = 1 + \mathrm{e}^{2}\left(\mathrm{e}^{2}-2\right)$.

$\boxed{}$, $\boxed{x = \dfrac{1}{3\mathrm{e}^{2}-1} \approx 0.0472}$, $\boxed{y = \ln 3}$, $\boxed{t = \ln\left[\mathrm{e}^{\frac{1}{2}} + \mathrm{e}^{-\frac{1}{2}}\right]}$



a) $\ln(x+1) = 2 + \ln 3x$

$\Rightarrow \ln(x+1) - \ln 3x = 2$

$\Rightarrow \ln\left(\frac{x+1}{3x}\right) = 2$

$\Rightarrow \frac{x+1}{3x} = e^{2}$

$\Rightarrow x+1 = 3xe^{2}$

$\Rightarrow 1 = 3xe^{2} - x$

$\Rightarrow 1 = x(3e^{2}-1)$

$\Rightarrow x = \frac{1}{3e^{2}-1} \simeq 0.0472$

b) $\frac{8e^{y}}{e^{2y}-1} = 3$

$\Rightarrow 8e^{y} = 3e^{2y} - 3$

$\Rightarrow 0 = 3e^{2y} - 8e^{y} - 3$

$\Rightarrow 0 = 3(e^{y})^{2} - 8(e^{y}) - 3$

$\Rightarrow 0 = (3e^{y}+1)(e^{y}-3)$

$\Rightarrow e^{y} = \cancel{-\frac{1}{3}}$ or 3

$\Rightarrow y = \ln 3$

c) $e^{2t+2}(e^{2t}-4) = 1 + e^{2}(e^{2}-2)$

$\Rightarrow e^{4t+2} - 4e^{2t+2} = 1 - 2e^{2} + e^{4}$

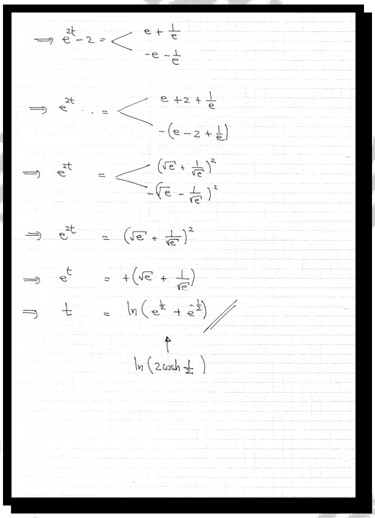
DIVIDE THE EQUATION BY e^{2}

$\Rightarrow e^{4t} - 4e^{2t} = \frac{1}{e^{2}} - 2 + e^{2}$

$\Rightarrow (e^{2t}-2)^{2} - 4 = \frac{1}{e^{2}} - 2 + e^{2}$

$\Rightarrow (e^{2t}-2)^{2} = \frac{1}{e^{2}} + 2 + e^{2}$

$\Rightarrow (e^{2t}-2)^{2} = \left(e + \frac{1}{e}\right)^{2}$



$\Rightarrow e^{2t} - 2 = e + \frac{1}{e}$ or $-e - \frac{1}{e}$

$\Rightarrow e^{2t} = e + 2 + \frac{1}{e}$ or $-\left(e - 2 + \frac{1}{e}\right)$

$\Rightarrow e^{2t} = \left(\sqrt{e} + \frac{1}{\sqrt{e}}\right)^{2}$ or $-\left(\sqrt{e} - \frac{1}{\sqrt{e}}\right)^{2}$

$\Rightarrow e^{2t} = \left(\sqrt{e} + \frac{1}{\sqrt{e}}\right)^{2}$

$\Rightarrow e^{t} = +\left(\sqrt{e} + \frac{1}{\sqrt{e}}\right)$

$\Rightarrow t = \ln\left(e^{\frac{1}{2}} + e^{-\frac{1}{2}}\right)$

$\ln\left(2\cosh\frac{1}{2}\right)$

Question 106 (*****)

Solve the following logarithmic equation.

$$\ln\left(\frac{1}{12}-\frac{1}{3x^2}\right)-1=\ln\left(\frac{1}{12}+\frac{1}{4x}+\frac{1}{6x^2}\right),$$

giving the value of x in exact form.

$\boxed{}$, $\boxed{x=\frac{2+e}{1-e}}$

MANIPULATE THE LOGS TO "EXTRACTION"

$\Rightarrow \ln\left(\frac{1}{12}-\frac{1}{3x^2}\right)-1=\ln\left(\frac{1}{12}+\frac{1}{4x}+\frac{1}{6x^2}\right)$

$\Rightarrow \ln\left(\frac{1}{12}-\frac{1}{3x^2}\right)-\ln\left(\frac{1}{12}+\frac{1}{4x}+\frac{1}{6x^2}\right)=1$

$\Rightarrow \ln\left(\dfrac{\frac{1}{12}-\frac{1}{3x^2}}{\frac{1}{12}+\frac{1}{4x}+\frac{1}{6x^2}}\right)=1$

$\Rightarrow \dfrac{\frac{1}{12}-\frac{1}{3x^2}}{\frac{1}{12}+\frac{1}{4x}+\frac{1}{6x^2}}=e$

MULTIPLY "TOP & BOTTOM" OF THE L.H.S BY $12x^2$

$\Rightarrow \dfrac{x^2-4}{x^2+3x+2}=e$

$\Rightarrow \dfrac{(x-2)(x+2)}{(x+2)(x+1)}=e$

AS $x\neq-2$, SINCE THE ARGUMENTS OF THE LOGS ARE ZERO, WE DIVIDE

$\Rightarrow \dfrac{x-2}{x+1}=e$

$\Rightarrow x-2=ex+e$

$\Rightarrow x-ex=e+2$

$\Rightarrow x(1-e)=2+e$

$\Rightarrow x=\dfrac{2+e}{1-e}$

Question 107 (*****)

On the 1st January 2000 a rare stamp was purchased at an auction for £16000 and by the 1st January 2010 its value was four times as large as its purchase price.

The future value of this stamp, £V, t years after the 1st January 2000 is modelled by the equation

$$V = Ae^{pt}, \quad t \geq 0,$$

where A and p are positive constants.

On the 1st January 1990 a different stamp was purchased for £2.

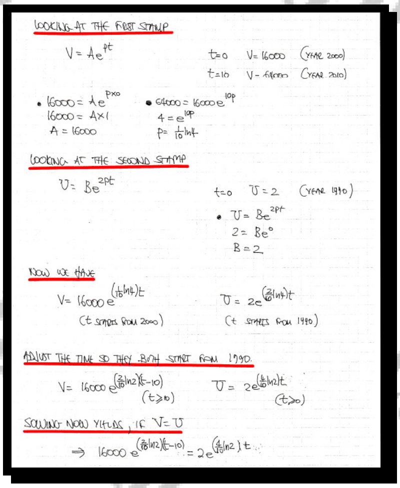
The future value of this stamp, £U, t years after the 1st January 1990 is modelled by the equation

$$U = Be^{2pt}, \quad t \geq 0,$$

where B is a positive constant.

Determine the year, during which the two stamps will achieve the same value, according to their modelling equations.

$\boxed{\text{MP1}}$, $\boxed{2044}$



LOOKING AT THE FIRST STAMP

$V = Ae^{pt}$

$t=0 \quad V = 16000 \quad$ (YEAR 2000)

$t=10 \quad V = 64000 \quad$ (YEAR 2010)

- $16000 = Ae^{p \times 0}$, $16000 = A \times 1$, $A = 16000$
- $64000 = 16000e^{10p}$, $4 = e^{10p}$, $p = \frac{1}{10}\ln 4$

LOOKING AT THE SECOND STAMP

$U = Be^{2pt}$

$t=0 \quad U = 2 \quad$ (YEAR 1990)

- $U = Be^{2pt}$, $2 = Be^{0}$, $B = 2$

NOW WE HAVE

$V = 16000e^{(\frac{1}{10}\ln 4)t}$ (t STARTS FROM 2000)

$U = 2e^{(\frac{2}{10}\ln 4)t}$ (t STARTS FROM 1990)

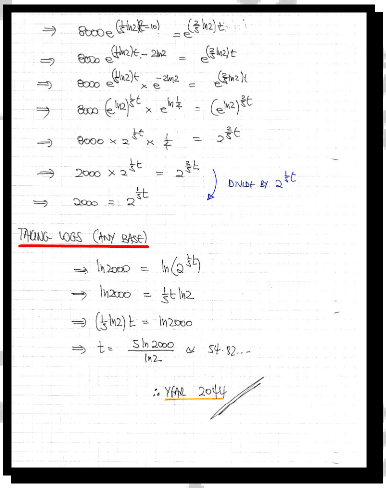
ADJUST THE TIME SO THEY BOTH START FROM 1990.

$V = 16000e^{(\frac{2}{10}\ln 2)(t-10)}$ $(t \geq 10)$

$U = 2e^{(\frac{4}{10}\ln 2)t}$ $(t \geq 0)$

SOLVING NOW YIELDS, IF $V = U$

$\Rightarrow 16000e^{(\frac{2}{10}\ln 2)(t-10)} = 2e^{(\frac{4}{10}\ln 2)t}$



$\Rightarrow 8000e^{(\frac{1}{5}\ln 2)(t-10)} = e^{(\frac{2}{5}\ln 2)t}$

$\Rightarrow 8000e^{(\frac{1}{5}\ln 2)t - 2\ln 2} = e^{(\frac{2}{5}\ln 2)t}$

$\Rightarrow 8000e^{(\frac{1}{5}\ln 2)t} \times e^{-2\ln 2} = e^{(\frac{2}{5}\ln 2)t}$

$\Rightarrow 8000(e^{\ln 2})^{\frac{1}{5}t} \times e^{\ln \frac{1}{4}} = (e^{\ln 2})^{\frac{2}{5}t}$

$\Rightarrow 8000 \times 2^{\frac{1}{5}t} \times \frac{1}{4} = 2^{\frac{2}{5}t}$

$\Rightarrow 2000 \times 2^{\frac{1}{5}t} = 2^{\frac{2}{5}t}$

DIVIDE BY $2^{\frac{1}{5}t}$

$\Rightarrow 2000 = 2^{\frac{1}{5}t}$

TAKING LOGS (ANY BASE)

$\Rightarrow \ln 2000 = \ln(2^{\frac{1}{5}t})$

$\Rightarrow \ln 2000 = \frac{1}{5}t\ln 2$

$\Rightarrow (\frac{1}{5}\ln 2)t = \ln 2000$

$\Rightarrow t = \frac{5\ln 2000}{\ln 2} \approx 54.82\ldots$

$\therefore$ YEAR 2044

Question 108 (*****)

Use algebra, to solve the following equation.

$$e^x + e^{1-x} = e + 1 .$$

$\boxed{x = 0, \quad x = 1}$

<u>REWRITE THIS AS A QUADRATIC IN e^x</u>

$\Rightarrow e^x + e^{1-x} = e+1$

$\Rightarrow e^x + \frac{e}{e^x} = e+1$

$\Rightarrow (e^x)^2 + e = (e+1)e^x$

$\Rightarrow e^{2x} - (e+1)e^x + e = 0$

<u>BY THE QUADRATIC FORMULA OR COMPLETING THE SQUARE</u>

$\Rightarrow \left[e^x - \frac{e+1}{2}\right]^2 - \left(\frac{e+1}{2}\right)^2 + e = 0$

$\Rightarrow \left[e^x - \frac{e+1}{2}\right]^2 - \frac{e^2+2e+1}{4} + e = 0$

$\Rightarrow \left[e^x - \frac{e+1}{2}\right]^2 = \frac{e^2+2e+1}{4} - e$

$\Rightarrow \left[e^x - \frac{e+1}{2}\right]^2 = \frac{e^2 - 2e + 1 - 4e}{4}$

$\Rightarrow \left[e^x - \frac{e+1}{2}\right]^2 = \frac{e^2 - 2e + 1}{4}$

$\Rightarrow \left[e^x - \frac{e+1}{2}\right]^2 = \frac{(e-1)^2}{4}$

$\Rightarrow \left[e^x - \frac{e+1}{2}\right]^2 = \pm \frac{e-1}{2}$

$\Rightarrow e^x = \frac{e+1}{2} \pm \frac{e-1}{2}$

$\Rightarrow e^x = \begin{cases} e \\ 1 \end{cases}$

$\Rightarrow x = \begin{cases} 1 \\ 0 \end{cases}$

Question 109 (*****)

It is given that

$$y = 2\ln\left(e^{x}+1\right) - \ln\left(e^{x}-1\right), \ x \in \mathbb{R}.$$

Express x in terms of y.

, $\boxed{x = \ln\left[\frac{1}{2}e^{y}\left(1 \pm \sqrt{1-8e^{-y}}\right) - 1\right]}$

PROCEED AS FOLLOWS

$\Rightarrow y = 2\ln(e^{x}+1) - \ln(e^{x}-1)$

$\Rightarrow y = \ln(e^{x}+1)^{2} - \ln(e^{x}-1)$

$\Rightarrow y = \ln\left(\frac{e^{2x}+2e^{x}+1}{e^{x}-1}\right)$

$\Rightarrow e^{y} = \frac{e^{2x}+2e^{x}+1}{e^{x}-1}$

$\Rightarrow e^{y}e^{x} - e^{y} = e^{2x} + 2e^{x} + 1$

$\Rightarrow 0 = e^{2x} + 2e^{x} - e^{y}e^{x} + e^{y} + 1$

$\Rightarrow e^{2x} + e^{x}(2-e^{y}) + (e^{y}+1) = 0$

BY THE QUADRATIC FORMULA OR BY COMPLETING THE SQUARE

$\Rightarrow e^{x} = \frac{-(2-e^{y}) \pm \sqrt{(2-e^{y})^{2} - 4\times1\times(e^{y}+1)}}{2\times1}$

$\Rightarrow e^{x} = \frac{e^{y} - 2 \pm \sqrt{e^{2y} - 4e^{y} + 4 - 4e^{y} - 4}}{2}$

$\Rightarrow e^{x} = \frac{e^{y} - 2 \pm \sqrt{e^{2y} - 8e^{y}}}{2}$

$\Rightarrow e^{x} = \frac{e^{y} - 2 \pm e^{y}\sqrt{1-8e^{-y}}}{2}$

$\Rightarrow e^{x} = \frac{e^{y}\left[1 \pm \sqrt{1-8e^{-y}}\right] - 2}{2}$

$\Rightarrow e^{x} = -1 + \frac{1}{2}e^{y}\left[1 \pm \sqrt{1-8e^{-y}}\right]$

$\Rightarrow x = \ln\left[-1 + \frac{1}{2}e^{y}\left[1 \pm \sqrt{1-8e^{-y}}\right]\right]$

Question 110 (*****)

Given that $x = \frac{1}{2}\left(e^{y} - e^{-y}\right)$ show that

$$y = \ln\left(x + \sqrt{x^{2}+1}\right).$$

proof

$x = \frac{1}{2}(e^{y} - e^{-y})$

$\Rightarrow 2x = e^{y} - e^{-y}$

$\Rightarrow 0 = e^{y} - 2x - e^{-y}$

$\Rightarrow 0 = e^{2y} - 2xe^{y} - 1$

(MULTIPLIED BY e^{y})

$\Rightarrow e^{2y} - 2xe^{y} - 1 = 0$

$\Rightarrow (e^{y} - x)^{2} - x^{2} - 1 = 0$

$\Rightarrow (e^{y} - x)^{2} = x^{2} + 1$

$\Rightarrow e^{y} - x = \pm\sqrt{x^{2}+1}$

$\Rightarrow e^{y} = x \pm \sqrt{x^{2}+1}$

BUT $e^{y} > 0$

$\therefore e^{y} \neq x - \sqrt{x^{2}+1}$

$\Rightarrow e^{y} = x + \sqrt{x^{2}+1}$

$\Rightarrow y = \ln\left(x + \sqrt{x^{2}+1}\right)$

Question 111 (*****)

Solve the following equation

$$\frac{e^{2x}+16^x}{(4e)^x}=\frac{4+e}{2\sqrt{e}}, \quad x\in\mathbb{R}.$$

$x=\pm\frac{1}{2}$

$$\boxed{\frac{e^{2x}+16^x}{(4e)^x}=\frac{4+e}{2\sqrt{e}}, \quad x\in\mathbb{R}}$$

- START BY AN INITIAL TIDY TO DECIDE UPON A METHOD

$$\Rightarrow \frac{e^{2x}+16^x}{4^x e^x}=\frac{4+e}{2\sqrt{e}}$$

$$\Rightarrow \frac{e^{2x}+4^{2x}}{4^x e^x}=\frac{4+e}{2\sqrt{e}}$$

$$\Rightarrow \frac{e^{2x}}{4^x e^x}+\frac{4^{2x}}{4^x e^x}=\frac{4+e}{2\sqrt{e}}$$

$$\Rightarrow \frac{e^x}{4^x}+\frac{4^x}{e^x}=\frac{4+e}{2\sqrt{e}}$$

- THIS IS A QUADRATIC $\frac{e^x}{4^x}$ (OR ITS RECIPROCAL)

LET $y=\frac{e^x}{4^x}$

$$\Rightarrow y+\frac{1}{y}=\frac{4+e}{2\sqrt{e}}$$

$$\Rightarrow y^2-\frac{4+e}{2\sqrt{e}}y+1=0$$

- SOLVE BY COMPLETING THE SQUARE

$$\Rightarrow \left[y-\frac{4+e}{4e^{\frac{1}{2}}}\right]^2-\frac{(4+e)^2}{16e}+1=0$$

$$\Rightarrow \left[y-\frac{4+e}{4e^{\frac{1}{2}}}\right]^2=\frac{(4+e)^2}{16e}-1$$

$$\Rightarrow \left[y-\frac{4+e}{4e^{\frac{1}{2}}}\right]^2=\frac{(16+8e+e^2)-16e}{16e}$$

$$\Rightarrow \left[y-\frac{4+e}{4e^{\frac{1}{2}}}\right]^2=\frac{e^2-8e+16}{16e} \quad \leftarrow \text{PERFECT SQUARE}$$

$$\Rightarrow \left[y-\frac{4+e}{4e^{\frac{1}{2}}}\right]^2=\frac{(e-4)^2}{16e}$$

$$\Rightarrow y-\frac{4+e}{4e^{\frac{1}{2}}}=\frac{\pm(e-4)}{4e^{\frac{1}{2}}}$$

$$\Rightarrow y=\frac{(4+e)\pm(e-4)}{4e^{\frac{1}{2}}}$$

$$\Rightarrow y=\begin{cases}\dfrac{(e+4)+(e-4)}{4e^{\frac{1}{2}}}=\dfrac{2e}{4e^{\frac{1}{2}}}=\dfrac{1}{2}e^{\frac{1}{2}}\\[2ex] \dfrac{(e+4)-(e-4)}{4e^{\frac{1}{2}}}=\dfrac{8}{4e^{\frac{1}{2}}}=\dfrac{2}{e^{\frac{1}{2}}}=2e^{-\frac{1}{2}}\end{cases}$$

- REVERSING THE SUBSTITUTIONS FOR EACH OF THE TWO SOLUTIONS

$$\Rightarrow \frac{e^x}{4^x}=\frac{1}{2}e^{\frac{1}{2}}$$

$$\Rightarrow \left(\frac{e}{4}\right)^x=\frac{1}{2}e^{\frac{1}{2}}$$

$$\Rightarrow \ln\left(\frac{e}{4}\right)^x=\ln\left(\frac{1}{2}e^{\frac{1}{2}}\right)$$

$$\Rightarrow x\left[\ln e-\ln 4\right]=\ln\frac{1}{2}+\ln e^{\frac{1}{2}}$$

$$\Rightarrow x\left[1-\ln 4\right]=-\ln 2+\frac{1}{2}$$

$$\Rightarrow x\left[1-\ln 4\right]=\frac{1}{2}\left[1-2\ln 2\right]$$

$$\Rightarrow x\left[1-\ln 4\right]=\frac{1}{2}\left[1-\ln 4\right]$$

$$\Rightarrow x=\frac{1}{2}$$

$$\Rightarrow \frac{e^x}{4^x}=2e^{-\frac{1}{2}}$$

$$\Rightarrow \left(\frac{e}{4}\right)^x=2e^{-\frac{1}{2}}$$

$$\Rightarrow \ln\left(\frac{e}{4}\right)^x=\ln\left(2e^{-\frac{1}{2}}\right)$$

$$\Rightarrow x\left[\ln e-\ln 4\right]=\ln 2+\ln e^{-\frac{1}{2}}$$

$$\Rightarrow x\left[1-\ln 4\right]=\ln 2-\frac{1}{2}$$

$$\Rightarrow x\left[1-\ln 4\right]=-\frac{1}{2}\left[1-2\ln 2\right]$$

$$\Rightarrow x\left[1-\ln 4\right]=-\frac{1}{2}\left[1-\ln 4\right]$$

$$\Rightarrow x=-\frac{1}{2}$$

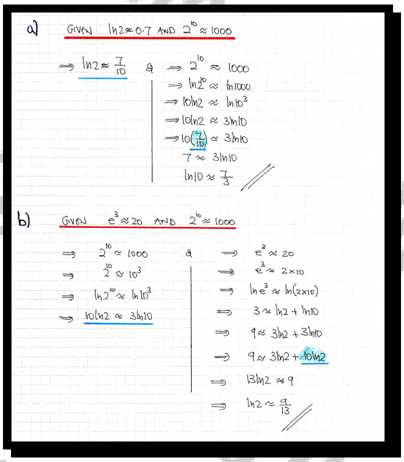
Question 112 (*****)

It is given that 2^{10} is approximately 1000.

a) Given further that $\ln 2$ is approximately 0.7, find the approximate value of $\ln 10$, giving the answer in the form $\frac{a}{b}$, where a and b are positive integers.

b) Given further that e^3 is approximately 20, show that the approximate value of $\ln 2$ is $\frac{9}{13}$.

$\boxed{}$, $\boxed{\ln 10 \approx \frac{7}{3}}$



a) GIVEN $\ln 2 \approx 0.7$ AND $2^{10} \approx 1000$

$\Rightarrow \ln 2 \approx \frac{7}{10}$ &

$\Rightarrow 2^{10} \approx 1000$
$\Rightarrow \ln 2^{10} \approx \ln 1000$
$\Rightarrow 10\ln 2 \approx \ln 10^3$
$\Rightarrow 10\ln 2 \approx 3\ln 10$
$\Rightarrow 10\left(\frac{7}{10}\right) \approx 3\ln 10$
$7 \approx 3\ln 10$
$\ln 10 \approx \frac{7}{3}$

b) GIVEN $e^3 \approx 20$ AND $2^{10} \approx 1000$

$\Rightarrow 2^{10} \approx 1000$
$\Rightarrow 2^{10} \approx 10^3$
$\Rightarrow \ln 2^{10} \approx \ln 10^3$
$\Rightarrow 10\ln 2 \approx 3\ln 10$ &

$\Rightarrow e^3 \approx 20$
$\Rightarrow e^3 \approx 2\times 10$
$\Rightarrow \ln e^3 \approx \ln(2\times 10)$
$\Rightarrow 3 \approx \ln 2 + \ln 10$
$\Rightarrow 9 \approx 3\ln 2 + 3\ln 10$
$\Rightarrow 9 \approx 3\ln 2 + 10\ln 2$
$\Rightarrow 13\ln 2 \approx 9$
$\Rightarrow \ln 2 \approx \frac{9}{13}$

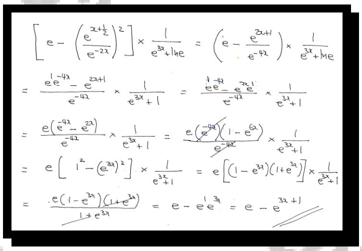
Question 113 (*****)

Show that the expression

$$\left[\mathrm{e}-\left(\frac{\mathrm{e}^{x+\frac{1}{2}}}{\mathrm{e}^{-2x}}\right)^{2}\right]\times\frac{1}{\mathrm{e}^{3x}+\ln\mathrm{e}}$$

simplifies to $\mathrm{e}-\mathrm{e}^{3x+1}$.

$\boxed{}$, $\boxed{\text{proof}}$



$$\left[\mathrm{e}-\left(\frac{\mathrm{e}^{x+\frac{1}{2}}}{\mathrm{e}^{-2x}}\right)^{2}\right]\times\frac{1}{\mathrm{e}^{3x}+\ln\mathrm{e}}=\left(\mathrm{e}-\frac{\mathrm{e}^{2x+1}}{\mathrm{e}^{-4x}}\right)\times\frac{1}{\mathrm{e}^{3x}+\ln\mathrm{e}}$$

$$=\frac{\mathrm{e}^{1}\mathrm{e}^{-4x}-\mathrm{e}^{2x+1}}{\mathrm{e}^{-4x}}\times\frac{1}{\mathrm{e}^{3x}+1}=\frac{\mathrm{e}^{1}\mathrm{e}^{-4x}-\mathrm{e}^{2x}\mathrm{e}^{1}}{\mathrm{e}^{-4x}}\times\frac{1}{\mathrm{e}^{3x}+1}$$

$$=\frac{\mathrm{e}\left(\mathrm{e}^{-4x}-\mathrm{e}^{2x}\right)}{\mathrm{e}^{-4x}}\times\frac{1}{\mathrm{e}^{3x}+1}=\frac{\mathrm{e}\cancel{\left(\mathrm{e}^{-4x}\right)}\left(1-\mathrm{e}^{6x}\right)}{\cancel{\mathrm{e}^{-4x}}}\times\frac{1}{\mathrm{e}^{3x}+1}$$

$$=\mathrm{e}\left[1^{2}-\left(\mathrm{e}^{3x}\right)^{2}\right]\times\frac{1}{\mathrm{e}^{3x}+1}=\mathrm{e}\left[\left(1-\mathrm{e}^{3x}\right)\left(1+\mathrm{e}^{3x}\right)\right]\times\frac{1}{\mathrm{e}^{3x}+1}$$

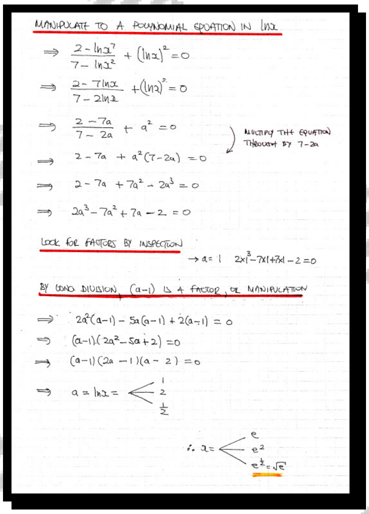
$$=\frac{\mathrm{e}\left(1-\mathrm{e}^{3x}\right)\cancel{\left(1+\mathrm{e}^{3x}\right)}}{\cancel{1+\mathrm{e}^{3x}}}=\mathrm{e}-\mathrm{e}^{1}\mathrm{e}^{3x}=\mathrm{e}-\mathrm{e}^{3x+1}$$

Question 114 (*****)

Find, in exact form, the solutions of the following equation

$$\frac{2-\ln x^{7}}{7-\ln x^{2}}+(\ln x)^{2}=0.$$

$\boxed{}$, $\boxed{x=\mathrm{e},\ \mathrm{e}^{2},\ \sqrt{\mathrm{e}}}$



MANIPULATE TO A POLYNOMIAL EQUATION IN $\ln x$

$$\Rightarrow \frac{2-\ln x^{7}}{7-\ln x^{2}}+(\ln x)^{2}=0$$

$$\Rightarrow \frac{2-7\ln x}{7-2\ln x}+(\ln x)^{2}=0$$

$$\Rightarrow \frac{2-7a}{7-2a}+a^{2}=0$$

MULTIPLY THE EQUATION THROUGH BY $7-2a$

$$\Rightarrow 2-7a+a^{2}(7-2a)=0$$

$$\Rightarrow 2-7a+7a^{2}-2a^{3}=0$$

$$\Rightarrow 2a^{3}-7a^{2}+7a-2=0$$

LOOK FOR FACTORS BY INSPECTION

$\rightarrow a=1 \quad 2\times1^{3}-7\times1+7\times1-2=0$

BY LONG DIVISION, $(a-1)$ IS A FACTOR, OR MANIPULATION

$$\Rightarrow 2a^{2}(a-1)-5a(a-1)+2(a-1)=0$$

$$\Rightarrow (a-1)(2a^{2}-5a+2)=0$$

$$\Rightarrow (a-1)(2a-1)(a-2)=0$$

$$\Rightarrow a=\ln x=\ 1,\ 2,\ \tfrac{1}{2}$$

$$\therefore x=\ \mathrm{e},\ \mathrm{e}^{2},\ \mathrm{e}^{\frac{1}{2}}=\sqrt{\mathrm{e}}$$

Question 115 (*****)

Solve the following equation

$$e^{4(x+1)^2} = \ln e^{-e} + \left(1+\frac{1}{e}\right)e^{2x^2+4x+3}.$$

$\boxed{}$, $\boxed{x=-1,\ \frac{1}{2}\left(-2\pm\sqrt{2}\right)}$

$$\boxed{e^{4(x+1)^2} = \ln e^{-e} + \left(1+\tfrac{1}{e}\right)e^{2x^2+4x+3}}$$

$\Rightarrow e^{4(x+1)^2} = -e\ln e + \left(1+\frac{1}{e}\right)e^{2x^2+4x+3}$

$\Rightarrow e^{4(x+1)^2} = -e + \left(1+\frac{1}{e}\right)e^{2(x+1)^2+1}$ $\quad \boxed{\begin{aligned}2x^2+4x+3 &= 2\left[x^2+2x+1\right]+1\\ &= 2(x+1)^2+1\end{aligned}}$

$\Rightarrow e^{4(x+1)^2} = -e + \left(1+\frac{1}{e}\right)\times e^{2(x+1)^2}\times e^1$

$\Rightarrow \left[e^{2(x+1)^2}\right]^2 = -e + (e+1)\left[e^{2(x+1)^2}\right]$

$\Rightarrow y^2 = -e + (e+1)y$ $\quad \boxed{\text{WHERE } y = e^{2(x+1)^2}}$

$\Rightarrow y^2 - (e+1)y + e = 0$

$\Rightarrow (y-1)(y-e) = 0$

$y = \begin{cases} 1 \\ e \end{cases}$

- $e^{2(x+1)^2} = 1$

  $2(x+1)^2 = 0$

  $x = -1$

- $e^{2(x+1)^2} = e$

  $2(x+1)^2 = 1$

  $(x+1)^2 = \frac{1}{2}$

  $x+1 = \pm\frac{1}{\sqrt{2}}$

  $x = -1 \pm \frac{1}{\sqrt{2}}$

Question 116 (*****)

$$y = 16e^{-\frac{t}{3}\ln 2},\ t \in \mathbb{R}.$$

Show clearly that when $t = 10$, $y = \sqrt[3]{4}$

$\boxed{}$, $\boxed{\text{proof}}$

$$\boxed{y = 16e^{-\frac{t}{3}\ln 2}}$$

<u>WHEN $t=10$</u>

$y = 16\times e^{-\frac{10}{3}\ln 2} = 16\times\left(e^{\ln 2}\right)^{-\frac{10}{3}} = 16\times 2^{-\frac{10}{3}}$

$= 16\times 2^{-\frac{9}{3}}\times 2^{-\frac{1}{3}} = 16\times 2^{-3}\times\frac{1}{2^{\frac{1}{3}}} = 16\times\frac{1}{8}\times\frac{1}{2^{\frac{1}{3}}}$

$= 2\times\frac{1}{2^{\frac{1}{3}}} = \frac{2}{2^{\frac{1}{3}}} = 2^{\frac{2}{3}} = \left(2^2\right)^{\frac{1}{3}} = 4^{\frac{1}{3}}$

$= \sqrt[3]{4}$

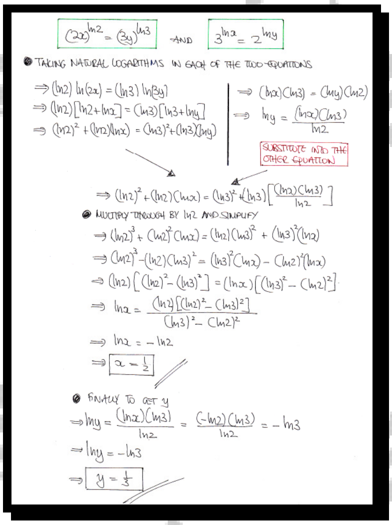
AS REQUIRED

Question 117 **(*****)**

Solve the following simultaneous equations.

$$(2x)^{\ln 2} = (3y)^{\ln 3} \quad \text{and} \quad 3^{\ln x} = 2^{\ln y} .$$

$\boxed{}$, $\boxed{x = \frac{1}{2}, \quad y = \frac{1}{3}}$



$\boxed{(2x)^{\ln 2} = (3y)^{\ln 3}}$ AND $\boxed{3^{\ln x} = 2^{\ln y}}$

- TAKING NATURAL LOGARITHMS IN EACH OF THE TWO EQUATIONS

$\Rightarrow (\ln 2)\ln(2x) = (\ln 3)\ln(3y)$

$\Rightarrow (\ln 2)[\ln 2 + \ln x] = (\ln 3)[\ln 3 + \ln y]$

$\Rightarrow (\ln 2)^2 + (\ln 2)(\ln x) = (\ln 3)^2 + (\ln 3)(\ln y)$

$\Rightarrow (\ln x)(\ln 3) = (\ln y)(\ln 2)$

$\Rightarrow \ln y = \frac{(\ln x)(\ln 3)}{\ln 2}$

SUBSTITUTE INTO THE OTHER EQUATION

$\Rightarrow (\ln 2)^2 + (\ln 2)(\ln x) = (\ln 3)^2 + (\ln 3)\left[\frac{(\ln x)(\ln 3)}{\ln 2}\right]$

- MULTIPLY THROUGH BY $\ln 2$ AND SIMPLIFY

$\Rightarrow (\ln 2)^3 + (\ln 2)^2(\ln x) = (\ln 2)(\ln 3)^2 + (\ln 3)^2(\ln x)$

$\Rightarrow (\ln 2)^3 - (\ln 2)(\ln 3)^2 = (\ln 3)^2(\ln x) - (\ln 2)^2(\ln x)$

$\Rightarrow (\ln 2)\left[(\ln 2)^2 - (\ln 3)^2\right] = (\ln x)\left[(\ln 3)^2 - (\ln 2)^2\right]$

$\Rightarrow \ln x = \frac{(\ln 2)\left[(\ln 2)^2 - (\ln 3)^2\right]}{(\ln 3)^2 - (\ln 2)^2}$

$\Rightarrow \ln x = -\ln 2$

$\Rightarrow \boxed{x = \frac{1}{2}}$

- FINALLY TO GET y

$\Rightarrow \ln y = \frac{(\ln x)(\ln 3)}{\ln 2} = \frac{(-\ln 2)(\ln 3)}{\ln 2} = -\ln 3$

$\Rightarrow \ln y = -\ln 3$

$\Rightarrow \boxed{y = \frac{1}{3}}$

Question 118 (*****)

A function f is defined as

$$f(x) = \mathrm{e}^{ax} + b, \quad -\ln 2 \le x \le \ln 2,$$

where a and b are positive constants.

It is given that $f\left(\ln\left(\frac{3}{2}\right)\right) = \frac{13}{4}$.

a) Show clearly that

$$b = \frac{13}{4} - \left(\frac{3}{2}\right)^a.$$

It is further given that

$$f\left(\ln\left(\frac{2}{3}\right)\right) = \frac{13}{9}.$$

b) Find the value of a and the value of b.

$\boxed{}$, $\boxed{a = 2, \; b = 1}$

(a) $f(x) = e^{ax} + b$

$\Rightarrow \frac{13}{4} = e^{a\ln\frac{3}{2}} + b$

$\Rightarrow \frac{13}{4} = \left(e^{\ln\frac{3}{2}}\right)^a + b$

$\Rightarrow \frac{13}{4} = \left(\frac{3}{2}\right)^a + b$

$\Rightarrow \boxed{b = \frac{13}{4} - \left(\frac{3}{2}\right)^a}$

AS REQUIRED

(b) SIMILARLY

$\Rightarrow \frac{13}{9} = e^{a\ln\frac{2}{3}} + b$

$\Rightarrow \frac{13}{9} = \left(e^{\ln\frac{2}{3}}\right)^a + b$

$\Rightarrow \frac{13}{9} = \left(\frac{2}{3}\right)^a + b$

$\Rightarrow b = \frac{13}{9} - \left(\frac{2}{3}\right)^a$

$\Rightarrow \frac{13}{4} - \left(\frac{3}{2}\right)^a = \frac{13}{9} - \left(\frac{2}{3}\right)^a$

$\Rightarrow \left(\frac{2}{3}\right)^a - \left(\frac{3}{2}\right)^a + \frac{65}{36} = 0$

$\Rightarrow \left(\frac{2}{3}\right)^a - \left(\frac{2}{3}\right)^{-a} + \frac{65}{36} = 0$

$\boxed{\text{LET } \left(\frac{2}{3}\right)^a = z}$

$\Rightarrow z - z^{-1} + \frac{65}{36} = 0$

$\Rightarrow z - \frac{1}{z} + \frac{65}{36} = 0$

$\Rightarrow z^2 - 1 + \frac{65}{36}z = 0$

$\Rightarrow 36z^2 + 65z - 36 = 0$

$\Rightarrow (9z - 4)(4z + 9) = 0$

$\Rightarrow z = \begin{cases} -9/4 \\ 4/9 \end{cases}$ ie $\left(\frac{2}{3}\right)^a = \begin{cases} \cancel{-\frac{9}{4}} \\ \frac{4}{9} \end{cases}$

$\therefore a = 2$

HENCE $b = \frac{13}{9} - \frac{4}{9} = 1$

ie $b = 1$

Question 119 (*****)

A curve has equation

$$y = e^{x} + 2e^{\frac{1}{2}x}.$$

Find a simplified expression for $\frac{dy}{dx}$ in terms of y.

$\square$, $\boxed{\frac{dy}{dx} = 1 + y - \sqrt{y+1} = (y+1) - (y+1)^{\frac{1}{2}}}$

DIFFERENTIATING IS STRAIGHT FORWARD

$$y = e^{x} + 2e^{\frac{1}{2}x}$$

$$\frac{dy}{dx} = e^{x} + e^{\frac{1}{2}x}$$

NOW PROCEED AS FOLLOWS

$$\frac{dy}{dx} = \left(y - 2e^{\frac{1}{2}x}\right) + e^{\frac{1}{2}x}$$

$$\frac{dy}{dx} = y - e^{\frac{1}{2}x}$$

MANIPULATE THE EQUATION GIVEN

$$y = e^{x} + 2e^{\frac{1}{2}x}$$

$$1 + y = e^{x} + 2e^{\frac{1}{2}x} + 1$$

$$1 + y = \left(e^{\frac{1}{2}x} + 1\right)^2$$

$$e^{\frac{1}{2}x} + 1 = \pm\sqrt{y+1}$$

$$e^{\frac{1}{2}x} = -1 \pm \sqrt{y+1}$$

BUT $y > 0$.

$$e^{\frac{1}{2}x} = -1 + \sqrt{y+1}$$

RETURNING TO THE DERIVATIVE

$$\frac{dy}{dx} = y - e^{\frac{1}{2}x}$$

$$\frac{dy}{dx} = y - \left(-1 + \sqrt{y+1}\right)$$

$$\therefore \frac{dy}{dx} = (y+1) - (y+1)^{\frac{1}{2}}$$

Question 120 (*****)

Find in exact form the solution of the following equation.

$$e^2 - e^{3x} - 1 = \left(\frac{e^{x+1}}{e^{-2x}}\right)^2, \quad x \in \mathbb{R}.$$

$$\boxed{}, \quad \boxed{x = \frac{1}{2}\ln\left(1+e^{-2}\right)}$$

PROCEED BY REARRANGING AS FOLLOWS

$$\Rightarrow e^2 - e^{3x} - 1 = \left(\frac{e^{x+1}}{e^{-2x}}\right)^2$$

$$\Rightarrow e^2 - e^{3x} - 1 = \left(e^{3x+1}\right)^2$$

$$\Rightarrow e^2 - e^{3x} - 1 = e^{6x+2}$$

$$\Rightarrow e^2 - e^{6x+2} = e^{3x} + 1$$

$$\Rightarrow e^2\left(1 - e^{6x}\right) = 1 + e^{3x}$$

NOW THE L.H.S "HIDES" A DIFFERENCE OF SQUARES

$$\Rightarrow e^2\left(1 - e^{3x}\right)\left(1 + e^{3x}\right) = 1 + e^{3x}$$

$$\Rightarrow e^2\left(1 - e^{3x}\right) = 1$$

AS $e^{3x}+1 \neq 0$

$$\Rightarrow 1 - e^{3x} = \frac{1}{e^2}$$

$$\Rightarrow 1 - \frac{1}{e^2} = e^{3x}$$

$$\Rightarrow 3x = \ln\left(1 - e^{-2}\right)$$

$$\Rightarrow x = \frac{1}{3}\ln\left(1 - e^{-2}\right)$$

Question 121 (*****)

$$f(x) \equiv (2+\mathrm{e})(2\mathrm{e})^{x-1} - \mathrm{e}^{2x-1} 4^{x-\frac{1}{2}}, \quad x \in \mathbb{R}.$$

a) Find in exact simplified form the solution of the equation $f(x)=0$.

b) Determine, in terms of $\ln 2$, the two solutions of the equation $f(x)=1$.

$\boxed{}$, $\boxed{x = \dfrac{\ln(2+\mathrm{e})}{1+\ln 2}}$, $\boxed{x = \dfrac{1}{1+\ln 2} \cup x = \dfrac{\ln 2}{1+\ln 2}}$

$$\boxed{f(x) = (2e)^{x-1}(e+2) - e^{2x-1}4^{x-\frac{1}{2}}}$$

a) $f(x) = 0$

$\Rightarrow (2e)^{x-1}(e+2) - e^{2x-1}4^{x-\frac{1}{2}} = 0$

$\Rightarrow (2e)^{x-1}(e+2) = e^{2x-1} \times (2^2)^{x-\frac{1}{2}}$

$\Rightarrow 2^{x-1} \times e^{x-1} \times (e+2) = e^{2x-1} \times 2^{2x-1}$

$\Rightarrow \dfrac{2^{x-1} \times e^{x-1} \times (e+2)}{2^{x-1} \times e^{x-1}} = \dfrac{e^{2x-1} \times 2^{2x-1}}{2^{x-1} e^{x-1}}$

$2^{x-1} \times e^{x-1} \neq 0$

$\Rightarrow e+2 = e^x \times 2^x$

$\Rightarrow e+2 = (2e)^x$

$\Rightarrow \ln(e+2) = \ln(2e)^x$

$\Rightarrow \ln(e+2) = x\ln(2e)$

$\Rightarrow x = \dfrac{\ln(2+e)}{\ln(2e)}$

OR

$x = \dfrac{\ln(2+e)}{1+\ln 2}$

b) $f(x) = 1$

$\Rightarrow (2e)^{x-1}(e+2) - e^{2x-1}4^{x-\frac{1}{2}} = 1$

$\Rightarrow 2^{x-1}e^{x-1}(e+2) - e^{2x-1}(2^2)^{x-\frac{1}{2}} = 1$

$\Rightarrow 2^{x-1}e^{x-1}(e+2) - e^{2x-1}2^{2x-1} = 1$ $\quad \times 2e$

$\Rightarrow 2^x e^x (e+2) - e^{2x}2^{2x} = 2e$

$\Rightarrow 0 = (2e)^{2x} + 2e - (2e)^x(e+2)$

$\Rightarrow (2e)^{2x} - (2e)^x(e+2) + 2e = 0$

$\Rightarrow y^2 - (e+2)y + 2e = 0$ $\quad \boxed{y = (2e)^x}$

$\Rightarrow (y-e)(y-2) = 0$

HENCE WE OBTAIN FOR EACH SOLUTION

- $y = (2e)^x = e$

  $\Rightarrow \ln(2e)^x = 1$

  $\Rightarrow x\ln(2e) = 1$

  $\Rightarrow x = \dfrac{1}{\ln 2e}$

  $\Rightarrow x = \dfrac{1}{1+\ln 2}$

- $y = (2e)^x = 2$

  $\Rightarrow \ln(2e)^x = \ln 2$

  $\Rightarrow x\ln 2e = \ln 2$

  $\Rightarrow x = \dfrac{\ln 2}{\ln 2e}$

  $\Rightarrow x = \dfrac{\ln 2}{1+\ln 2}$

Question 122 (*****)

The distinct points A and B lie on the curve with equation

$$\ln(x+y) = \ln x + \ln y, \quad x \in (0,\infty), \quad y \in (0,\infty).$$

a) Determine possible coordinates for A and B, further verifying that these coordinates indeed satisfy the above given equation.

b) Sketch the curve, showing clearly all the relevant details.

$\boxed{A\left(4,\frac{4}{3}\right)}$, $\boxed{B\left(\frac{3}{2},3\right)}$

START BY EXPONENTIATION ON BOTH SIDES

$$\ln(x+y) = \ln(x) + \ln(y) \qquad x \neq 0,\ y \neq 0$$
$$e^{\ln(x+y)} = e^{\ln x + \ln y}$$
$$x+y = e^{\ln x} \times e^{\ln y}$$
$$x+y = xy$$

NOTICE THAT x & y ARE SWAPABLE, SO SOLVE FOR EITHER

$$x = xy - y$$
$$x = y(x-1)$$
$$y = \frac{x}{x-1} \qquad \text{(DEFINED FOR ALL } x \text{ \& } y \text{ EXCEPT } x \neq 1, y \neq 1)$$

THUS PICKING ANY x, EXCEPT 1 OR ZERO

$x = 4 \quad y = \frac{4}{3} \qquad \therefore A\left(4,\frac{4}{3}\right)$

- $\ln(x+y) = \ln\left(4+\frac{4}{3}\right) = \ln\left(\frac{16}{3}\right)$
- $\ln x + \ln y = \ln 4 + \ln\frac{4}{3} = \ln\left(4 \times \frac{4}{3}\right) = \ln\left(\frac{16}{3}\right)$

$x = \frac{3}{2} \quad y = \frac{3/2}{3/2-1} = \frac{3/2}{1/2} = 3 \qquad \therefore B\left(\frac{3}{2},3\right)$

- $\ln(x+y) = \ln\left(\frac{3}{2}+3\right) = \ln\left(\frac{9}{2}\right)$
- $\ln y + \ln y = \ln\frac{3}{2} + \ln 3 = \ln\left(\frac{3}{2}\times 3\right) = \ln\left(\frac{9}{2}\right)$

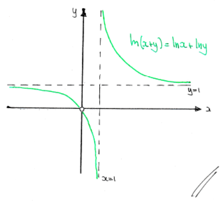
INDEED POSSIBLE COORDINATES

b) FINALLY THE SKETCH OF THE CURVE

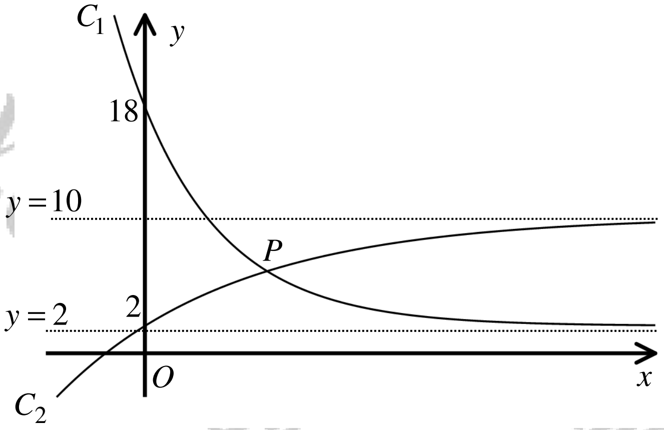
$\ln x + \ln y = \ln(x+y)$ IS THE SAME AS $y = \frac{x}{x-1}$

EXCEPT THE ORIGIN!!!

$y = \frac{x}{x-1} = \frac{x-1+1}{x-1} = 1 + \frac{1}{x-1}$, IE TRANSLATION OF $y = \frac{1}{x}$ BY $\binom{1}{1}$



Question 123 (*****)



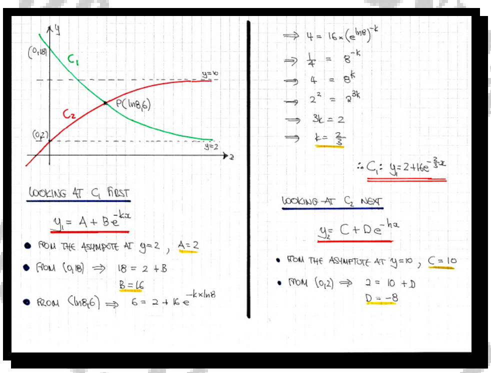
Two exponential curves, C_1 and C_2, intersect at the point $P(\ln 8, 6)$.

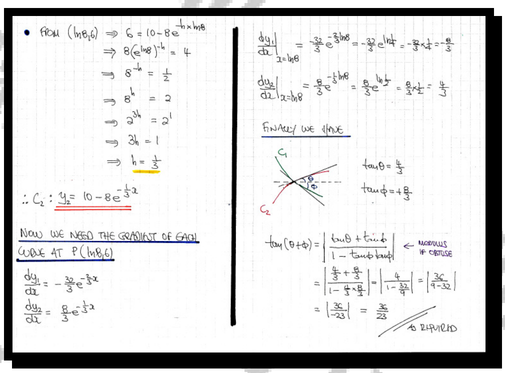
- C_1 meets the y axis at $(0,18)$ and the straight line with equation $y=2$ is an asymptote to C_1.

- C_2 meets the y axis at $(0,2)$ and the straight line with equation $y=10$ is an asymptote to C_2.

Show that at P, C_1 and C_1 cross each other at an acute angle of $\arctan\left(\frac{36}{23}\right)$.

$\boxed{}$, $\boxed{\text{proof}}$





Question 124 (*****)

Solve the following equation.

$$3\mathrm{e}^{2(x+1)} - (2\mathrm{e})^{x}\left(\mathrm{e}^{4}+9\right) + 3\mathrm{e}^{2}\times 4^{x}\ ,\ x \in \mathbb{R}\ .$$

Give the two solutions of the equation in the form $x = \pm A$, where A is in the form $\dfrac{a - \ln 3}{b - \ln 2}$, where a and b are positive integers.

$\boxed{}$, $\boxed{x = \pm\dfrac{2-\ln 3}{1-\ln 2}}$

$$\boxed{3e^{2(x+1)} - (2e)^{x}(e^{4}+9) + 3e^{2}4^{x} = 0}$$

- PROCEED AS FOLLOWS

$$\Rightarrow 3e^{2x+2} - 2^{x}e^{x}(e^{4}+9) + 3e^{2}4^{x} = 0$$
$$\Rightarrow 3e^{2x}e^{2} - 2^{x}e^{x}(e^{4}+9) + 3e^{2}(2^{x})^{2} = 0$$

- DIVIDE THE EQUATION THROUGH BY $3e^{2}e^{x}(2^{x}) \neq 0$

$$\Rightarrow \frac{\cancel{3}e^{2x}\cancel{e^{2}}}{\cancel{3}\cancel{e^{2}}e^{x}(2^{x})} - \frac{\cancel{2^{x}}\cancel{e^{x}}(e^{4}+9)}{3e^{2}\cancel{e^{x}}\cancel{(2^{x})}} + \frac{\cancel{3e^{2}}(2^{x})^{2}}{\cancel{3}\cancel{e^{2}}e^{x}(2^{x})} = 0$$
$$\Rightarrow \frac{e^{x}}{2^{x}} - \frac{e^{4}+9}{3e^{2}} + \frac{2^{x}}{e^{x}} = 0$$

- LET $A = \frac{e^{x}}{2^{x}}$ TO TRANSFORM THE ABOVE EQUATION INTO A QUADRATIC IN $\frac{e^{x}}{2^{x}}$

$$\Rightarrow A - \frac{e^{4}+9}{3e^{2}} + \frac{1}{A} = 0$$
$$\Rightarrow A^{2} - A\left(\frac{e^{4}+9}{3e^{2}}\right) + 1 = 0$$

- COMPLETING THE SQUARE (OR ATTEMPT TO FACTORIZE)

$$\Rightarrow \left[A - \frac{e^{4}+9}{6e^{2}}\right]^{2} - \frac{(e^{4}+9)^{2}}{36e^{4}} + 1 = 0$$
$$\Rightarrow \left[A - \frac{e^{4}+9}{6e^{2}}\right]^{2} = \frac{(e^{4}+9)^{2}}{36e^{4}} - 1$$

- TIDY THE R.H.S FIRST

$$\frac{(e^{4}+9)^{2}}{36e^{4}} - 1 = \frac{e^{8}+18e^{4}+81}{36e^{4}} - \frac{36e^{4}}{36e^{4}} = \frac{e^{8}-18e^{4}+81}{36e^{4}}$$
$$= \frac{(e^{4}-9)^{2}}{36e^{4}} = \left(\frac{e^{4}-9}{6e^{2}}\right)^{2}$$

- THUS WE HAVE

$$\Rightarrow \left[A - \frac{e^{4}+9}{6e^{2}}\right]^{2} = \left(\frac{e^{4}-9}{6e^{2}}\right)^{2}$$
$$\Rightarrow A - \frac{e^{4}+9}{6e^{2}} = \begin{cases} \frac{e^{4}-9}{6e^{2}} \\ \frac{9-e^{4}}{6e^{2}} \end{cases}$$
$$\Rightarrow A = \begin{cases} \frac{2e^{4}}{6e^{2}} \\ \frac{18}{6e^{2}} \end{cases}$$
$$\Rightarrow A = \begin{cases} \frac{1}{3}e^{2} \\ 3e^{-2} \end{cases}$$

- SOLVING EACH CASE SEPARATELY

$\Rightarrow \frac{e^{x}}{2^{x}} = \frac{1}{3}e^{2}$	$\Rightarrow \frac{e^{x}}{2^{x}} = \frac{3}{e^{2}}$
$\Rightarrow \left(\frac{1}{2}e\right)^{x} = \frac{1}{3}e^{2}$	$\Rightarrow \left(\frac{e}{2}\right)^{x} = \frac{3}{e^{2}}$
$\Rightarrow \ln\left(\frac{1}{2}e\right)^{x} = \ln\left(\frac{1}{3}e^{2}\right)$	$\Rightarrow \ln\left(\frac{1}{2}e\right)^{x} = \ln\left(\frac{3}{e^{2}}\right)$
$\Rightarrow x\ln\left(\frac{1}{2}e\right) = \ln\frac{1}{3}e^{2}$	$\Rightarrow x\ln\left(\frac{1}{2}e\right) = \ln\left(\frac{3}{e^{2}}\right)$
$\Rightarrow x\left[\ln\frac{1}{2} + \ln e\right] = \ln\frac{1}{3} + \ln e^{2}$	$\Rightarrow x\left[\ln\frac{1}{2} + \ln e\right] = \ln 3 - \ln e^{2}$
$\Rightarrow x\left[-\ln 2 + 1\right] = -\ln 3 + 2$	$\Rightarrow x\left[-\ln 2 + 1\right] = \ln 3 - 2$
$\Rightarrow x = \frac{2-\ln 3}{1-\ln 2}$	$\Rightarrow x = \frac{\ln 3 - 2}{1 - \ln 2}$
	$\Rightarrow x = -\frac{2-\ln 3}{1-\ln 2}$

Question 125 (*****)

Solve the following equation.

$$\left(1+e^{-2}\right)\left(e^{2}\right)^{x^2-4x+5} = e^{2}+e^{4(x-2)^2},\ x\in\mathbb{R}.$$

$\boxed{}$, $\boxed{x=1,\quad x=2,\quad x=3}$

$$\left(1+e^{-2}\right)\left(e^{2}\right)^{x^2-4x+5} = e^{2}+e^{4(x-2)^2}$$

TIDY THE EQUATION AS FOLLOWS

$$\Rightarrow \left(1+e^{-2}\right)e^{2(x^2-4x+5)} = e^{2}+e^{4(x-2)^2}$$

$$\Rightarrow \left(1+e^{-2}\right)e^{2[(x-2)^2+1]} = e^{2}+e^{4(x-2)^2}$$

$$\Rightarrow \left(1+e^{-2}\right)e^{2(x-2)^2+2} = e^{2}+e^{2\times 2(x-2)^2}$$

$$\Rightarrow \left(1+e^{-2}\right)e^{2(x-2)^2}\times e^{2} = e^{2}+\left[e^{2(x-2)^2}\right]^2$$

$$\Rightarrow \left(e^{2}+1\right)e^{2(x-2)^2} = e^{2}+\left[e^{2(x-2)^2}\right]^2$$

$$\Rightarrow \left(e^{2}+1\right)A = e^{2}+A^2 \qquad A=e^{2(x-2)^2}$$

REGROUP & FACTORIZE THE QUADRATIC

$$\Rightarrow 0 = A^2-\left(e^{2}+1\right)A+e^{2}$$

$$\Rightarrow 0 = \left(A-e^{2}\right)\left(A-1\right)$$

$$\Rightarrow A = \begin{cases}1\\ e^{2}\end{cases} \Rightarrow e^{2(x-2)^2} = \begin{cases}1\\ e^{2}\end{cases}$$

$$\Rightarrow \text{EITHER } x=2$$

$$\text{OR } 2(x-2)^2=2$$

$$(x-2)^2=1$$

$$x-2=\begin{cases}-1\\ +1\end{cases}$$

$$x=\begin{cases}1\\ 3\end{cases}$$

Question 126 (*****)

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

Show, with a detailed method, that the real solution of the following exponential equation

$$9^x + 12^x = 16^x,$$

can be written in the form

$$\frac{\ln(\varphi - 1)}{\ln 3 - \ln 4},$$

SPX-P, proof

WORK WITH PRIME FACTORS TO START WITH

$$\Rightarrow 9^x + 12^x = 16^x$$
$$\Rightarrow (3^2)^x + (2^2 \times 3)^x = (2^4)^x$$
$$\Rightarrow 3^{2x} + 2^{2x} \times 3^x = 2^{4x}$$
$$\Rightarrow \frac{3^{2x}}{2^{4x}} + \frac{2^{2x} \times 3^x}{2^{4x}} = 1$$
$$\Rightarrow \frac{3^{2x}}{4^{2x}} + \frac{3^x}{2^{2x}} = 1$$
$$\Rightarrow \frac{3^{2x}}{4^{2x}} + \frac{3^x}{4^x} = 1$$
$$\Rightarrow \left(\left(\frac{3}{4}\right)^2\right)^x + \left(\frac{3}{4}\right)^x = 1$$

LET $A = \left(\frac{3}{4}\right)^x$, TO SOLVE THE QUADRATIC

$$\Rightarrow \left[\left(\frac{3}{4}\right)^x\right]^2 + \left(\frac{3}{4}\right)^x - 1 = 0$$
$$\Rightarrow A^2 + A - 1 = 0$$

DOES NOT FACTORIZE NICELY

$$\Rightarrow 4A^2 + 4A - 4 = 0$$
$$\Rightarrow 4A^2 + 4A + 1 = 5$$
$$\Rightarrow (2A+1)^2 = 5$$
$$\Rightarrow 2A + 1 = \pm\sqrt{5}$$
$$\Rightarrow 2A = -1 \pm \sqrt{5}$$
$$\Rightarrow A = \frac{1}{2}(-1 \pm \sqrt{5})$$

BUT RETURNING TO x, DOES NOT "ALLOW" ONE OF THE SOLUTION

$$\Rightarrow \left(\frac{3}{4}\right)^x = \begin{cases} \cancel{\frac{1}{2}(-1-\sqrt{5})} \\ \frac{1}{2}(-1+\sqrt{5}) \end{cases}$$

$u^2 - u - 1 = 0$
$u = \frac{1 \pm \sqrt{5}}{2}$
$\phi = \frac{1+\sqrt{5}}{2}$

$$\Rightarrow \left(\frac{3}{4}\right)^x = \frac{-1+\sqrt{5}}{2}$$
$$\Rightarrow \left(\frac{3}{4}\right)^x = \frac{1+\sqrt{5}-2}{2} = \frac{1+\sqrt{5}}{2} - 1 = \phi - 1$$
$$\Rightarrow \ln\left(\frac{3}{4}\right)^x = \ln(\phi - 1)$$
$$\Rightarrow x \ln\frac{3}{4} = \ln(\phi - 1)$$
$$\Rightarrow x = \frac{\ln(\phi-1)}{\ln(3/4)} \quad \text{OR} \quad x = \frac{\ln(\phi-1)}{\ln 3 - \ln 4}$$

IMPORTANT NOTE – ONCE THE "STRUCTURE" HAS BEEN EXPOSED THERE IS A QUICKER MANIPULATION

$$\Rightarrow 9^x + 12^x = 16^x$$
$$\Rightarrow \frac{9^x}{16^x} + \frac{12^x}{16^x} = \frac{16^x}{16^x}$$
$$\Rightarrow \left(\frac{9}{16}\right)^x + \left(\frac{12}{16}\right)^x = \left(\frac{16}{16}\right)^x$$
$$\Rightarrow \left(\frac{9}{16}\right)^x + \left(\frac{3}{4}\right)^x = 1^x$$
$$\Rightarrow \left(\frac{3}{4}\right)^{2x} + \left(\frac{3}{4}\right)^x = 1$$

ETC

Question 127 (*****)

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k \, .$$

Given that e is Euler's number, use a detailed method to find the exact value of

$$\prod_{r=1}^{\infty}\left[\frac{\sqrt[(2r)]{e}}{\sqrt[(2r+1)]{e}}\right].$$

$\boxed{}$, $\boxed{\frac{1}{2}e}$

GENERATE SOME TERMS

$$\prod_{r=1}^{\infty}\frac{\sqrt[2r]{e}}{\sqrt[2r+1]{e}} = \frac{\sqrt[2]{e}}{\sqrt[3]{e}} \times \frac{\sqrt[4]{e}}{\sqrt[5]{e}} \times \frac{\sqrt[6]{e}}{\sqrt[7]{e}} \times \frac{\sqrt[8]{e}}{\sqrt[9]{e}} \times \ldots$$

$$= \frac{e^{\frac{1}{2}}}{e^{\frac{1}{3}}} \times \frac{e^{\frac{1}{4}}}{e^{\frac{1}{5}}} \times \frac{e^{\frac{1}{6}}}{e^{\frac{1}{7}}} \times \frac{e^{\frac{1}{8}}}{e^{\frac{1}{9}}} \times \ldots$$

$$= e^{\frac{1}{2}} \times e^{-\frac{1}{3}} \times e^{\frac{1}{4}} \times e^{-\frac{1}{5}} \times e^{\frac{1}{6}} \times e^{-\frac{1}{7}} \times \ldots$$

$$= e^{\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\ldots}$$

THIS IS "ALMOST" THE ALTERNATING HARMONIC WHICH CONVERGES TO $\ln 2$

$$\Rightarrow \frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\ldots$$

$$= -\left[-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots\right]$$

$$= 1-\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots\right]$$

$$= 1-\ln 2$$

THUS WE NOW HAVE

$$\prod_{r=1}^{\infty}\frac{\sqrt[2r]{e}}{\sqrt[2r+1]{e}} = e^{1-\ln 2} = e^{1} \times e^{-\ln 2} = e \times \frac{1}{2} = \frac{1}{2}e$$

Question 128 (*****)

The distinct points A and B lie on the curve with equation

$$\ln(x-y) = \ln x + \ln y\,, \quad x \in (0,\infty)\,, \quad y \in (0,\infty)\,.$$

a) Determine possible coordinates for A and B, further verifying that these coordinates indeed satisfy the above given equation.

b) Sketch the curve, showing clearly all the relevant details.

$\boxed{A(4,2)}$, $\boxed{B\left(\frac{9}{2},\frac{3}{2}\right)}$

a) EXPONENTIATING BOTH SIDES OF THE EQUATION

$$\ln(x-y) = \ln x - \ln y$$
$$e^{\ln(x-y)} = e^{\ln x - \ln y}$$
$$x-y = e^{\ln x} \times e^{-\ln y}$$
$$x-y = x \times \frac{1}{y}$$
$$xy - y^2 = x$$
$$xy - x = y^2$$
$$x(y-1) = y^2$$
$$x = \frac{y^2}{y-1} \quad \text{(DEFINED BUT WITH RESTRICTIONS)}$$

PICKING SOME SENSIBLE VALUES OF y

$y=2 \quad x = \frac{4}{2-1} = 4 \quad \therefore A(4,2)$

- $\ln(x-y) = \ln(4-2) = \ln 2$
- $\ln x - \ln 2 = \ln 4 - \ln 2 = \ln\left(\frac{4}{2}\right) = \ln 2$

$y = \frac{3}{2} \quad x = \frac{\frac{9}{4}}{\frac{3}{2}-1} = \frac{\frac{9}{4}}{\frac{1}{2}} = \frac{9}{2} \quad \therefore B\left(\frac{9}{2},\frac{3}{2}\right)$

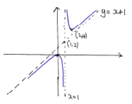
- $\ln(x-y) = \ln\left(\frac{9}{2}-\frac{3}{2}\right) = \ln 3$
- $\ln x - \ln y = \ln\frac{9}{2} - \ln\frac{3}{2} = \ln\left(\frac{9/2}{3/2}\right) = \ln 3$

INDEED THE ABOVE ARE POSSIBLE CO-ORDINATES

b) TO SKETCH THIS CURVE WE NEED TO CONSIDER THAT $x > y > 0$

$$x = \frac{y^2}{y-1} = \frac{y(y-1)+(y-1)+1}{y-1} = y+1+\frac{1}{y-1}$$

NOW CONSIDER THE GRAPH OF $y = x+1+\frac{1}{x-1}$

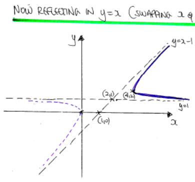


- FOR $x \gg 1$ $y \sim x+1$
- FOR $x \ll -1$ $y \sim x+1$
- AS $x \to 1^+$ $y \to +\infty$
- AS $x \to 1^-$ $y \to -\infty$
- $y' = 1 - \frac{1}{(x-1)^2}$

$0 = 1 - \frac{1}{(x-1)^2}$

$x = \begin{cases} 0 \\ 2 \end{cases} \quad y = \begin{cases} 0 \\ 4 \end{cases}$

NOW REFLECTING IN $y=x$ (SWAPPING x & y)



- $x > 0$
- $y > 0$
- $x > y$

$\left.\begin{aligned} \frac{y^2}{y-1} &> y \\ \frac{y}{y-1} &> 1 \end{aligned}\right\} y \neq 0$

$y > y-1$ TRUE.

$\therefore$ SKETCH IS SHOWN IN BOLD IN THE FIRST QUADRANT