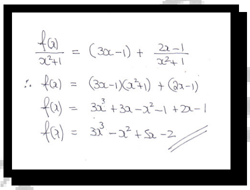
POLYNOMIALS

Question 1 (**)

When $f(x)$ is divided by $\left(x^2+1\right)$ the quotient is $(3x-1)$ and the remainder is $(2x-1)$.

Determine an expression for $f(x)$.

$$\boxed{f(x)=3x^3-x^2+5x-2}$$

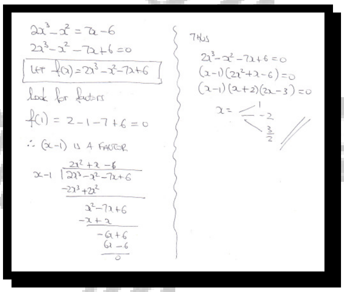


Question 2 (***)

Find the three solutions of the cubic equation

$$2x^3-x^2=7x-6\,.$$

$$\boxed{x=-2,1,\frac{3}{2}}$$

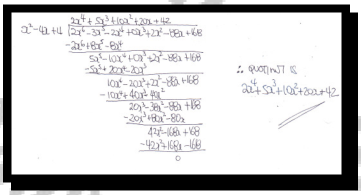


Question 3 (***)

Find the quotient of the division of

$$2x^6-3x^5-2x^4+2x^2-88x+168 \quad \text{by} \quad x^2-4x+4.$$

$\boxed{2x^4+5x^3+10x^2+20x+42}$

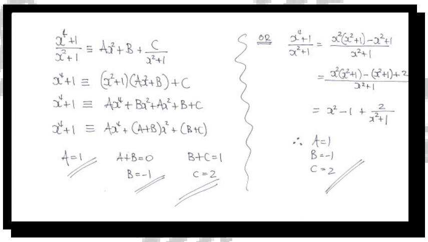


Question 4 (***)

$$\frac{x^4+1}{x^2+1} \equiv Ax^2+B+\frac{C}{x^2+1}.$$

Find the value of each of the constants A, B and C.

$\boxed{A=1}$, $\boxed{B=-1}$, $\boxed{C=2}$



Question 5 (***+)

A quintic polynomial is defined, in terms of the constants a and b, by

$$f(x) = x^5 + ax^4 + bx^3 - x^2 + 4x - 3.$$

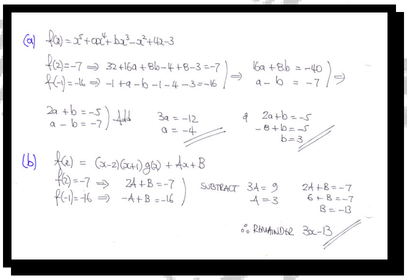
When $f(x)$ is divided by $(x-2)$ the remainder is -7.

When $f(x)$ is divided by $(x+1)$ the remainder is -16.

a) Determine in any order the value of a and the value of b.

b) Find the remainder when $f(x)$ is divided by $(x-2)(x+1)$.

[ ], $\boxed{a=-4}$, $\boxed{b=3}$, $\boxed{3x-13}$



Question 6 (***+)

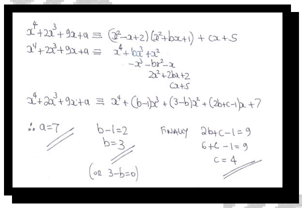
A polynomial $p(x)$ is defined, in terms of a constant a, by

$$p(x) = x^4 + 2x^3 + 9x + a.$$

When $p(x)$ is divided by $x^2 - x + 2$ the quotient is $x^2 + bx + 1$ and the remainder is $cx + 5$, where b and c are constants.

Find the value of a, b and c.

$\boxed{a = 7,\ b = 3,\ c = 4}$



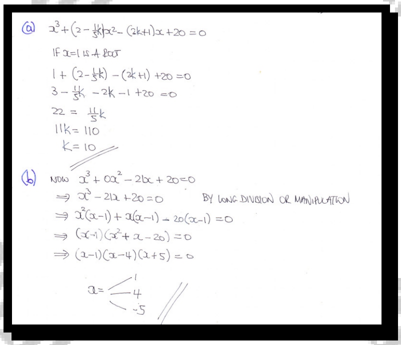
Question 7 (***+)

$$x^3+\left(2-\tfrac{1}{5}k\right)x^2+(2k+1)x+20=0.$$

a) Determine the value of the real constant k, if the above equation is to have $x=1$ as one of its roots.

b) Solve the equation for the value of k, found in part (**a**).

$\boxed{k=10}$, $\boxed{x=-5,4,1}$


(a) $x^3+(2-\frac{1}{5}k)x^2-(2k+1)x+20=0$

IF $x=1$ IS A ROOT

$1+(2-\frac{1}{5}k)-(2k+1)+20=0$

$3-\frac{1}{5}k-2k-1+20=0$

$22=\frac{11}{5}k$

$11k=110$

$k=10$

(b) NOW $x^3+0x^2-21x+20=0$

$\Rightarrow x^3-21x+20=0$ BY LONG DIVISION OR MANIPULATION

$\Rightarrow x^2(x-1)+x(x-1)-20(x-1)=0$

$\Rightarrow (x-1)(x^2+x-20)=0$

$\Rightarrow (x-1)(x-4)(x+5)=0$

$x=1, 4, -5$

Question 5 (***+)

The following information is given for a polynomial $f(x)$.

- When $f(x)$ is divided by $(x-2)$ the remainder is 5.
- When $f(x)$ is divided by $(x+2)$ the remainder is -11.
- When $f(x)$ is divided by $(x+2)(x-2)$ the remainder is $ax+b$, and the quotient is $g(x)$, where a and b are constants.

a) Determine the value of a and the value of b.

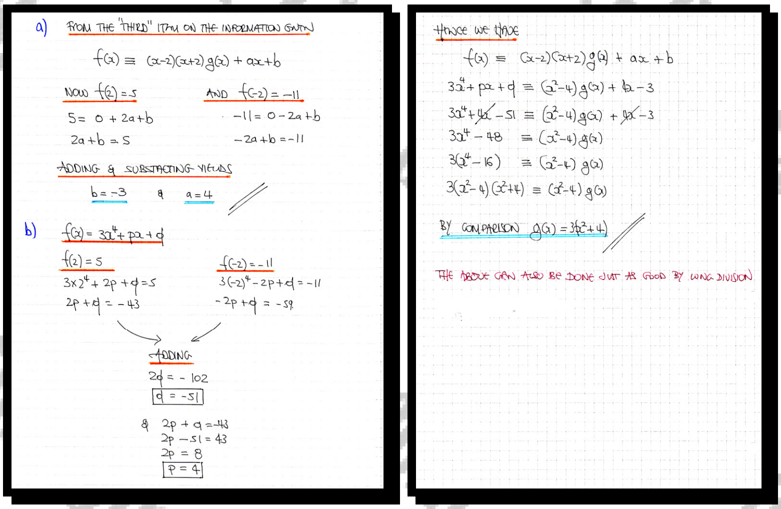
It is further given that

$$f(x)=3x^4+px+q,$$

where p and q are constants.

b) Find a simplified expression for $g(x)$.

$\boxed{}$, $\boxed{a=-4}$, $\boxed{b=3}$, $\boxed{g(x)=3\left(x^2+4\right)}$



Question 9 (****)

$$f(x) = x^3 + (a+2)x^2 - 2x + b,$$

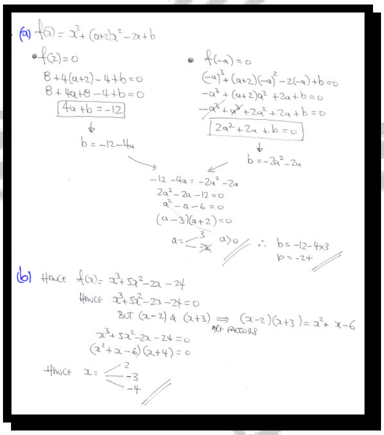
where a and b are non zero constants.

It is given that $(x-2)$ and $(x+a)$ are factors of $f(x)$, $a > 0$.

a) By forming two equations show that $a = 3$ and find the value of b.

b) Solve the equation $f(x) = 0$.

$\boxed{b = -24}$, $\boxed{x = -4, -3, 2}$



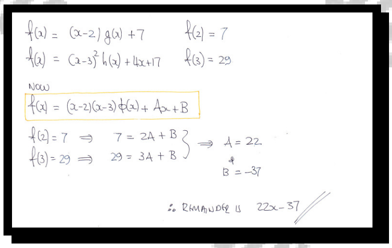
Question 10 (****)

When the polynomial $f(x)$ is divided by $(x-2)$ the remainder is 7.

When $f(x)$ is divided by $(x-3)^2$ the remainder is $(4x+17)$.

Find the remainder when $f(x)$ is divided by $(x-2)(x-3)$.

$\boxed{22x-37}$



Question 11 (****)

A polynomial $p(x)$ is given by

$$p(x) = 4x^3 - 2x^2 + x + 5 .$$

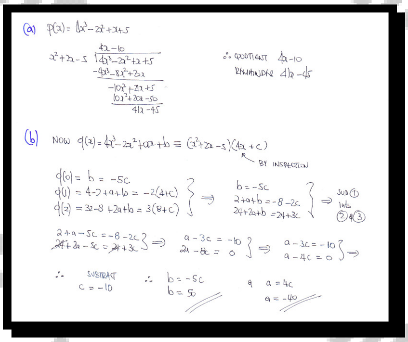
a) Find the remainder and the quotient when $p(x)$ is divided by $x^2 + 2x - 5$.

A different polynomial $q(x)$ is defined as

$$q(x) = 4x^3 - 2x^2 + ax + b .$$

b) Find the value of each of the constants a and b so that when $q(x)$ is divided by $x^2 + 2x - 5$ there is no remainder.

$\boxed{R = 41x - 45}$, $\boxed{Q = 4x - 10}$, $\boxed{a = -40,\ b = 50}$



Question 12 (****+)

$$f(x) \equiv x^4 - 9x^3 + 30x^2 - 44x .$$

The polynomial $f(x)$ satisfies the relationship

$$f(x) \equiv (x-3)(x-A)^3 + B ,$$

where A and B are constants.

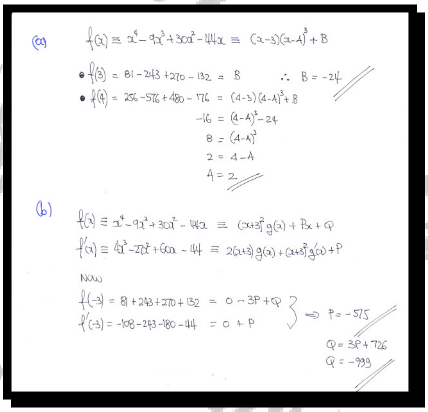
a) Find the value of A and the value of B.

The polynomial $f(x)$ also satisfies the relationship

$$f(x) \equiv (x+3)^2 g(x) + Px + Q \text{, where } P \text{ and } Q \text{ are constants.}$$

b) Find the value of each of the constants P and Q.

$\boxed{A=2}$, $\boxed{B=-24}$, $\boxed{P=-575}$, $\boxed{Q=-999}$


(a) $f(x) \equiv x^4 - 9x^3 + 30x^2 - 44x \equiv (x-3)(x-A)^3 + B$

• $f(3) = 81 - 243 + 270 - 132 = B$ ∴ $B = -24$

• $f(4) = 256 - 576 + 480 - 176 = (4-3)(4-A)^3 + B$

$-16 = (4-A)^3 - 24$

$8 = (4-A)^3$

$2 = 4 - A$

$A = 2$

(b) $f(x) \equiv x^4 - 9x^3 + 30x^2 - 44x \equiv (x+3)^2 g(x) + Px + Q$

$f'(x) \equiv 4x^3 - 27x^2 + 60x - 44 \equiv 2(x+3)g(x) + (x+3)^2 g'(x) + P$

Now

$f(-3) = 81 + 243 + 270 + 132 = 0 - 3P + Q$

$f'(-3) = -108 - 243 - 180 - 44 = 0 + P$

$\Rightarrow P = -575$

$Q = 3P + 726$

$Q = -999$

Question 13 (****+)

$$f(x) \equiv 9x^4 + 24x^3 - 32x - 16 .$$

The polynomial $f(x)$ has linear factors

$$(x+2) \quad \text{and} \quad (3x+2)$$

a) Show that the roots of the equation $f(x)=0$ are

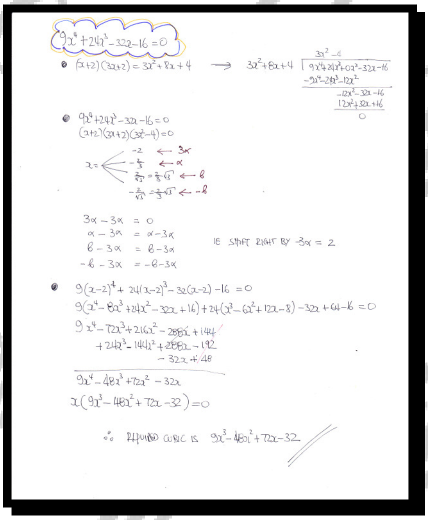
$$\alpha, \ 3\alpha, \ \beta \text{ and } -\beta,$$

where α and β must be stated in exact form if appropriate.

b) Hence determine a cubic equation with integer coefficients with roots

$$-2\alpha, \ \beta - 3\alpha \text{ and } -\beta - 3a .$$

$\boxed{\alpha = -\frac{2}{3}}$, $\boxed{\beta = -\frac{2}{3}\sqrt{3}}$, $\boxed{9x^3 - 48x^2 + 72x - 32 = 0}$

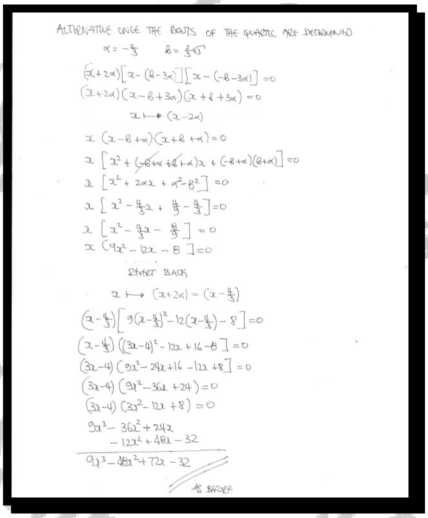


$9x^4 + 24x^3 - 32x - 16 = 0$

- $(x+2)(3x+2) = 3x^2 + 8x + 4 \implies 3x^2+8x+4 \,\big)\, 9x^4+24x^3+0x^2-32x-16$ gives quotient $3x^2-4$, remainder 0
- $9x^4+24x^3-32x-16=0$
  $(x+2)(3x+2)(3x^2-4)=0$
  $x = -2 \leftarrow 3\alpha$; $-\frac{2}{3} \leftarrow \alpha$; $\frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3} \leftarrow \beta$; $-\frac{2}{\sqrt{3}} = -\frac{2}{3}\sqrt{3} \leftarrow -\beta$

  $3\alpha - 3\alpha = 0$
  $\alpha - 3\alpha = \alpha - 3\alpha$
  $\beta - 3\alpha = \beta - 3\alpha$
  $-\beta - 3\alpha = -\beta - 3\alpha$
  IE SHIFT RIGHT BY $-3\alpha = 2$
- $9(x-2)^4 + 24(x-2)^3 - 32(x-2) - 16 = 0$
  $9(x^4 - 8x^3 + 24x^2 - 32x + 16) + 24(x^3 - 6x^2 + 12x - 8) - 32x + 64 - 16 = 0$
  $9x^4 - 72x^3 + 216x^2 - 288x + 144$
  $+ 24x^3 - 144x^2 + 288x - 192$
  $- 32x + 48$
  $9x^4 - 48x^3 + 72x^2 - 32x$
  $x(9x^3 - 48x^2 + 72x - 32) = 0$

$\therefore$ REQUIRED CUBIC IS $9x^3 - 48x^2 + 72x - 32$



ALTERNATIVE ONCE THE ROOTS OF THE QUARTIC ARE DETERMINED

$\alpha = -\frac{2}{3} \quad \beta = \frac{2}{3}\sqrt{3}$

$(x+2\alpha)[x-(\beta-3\alpha)][x-(-\beta-3\alpha)] = 0$
$(x+2\alpha)(x-\beta+3\alpha)(x+\beta+3\alpha) = 0$
$x \mapsto (x-2\alpha)$
$x(x-\beta+\alpha)(x+\beta+\alpha) = 0$
$x[x^2 + (-\beta+\alpha+\beta+\alpha)x + (-\beta+\alpha)(\beta+\alpha)] = 0$
$x[x^2 + 2\alpha x + \alpha^2 - \beta^2] = 0$
$x[x^2 - \frac{4}{3}x + \frac{4}{9} - \frac{4}{3}] = 0$
$x[x^2 - \frac{4}{3}x - \frac{8}{9}] = 0$
$x(9x^2 - 12x - 8) = 0$

REVERT BACK

$x \mapsto (x+2\alpha) = (x - \frac{4}{3})$
$(x-\frac{4}{3})[9(x-\frac{4}{3})^2 - 12(x-\frac{4}{3}) - 8] = 0$
$(x-\frac{4}{3})[(3x-4)^2 - 12x + 16 - 8] = 0$
$(3x-4)(9x^2 - 24x + 16 - 12x + 8) = 0$
$(3x-4)(9x^2 - 36x + 24) = 0$
$(3x-4)(3x^2 - 12x + 8) = 0$
$9x^3 - 36x^2 + 24x$
$- 12x^2 + 48x - 32$
$9x^3 - 48x^2 + 72x - 32$

AS BEFORE

Question 14 **(****+)**

A polynomial $f(x)$ is defined by

$$f(x) \equiv 2x^6 + ax^5 + bx^4 + 2x^2,$$

where a and b are constants.

When $f(x)$ is divided by $(x-2)(2x+1)$ the remainder is $(3x+2)$.

a) Determine the value of a and the value of b.

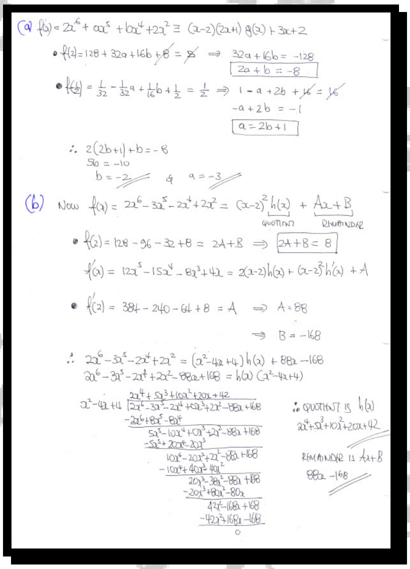
When $f(x)$ is divided by $(x-2)^2$ the quotient is $h(x)$ and the remainder is $(Ax+B)$, where A and B are constants..

b) Find …

i. … the value of A and the value of B.

ii. … an expression for $h(x)$.

$\boxed{a=-3}$, $\boxed{b=-2}$, $\boxed{A=88}$, $\boxed{B=-168}$, $\boxed{h(x) = 2x^4 + 5x^3 + 10x^2 + 20x + 42}$



(a) $f(x) = 2x^6 + ax^5 + bx^4 + 2x^2 \equiv (x-2)(2x+1)g(x) + 3x + 2$

• $f(2) = 128 + 32a + 16b + \cancel{8} = \cancel{8} \Rightarrow 32a + 16b = -128$
$\boxed{2a + b = -8}$

• $f(-\frac{1}{2}) = \frac{1}{32} - \frac{1}{32}a + \frac{1}{16}b + \frac{1}{2} = \frac{1}{2} \Rightarrow 1 - a + 2b + \cancel{16} = \cancel{16}$
$-a + 2b = -1$
$\boxed{a = 2b + 1}$

$\therefore 2(2b+1) + b = -8$
$5b = -10$
$b = -2$ & $a = -3$

(b) Now $f(x) = 2x^6 - 3x^5 - 2x^4 + 2x^2 = (x-2)^2 \underbrace{h(x)}_{\text{QUOTIENT}} + \underbrace{Ax + B}_{\text{REMAINDER}}$

• $f(2) = 128 - 96 - 32 + 8 = 2A + B \Rightarrow \boxed{2A + B = 8}$

$f'(x) = 12x^5 - 15x^4 - 8x^3 + 4x = 2(x-2)h(x) + (x-2)^2 h'(x) + A$

• $f'(2) = 384 - 240 - 64 + 8 = A \Rightarrow A = 88$
$\Rightarrow B = -168$

$\therefore 2x^6 - 3x^5 - 2x^4 + 2x^2 = (x^2 - 4x + 4)h(x) + 88x - 168$
$2x^6 - 3x^5 - 2x^4 + 2x^2 - 88x + 168 = h(x)(x^2 - 4x + 4)$

$$\begin{array}{r|l} & 2x^4 + 5x^3 + 10x^2 + 20x + 42 \\ x^2 - 4x + 4 & 2x^6 - 3x^5 - 2x^4 + 0x^3 + 2x^2 - 88x + 168 \\ & -2x^6 + 8x^5 - 8x^4 \\ & 5x^5 - 10x^4 + 0x^3 + 2x^2 - 88x + 168 \\ & -5x^5 + 20x^4 - 20x^3 \\ & 10x^4 - 20x^3 + 2x^2 - 88x + 168 \\ & -10x^4 + 40x^3 - 40x^2 \\ & 20x^3 - 38x^2 - 88x + 168 \\ & -20x^3 + 80x^2 - 80x \\ & 42x^2 - 168x + 168 \\ & -42x^2 + 168x - 168 \\ & 0 \end{array}$$

$\therefore$ QUOTIENT IS $h(x)$
$2x^4 + 5x^3 + 10x^2 + 20x + 42$

REMAINDER IS $Ax + B$
$88x - 168$

Question 15 (*****)

A polynomial in x satisfies the relationship

$$f(x) = \left(x^2 - 4\right) g(x) + Ax + B,$$

where A and B are constants.

a) Find the value of A and the value of B, given that $f(2) = 5$ and $f(-2) = -7$.

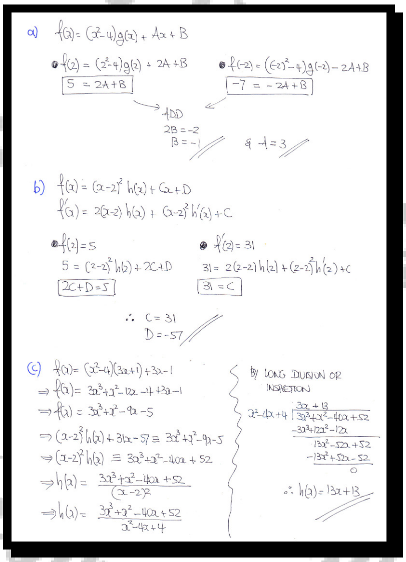
It is now given that the polynomial in x also satisfies the relationship

$$f(x) = (x-2)^2 h(x) + Cx + D.$$

b) Find the value of each of the constants C and D, given that $f'(2) = 31$.

c) Given further that $g(x) = 3x+1$, find $h(x)$.

$\boxed{A=3}$, $\boxed{B=-1}$, $\boxed{C=31}$, $\boxed{D=-57}$, $\boxed{h(x)=3x+13}$


a) $f(x) = (x^2-4)g(x) + Ax + B$

- $f(2) = (2^2-4)g(2) + 2A + B$, $\boxed{5 = 2A + B}$
- $f(-2) = ((-2)^2-4)g(-2) - 2A + B$, $\boxed{-7 = -2A + B}$

ADD

$2B = -2$

$B = -1$ & $A = 3$

b) $f(x) = (x-2)^2 h(x) + Cx + D$

$f'(x) = 2(x-2)h(x) + (x-2)^2 h'(x) + C$

- $f(2) = 5$: $5 = (2-2)^2 h(2) + 2C + D$, $\boxed{2C + D = 5}$
- $f'(2) = 31$: $31 = 2(2-2)h(2) + (2-2)^2 h'(2) + C$, $\boxed{31 = C}$

$\therefore C = 31$, $D = -57$

(c) $f(x) = (x^2-4)(3x+1) + 3x - 1$

$\Rightarrow f(x) = 3x^3 + x^2 - 12x - 4 + 3x - 1$

$\Rightarrow f(x) = 3x^3 + x^2 - 9x - 5$

$\Rightarrow (x-2)^2 h(x) + 31x - 57 \equiv 3x^3 + x^2 - 9x - 5$

$\Rightarrow (x-2)^2 h(x) \equiv 3x^3 + x^2 - 40x + 52$

$\Rightarrow h(x) = \frac{3x^3 + x^2 - 40x + 52}{(x-2)^2}$

$\Rightarrow h(x) = \frac{3x^3 + x^2 - 40x + 52}{x^2 - 4x + 4}$

BY LONG DIVISION OR INSPECTION

$x^2 - 4x + 4 \,\big)\, 3x^3 + x^2 - 40x + 52$, quotient $3x + 13$

$-3x^3 + 12x^2 - 12x$

$13x^2 - 52x + 52$

$-13x^2 + 52x - 52$

0

$\therefore h(x) = 3x + 13$

Question 16 (*****)

A polynomial in x is given by

$$f(x) = x^8 + kx^5 - 27x^2 - 13, \text{ where } k \text{ is a constant.}$$

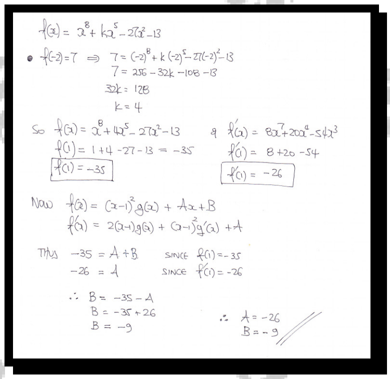
The polynomial also satisfies the relationship

$$f(x) = (x-1)^2 g(x) + Ax + B,$$

where A and B are constants.

Find the value of A and the value of B, given that $f(2) = 7$

$\boxed{A = -26}$, $\boxed{B = -9}$



$f(x) = x^8 + kx^5 - 27x^2 - 13$

- $f(-2) = 7 \Rightarrow 7 = (-2)^8 + k(-2)^5 - 27(-2)^2 - 13$

$7 = 256 - 32k - 108 - 13$

$32k = 128$

$k = 4$

So $f(x) = x^8 + 4x^5 - 27x^2 - 13$ & $f'(x) = 8x^7 + 20x^4 - 54x^3$

$f(1) = 1 + 4 - 27 - 13 = -35$ $f'(1) = 8 + 20 - 54$

$\boxed{f(1) = -35}$ $\boxed{f'(1) = -26}$

Now $f(x) = (x-1)^2 g(x) + Ax + B$

$f'(x) = 2(x-1)g(x) + (x-1)^2 g'(x) + A$

Thus $-35 = A + B$ SINCE $f(1) = -35$

$-26 = A$ SINCE $f'(1) = -26$

$\therefore B = -35 - A$

$B = -35 + 26$

$B = -9$

$\therefore A = -26$

$B = -9$

Question 17 (*****)

$$f(x) \equiv Ax^5 + Bx^4 + 8x^2,$$

where A and B are non zero constants.

The polynomial $f(x)$ satisfies the relationship

$$f(x) \equiv (2x-1)(x-2)g(x) + 169x - 82.$$

a) Find the value of A and the value of B.

b) Determine the polynomial $g(x)$.

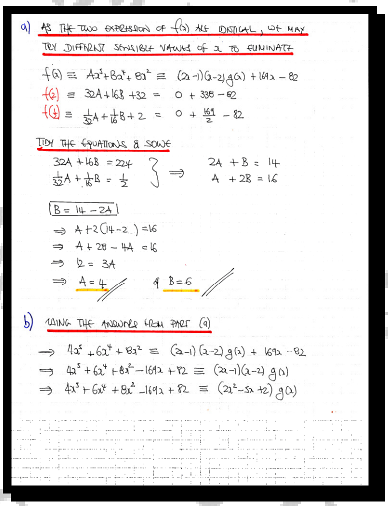
The polynomial $f(x)$ also satisfies the relationship

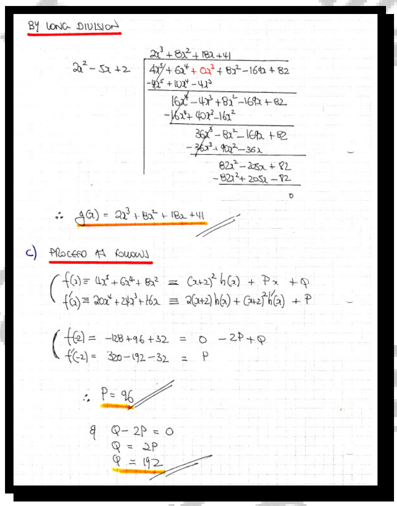
$$f(x) \equiv (x+2)^2 h(x) + Px + Q,$$

where P and Q are constants.

c) Find the value of each of the constants P and Q.

$\boxed{}$, $\boxed{A=4}$, $\boxed{B=6}$, $\boxed{g(x) \equiv 2x^3 + 8x^2 + 18x + 41}$, $\boxed{P=96}$, $\boxed{Q=192}$





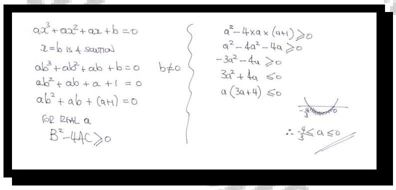
Question 18 (*****)

$$ax^3 + ax^2 + ax + b = 0,$$

where a and b are non zero real constants.

Given that $x = b$ is a root of the above cubic equation, determine the range of possible values of a.

$$\boxed{-\frac{4}{3} \le a \le 0}$$



Question 19 (*****)

$$f(x) \equiv x^5 + 3x^4 - 40x^3 - 47x^2 .$$

The polynomial $f(x)$ satisfies the relationship

$$f(x) \equiv (x-2)(x+A)(x+B)\left(x^2+3x-1\right) - 249x + 70 ,$$

where A and B are integer constants.

a) Find the value of A and the value of B.

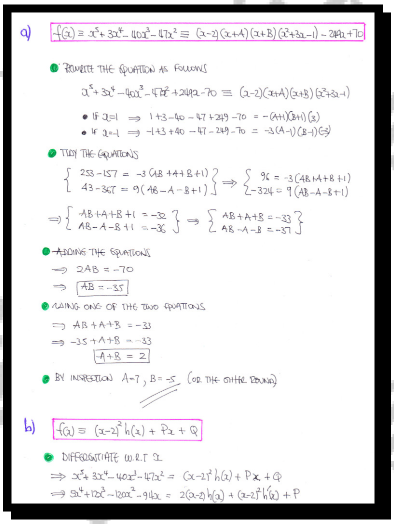
The polynomial $f(x)$ also satisfies the relationship

$$f(x) \equiv (x-2)^2 h(x) + Px + Q ,$$

where P and Q are constants.

b) Find the value of each of the constants P and Q.

$\boxed{\text{SP}}$, $\boxed{A=7}$, $\boxed{B=-5}$, $\boxed{P=-492}$, $\boxed{Q=556}$



a) $f(x) \equiv x^5+3x^4-40x^3-47x^2 \equiv (x-2)(x+A)(x+B)(x^2+3x-1) - 249x+70$

- REWRITE THE EQUATION AS FOLLOWS

$x^5+3x^4-40x^3-47x^2+249x-70 \equiv (x-2)(x+A)(x+B)(x^2+3x-1)$

- IF $x=1 \Rightarrow 1+3-40-47+249-70 = -(A+1)(B+1)(3)$
- IF $x=-1 \Rightarrow -1+3+40-47-249-70 = -3(A-1)(B-1)(-3)$

- TIDY THE EQUATIONS

$\left\{\begin{array}{l} 253-157 = -3(AB+A+B+1) \\ 43-367 = 9(AB-A-B+1) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 96 = -3(AB+A+B+1) \\ -324 = 9(AB-A-B+1) \end{array}\right.$

$\Rightarrow \left\{\begin{array}{l} AB+A+B+1 = -32 \\ AB-A-B+1 = -36 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} AB+A+B = -33 \\ AB-A-B = -37 \end{array}\right\}$

- ADDING THE EQUATIONS

$\Rightarrow 2AB = -70$

$\Rightarrow \boxed{AB=-35}$

- USING ONE OF THE TWO EQUATIONS

$\Rightarrow AB+A+B = -33$

$\Rightarrow -35+A+B = -33$

$\boxed{A+B=2}$

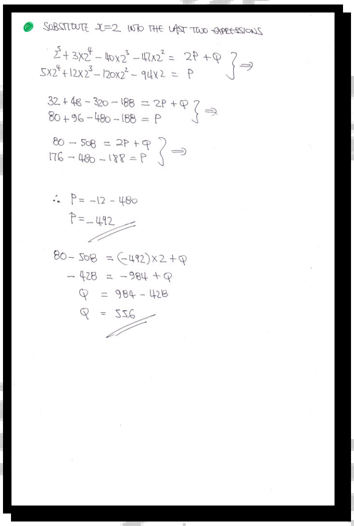
- BY INSPECTION $A=7$, $B=-5$ (OR THE OTHER ROUND)

b) $f(x) \equiv (x-2)^2 h(x) + Px + Q$

- DIFFERENTIATE W.R.T x

$\Rightarrow x^5+3x^4-40x^3-47x^2 = (x-2)^2 h(x) + Px + Q$

$\Rightarrow 5x^4+12x^3-120x^2-94x = 2(x-2)h(x) + (x-2)^2 h'(x) + P$



- SUBSTITUTE $x=2$ INTO THE LAST TWO EXPRESSIONS

$\left.\begin{array}{l} 2^5+3\times2^4-40\times2^3-47\times2^2 = 2P+Q \\ 5\times2^4+12\times2^3-120\times2^2-94\times2 = P \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} 32+48-320-188 = 2P+Q \\ 80+96-480-188 = P \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} 80-508 = 2P+Q \\ 176-480-188 = P \end{array}\right\} \Rightarrow$

$\therefore P = -12-480$

$P = -492$

$80-508 = (-492)\times2+Q$

$-428 = -984+Q$

$Q = 984-428$

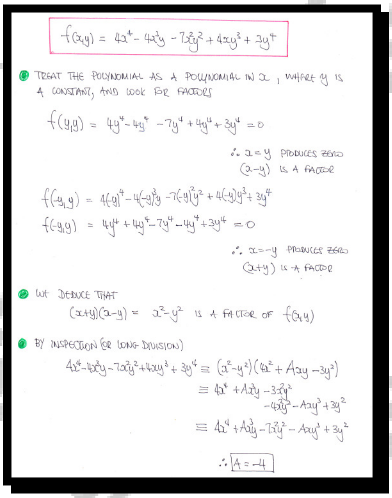
$Q = 556$

Question 20 (*****)

$$f(x, y) \equiv 4x^4 - 4x^3y - 7x^2y^2 + 4xy^3 + 3y^4 .$$

Express $f(x, y)$ as a product of 4 linear factors.

$\boxed{}$, $\boxed{f(x, y) \equiv (x+y)(x-y)(2x+y)(2x-3y)}$



$f(x,y) = 4x^4 - 4x^3y - 7x^2y^2 + 4xy^3 + 3y^4$

- TREAT THE POLYNOMIAL AS A POLYNOMIAL IN x, WHERE y IS A CONSTANT, AND LOOK FOR FACTORS

$f(y,y) = 4y^4 - 4y^4 - 7y^4 + 4y^4 + 3y^4 = 0$

$\therefore x = y$ PRODUCES ZERO, $(x-y)$ IS A FACTOR

$f(-y,y) = 4(-y)^4 - 4(-y)^3y - 7(-y)^2y^2 + 4(-y)y^3 + 3y^4$

$f(-y,y) = 4y^4 + 4y^4 - 7y^4 - 4y^4 + 3y^4 = 0$

$\therefore x = -y$ PRODUCES ZERO, $(x+y)$ IS A FACTOR

- WE DEDUCE THAT

$(x+y)(x-y) = x^2 - y^2$ IS A FACTOR OF $f(x,y)$

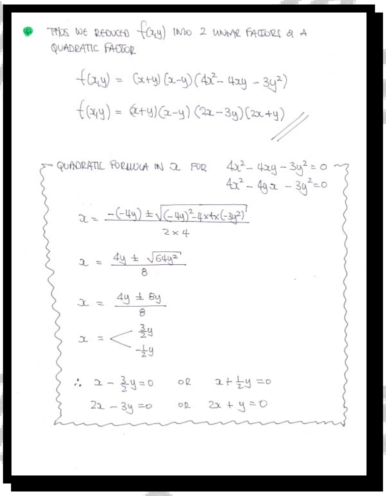
- BY INSPECTION (OR LONG DIVISION)

$4x^4 - 4x^3y - 7x^2y^2 + 4xy^3 + 3y^4 \equiv (x^2-y^2)(4x^2 + Axy - 3y^2)$

$\equiv 4x^4 + Ax^3y - 3x^2y^2 - 4x^2y^2 - Axy^3 + 3y^4$

$\equiv 4x^4 + Ax^3y - 7x^2y^2 - Axy^3 + 3y^4$

$\therefore \boxed{A = -4}$



- THUS WE REDUCED $f(x,y)$ INTO 2 LINEAR FACTORS & A QUADRATIC FACTOR

$f(x,y) = (x+y)(x-y)(4x^2 - 4xy - 3y^2)$

$f(x,y) = (x+y)(x-y)(2x-3y)(2x+y)$ //

QUADRATIC FORMULA IN x FOR $4x^2 - 4xy - 3y^2 = 0$

$4x^2 - 4yx - 3y^2 = 0$

$x = \frac{-(-4y) \pm \sqrt{(-4y)^2 - 4 \times 4 \times (-3y^2)}}{2 \times 4}$

$x = \frac{4y \pm \sqrt{64y^2}}{8}$

$x = \frac{4y \pm 8y}{8}$

$x = \frac{3}{2}y$ or $-\frac{1}{2}y$

$\therefore x - \frac{3}{2}y = 0$ OR $x + \frac{1}{2}y = 0$

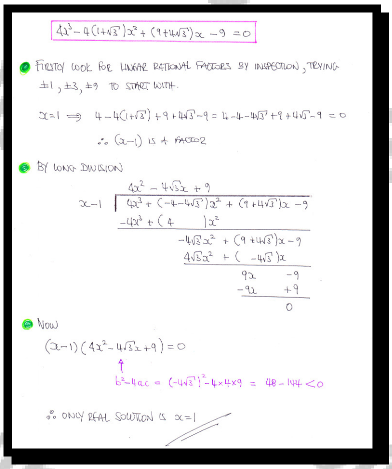
$2x - 3y = 0$ OR $2x + y = 0$

Question 21 (*****)

Solve the cubic equation

$$4x^3 - 4\left(1+\sqrt{3}\right)x^2 + \left(9+4\sqrt{3}\right)x - 9 = 0, \ x \in \mathbb{R}.$$

$\boxed{}$, $\boxed{x=1}$


$4x^3 - 4(1+\sqrt{3})x^2 + (9+4\sqrt{3})x - 9 = 0$

- FIRSTLY LOOK FOR LINEAR RATIONAL FACTORS BY INSPECTION, TRYING $\pm 1, \pm 3, \pm 9$ TO START WITH.

$x=1 \Rightarrow 4 - 4(1+\sqrt{3}) + 9 + 4\sqrt{3} - 9 = 4 - 4 - 4\sqrt{3} + 9 + 4\sqrt{3} - 9 = 0$

$\therefore (x-1)$ IS A FACTOR

- BY LONG DIVISION

$$\begin{array}{r|l} & 4x^2 - 4\sqrt{3}x + 9 \\ \hline x-1 & 4x^3 + (-4-4\sqrt{3})x^2 + (9+4\sqrt{3})x - 9 \\ & -4x^3 + (4 \qquad)x^2 \\ \hline & -4\sqrt{3}x^2 + (9+4\sqrt{3})x - 9 \\ & 4\sqrt{3}x^2 + (\ -4\sqrt{3})x \\ \hline & 9x \qquad -9 \\ & -9x \qquad +9 \\ \hline & 0 \end{array}$$

- NOW

$(x-1)(4x^2 - 4\sqrt{3}x + 9) = 0$

$b^2 - 4ac = (-4\sqrt{3})^2 - 4 \times 4 \times 9 = 48 - 144 < 0$

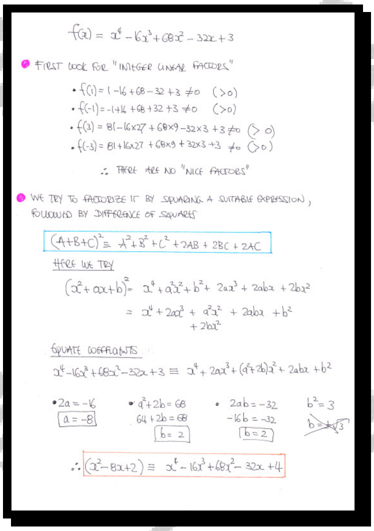
$\therefore$ ONLY REAL SOLUTION IS $x = 1$

Question 22 (*****)

$$f(x) \equiv x^4 - 16x^3 + 68x^2 - 32x + 3, \quad x \in \mathbb{R}.$$

Factorize $f(x)$ into a product of 4 linear factors.

$$f(x) \equiv \left(x - 4 - \sqrt{15}\right)\left(x - 4 + \sqrt{15}\right)\left(x - 4 - \sqrt{13}\right)\left(x - 4 + \sqrt{13}\right)$$



$f(x) = x^4 - 16x^3 + 68x^2 - 32x + 3$

- FIRST LOOK FOR "INTEGER LINEAR FACTORS"
  - $f(1) = 1 - 16 + 68 - 32 + 3 \neq 0 \quad (>0)$
  - $f(-1) = -1 + 16 + 68 + 32 + 3 \neq 0 \quad (>0)$
  - $f(3) = 81 - 16 \times 27 + 68 \times 9 - 32 \times 3 + 3 \neq 0 \quad (>0)$
  - $f(-3) = 81 + 16 \times 27 + 68 \times 9 + 32 \times 3 + 3 \neq 0 \quad (>0)$

  $\therefore$ THERE ARE NO "NICE FACTORS"

- WE TRY TO FACTORIZE IT BY SQUARING A SUITABLE EXPRESSION, FOLLOWED BY DIFFERENCE OF SQUARES

$(A+B+C)^2 \equiv A^2 + B^2 + C^2 + 2AB + 2BC + 2AC$

HERE WE TRY

$(x^2 + ax + b)^2 = x^4 + a^2x^2 + b^2 + 2ax^3 + 2abx + 2bx^2$

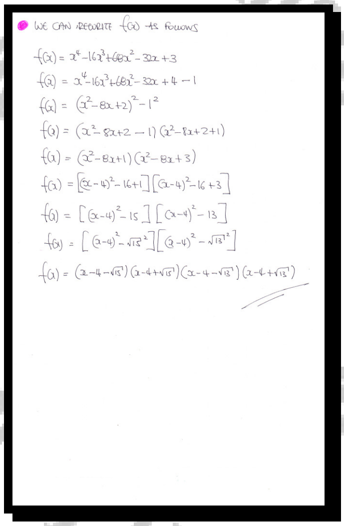
$= x^4 + 2ax^3 + a^2x^2 + 2abx + b^2 + 2bx^2$

EQUATE COEFFICIENTS

$x^4 - 16x^3 + 68x^2 - 32x + 3 \equiv x^4 + 2ax^3 + (a^2 + 2b)x^2 + 2abx + b^2$

- $2a = -16$, $a = -8$
- $a^2 + 2b = 68$, $64 + 2b = 68$, $b = 2$
- $2ab = -32$, $-16b = -32$, $b = 2$
- $b^2 = 3$, ~~$b = \pm\sqrt{3}$~~

$\therefore (x^2 - 8x + 2) \equiv x^4 - 16x^3 + 68x^2 - 32x + 4$



- WE CAN REWRITE $f(x)$ AS FOLLOWS

$f(x) = x^4 - 16x^3 + 68x^2 - 32x + 3$

$f(x) = x^4 - 16x^3 + 68x^2 - 32x + 4 - 1$

$f(x) = (x^2 - 8x + 2)^2 - 1^2$

$f(x) = (x^2 - 8x + 2 - 1)(x^2 - 8x + 2 + 1)$

$f(x) = (x^2 - 8x + 1)(x^2 - 8x + 3)$

$f(x) = \left[(x-4)^2 - 16 + 1\right]\left[(x-4)^2 - 16 + 3\right]$

$f(x) = \left[(x-4)^2 - 15\right]\left[(x-4)^2 - 13\right]$

$f(x) = \left[(x-4)^2 - \sqrt{15}^2\right]\left[(x-4)^2 - \sqrt{13}^2\right]$

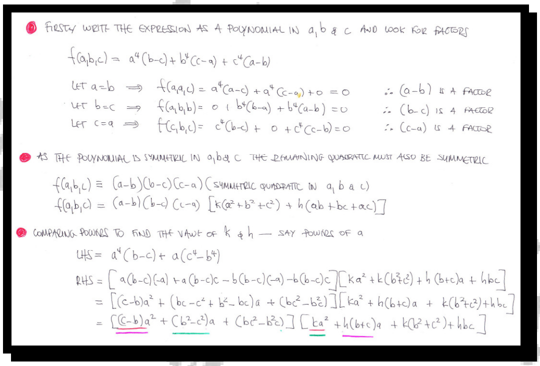
$f(x) = (x - 4 - \sqrt{15})(x - 4 + \sqrt{15})(x - 4 - \sqrt{13})(x - 4 + \sqrt{13})$

Question 23 (*****)

$$f(a,b,c) \equiv a^4(b-c)+b^4(c-a)+c^4(a-b).$$

Factorize $f(a,b,c)$ into a product of 3 linear factors and 1 quadratic factor.

$$f(a,b,c) \equiv -(a-b)(b-c)(c-a)\left(a^2+b^2+c^2+ab+bc+ca\right)$$



• FIRSTLY WRITE THE EXPRESSION AS A POLYNOMIAL IN a, b & c AND LOOK FOR FACTORS

$f(a,b,c) = a^4(b-c)+b^4(c-a)+c^4(a-b)$

LET $a=b \implies f(a,a,c) = a^4(a-c)+a^4(c-a)+0=0$ ∴ $(a-b)$ IS A FACTOR

LET $b=c \implies f(a,b,b) = 0 + b^4(b-a)+b^4(a-b)=0$ ∴ $(b-c)$ IS A FACTOR

LET $c=a \implies f(c,b,c) = c^4(b-c)+0+c^4(c-b)=0$ ∴ $(c-a)$ IS A FACTOR

• AS THE POLYNOMIAL IS SYMMETRIC IN a, b & c THE REMAINING QUADRATIC MUST ALSO BE SYMMETRIC

$f(a,b,c) \equiv (a-b)(b-c)(c-a)$(SYMMETRIC QUADRATIC IN a, b & c)

$f(a,b,c) = (a-b)(b-c)(c-a)\left[k(a^2+b^2+c^2)+h(ab+bc+ac)\right]$

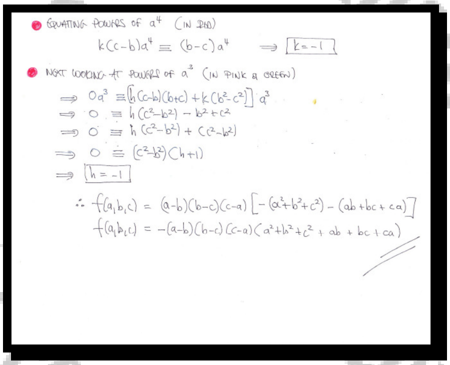
• COMPARING POWERS TO FIND THE VALUE OF k & h — SAY POWERS OF a

$LHS = a^4(b-c)+a(c^4-b^4)$

$RHS = \left[a(b-c)(-a)+a(b-c)c-b(b-c)(-a)-b(b-c)c\right]\left[ka^2+k(b^2+c^2)+h(b+c)a+hbc\right]$

$= \left[(c-b)a^2+(bc-c^2+b^2-bc)a+(bc^2-b^2c)\right]\left[ka^2+h(b+c)a+k(b^2+c^2)+hbc\right]$

$= \left[(c-b)a^2+(b^2-c^2)a+(bc^2-b^2c)\right]\left[ka^2+h(b+c)a+k(b^2+c^2)+hbc\right]$



• EQUATING POWERS OF a^4 (IN RED)

$k(c-b)a^4 \equiv (b-c)a^4 \implies k=-1$

• NEXT LOOKING AT POWERS OF a^3 (IN PINK & GREEN)

$\implies 0a^3 \equiv \left[h(c-b)(b+c)+k(b^2-c^2)\right]a^3$

$\implies 0 \equiv h(c^2-b^2)-b^2+c^2$

$\implies 0 \equiv h(c^2-b^2)+(c^2-b^2)$

$\implies 0 \equiv (c^2-b^2)(h+1)$

$\implies h=-1$

∴ $f(a,b,c) = (a-b)(b-c)(c-a)\left[-(a^2+b^2+c^2)-(ab+bc+ca)\right]$

$f(a,b,c) = -(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$

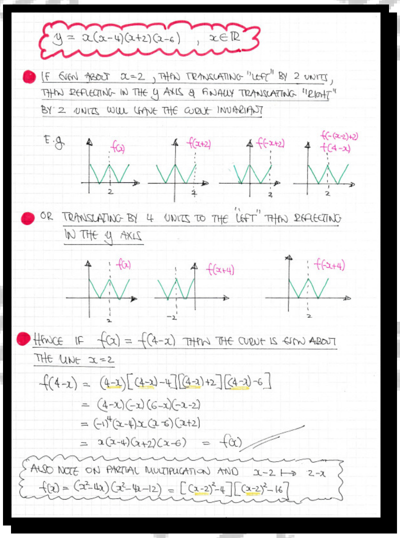
Question 24 (*****)

A quartic curve C has the following equation.

$$y = x(x-4)(x+2)(x-6), \quad x \in \mathbb{R}.$$

By considering suitable transformations, show that C is even about the straight line with equation $x = 2$.

[ ], [proof]


$y = x(x-4)(x+2)(x-6)$, $x \in \mathbb{R}$

- IF EVEN ABOUT $x=2$, THEN TRANSLATING "LEFT" BY 2 UNITS, THEN REFLECTING IN THE y AXIS & FINALLY TRANSLATING "RIGHT" BY 2 UNITS WILL LEAVE THE CURVE INVARIANT

- OR TRANSLATING BY 4 UNITS TO THE "LEFT" THEN REFLECTING IN THE y AXIS

- HENCE IF $f(x) = f(4-x)$ THEN THE CURVE IS EVEN ABOUT THE LINE $x=2$

$$\begin{aligned} f(4-x) &= (4-x)[(4-x)-4][(4-x)+2][(4-x)-6] \\ &= (4-x)(-x)(6-x)(-x-2) \\ &= (-1)^4(x-4)x(x-6)(x+2) \\ &= x(x-4)(x+2)(x-6) = f(x) \end{aligned}$$

ALSO NOTE ON PARTIAL MULTIPLICATION AND $x-2 \mapsto 2-x$

$f(x) = (x^2-4x)(x^2-4x-12) = [(x-2)^2-4][(x-2)^2-16]$

Question 25 (*****)

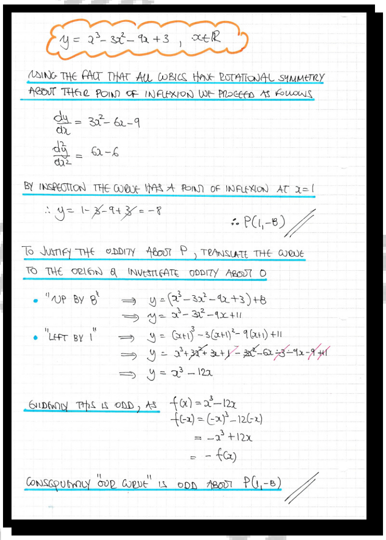
A cubic curve with equation

$$y = x^3 - 3x^2 - 9x + 3, \quad x \in \mathbb{R},$$

is odd about some point P.

Find the coordinates of P and use transformation arguments to justify the assertion that the curve is odd about P.

$\boxed{}$, $\boxed{P(1,-8)}$



$y = x^3 - 3x^2 - 9x + 3, \quad x \in \mathbb{R}$

USING THE FACT THAT ALL CUBICS HAVE ROTATIONAL SYMMETRY ABOUT THEIR POINT OF INFLEXION WE PROCEED AS FOLLOWS

$\frac{dy}{dx} = 3x^2 - 6x - 9$

$\frac{d^2y}{dx^2} = 6x - 6$

BY INSPECTION THE CURVE HAS A POINT OF INFLEXION AT $x = 1$

$\therefore y = 1 - 3 - 9 + 3 = -8$

$\therefore P(1,-8)$

TO JUSTIFY THE ODDITY ABOUT P, TRANSLATE THE CURVE TO THE ORIGIN & INVESTIGATE ODDITY ABOUT O

- "UP BY 8" $\Rightarrow y = (x^3 - 3x^2 - 9x + 3) + 8$
  $\Rightarrow y = x^3 - 3x^2 - 9x + 11$
- "LEFT BY 1" $\Rightarrow y = (x+1)^3 - 3(x+1)^2 - 9(x+1) + 11$
  $\Rightarrow y = x^3 + 3x^2 + 3x + 1 - 3x^2 - 6x - 3 - 9x - 9 + 11$
  $\Rightarrow y = x^3 - 12x$

EVIDENTLY THIS IS ODD, AS $f(x) = x^3 - 12x$

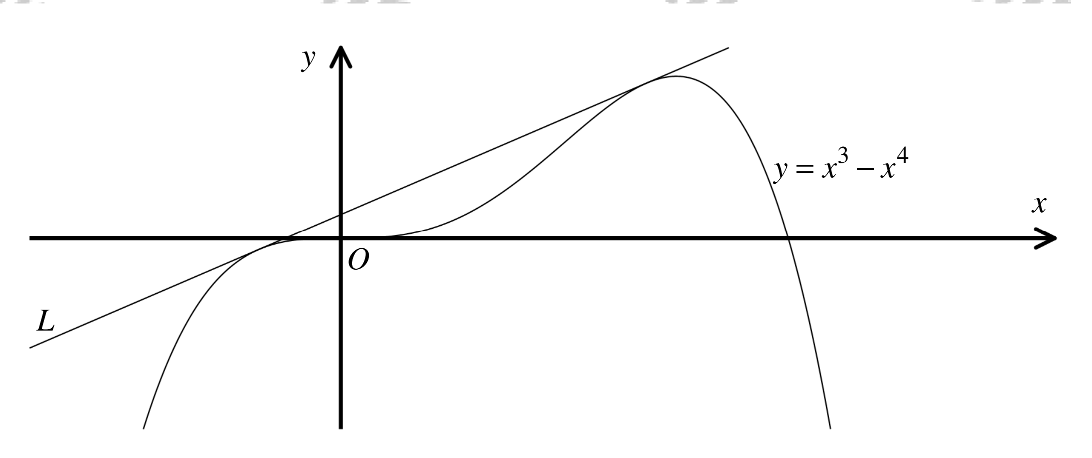
$f(-x) = (-x)^3 - 12(-x)$

$= -x^3 + 12x$

$= -f(x)$

CONSEQUENTLY "OUR CURVE" IS ODD ABOUT $P(1,-8)$

Question 26 (*****)



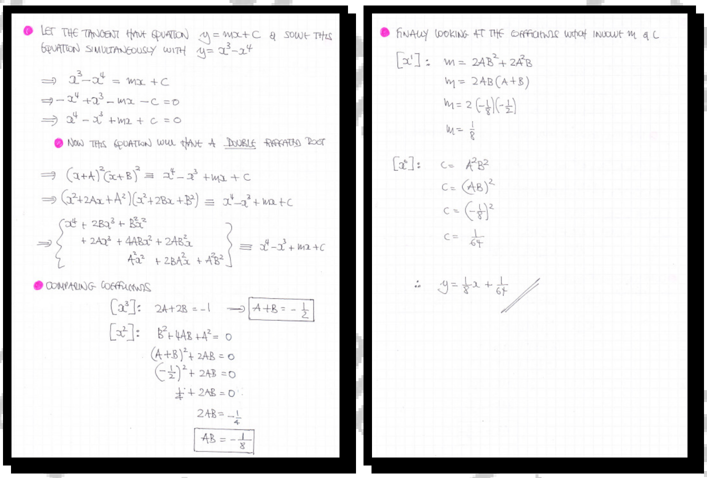
The figure above shows the curve C with equation

$$y = x^3 - x^4.$$

The straight line L is a tangent to the C, at two distinct points.

Determine an equation of L.

$\boxed{}$, $\boxed{y = \frac{1}{8}x + \frac{1}{64}}$



Question 27 (*****)

Solve the cubic equation

$$16x^3 - 48x^2 + 60x - 31 = 0, \; x \in \mathbb{R}.$$

You may assume that this cubic equation only has one real root.

$\boxed{}$, $\boxed{x = 1 + 2^{-\frac{2}{3}} - 2^{-\frac{4}{3}}}$

START BY WRITING THE CUBIC IN REDUCED FORM

$$16x^3 - 48x^2 + 60x - 31 = 0$$
$$x^3 - 3x^2 + \frac{15}{4}x - \frac{31}{16} = 0$$

LET $x = y - \frac{a}{3} = y - \frac{-3}{3} \Rightarrow \boxed{x = y + 1}$

SUBSTITUTE BACK INTO THE CUBIC

$\Rightarrow 16(y+1)^3 - 48(y+1)^2 + 60(y+1) - 31 = 0$

$\Rightarrow 16(y^3 + 3y^2 + 3y + 1) - 48(y^2 + 2y + 1) + 60(y+1) - 31 = 0$

$\Rightarrow 16y^3 + 48y^2 + 48y + 16 - 48y^2 - 96y - 48 + 60y + 60 - 31 = 0$

$\Rightarrow 16y^3 + 12y - 3 = 0$

$\Rightarrow 16y^3 + 12y = 3$

WE NOW USE THE IDENTITY OF $\sinh 3t$ AS THE COEFFICIENT OF y IS POSITIVE

$$\boxed{\sinh 3t = 3\sinh t + 4\sinh^3 t}$$

$\Rightarrow 4y^3 + 3y = 3/4$

$\Rightarrow 4\sinh^3 t + 3\sinh t = \frac{3}{4}$ $\quad \{ y = \sinh t \}$

$\Rightarrow \sinh 3t = \frac{3}{4}$

$\Rightarrow 3t = \text{arsinh}\,\frac{3}{4}$

$\Rightarrow t = \frac{1}{3}\ln\left[\frac{3}{4} + \sqrt{\frac{9}{16} + 1}\right]$

$\Rightarrow t = \frac{1}{3}\ln\left[\frac{3}{4} + \sqrt{\frac{25}{16}}\right]$

$\Rightarrow t = \frac{1}{3}\ln\left[\frac{3}{4} + \frac{5}{4}\right]$

$\Rightarrow t = \frac{1}{3}\ln 2$

$\Rightarrow t = \ln 2^{\frac{1}{3}}$

HENCE THE REQUIRED SOLUTION CAN BE FOUND

$\Rightarrow x = y + 1$

$\Rightarrow x = 1 + \sinh t$

$\Rightarrow x = 1 + \sinh\left(\ln 2^{\frac{1}{3}}\right)$

$\Rightarrow x = 1 + \frac{1}{2}\left[e^{\ln 2^{\frac{1}{3}}} - e^{-\ln 2^{\frac{1}{3}}}\right]$

$\Rightarrow x = 1 + 2^{-1}\left[2^{\frac{1}{3}} - 2^{-\frac{1}{3}}\right]$

$\Rightarrow x = 1 + 2^{-\frac{2}{3}} - 2^{-\frac{4}{3}}$

Question 28 (*****)

Solve the cubic equation

$$16x^3 + 96x^2 + 180x + 99 = 0, \quad x \in \mathbb{R}.$$

You may assume that this cubic equation only has one real root.

$$x = -2 + 2^{-\frac{2}{3}} - 2^{-\frac{4}{3}}$$

- START BY WRITING THE CUBIC IN REDUCED FORM

$$16x^3 + 96x^2 + 180x + 99 = 0$$
$$x^3 + 6x^2 + \frac{45}{4}x + \frac{99}{16} = 0 \quad (a = 6)$$

LET $x = y - \frac{a}{3} = y - \frac{6}{3} \Rightarrow \boxed{x = y - 2 \\ y = x + 2}$

- SUBSTITUTING INTO THE CUBIC YIELDS

$$\Rightarrow 16(y-2)^3 + 96(y-2)^2 + 180(y-2) + 99 = 0$$
$$\Rightarrow 16(y^3 - 6y^2 + 12y - 8) + 96(y^2 - 4y + 4) + 180(y-2) + 99 = 0$$
$$\Rightarrow \left\{ \begin{array}{r} 16y^3 - 96y^2 + 192y - 128 \\ + 96y^2 - 384y + 384 \\ + 180y - 360 \\ + 99 \end{array} \right\} = 0$$
$$\Rightarrow 16y^3 \quad - 12y - 5 = 0$$
$$\Rightarrow 16y^3 - 12y - 5 = 0$$

- NOW WE USE THE IDENTITY (COEFFICIENT OF y NEGATIVE)

$$\boxed{\cos 3t \equiv 4\cos^3 t - 3\cos t \\ \text{OR} \\ \cosh 3t \equiv 4\cosh^3 t - 3\cosh t}$$

$$\Rightarrow 16y^3 - 12y = 5$$
$$\Rightarrow 4y^3 - 3y = \frac{5}{4}$$
$$\Rightarrow 4\cosh^3 t - 3\cosh t = \frac{5}{4} \qquad y = \cosh t$$
$$\Rightarrow \cosh 3t = \frac{5}{4}$$
$$\Rightarrow 3t = \pm \text{arccosh}\left(\frac{5}{4}\right) = \pm \ln\left[\frac{5}{4} + \sqrt{\frac{25}{16} - 1}\right]$$
$$\Rightarrow 3t = \pm \ln\left(\frac{5}{4} + \sqrt{\frac{9}{16}}\right) = \pm \ln\left(\frac{5}{4} + \frac{3}{4}\right) = \pm \ln 2$$
$$\Rightarrow t = \pm \frac{1}{3}\ln 2$$

- FINALLY WE HAVE A REAL SOLUTION

$$\Rightarrow x = y - 2$$
$$\Rightarrow x = \cosh t - 2$$
$$\Rightarrow x = \cosh\left[\pm \frac{1}{3}\ln 2\right] - 2$$
$$\Rightarrow x = \cosh\left(\frac{1}{3}\ln 2\right) - 2$$
$$\Rightarrow x = \cosh\left(\ln 2^{\frac{1}{3}}\right) - 2$$
$$\Rightarrow x = \frac{1}{2}\left[e^{\ln 2^{\frac{1}{3}}} + e^{-\ln 2^{\frac{1}{3}}}\right] - 2$$
$$\Rightarrow x = 2^{-1}\left[2^{\frac{1}{3}} + 2^{-\frac{1}{3}}\right] - 2$$
$$\Rightarrow x = 2^{-\frac{2}{3}} + 2^{-\frac{4}{3}} - 2$$

Question 29 (*****)

Solve the cubic equation

$$x^3 - 9x^2 + 3x - 3 = 0, \; x \in \mathbb{R}.$$

You may assume that this cubic equation only has one real root.

$\boxed{}$, $\boxed{x = 3 + 2^{\frac{5}{3}} + 2^{\frac{4}{3}}}$

START BY WRITING THE CUBIC IN REDUCED FORM

$$x^3 - 9x^2 + 3x - 3 = 0$$

LET $x = y - \frac{a}{3} = y - \frac{-9}{3} \Rightarrow x = y + 3$

SUBSTITUTE INTO THE CUBIC

$\Rightarrow (y+3)^3 - 9(y+3)^2 + 3(y+3) - 3 = 0$

$\Rightarrow y^3 + 9y^2 + 27y + 27 - 9(y^2 + 6y + 9) + 3y + 9 - 3 = 0$

$\Rightarrow \begin{bmatrix} y^3 + 9y^2 + 27y + 27 \\ -9y^2 - 54y - 81 \\ 3y + 6 \end{bmatrix} = 0$

$\Rightarrow y^3 - 24y - 48 = 0$

$\Rightarrow y^3 - 24y = 48$

WE USE THE IDENTITY (COEFFICIENT OF y NEGATIVE)

$$\boxed{\begin{gathered}\cos 3t \equiv 4\cos^3 t - 3\cos t \\ \text{OR} \\ \cosh 3t \equiv 4\cosh^3 t - 3\cosh t\end{gathered}}$$

LET $y = \lambda \cosh t, \; \lambda \neq 0$

$$\begin{pmatrix} \lambda^3 \cosh^3 t - 24\lambda \cosh t = 48 \\ 4\cosh^3 t - 3\cosh t = \cosh 3t \end{pmatrix}$$

$$\frac{\lambda^3}{4} = \frac{-24\lambda}{-3} = \frac{48}{\cosh 3t}$$

FROM THE "FIRST TWO" WE OBTAIN

$$\frac{\lambda^2}{4} = 8$$

$\Rightarrow \lambda = \pm\sqrt{32} = \pm 4\sqrt{2}$

THUS WE NOW HAVE

$\Rightarrow \frac{48}{\cosh 3t} = 8\lambda = \pm 32\sqrt{2}$

$\Rightarrow \cosh 3t = \pm\frac{48}{32\sqrt{2}} = \pm\frac{3}{2\sqrt{2}} = \pm\frac{3\sqrt{2}}{4}$

$\Rightarrow 3t = \pm \text{arccosh}\left(\pm\frac{3}{4}\sqrt{2}\right)$

$\Rightarrow 3t = \pm \text{arccos}\left(\frac{3}{4}\sqrt{2}\right)$

$\Rightarrow t = \pm\frac{1}{3}\ln\left[\frac{3}{4}\sqrt{2} + \sqrt{\frac{9}{8} - 1}\right]$

$\Rightarrow t = \pm\frac{1}{3}\ln\left[\frac{3}{4}\sqrt{2} + \sqrt{\frac{1}{8}}\right]$

$\Rightarrow t = \pm\frac{1}{3}\ln\left[\frac{3}{4}\sqrt{2} + \sqrt{\frac{2}{16}}\right]$

$\Rightarrow t = \pm\frac{1}{3}\ln\left[\frac{3}{4}\sqrt{2} + \frac{1}{4}\sqrt{2}\right]$

$\Rightarrow t = \pm\frac{1}{3}\ln\sqrt{2}$

FINALLY WE HAVE

$x = 3 + y = 3 + \lambda\cosh t = 3 + 4\sqrt{2}\cosh\left[\pm\frac{1}{3}\ln\sqrt{2}\right]$

$x = 3 + 4\sqrt{2}\cosh\left(\frac{1}{3}\ln\sqrt{2}\right) = 3 + 4\sqrt{2} \times \frac{1}{2}\left[e^{\frac{1}{3}\ln\sqrt{2}} + e^{-\frac{1}{3}\ln\sqrt{2}}\right]$

$x = 3 + 2\sqrt{2} \times \left[2^{\frac{1}{6}} + 2^{-\frac{1}{6}}\right] = 3 + 2^{\frac{3}{2}}\left[2^{\frac{1}{6}} + 2^{-\frac{1}{6}}\right]$

$x = 3 + 2^{\frac{5}{3}} + 2^{\frac{4}{3}}$