DISCRETE RANDOM VARIABLES

Question 1 (**)

The probability distribution of a discrete random variable X is given by

x	0	1	2	3	4
$\mathrm{P}(X=x)$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{1}{4}$	a	$\frac{1}{24}$

where a is a positive constant.

a) Explain why $a=0$.

b) Find the value of $\mathrm{E}(X)$.

c) Calculate $\mathrm{Var}(X)$.

$\boxed{}$, $\boxed{\sum \mathrm{P}(X=x)=1}$, $\boxed{\mathrm{E}(X)=1}$, $\boxed{\mathrm{Var}(X)=1}$

x	0	1	2	3	4
$P(X=x)$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{1}{4}$	a	$\frac{1}{24}$

a) $\sum P(X=x)=1$

$\frac{3}{8}+\frac{1}{3}+\frac{1}{4}+a+\frac{1}{24}=1$

$1+a=1$

$a=0$

b) $E(X)=\sum x\,P(X=x)$

$\Rightarrow E(X)=\left(0\times\frac{3}{8}\right)+\left(1\times\frac{1}{3}\right)+\left(2\times\frac{1}{4}\right)+(3\times 0)+\left(4\times\frac{1}{24}\right)$

$\Rightarrow E(X)=0+\frac{1}{3}+\frac{1}{2}+0+\frac{1}{6}$

$\Rightarrow E(X)=1$

c) $E(X^2)=\sum x^2 P(X=x)$

$\Rightarrow E(X^2)=\left(0^2\times\frac{3}{8}\right)+\left(1^2\times\frac{1}{3}\right)+\left(2^2\times\frac{1}{4}\right)+\left(3^2\times 0\right)+\left(4^2\times\frac{1}{24}\right)$

$\Rightarrow E(X^2)=0+\frac{1}{3}+1+0+\frac{2}{3}$

$\Rightarrow E(X^2)=2$

$\mathrm{Var}(X)=E(X^2)-[E(X)]^2$

$\mathrm{Var}(X)=2-1^2$

$\mathrm{Var}(X)=1$

Question 2 (**)

The probability distribution of a discrete random variable X is given by

x	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$

Find, showing full workings where appropriate, the value of

a) $\mathrm{P}(1 < X \le 3)$.

b) $\mathrm{F}(1.8)$.

c) $\mathrm{E}(X)$.

d) $\mathrm{Var}(X)$.

e) $\mathrm{E}(2X - 3)$.

f) $\mathrm{Var}(2X - 3)$.

$\boxed{}$, $\boxed{P(1 < X \le 3) = \frac{2}{3}}$, $\boxed{\mathrm{F}(1.8) = \frac{1}{3}}$, $\boxed{\mathrm{E}(X) = \frac{23}{12}}$, $\boxed{\mathrm{Var}(X) = \frac{131}{144} \approx 0.910}$,

$\boxed{\mathrm{E}(2X - 3) = \frac{5}{6}}$, $\boxed{\mathrm{Var}(2X - 3) = \frac{131}{36} \approx 3.639}$

x	0	1	2	3
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$

a) $P(1 < X \le 3) = P(X = 2, 3) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

b) $F(1.8) = P(X \le 1.8) = P(X = 0, 1) = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$

c) $E(X) = \sum x P(X = x)$
$= \left(0 \times \frac{1}{12}\right) + \left(1 \times \frac{1}{4}\right) + \left(2 \times \frac{1}{3}\right) + \left(3 \times \frac{1}{3}\right)$
$= 0 + \frac{1}{4} + \frac{2}{3} + 1$
$= \frac{23}{12} \approx 1.9166\ldots$

d) $E(X^2) = \sum x^2 P(X = x)$
$= \left(0^2 \times \frac{1}{12}\right) + \left(1^2 \times \frac{1}{4}\right) + \left(2^2 \times \frac{1}{3}\right) + \left(3^2 \times \frac{1}{3}\right)$
$= 0 + \frac{1}{4} + \frac{4}{3} + 3$
$= \frac{55}{12} \approx 4.583\ldots$

$\mathrm{Var}(X) = E(X^2) - (E(X))^2$
$= \frac{55}{12} - \left(\frac{23}{12}\right)^2$
$= \frac{131}{144} \approx 0.9097\ldots$

e) $E(2X - 3) = 2E(X) - 3$
$= 2 \times \frac{23}{12} - 3$
$= \frac{23}{6} - 3$
$= \frac{5}{6} \approx 0.8333\ldots$

f) $\mathrm{Var}(2X - 3) = 2^2\,\mathrm{Var}(X)$
$= 4\,\mathrm{Var}(X)$
$= 4 \times \frac{131}{144}$
$= \frac{131}{36} \approx 3.639\ldots$

Question 3 (**)

The probability distribution of a discrete random variable X is given by

$$\mathrm{P}(X = x) = \begin{cases} kx^2 & x = 3,4,5 \\ 0 & \text{otherwise} \end{cases}$$

a) Determine the value of the constant k.

b) Find the value of …

  i. … $\mathrm{E}(X)$.

  ii. … $\mathrm{Var}(X)$.

c) Determine …

  i. … $\mathrm{E}(5X-4)$.

  ii. … $\mathrm{Var}(5X-4)$.

$\boxed{}$, $\boxed{k = \frac{1}{50}}$, $\boxed{\mathrm{E}(X) = 4.32}$, $\boxed{\mathrm{Var}(X) = 0.5776}$, $\boxed{\mathrm{E}(5X-4) = 17.6}$, $\boxed{\mathrm{Var}(5X-4) = 14.44}$

$$\mathrm{P}(X=x) = \begin{cases} kx^2 & x = 3,4,5 \\ 0 & \text{OTHERWISE} \end{cases}$$

a) WRITE THE FORMULA AS A TABLE

x	3	4	5
$\mathrm{P}(X=x)$	$9k$	$16k$	$25k$

$9k + 16k + 25k = 1$

$50k = 1$

$k = \frac{1}{50}$

b) I) $\mathrm{E}(X) = \sum x\,\mathrm{P}(X=x)$

$\Rightarrow \mathrm{E}(X) = (3\times 9k) + (4\times 16k) + (5\times 25k)$

$\Rightarrow \mathrm{E}(X) = 27k + 64k + 125k$

$\Rightarrow \mathrm{E}(X) = 216k$

$\Rightarrow \mathrm{E}(X) = 4.32$

II) $\mathrm{E}(X^2) = \sum x^2\,\mathrm{P}(X=x)$

$\Rightarrow \mathrm{E}(X^2) = (3^2\times 9k) + (4^2\times 16k) + (5^2\times 25k)$

$\Rightarrow \mathrm{E}(X^2) = 81k + 256k + 625k$

$\Rightarrow \mathrm{E}(X^2) = 962k$

$\Rightarrow \mathrm{E}(X^2) = 19.24$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

$\Rightarrow \mathrm{Var}(X) = 19.24 - 4.32^2$

$\Rightarrow \mathrm{Var}(X) = 0.5776$

c) I) $\mathrm{E}(5X-4) = 5\,\mathrm{E}(X) - 4$

$= 5\times 4.32 - 4$

$= 17.6$

II) $\mathrm{Var}(5X-4) = 5^2\,\mathrm{Var}(X)$

$= 25 \times 0.5776$

$= 14.44$

Question 4 (**)

The discrete random variable X has mean 7 and variance 11.

a) Calculate $\mathrm{E}\left(X^2\right)$.

b) Given that $Y = 2X - 4$, determine the mean and variance of Y.

$\boxed{}$, $\boxed{\mathrm{E}\left(X^2\right) = 60}$, $\boxed{\mathrm{E}(Y) = 10}$, $\boxed{\mathrm{Var}(Y) = 44}$

a) $\boxed{E(X) = 7 \quad \mathrm{Var}(X) = 11}$

USING $\mathrm{Var}(X) = E(X^2) - (E(X))^2$

$11 = E(X^2) - 7^2$

$E(X^2) = 60$

b) $\underline{Y = 2X - 4}$

$E(Y) = E(2X-4)$

$E(Y) = 2E(X) - 4$

$E(Y) = 2\times 7 - 4$

$E(Y) = 10$

$\underline{Y = 2X - 4}$

$\mathrm{Var}(Y) = \mathrm{Var}(2X-4)$

$\mathrm{Var}(Y) = 2^2\,\mathrm{Var}(X)$

$\mathrm{Var}(Y) = 4 \times 11$

$\mathrm{Var}(Y) = 44$

Question 5 (**)

The probability distribution of a discrete random variable X is given by

$$\mathrm{P}(X=x)=\begin{cases} k(2-x)^2 & x=-2,-1,0,1,2 \\ 0 & \text{otherwise} \end{cases}$$

a) Determine the value of the constant k.

b) Find the value of …

i. … $\mathrm{E}(X)$.

ii. … $\mathrm{E}(X^2)$.

c) Determine …

i. … $\mathrm{E}(1-15X)$.

ii. … $\mathrm{Var}(1-15X)$.

$\boxed{}$, $\boxed{k=\frac{1}{30}}$, $\boxed{\mathrm{E}(X)=-\frac{4}{3}}$, $\boxed{\mathrm{E}(X^2)=\frac{37}{15}}$, $\boxed{\mathrm{E}(1-15X)=21}$, $\boxed{\mathrm{Var}(1-15X)=155}$

a) WRITE THE FORMULA INTO A TABLE FORM

x	-2	-1	0	1	$\cancel{2}$
$P(X=x)$	$16k$	$9k$	$4k$	k	$\cancel{0}$

$$16k+9k+4k+k=1$$
$$30k=1$$
$$k=\frac{1}{30}$$

b) USING THE STANDARD FORMULAE

I) $E(X)=\sum xP(X=x)$

$E(X)=(-2\times16k)+(-1\times9k)+(0\times4k)+(1\times k)$

$E(X)=-32k-9k+k$

$E(X)=-40k$

$E(X)=-\frac{4}{3}$

II) $E(X^2)=\sum x^2 P(X=x)$

$E(X^2)=(-2)^2\times16k+(-1)^2\times9k+0^2\times4k+1^2\times k$

$E(X^2)=64k+9k+k$

$E(X^2)=74k$

$E(X^2)=\frac{37}{15}\simeq 2.4666\ldots$

c) I) USING THE STANDARD TRANSFORMATION EQUATIONS

$E(1-15X)=E(-15X+1)$

$=-15E(X)+1$

$=-15\times\left(-\frac{4}{3}\right)+1$

$=21$

II) NEED THE VARIANCE OF X FIRST

$\mathrm{Var}(X)=E(X^2)-(E(X))^2$

$\mathrm{Var}(X)=\frac{37}{15}-\left(-\frac{4}{3}\right)^2$

$\mathrm{Var}(X)=\frac{37}{15}-\frac{16}{9}$

$\mathrm{Var}(X)=\frac{31}{45}$

HENCE WE CAN NOW TRANSFORM

$\mathrm{Var}(1-15X)=\mathrm{Var}(-15X+1)$

$=(-15)^2\,\mathrm{Var}(X)$

$=225\times\frac{31}{45}$

$=155$

Question 6 (**)

The probability distribution of a discrete random variable X is given by

$$\mathrm{P}(X = x) = \begin{cases} kx(5-x) & x = 1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

a) Determine the value of the constant k.

b) State the value of $\mathrm{E}(X)$.

c) Calculate $\mathrm{Var}(X)$.

d) Determine the value of $\mathrm{E}(4X-5)$.

$\boxed{\text{FS-R}}$, $\boxed{k = \frac{1}{20}}$, $\boxed{\mathrm{E}(X) = 2.5}$, $\boxed{\mathrm{Var}(X) = 1.05}$, $\boxed{\mathrm{E}(4X-5) = 5}$

a) WRITE THE DISTRIBUTION IN "TABLE" FORM

x	1	2	3	4
$P(X=x)$	$4k$	$6k$	$6k$	$4k$

$4k + 6k + 6k + 4k = 1$

$20k = 1$

$k = \frac{1}{20}$

b) AS THE PROBABILITIES ARE SYMMETRICAL & THE "GAPS" IN x ARE EQUAL, BY SYMMETRY

$E(X) = 2.5$

c) FIND $E(X^2) = \sum x^2 P(X=x)$

$E(X^2) = (1^2 \times 4k) + (2^2 \times 6k) + (3^2 \times 6k) + (4^2 \times 4k)$

$= 4k + 24k + 54k + 64k$

$= 146k$

$= 7.3$

$\mathrm{Var}(X) = E(X^2) - (E(X))^2$

$= 7.3 - (2.5)^2$

$= 1.05$

d) $E(4X-5) = 4E(X) - 5$

$= 4 \times 2.5 - 5$

$= 5$

Question 7 (***)

The probability distribution of a discrete random variable X is given by

x	0	1	2	3
$P(X=x)$	0.5	0.35	a	b

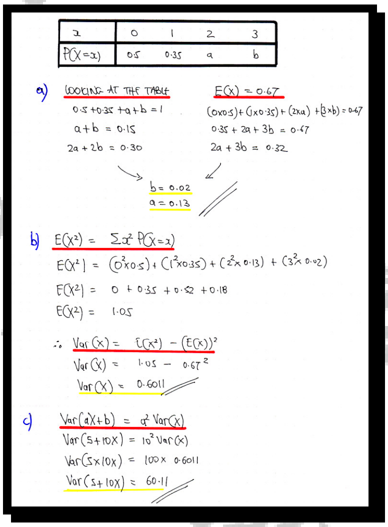
where a and b are positive constants.

a) Given that $E(X)=0.67$, find the value of a and the value of b.

b) Determine the variance of X.

c) Calculate $\text{Var}(5+10X)$.

$\boxed{}$, $\boxed{a=0.13,\ b=0.02}$, $\boxed{\text{Var}(X)=0.6011}$, $\boxed{\text{Var}(5+10X)=60.11}$



x	0	1	2	3
$P(X=x)$	0.5	0.35	a	b

a) LOOKING AT THE TABLE
$0.5+0.35+a+b=1$
$a+b=0.15$
$2a+2b=0.30$

$E(X)=0.67$
$(0\times0.5)+(1\times0.35)+(2\times a)+(3\times b)=0.67$
$0.35+2a+3b=0.67$
$2a+3b=0.32$

$b=0.02$
$a=0.13$

b) $E(X^2)=\sum x^2 P(X=x)$
$E(X^2)=(0^2\times0.5)+(1^2\times0.35)+(2^2\times0.13)+(3^2\times0.02)$
$E(X^2)=0+0.35+0.52+0.18$
$E(X^2)=1.05$

$\therefore\ \text{Var}(X)=E(X^2)-(E(X))^2$
$\text{Var}(X)=1.05-0.67^2$
$\text{Var}(X)=0.6011$

c) $\text{Var}(aX+b)=a^2\,\text{Var}(X)$
$\text{Var}(5+10X)=10^2\,\text{Var}(X)$
$\text{Var}(5+10X)=100\times0.6011$
$\text{Var}(5+10X)=60.11$

Question 8 (***)

The probability distribution of a discrete random variable X is given by

x	1	3	5	7	9
$\mathrm{P}(X=x)$	0.2	a	0.2	b	0.15

where a and b are positive constants.

a) Given that $\mathrm{E}(X)=4.5$, find the value of a and the value of b.

b) Determine $\mathrm{E}(29-6X)$.

$\boxed{a=0.3,\ \ b=0.15}$, $\boxed{\mathrm{E}(29-6X)=2}$

(a)

x	1	3	5	7	9
$P(X=x)$	0.2	a	0.2	b	0.15

• $0.2+a+0.2+b+0.15=1$

$\boxed{a+b=0.45}$

$\boxed{a=0.45-b}$

• $E(X)=4.5$

$(1\times0.2)+(3a)+(5\times0.2)+(7b)+(9\times0.15)=4.5$

$0.2+3a+1+7b+1.35=4.5$

$\boxed{3a+7b=1.95}$

$3(0.45-b)+7b=1.95$

$1.35-3b+7b=1.95$

$4b=0.6$

$b=0.15$

& $a=0.3$

(b) $E(29-6X)=E(-6X+29)$

$=-6E(X)+29=-6\times4.5+29=2$

Question 9 (***)

Two fair spinners, both numbered with 0, 1, 2 and 3, are spun together and the **product** of their scores is recorded.

The discrete random variable X represents the product of the scores of these spinners and its probability distribution is summarized in the table below

x	0	1	2	3	4	6	9
$\mathrm{P}(X=x)$	$\frac{7}{16}$	$\frac{1}{16}$	a	b	c	$\frac{2}{16}$	$\frac{1}{16}$

a) Find the value of a, b and c.

b) Determine $\mathrm{E}(X)$.

c) Find the value of $\mathrm{Var}(X)$.

d) Calculate $\mathrm{E}(4X-1)$.

e) Calculate $\mathrm{Var}(4X-1)$.

$\square$, $\boxed{a=\frac{1}{8},\ \ b=\frac{1}{8},\ \ c=\frac{1}{16}}$, $\boxed{\mathrm{E}(X)=\frac{9}{4}}$, $\boxed{\mathrm{Var}(X)=\frac{115}{16}}$, $\boxed{\mathrm{E}(4X-1)=8}$, $\boxed{\mathrm{Var}(4X-1)=115}$

a)

x	0	1	2	3	4	6	9
$P(X=x)$	$\frac{7}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

"as in a"

$2\text{-}2:\ \frac{1}{4}\times\frac{1}{4}$

$2\text{-}1:\ \frac{1}{4}\times\frac{1}{4}=\frac{1}{16}$

$1\text{-}2:\ \frac{1}{4}\times\frac{1}{4}=\frac{1}{16}$

$a=\frac{1}{16}+\frac{1}{16}=\frac{2}{16}$

$\therefore\ a=\frac{1}{8},\ b=\frac{1}{8},\ c=\frac{1}{16}$

b)

$$E(X)=\sum x\,P(X=x)$$
$$=\left(0\times\tfrac{7}{16}\right)+\left(1\times\tfrac{1}{16}\right)+\dots+\left(9\times\tfrac{1}{16}\right)$$
$$=\tfrac{1}{16}+\tfrac{4}{16}+\tfrac{6}{16}+\tfrac{4}{16}+\tfrac{12}{16}+\tfrac{9}{16}$$
$$=\tfrac{36}{16}=\tfrac{9}{4}=2.25$$

c)

$$E(X^2)=\sum x^2\,P(X=x)$$
$$=\left(0^2\times\tfrac{7}{16}\right)+\left(1^2\times\tfrac{1}{16}\right)+\dots+\left(9^2\times\tfrac{1}{16}\right)$$
$$=\tfrac{1}{16}+\tfrac{8}{16}+\tfrac{18}{16}+\tfrac{16}{16}+\tfrac{72}{16}+\tfrac{81}{16}$$
$$=\tfrac{196}{16}=\tfrac{49}{4}$$

$$\therefore\ \mathrm{Var}(X)=E(X^2)-\left(E(X)\right)^2$$
$$=\tfrac{49}{4}-\left(\tfrac{9}{4}\right)^2$$
$$=\tfrac{115}{16}$$

d)

$$E(4X-1)=4E(X)-1$$
$$=4\times\tfrac{9}{4}-1$$
$$=8$$

e)

$$\mathrm{Var}(4X-1)=4^2\,\mathrm{Var}(X)$$
$$=16\times\tfrac{115}{16}$$
$$=115$$

Question 10 (***)

The probability distribution of a discrete random variable X is summarised in the table below.

x	1	2	3	4	5
$\mathrm{P}(X = x)$	0.2	0.1	0.2	0.25	0.25

a) Find the value of …

i. … $\mathrm{E}(X)$.

ii. … $\mathrm{E}(X^2)$.

iii. … $\mathrm{Var}(X)$.

b) Calculate …

i. … $\mathrm{E}(3-X)$.

ii. … $\mathrm{Var}(3-X)$

c) Determine the value of

$$\mathrm{P}\left[4X-3 \geq 2(X+1)\right].$$

$\boxed{}$, $\boxed{\mathrm{E}(X)=3.25}$, $\boxed{\mathrm{E}(X^2)=12.65}$, $\boxed{\mathrm{Var}(X)=2.0875}$, $\boxed{\mathrm{E}(3-X)=-0.25}$, $\boxed{\mathrm{Var}(3-X)=2.0875}$, $\boxed{\mathrm{P}\left[4X-3 \geq 2(X+1)\right]=0.7}$

x	1	2	3	4	5
$P(X=x)$	0.2	0.1	0.2	0.25	0.25

a)I) $E(X) = (1\times0.2)+(2\times0.1)+(3\times0.2)+(4\times0.25)+(5\times0.25)$
$= 0.2 + 0.2 + 0.6 + 1 + 1.25$
$= 3.25$

a)II) $E(X^2) = (1^2\times0.2)+(2^2\times0.1)+(3^2\times0.2)+(4^2\times0.25)+(5^2\times0.25)$
$= 0.2 + 0.4 + 1.8 + 4 + 6.25$
$= 12.65$

a)III) $\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 12.65 - 3.25^2 = 2.0875$

b)I) $E(3-X) = E(-X+3) = -1\times E(X) + 3 = -3.25+3$
$= -0.25$

b)II) $\mathrm{Var}(3-X) = \mathrm{Var}(-X+3) = (-1)^2\mathrm{Var}(X) = \mathrm{Var}(X) = 2.0875$

c) $P[4X-3 \geqslant 2(X+1)] = P(4X-3 \geqslant 2X+2)$
$= P(2X \geqslant 5)$
$= P(X \geqslant \frac{5}{2})$
$= P(X = 3,4,5)$
$= 0.2 + 0.25 + 0.25$
$= 0.7$

Question 11 (***)

The cumulative distribution $\mathrm{F}(x)$, of a discrete random variable X is given by

x	1	2	3	4	5	6	7	8
$\mathrm{F}(x)$	0.25	0.40	0.55	0.65	0.75	0.85	0.95	1

a) Find the value …

i. … $\mathrm{E}(X)$.

ii. … $\mathrm{Var}(X)$.

The discrete random variable Y is defined as $Y = 5X - 3$.

b) Determine the mean and variance of Y.

$\boxed{}$, $\boxed{\mathrm{E}(X)=3.6}$, $\boxed{\mathrm{Var}(X)=5.04}$, $\boxed{\mathrm{E}(Y)=15}$, $\boxed{\mathrm{Var}(Y)=126}$

a)

x	1	2	3	4	5	6	7	8
$F(x)$	0.25	0.40	0.55	0.65	0.75	0.85	0.95	1
$P(X=x)$	0.25	0.15	0.15	0.10	0.10	0.10	0.10	0.05

I) $E(X) = \sum x P(X=x)$

$= (1\times0.25)+(2\times0.15)+(3\times0.15)+\dots+(8\times0.05)$

$= 0.25+0.3+0.45+0.4+0.5+0.6+0.7+0.4$

$= 3.6$

II) $E(X^2) = \sum x^2 P(X=x)$

$= (1^2\times0.25)+(2^2\times0.15)+(3^2\times0.15)+\dots+(8^2\times0.05)$

$= 0.25+0.6+1.35+1.6+2.5+3.6+4.9+3.2$

$= 18$

$\mathrm{Var}(X) = E(X^2)-(E(X))^2$

$= 18-3.6^2$

$= 18-12.96$

$= 5.04$

b) • $E(Y) = E(5X-3) = 5E(X)-3 = 5\times3.6-3 = 15$

• $\mathrm{Var}(Y) = \mathrm{Var}(5X-3) = 5^2\times\mathrm{Var}(X) = 25\times5.04 = 126$

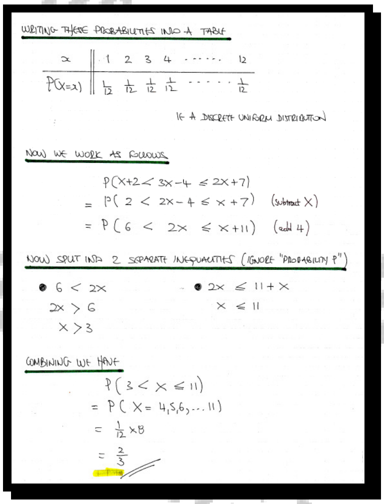
Question 12 (***)

The probability distribution of a discrete random variable X is given by

$$\mathrm{P}(X = x) = \begin{cases} \dfrac{1}{12} & x = 1, 2, 3, \ldots, 12 \\ 0 & \text{otherwise} \end{cases}$$

Determine $\mathrm{P}(X + 2 < 3X - 4 \le 2X + 7)$.

$\boxed{}$, $\boxed{\mathrm{P}(X + 2 < 3X - 4 \le 2X + 7) = \dfrac{2}{3}}$



WRITING THESE PROBABILITIES INTO A TABLE

x	1	2	3	4	......	12
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	......	$\frac{1}{12}$

IE A DISCRETE UNIFORM DISTRIBUTION

NOW WE WORK AS FOLLOWS

$$P(X+2 < 3X-4 \le 2X+7)$$
$$= P(2 < 2X-4 \le X+7) \quad \text{(subtract } X\text{)}$$
$$= P(6 < 2X \le X+11) \quad \text{(add 4)}$$

NOW SPLIT INTO 2 SEPARATE INEQUALITIES (IGNORE "PROBABILITY P")

- $6 < 2X$, $2X > 6$, $X > 3$
- $2X \le 11 + X$, $X \le 11$

COMBINING WE HAVE

$$P(3 < X \le 11)$$
$$= P(X = 4, 5, 6, \ldots 11)$$
$$= \tfrac{1}{12} \times 8$$
$$= \tfrac{2}{3}$$

Question 13 (***+)

A sixth form class consists of 6 boys and 4 girls.

Three students are selected at random from this class and the variable X represents the number of girls selected.

Show that the probability distribution of X is given by

x	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$

[illegible], proof

WORK WITH OR WITHOUT A TREE DIAGRAM

- $P(X=0) = P(\text{no girl}) = P(B,B,B) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{120}{720}$
- $P(X=3) = P(\text{all girls}) = P(G,G,G) = \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{24}{720}$
- $P(X=1) = P(\text{1 girl \& 2 boys})$

$$= P(G,B,B) + P(B,G,B) + P(B,B,G)$$
$$= \left(\frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}\right) + \left(\frac{6}{10} \times \frac{4}{9} \times \frac{5}{8}\right) + \left(\frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}\right)$$
$$= \frac{120}{720} \times 3$$
$$= \frac{360}{720}$$

- $P(X=2) = 1 - P(X=0,1,3) = 1 - \left(\frac{120}{720} + \frac{24}{720} + \frac{360}{720}\right) = \frac{216}{720}$

HENCE WE OBTAIN

x	0	1	2	3
$P(X=x)$	$\frac{120}{720}$	$\frac{360}{720}$	$\frac{216}{720}$	$\frac{24}{720}$
	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$
	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

$\div 24$
$\div 24$

OR IN ITS SIMPLEST FORM

Question 14 (***+)

The cumulative distribution of a discrete random variable X is given by

x	1	2	4	5
$\mathrm{F}(x)$	$\frac{3}{20}$	$\frac{2k+3}{20}$	$\frac{k+5}{10}$	$\frac{k+2}{4}$

where k is a positive constant.

a) Show clearly that $k = 2$.

b) Find the value of …

i. … $\mathrm{E}(X)$.

ii. … $\mathrm{E}(X^2)$.

c) Calculate $\mathrm{Var}(20X - 2)$.

$\boxed{}$, $\boxed{\mathrm{E}(X) = 3.45}$, $\boxed{\mathrm{E}(X^2) = 14.05}$, $\boxed{\mathrm{Var}(20X - 2) = 859}$

a)

x	1	2	4	5
$F(x)$	$\frac{3}{20}$	$\frac{2k+3}{20}$	$\frac{k+5}{10}$	$\frac{k+2}{4}$
$P(X=x)$	$\frac{3}{20}$	$\left(\frac{1}{5}\right)$	$\left(\frac{7}{20}\right)$	$\left(\frac{3}{10}\right)$

$F(5) = P(X \leqslant 5) = 1$

$\frac{k+2}{4} = 1$

$k+2 = 4$

$k = 2$

b) FILL IN THE ABOVE TABLE (IN TERMS OF k, OR WITH NUMBERS)

- $\frac{2k+3}{20} - \frac{3}{20} = \frac{7}{20} - \frac{3}{20} = \frac{4}{20} = \frac{1}{5}$
- $\frac{k+5}{10} - \frac{2k+3}{20} = \frac{7}{10} - \frac{7}{20} = \frac{7}{20}$
- $\frac{k+2}{4} - \frac{k+5}{10} = 1 - \frac{7}{10} = \frac{3}{10}$

I) $E(X) = \sum x P(X=x)$

$E(X) = \left(1 \times \frac{3}{20}\right) + \left(2 \times \frac{1}{5}\right) + \left(4 \times \frac{7}{20}\right) + \left(5 \times \frac{3}{10}\right)$

$E(X) = \frac{3}{20} + \frac{2}{5} + \frac{7}{5} + \frac{3}{2}$

$E(X) = 3.45$

II) $E(X^2) = \sum x^2 P(X=x)$

$E(X^2) = \left(1^2 \times \frac{3}{20}\right) + \left(2^2 \times \frac{1}{5}\right) + \left(4^2 \times \frac{7}{20}\right) + \left(5^2 \times \frac{3}{10}\right)$

$= \frac{3}{20} + \frac{4}{5} + \frac{28}{5} + \frac{15}{2}$

$= 14.05$

c) FIND THE VARIANCE FIRST

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$\mathrm{Var}(X) = 14.05 - 3.45^2$

$\mathrm{Var}(X) = 2.1475$

HENCE USING $\mathrm{Var}(aX+b) = a^2 \mathrm{Var}(X)$

$\mathrm{Var}(20X-2) = 20^2 \mathrm{Var}(X)$

$= 400 \times 2.1475$

$= 859$

Question 15 (***+)

A biased six sided die has the following probability distribution

x	1	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$

where the random variable X represents the number shown on its uppermost face when it comes to rest after it is rolled.

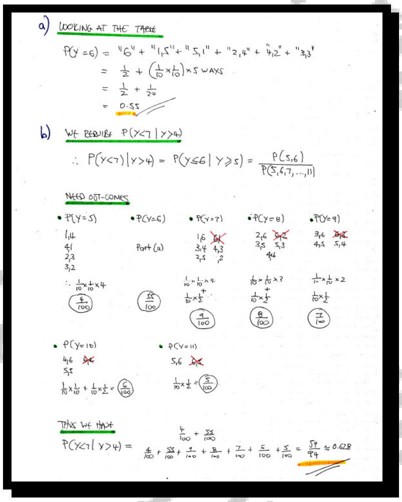
The die is rolled twice and the two independent observations of X, X_1 and X_2, produce the score Y defined as

$$Y = \begin{cases} 6 & \text{if } X_1 = 6 \\ X_1 + X_2 & \text{if } X_1 \neq 6 \end{cases}$$

a) Find the value of $\mathrm{P}(Y=6)$.

b) Find the value of $\mathrm{P}(Y<7 \mid Y>4)$.

$\boxed{}$, $\boxed{\mathrm{P}(Y=6)=0.55}$, $\boxed{\mathrm{P}(Y<7 \mid Y>4)=\frac{59}{94}}$



a) LOOKING AT THE TABLE

$\mathrm{P}(Y=6) = \text{"6"} + \text{"1,5"} + \text{"5,1"} + \text{"2,4"} + \text{"4,2"} + \text{"3,3"}$
$= \frac{1}{2} + \left(\frac{1}{10}\times\frac{1}{10}\right)\times 5 \text{ WAYS}$
$= \frac{1}{2} + \frac{1}{20}$
$= 0.55$

b) WE REQUIRE $\mathrm{P}(Y<7 \mid Y>4)$

$\therefore \mathrm{P}(Y<7 \mid Y>4) = \mathrm{P}(Y\leq 6 \mid Y \geq 5) = \frac{\mathrm{P}(5,6)}{\mathrm{P}(5,6,7,\dots,11)}$

NEED OUT-COMES

- $\mathrm{P}(Y=5)$: 1,4 4,1 2,3 3,2 $\therefore \frac{1}{10}\times\frac{1}{10}\times 4$ $\frac{4}{100}$
- $\mathrm{P}(Y=6)$: Part (a) $\frac{55}{100}$
- $\mathrm{P}(Y=7)$: 1,6 ~~6,1~~ 3,4 4,3 2,5 5,2 $\frac{1}{10}\times\frac{1}{10}\times 4 + \frac{1}{10}\times\frac{1}{2}$ $\frac{9}{100}$
- $\mathrm{P}(Y=8)$: 2,6 ~~6,2~~ 3,5 5,3 4,4 $\frac{1}{10}\times\frac{1}{10}\times 3 + \frac{1}{10}\times\frac{1}{2}$ $\frac{8}{100}$
- $\mathrm{P}(Y=9)$: 3,6 ~~6,3~~ 4,5 5,4 $\frac{1}{10}\times\frac{1}{10}\times 2 + \frac{1}{10}\times\frac{1}{2}$ $\frac{7}{100}$
- $\mathrm{P}(Y=10)$: 4,6 ~~6,4~~ 5,5 $\frac{1}{10}\times\frac{1}{10} + \frac{1}{10}\times\frac{1}{2} = \frac{6}{100}$
- $\mathrm{P}(Y=11)$: 5,6 ~~6,5~~ $\frac{1}{10}\times\frac{1}{2} = \frac{5}{100}$

THUS WE HAVE

$$\mathrm{P}(Y<7 \mid Y>4) = \frac{\frac{4}{100}+\frac{55}{100}}{\frac{4}{100}+\frac{55}{100}+\frac{9}{100}+\frac{8}{100}+\frac{7}{100}+\frac{6}{100}+\frac{5}{100}} = \frac{59}{94} \approx 0.628$$

Question 16 (***+)

The probability distribution of a discrete random variable X is given by

x	0	1	2	3	4	5	6
$\mathrm{P}(X=x)$	0.05	0.1	0.15	0.2	0.25	0.2	0.05

a) Find the value of $\mathrm{E}(X)$.

b) Calculate $\mathrm{Var}(X)$.

c) Determine $\mathrm{P}(\mu-\sigma \le X \le \mu+\sigma)$.

d) Find the value of $\mathrm{E}\left(4X^2-3.2\right)$.

$\boxed{}$, $\boxed{\mathrm{E}(X)=3.3}$, $\boxed{\mathrm{Var}(X)=2.41}$, $\boxed{\mathrm{P}(\mu-\sigma \le X \le \mu+\sigma)=0.6}$, $\boxed{\mathrm{E}\left(4X^2-3.2\right)=50}$

a)

x	0	1	2	3	4	5	6
$P(X=x)$	0.05	0.1	0.15	0.2	0.25	0.2	0.05

$E(X) = \sum x P(X=x)$
$= (0\times0.05)+(1\times0.1)+(2\times0.15)+\dots+(6\times0.05)$
$= 0+0.1+0.3+0.6+1+1+0.3$
$= 3.3$

b) $E(X^2) = \sum x^2 P(X=x)$
$= (0^2\times0.05)+(1^2\times0.1)+(2^2\times0.15)+\dots+(6^2\times0.05)$
$= 0+0.1+0.6+1.8+4+5+1.8$
$= 13.3$

$\mathrm{Var}(X) = E(X^2)-[E(X)]^2$
$= 13.3 - 3.3^2$
$= 13.3 - 10.89$
$= 2.41$

c) CALCULATE THE BOUNDS

$\mu = E(X) = 3.3$
$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{2.41} \approx 1.55$

$\therefore P(\mu-\sigma < X < \mu+\sigma) = P(1.75 < X < 4.85)$
$= P(X=2,3,4)$
$= P(X=2)+P(X=3)+P(X=4)$
$= 0.15+0.2+0.25$
$= 0.6$

d) $E(aX+b) \equiv aE(X)+b$
$E(aX^2+b) \equiv aE(X^2)+b$
$E(4X^2-3.2) \equiv 4E(X^2)-3.2$
$E(4X^2-3.2) = 4\times13.3-3.2$
$E(4X^2-3.2) = 50$

Question 17 (***+)

A box contains three blue discs and two red discs.

Three discs are selected at random from the box without replacement.

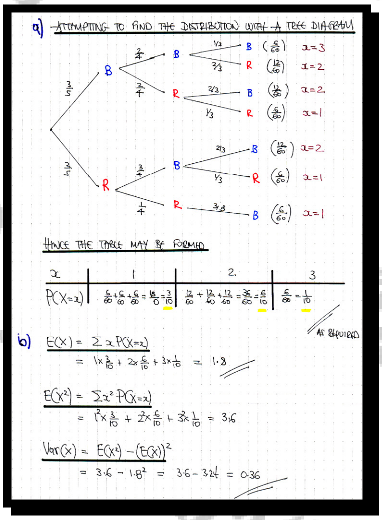
The variable X represents the number of blue discs selected.

a) Show that the probability distribution of X is given by

x	1	2	3
$\mathrm{P}(X=x)$	$\frac{3}{10}$	$\frac{6}{10}$	$\frac{1}{10}$

b) Determine $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

$\boxed{}$, $\boxed{\mathrm{E}(X)=\frac{9}{5}=1.8}$, $\boxed{\mathrm{Var}(X)=\frac{9}{25}=0.36}$



Question 18 (***+)

A sixth form class consists of 6 boys and 4 girls.

Three students are selected at random from this class and the variable X represents the number of girls selected.

a) Show that the probability distribution of X is given by

x	0	1	2	3
$\text{P}(X = x)$	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$

b) Determine $\text{E}(X)$ and $\text{Var}(X)$.

$\square$, $\boxed{\text{E}(X) = \frac{6}{5} = 1.2}$, $\boxed{\text{Var}(X) = \frac{14}{25} = 0.56}$

a) X = NUMBER OF GIRLS SELECTED

- $P(X=0) = \text{"BOY"} \times \text{"BOY"} \times \text{"BOY"}$
  $= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}$
  $= \frac{1}{6} = \frac{5}{30}$

- $P(X=3) = \text{"GIRL"} \times \text{"GIRL"} \times \text{"GIRL"}$
  $= \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8}$
  $= \frac{1}{30}$

- $P(X=1) = \left.\begin{array}{l} \text{"BOY"} \times \text{"BOY"} \times \text{"GIRL"} = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{1}{6} \\ \text{"BOY"} \times \text{"GIRL"} \times \text{"BOY"} = \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} = \frac{1}{6} \\ \text{"GIRL"} \times \text{"BOY"} \times \text{"BOY"} = \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} = \frac{1}{6} \end{array}\right\} \frac{1}{2} = \frac{15}{30}$

- $P(X=2) = 1 - \frac{5}{30} - \frac{1}{30} - \frac{15}{30} = \frac{9}{30}$

$\therefore$

x	0	1	2	3
$P(X=x)$	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$

b) $E(X) = \sum x P(X=x)$
$= 0 \times \frac{5}{30} + 1 \times \frac{15}{30} + 2 \times \frac{9}{30} + 3 \times \frac{1}{30} = 0 + \frac{15}{30} + \frac{18}{30} + \frac{3}{30} = \frac{6}{5}$

$E(X^2) = \sum x^2 P(X=x)$
$= 0^2 \times \frac{5}{30} + 1^2 \times \frac{15}{30} + 2^2 \times \frac{9}{30} + 3^2 \times \frac{1}{30} = 0 + \frac{15}{30} + \frac{36}{30} + \frac{9}{30} = \frac{60}{30} = 2$

$\text{Var}(X) = E(X^2) - (E(X))^2$
$= 2 - \left(\frac{6}{5}\right)^2 = \frac{50}{25} - \frac{36}{25} = \frac{14}{25}$

Question 19 (***+)

The probability distribution of the discrete random variable X is given by

x	2	3	4
$P(X = x)$	$0.4 - a$	$2a$	$0.6 - a$

where a is a constant.

a) State the range of the possible values of a.

Two independent observations of X, denoted by X_1 and X_2, are considered.

b) Determine, in terms of a, a simplified expression for $P(X_1 + X_2 = 6)$.

$\boxed{}$, $\boxed{0 \le a \le 0.4}$, $\boxed{P(X_1 + X_2 = 6) = 6a^2 - 2a + 0.48}$

a) LOOKING AT THE TABLE, THE PROBABILITIES MUST BE POSITIVE OR ZERO

$\therefore\ 0.4 - a \geqslant 0 \qquad 2a \geqslant 0 \qquad 0.6 - a \geqslant 0$

$a \leqslant 0.4 \qquad a \geqslant 0 \qquad a \leqslant 0.6$

$\therefore\ 0 \leqslant a \leqslant 0.4$

b) COLLECTING ALL THE OUTCOMES FOR $X_1 + X_2 = 6$

$$\left.\begin{array}{lll} 2,4 \Rightarrow (0.4-a)(0.6-a) & = & 0.24 - a + a^2 \\ 4,2 \Rightarrow (0.6-a)(0.4-a) & = & 0.24 - a + a^2 \\ 3,3 \rightarrow 2a \times 2a & = & 4a^2 \end{array}\right\} \text{ADDING}$$

$\therefore\ P(X_1 + X_2 = 6) = 6a^2 - 2a + 0.48$

Question 20 (***+)

Two standard fair cubical dice, numbered 1 to 6 are such rolled and the random variable X represents the sum of the scores of the two dice.

Determine the value of $\text{Var}(X)$.

$\boxed{}$, $\boxed{\text{Var}(X) = \frac{35}{6}}$

- DETERMINE THE PROBABILITY SPACE DIAGRAM

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

- THE PROBABILITY DISTRIBUTION OF X IS GIVEN BY

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

- BY SYMMETRY $E(X) = 7$

- DETERMINE THE $E(X^2)$

$$E(X^2) = \left(2^2 \times \frac{1}{36}\right) + \left(3^2 \times \frac{2}{36}\right) + \left(4^2 \times \frac{3}{36}\right) + \cdots + \left(12^2 \times \frac{1}{36}\right)$$
$$= \frac{(2^2 \times 1) + (3^2 \times 2) + (4^2 \times 3) + \cdots + (12^2 \times 1)}{36}$$
$$= \frac{1974}{36} = \frac{329}{6}$$

- FINALLY THE VARIANCE

$$\text{Var}(X) = E(X^2) - (E(X))^2$$
$$= \frac{329}{6} - 7^2 = \frac{35}{6}$$

Question 21 **(***+)**

The discrete random variable X has the following probability distribution

x	0	2	3
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

a) Determine $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

A game in a fun fair consists of throwing 5 darts on a small target.

If a dart lands on the central portion of the target the dart scores 3 points.

If a dart lands on the outer portion of the target the dart scores 2 points, otherwise the dart scores no points.

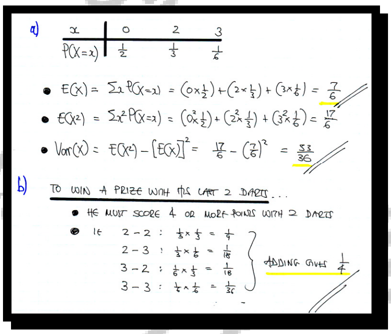
To win a prize, 10 or more points must be scored with 5 darts.

Paul has scored 6 points with his first 3 darts.

The likelihood of Paul scoring 0, 2 or 3 points is given by the probability distribution of part **(a)**.

b) Find the probability that Paul will win a prize after he throws his last 2 darts.

$\boxed{}$, $\boxed{\mathrm{E}(X) = \frac{7}{6}}$, $\boxed{\mathrm{Var}(X) = \frac{53}{36}}$, $\boxed{\frac{1}{4}}$



a)

x	0	2	3
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

- $E(X) = \sum x P(X=x) = \left(0\times\frac{1}{2}\right)+\left(2\times\frac{1}{3}\right)+\left(3\times\frac{1}{6}\right) = \frac{7}{6}$
- $E(X^2) = \sum x^2 P(X=x) = \left(0^2\times\frac{1}{2}\right)+\left(2^2\times\frac{1}{3}\right)+\left(3^2\times\frac{1}{6}\right) = \frac{17}{6}$
- $\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \frac{17}{6} - \left(\frac{7}{6}\right)^2 = \frac{53}{36}$

b) TO WIN A PRIZE WITH HIS LAST 2 DARTS...

- HE MUST SCORE 4 OR MORE POINTS WITH 2 DARTS
- IE

$$\left.\begin{array}{l} 2-2:\ \frac{1}{3}\times\frac{1}{3} = \frac{1}{9} \\ 2-3:\ \frac{1}{3}\times\frac{1}{6} = \frac{1}{18} \\ 3-2:\ \frac{1}{6}\times\frac{1}{3} = \frac{1}{18} \\ 3-3:\ \frac{1}{6}\times\frac{1}{6} = \frac{1}{36} \end{array}\right\}\ \text{ADDING GIVES } \frac{1}{4}$$

Question 22 (***+)

The probability distribution of a discrete random variable X is given by

$$\mathrm{P}(X=x)=\begin{cases} k(4-x) & x=0,1,2,3 \\ \frac{1}{2} & x=4 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $k=\frac{1}{20}$.

Two independent observations of X are made, denoted by X_1 and X_2.

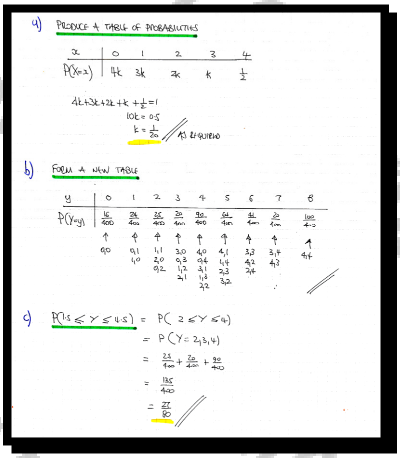
b) Find the probability distribution of Y, where $Y=X_1+X_2$.

c) Calculate $P(1.5 \le Y \le 4.5)$.

$\boxed{}$,

y	0	1	2	3	4	5	6	7	8
$\mathrm{P}(Y=y)$	$\frac{16}{400}$	$\frac{24}{400}$	$\frac{25}{400}$	$\frac{20}{400}$	$\frac{90}{400}$	$\frac{64}{400}$	$\frac{41}{400}$	$\frac{20}{400}$	$\frac{100}{400}$

$$\boxed{P(1.5 \le Y \le 4.5)=\frac{27}{80}}$$



a) PRODUCE A TABLE OF PROBABILITIES

x	0	1	2	3	4
$P(X=x)$	$4k$	$3k$	$2k$	k	$\frac{1}{2}$

$4k+3k+2k+k+\frac{1}{2}=1$

$10k=0.5$

$k=\frac{1}{20}$ // AS REQUIRED

b) FORM A NEW TABLE

y	0	1	2	3	4	5	6	7	8
$P(Y=y)$	$\frac{16}{400}$	$\frac{24}{400}$	$\frac{25}{400}$	$\frac{20}{400}$	$\frac{90}{400}$	$\frac{64}{400}$	$\frac{41}{400}$	$\frac{20}{400}$	$\frac{100}{400}$
	0,0	0,1 1,0	1,1 2,0 0,2	3,0 0,3 1,2 2,1	4,0 0,4 3,1 1,3 2,2	4,1 1,4 2,3 3,2	3,3 4,2 2,4	3,4 4,3	4,4

c) $P(1.5 \le Y \le 4.5) = P(2 \le Y \le 4)$

$= P(Y=2,3,4)$

$= \frac{25}{400}+\frac{20}{400}+\frac{90}{400}$

$= \frac{135}{400}$

$= \frac{27}{80}$ //

Question 23 (***+)

The probability distribution of a discrete random variable X is given by

x	1	2	3	4
$\mathrm{P}(X=x)$	a	b	b	c

where a, b and c are constants.

The cumulative distribution function of X is given by

x	1	2	3	4
$F(x)$	$\frac{1}{6}$	d	$\frac{2}{3}$	e

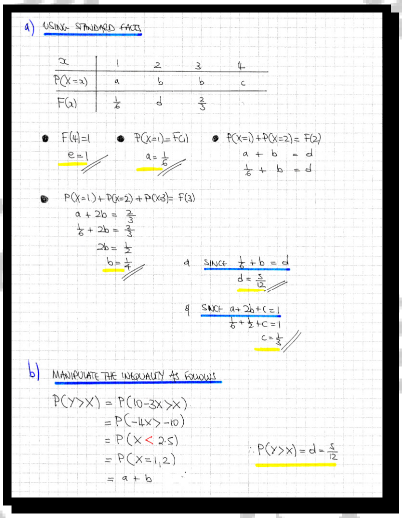
where d and e are constants.

a) Determine the value of each of the constants a, b, c, d and e.

The discrete random variable Y is defined as $Y = 10 - 3X$.

b) Find the value of $P(Y > X)$.

$\boxed{}$, $\boxed{(a,b,c,d,e) = \left(\frac{1}{6},\frac{1}{4},\frac{1}{3},\frac{5}{12},1\right)}$, $\boxed{P(Y > X) = \frac{5}{12}}$



a) USING STANDARD FACTS

x	1	2	3	4
$P(X=x)$	a	b	b	c
$F(x)$	$\frac{1}{6}$	d	$\frac{2}{3}$	

- $F(4)=1$
  $e=1$
- $P(X=1)=F(1)$
  $a=\frac{1}{6}$
- $P(X=1)+P(X=2)=F(2)$
  $a+b=d$
  $\frac{1}{6}+b=d$
- $P(X=1)+P(X=2)+P(X=3)=F(3)$
  $a+2b=\frac{2}{3}$
  $\frac{1}{6}+2b=\frac{2}{3}$
  $2b=\frac{1}{2}$
  $b=\frac{1}{4}$

& SINCE $\frac{1}{6}+b=d$
$d=\frac{5}{12}$

& SINCE $a+2b+c=1$
$\frac{1}{6}+\frac{1}{2}+c=1$
$c=\frac{1}{3}$

b) MANIPULATE THE INEQUALITY AS FOLLOWS

$P(Y>X)=P(10-3X>X)$
$=P(-4X>-10)$
$=P(X<2.5)$
$=P(X=1,2)$
$=a+b$

$\therefore P(Y>X)=d=\frac{5}{12}$

Question 24 **(***+)**

The discrete random variable X has the following probability distribution

x	0	1	3
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

Three independent observations of X are made, denoted by X_1, X_2 and X_3.

Calculate $P(X_1 + X_2 + X_3 \geq 4)$.

$$\boxed{P(X_1 + X_2 + X_3 \geq 4) = \frac{5}{8}}$$

COLLECTING OUTCOMES GREATER OR EQUAL TO 4

0, 1, 3 GIVES 4 : $\frac{1}{6} \times \frac{1}{3} \times \frac{1}{2} \times 3$ WAYS $= \frac{1}{12}$

1, 1, 3 GIVES 5 : $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{2} \times 3$ WAYS $= \frac{1}{24}$

1, 3, 3 GIVES 7 : $\frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} \times 3$ WAYS $= \frac{1}{4}$

0, 3, 3 GIVES 6 : $\frac{1}{6} \times \frac{1}{2} \times \frac{1}{2} \times 3$ WAYS $= \frac{1}{8}$

3, 3, 3 GIVES 9 : $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

$\therefore P(X_1 + X_2 + X_3 \geq 4) = \frac{5}{8} = 0.625$

Question 25 (****)

The discrete random variable X has the following probability distribution

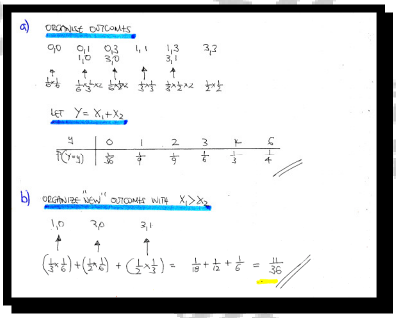
x	0	1	3
$\mathrm{P}(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

Two independent observations of X are made, denoted by X_1 and X_2.

a) Find the probability distribution of $X_1 + X_2$.

b) Calculate $P(X_1 > X_2)$.

$$\mathrm{P}(X_1 + X_2 = r) = \begin{cases} \frac{1}{36} & r=0 \\ \frac{1}{9} & r=1,2 \\ \frac{1}{4} & r=6 \\ \frac{1}{3} & r=4 \\ 0 & \text{otherwise} \end{cases}, \quad \boxed{P(X_1 > X_2) = \frac{11}{36}}$$



a) ORGANISE OUTCOMES

0,0   0,1 1,0   0,3 3,0   1,1   1,3 3,1   3,3

$\frac{1}{6}\times\frac{1}{6}$ $\frac{1}{6}\times\frac{1}{3}\times2$ $\frac{1}{6}\times\frac{1}{2}\times2$ $\frac{1}{3}\times\frac{1}{3}$ $\frac{1}{3}\times\frac{1}{2}\times2$ $\frac{1}{2}\times\frac{1}{2}$

LET $Y = X_1 + X_2$

y	0	1	2	3	4	6
$P(Y=y)$	$\frac{1}{36}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{4}$

b) ORGANIZE "NEW" OUTCOMES WITH $X_1 > X_2$

1,0   3,0   3,1

$\left(\frac{1}{3}\times\frac{1}{6}\right)+\left(\frac{1}{2}\times\frac{1}{6}\right)+\left(\frac{1}{2}\times\frac{1}{3}\right) = \frac{1}{18}+\frac{1}{12}+\frac{1}{6} = \frac{11}{36}$

Question 26 **(****)**

The probability distribution of a discrete random variable X is given by

$$\mathrm{P}(X=x)=\begin{cases} k(2-x) & x=0,1,2 \\ \dfrac{1}{4} & x=3 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $k=\frac{1}{4}$.

b) Find the value of $\mathrm{E}(X)$ and $\mathrm{E}\left(X^2\right)$.

c) Determine $\mathrm{Var}(3-X)$.

Two independent observations of X are made, denoted by X_1 and X_2.

d) Find the probability distribution of Y, where $Y=X_1+X_2$.

e) Calculate $P(1.5 \le Y \le 3.5)$.

$\boxed{\text{TS}}$, $\boxed{\mathrm{E}(X)=1}$, $\boxed{\mathrm{E}\left(X^2\right)=2.5}$, $\boxed{\mathrm{Var}(3-X)=1.5}$, $\boxed{\mathrm{P}(Y=y)=\begin{cases} \frac{1}{16} & y=2,6 \\ \frac{1}{8} & y=4 \\ \frac{1}{4} & y=0,1,3 \\ 0 & \text{otherwise} \end{cases}}$,

$\boxed{P(1.5 \le Y \le 3.5)=\frac{5}{16}}$

a) GENERATING A TABLE FOR PROBABILITIES

x	0	1	2	3
$P(X=x)$	$2k$	k	0	$\frac{1}{4}$

$2k+k+\frac{1}{4}=1$

$3k=\frac{3}{4}$

$k=\frac{1}{4}$

b) $E(X)=\sum x\,P(X=x)$

$=(0\times 2k)+(1\times k)+(2\times 0)+(3\times\frac{1}{4})$

$=0+k+0+\frac{3}{4}$

$=1$

$E(X^2)=\sum x^2\,P(X=x)$

$=(0^2\times 2k)+(1^2\times k)+(2^2\times 0)+(3^2\times\frac{1}{4})$

$=0+k+0+\frac{9}{4}$

$=\frac{5}{2}$

c) $\mathrm{Var}(3-X)=\mathrm{Var}(-X+3)=(-1)^2\,\mathrm{Var}(X)$

$=\mathrm{Var}(X)$

$=E(X^2)-(E(X))^2$

$=2.5-1^2=1.5$

d) If $Y=X_1+X_2$

x	0	1	3
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

y	0	1	2	3	4	6
$P(Y=y)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
	$(\frac{1}{2}\times\frac{1}{2})$	$(\frac{1}{2}\times\frac{1}{4})$ $\times 2$ WAYS	$(\frac{1}{4}\times\frac{1}{4})$	$(\frac{1}{2}\times\frac{1}{4})$ $\times 2$ WAYS	$(\frac{1}{4}\times\frac{1}{4})$ $\times 2$ WAYS	$(\frac{1}{4}\times\frac{1}{4})$
	0–0	1–0, 0–1	1–1	3–0, 0–3	3–1, 1–3	3–3

e) $P(1.5<Y\le 3.5)=P(Y=2,3)$

$=\frac{1}{16}+\frac{1}{4}$

$=\frac{5}{16}$

Question 27 (****)

The probability distribution of a discrete random variable X is given by

x	1	2	3
$\mathrm{P}(X=x)$	$0.3-k$	$2k$	$0.7-k$

a) Find the range of possible values of the constant k.

b) Determine $\mathrm{E}(X)$.

c) Given that $\mathrm{Var}(X)=0.72$, find the value of k.

X_1 and X_2 are two independent observations of X.

d) Find $P(X_1 = X_2)$.

$\boxed{\quad}$, $\boxed{0 \le k \le 0.3}$, $\boxed{\mathrm{E}(X)=2.4}$, $\boxed{k=0.06}$, $\boxed{0.4816}$

a)

x	1	2	3
$P(X=x)$	$0.3-k$	$2k$	$0.7-k$

- $0.3-k \geqslant 0$; $-k \geqslant -0.3$; $k \leqslant 0.3$
- $2k \geqslant 0$; $k \geqslant 0$
- $0.7-k \geqslant 0$; $-k \geqslant -0.7$; $(k \leqslant 0.7)$

$\therefore \; 0 \leqslant k \leqslant 0.3$

b) $E(X) = \sum x P(X=x)$

$\Rightarrow E(X) = 1\times(0.3-k) + 2\times 2k + 3\times(0.7-k)$

$\Rightarrow E(X) = 0.3 \cancel{-k} + \cancel{4k} + 2.1 \cancel{-3k}$

$\Rightarrow E(X) = 2.4$

c) START WITH $E(X^2) = \sum x^2 P(X=x)$

$\Rightarrow E(X^2) = 1^2\times(0.3-k) + 2^2\times 2k + 3^2\times(0.7-k)$

$\Rightarrow E(X^2) = 0.3-k+8k+9\times(0.7-k)$

$\Rightarrow E(X^2) = 0.3-k+8k+6.3-9k$

$\Rightarrow E(X^2) = -2k+6.6$

$\Rightarrow \mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$\Rightarrow 0.72 = -2k+6.6-2.4^2$

$\Rightarrow 2k = 0.12$

$\Rightarrow k = 0.06$

d) $P(X_1=X_2) = P(1,1) + P(2,2) + P(3,3)$

$= (0.3-0.06)^2 + (2\times 0.06)^2 + (0.7-0.06)^2$

$= 0.24^2 + 0.12^2 + 0.64^2$

$= 0.0576 + 0.0144 + 0.4096$

$= 0.4816$

Question 28 (****)

The probability distribution of the discrete random variable X is given by

x	2	3	4
$\mathrm{P}(X = x)$	$0.4 - a$	$2a$	$0.6 - a$

where a is a constant.

a) State the range of the possible values of a.

b) Show that $\mathrm{E}(X)$ is independent of a.

c) Given that $\mathrm{Var}(X) = 0.56$ show that $a = 0.2$.

Two independent observations of X, denoted by X_1 and X_2 are considered.

d) Calculate $P(X_1 + X_2 = 6)$.

$\boxed{}$, $\boxed{0 \le a \le 0.4}$, $\boxed{P(X_1 + X_2 = 6) = 0.32}$

x	2	3	4
$P(X=x)$	$0.4-a$	$2a$	$0.6-a$

a) • $0.4 - a \geqslant 0$
$-a \geqslant -0.4$
$a \leqslant 0.4$

• $0.6 - a \geqslant 0$
$-a \geqslant -0.6$
$a \leqslant 0.6$

NOT RELEVANT

$\therefore\ 0 \leqslant a \leqslant 0.4$

b) $E(X) = \sum x P(X=x)$
$= 2 \times (0.4 - a) + 3 \times 2a + 4 \times (0.6 - a)$
$= 0.8 - \cancel{2a} + \cancel{6a} + 2.4 - \cancel{4a}$
$= 3.2$

IE INDEPENDENT OF a

c) STARTING WITH $E(X^2)$

$E(X^2) = \sum x^2 P(X=x)$
$= 2^2 \times (0.4 - a) + 3^2 \times 2a + 4^2 \times (0.6 - a)$
$= 4(0.4 - a) + 9 \times 2a + 16(0.6 - a)$
$= 1.6 - 4a + 18a + 9.6 - 16a$
$= 11.2 - 2a$

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$
$0.56 = (11.2 - 2a) - 3.2^2$
$0.56 = 11.2 - 2a - 10.24$
$2a = 0.4$
$a = 0.2$

d)

x	2	3	4
$P(X=x)$	0.2	0.4	0.4

$P(X_1 + X_2 = 6) = P(3,3) + P(2,4) + P(4,2)$
$= (0.4 \times 0.4) + (0.2 \times 0.4) + (0.4 \times 0.2)$
$= 0.16 + 0.08 + 0.08$
$= 0.32$

Question 29 (****)

A biased spinner can show whole numbers from 1 to 8.

The probability of showing an 8 is 0.05 and the probability of showing a 7 is 0.11.

The probabilities of showing any of the other six whole numbers are all equal to one another.

Players in a gambling parlour pay £5 for a single spin.

A score of 8 wins the player £50, a score of 7 wins the player £20, otherwise the player wins no money.

In a typical day, a gambling addict has 150 spins on this spinner.

Find the expected loss of the gambling addict in a typical day.

$\boxed{}$, $\boxed{£45}$

x	1	2	3	4	5	6	7	8
$P(X=x)$	0.14	0.14	0.14	0.14	0.14	0.14	0.11	0.05

TRANSFORM THE ABOVE TABLE INTO MONEY

y	0	20	50
$P(Y=y)$	0.84	0.11	0.05

$$E(Y) = (0 \times 0.84) + (20 \times 0.11) + (50 \times 0.05)$$
$$= 0 + 2.2 + 2.5$$
$$= 4.7$$

$\therefore$ TYPICAL WINNING RETURN IS £4.70 PER SPIN

$\therefore$ A LOSS OF £5.00 − £4.70 = 0.30

$\therefore$ IN 150 SPINS

$$150 \times 0.3 = £45$$

Question 30 (****)

The discrete random variable X has the following probability distribution

x	0	1	3
$\mathrm{P}(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

a) Determine $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

Two independent observations of X are made, denoted by X_1 and X_2.

b) Find the probability distribution of X_1+X_2.

c) Calculate $P(X_1>X_2)$.

$\boxed{\mathrm{E}(X)=\frac{11}{6}}$, $\boxed{\mathrm{Var}(X)=\frac{53}{36}}$, $\boxed{\mathrm{P}(X_1+X_2=r)=\begin{cases}\frac{1}{36} & r=0\\ \frac{1}{9} & r=1,2\\ \frac{1}{4} & r=6\\ \frac{1}{3} & r=4\\ 0 & \text{otherwise}\end{cases}}$,

$\boxed{P(X_1>X_2)=\frac{11}{36}}$

a)

x	0	1	3
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

$$\underline{E(X) = \Sigma x P(X=x)}$$
$$= \left(0\times\tfrac{1}{6}\right) + \left(1\times\tfrac{1}{3}\right) + \left(3\times\tfrac{1}{2}\right)$$
$$= 0 + \tfrac{1}{3} + \tfrac{3}{2}$$
$$= \tfrac{11}{6}$$

$$\underline{E(X^2) = \Sigma x^2 P(X=x)}$$
$$= \left(0^2\times\tfrac{1}{6}\right) + \left(1^2\times\tfrac{1}{3}\right) + \left(3^2\times\tfrac{1}{2}\right)$$
$$= 0 + \tfrac{1}{3} + \tfrac{9}{2}$$
$$= \tfrac{29}{6}$$

$$\underline{\mathrm{Var}(X) = E(X^2) - [E(X)]^2}$$
$$= \tfrac{29}{6} - \left(\tfrac{11}{6}\right)^2$$
$$= \tfrac{29}{6} - \tfrac{121}{36}$$
$$= \tfrac{53}{36}$$

b) DETERMINE ALL THE OUTCOMES FOR X_1+X_2

$0+0 = 0 \quad \frac{1}{6}\times\frac{1}{6} = \frac{1}{36}$
$0+1 = 1 \quad \frac{1}{6}\times\frac{1}{3} = \frac{1}{18}$
$0+3 = 3 \quad \frac{1}{6}\times\frac{1}{2} = \frac{1}{12}$
$1+0 = 1 \quad \frac{1}{3}\times\frac{1}{6} = \frac{1}{18}$
$1+1 = 2 \quad \frac{1}{3}\times\frac{1}{3} = \frac{1}{9}$
$1+3 = 4 \quad \frac{1}{3}\times\frac{1}{2} = \frac{1}{6}$
$3+0 = 3 \quad \frac{1}{6}\times\frac{1}{2} = \frac{1}{12}$
$3+1 = 4 \quad \frac{1}{2}\times\frac{1}{3} = \frac{1}{6}$
$3+3 = 6 \quad \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$

LET $Y = X_1+X_2$

y	0	1	2	3	4	6
$P(Y=y)$	$\frac{1}{36}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{4}$

c) USING THE OUTCOMES FROM PART (b)

$$\left.\begin{array}{ll} 1,0: & \frac{1}{18}\\ 3,0: & \frac{1}{12}\\ 3,1: & \frac{1}{6}\end{array}\right\}\ \text{Adding} = \frac{11}{36}$$

Question 31 (****)

The probability distribution of a discrete random variable X is given by

$$\mathrm{P}(X=x)=\begin{cases}\frac{1}{20}x & x=1,2,3,4,5\\ \frac{1}{4} & x=6\\ 0 & \text{otherwise}\end{cases}$$

a) Find $\mathrm{P}(X>4)$.

b) Calculate $\mathrm{E}\left(\frac{1}{X}\right)$.

c) Show that $\mathrm{Var}\left(\frac{1}{X}\right)=\frac{173}{4800}$.

The discrete random variable Y is defined as $Y=\frac{X+3}{X}$.

d) Determine the value of $\mathrm{E}(Y)$ and the value of $\mathrm{Var}(Y)$.

$\boxed{}$, $\boxed{\mathrm{P}(X>4)=0.5}$, $\boxed{\mathrm{E}\left(\frac{1}{X}\right)=\frac{7}{24}}$, $\boxed{\mathrm{E}(Y)=\frac{15}{8}}$, $\boxed{\mathrm{Var}(Y)=\frac{519}{1600}}$

a) PUTTING THE DISTRIBUTION IN A TABLE

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{3}{20}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{4}$

$P(X>4) = P(X=5,6) = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$

b) $E\left(\frac{1}{X}\right) = \sum \frac{1}{x}P(X=x)$

$\frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
$P(X=x)$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{3}{20}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{4}$

$\Rightarrow E\left(\frac{1}{X}\right) = \left(1\times\frac{1}{20}\right)+\left(\frac{1}{2}\times\frac{1}{10}\right)+\left(\frac{1}{3}\times\frac{3}{20}\right)+\dots+\left(\frac{1}{6}\times\frac{1}{4}\right)$

$\Rightarrow E\left(\frac{1}{X}\right) = \frac{1}{20}+\frac{1}{20}+\frac{1}{20}+\frac{1}{20}+\frac{1}{20}+\frac{1}{24}$

$\Rightarrow E\left(\frac{1}{X}\right) = \frac{7}{24}$

c) $\mathrm{Var}(X) = E(X^2)-[E(X)]^2$

$\mathrm{Var}\left(\frac{1}{X}\right) = E\left(\frac{1}{X^2}\right) - \left[E\left(\frac{1}{X}\right)\right]^2$

$\mathrm{Var}\left(\frac{1}{X}\right) = \underbrace{\left(1^2\times\frac{1}{20}\right)+\left(\frac{1}{4}\times\frac{1}{10}\right)+\left(\frac{1}{9}\times\frac{3}{20}\right)+\dots+\left(\frac{1}{36}\times\frac{1}{4}\right)}_{E\left(\frac{1}{X^2}\right)\equiv\sum\frac{1}{x^2}P(X=x)}-\left[\frac{7}{24}\right]^2$

$\mathrm{Var}\left(\frac{1}{X}\right) = \frac{1}{20}+\frac{1}{40}+\frac{1}{60}+\frac{1}{80}+\frac{1}{100}+\frac{1}{144}-\frac{49}{576}$

$\mathrm{Var}\left(\frac{1}{X}\right) = \frac{173}{4800}$

d) $Y = \frac{X+3}{X} = 1+\frac{3}{X}$

- $E(Y) = E\left(1+\frac{3}{X}\right) = E\left[3\times\frac{1}{X}+1\right]$
  $= 3E\left(\frac{1}{X}\right)+1 = 3\times\frac{7}{24}+1$
  $= \frac{15}{8}$

- $\mathrm{Var}(Y) = \mathrm{Var}\left(1+\frac{3}{X}\right) = \mathrm{Var}\left[3\times\frac{1}{X}+1\right]$
  $= 3^2\,\mathrm{Var}\left(\frac{1}{X}\right) = 9\times\frac{173}{4800}$
  $= \frac{519}{1600}$

Question 32 (****)

A sixth form class consists of 3 boys and 7 girls.

Three students are selected at random from this class and the variable X represents the number of boys selected.

Show clearly that $\mathrm{E}(X) = 0.9$.

$\boxed{}$, $\boxed{\text{proof}}$

X = NO OF BOYS SELECTED (3 GOES WITHOUT REPLACEMENT)

WORKING INDIVIDUAL PROBABILITIES

- $P(X=0) = P(\text{girl, girl, girl}) = \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} = \frac{210}{720}$
- $P(X=3) = P(\text{boy, boy, boy}) = \frac{3}{10} \times \frac{2}{9} \times \frac{1}{8} = \frac{6}{720}$
- $P(X=1) = P(\text{boy, girl, girl}) + P(\text{girl, boy, girl}) + P(\text{girl, girl, boy}) = \frac{3}{10} \times \frac{7}{9} \times \frac{6}{8} + \frac{7}{10} \times \frac{3}{9} \times \frac{6}{8} + \frac{7}{10} \times \frac{6}{9} \times \frac{3}{8} = \frac{126}{720} + \frac{126}{720} + \frac{126}{720} = \frac{378}{720}$
- $P(X=2) = 1 - \left(\frac{210}{720} + \frac{6}{720} + \frac{378}{720}\right) = \frac{126}{720}$

FORMING THE TABLE OF THE DISTRIBUTION OF X

x	0	1	2	3
$P(X=x)$	$\frac{210}{720}$	$\frac{378}{720}$	$\frac{126}{720}$	$\frac{6}{720}$

$$\mathrm{E}(X) = \sum x P(X=x)$$
$$= 0 \times \frac{210}{720} + 1 \times \frac{378}{720} + 2 \times \frac{126}{720} + 3 \times \frac{6}{720}$$
$$= \frac{648}{720} = \frac{9}{10}$$

Question 33 (****)

The probability distribution of a discrete random variable X is given by

$$\mathrm{P}(X = x) = \begin{cases} \dfrac{1}{10} & x = 1, 2, 3, \dots, 10 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of $\mathrm{E}\left(X^2\right)$.

b) Determine $\mathrm{P}(X + 2 < 3X - 4 < X + 7)$.

It is further given that

$$\mathrm{E}(kX + 5) = 6.1,$$

where k is a constant.

c) Find the value of $\mathrm{Var}(kX + 5)$

$\boxed{}$, $\boxed{\mathrm{E}\left(X^2\right) = 38.5}$, $\boxed{\mathrm{P}(X + 2 < 3X - 4 < X + 7) = \frac{1}{5}}$, $\boxed{\mathrm{Var}(kX + 5) = 0.33}$

$\boxed{P(X = x) = \frac{1}{10} \qquad x = 1, 2, 3, \dots, 10}$

a) USING $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$\Rightarrow \frac{n^2 - 1}{12} = E(X^2) - \left(\frac{n+1}{2}\right)^2$

$\Rightarrow \frac{99}{12} = E(X^2) - \left(\frac{11}{2}\right)^2$

$\Rightarrow 8.25 = E(X^2) - 30.25$

$\Rightarrow E(X^2) = 38.5$

b) SIMPLIFYING THE PROBABILITY EXPRESSION

$P[X + 2 < 3X - 4 < X + 7)$

$= P[2 < 2X - 4 < 7]$

$= P[6 < 2X < 11]$

$= P[3 < X < \frac{11}{2}]$

$= P[X = 4, 5]$

$= \frac{1}{10} + \frac{1}{10}$

$= \frac{1}{5}$

c) FIRSTLY WE NEED THE EXPECTATION & VARIANCE

$E(X) = \frac{n+1}{2} = \frac{10+1}{2} = 5.5$ & $\mathrm{Var}(X) = \frac{n^2 - 1}{12} = \frac{10^2 - 1}{12} = \frac{33}{4}$

$\Rightarrow E(kX + 5) = 6.1$

$\Rightarrow kE(X) + 5 = 6.1$

$\Rightarrow (k \times 5.5) + 5 = 6.1$

$\Rightarrow 5.5k = 1.1$

$\Rightarrow k = \frac{1}{5}$

$\Rightarrow \mathrm{Var}(kX + 5) = k^2 \mathrm{Var}(X)$

$= \left(\frac{1}{5}\right)^2 \times \frac{33}{4}$

$= \frac{1}{25} \times \frac{33}{4}$

$= 0.33$

Question 34 (****+)

The probability distribution of a discrete random variable X is given by

x	1	2	3	4	5
$\mathrm{P}(X=x)$	k	0.1	0.2	0.3	$0.4-k$

where k is a positive constant.

a) Determine the range of values of $\mathrm{E}(X)$.

b) Given that $\mathrm{Var}(X) = 1.36$, find the value of k.

$\boxed{2.4 \le \mathrm{E}(X) \le 4}$, $\boxed{k = 0.05}$

x	1	2	3	4	5
$P(X=x)$	k	0.1	0.2	0.3	$0.4-k$

a) AS ALL PROBABILITIES MUST BE POSITIVE

- $k \geqslant 0$
- $0.4 - k \geqslant 0$, $k \leqslant 0.4$

$\therefore \; 0 \leqslant k \leqslant 0.4$

$E(X) = \sum x P(X=x)$

$E(X) = (1\times k) + (2\times 0.1) + (3\times 0.2) + (4\times 0.3) + 5\times(0.4-k)$

$E(X) = k + 0.2 + 0.6 + 1.2 + 2 - 5k$

$E(X) = 4 - 4k$

NEXT PROCEED AS FOLLOWS

$\Rightarrow \; 0 \leqslant k \leqslant 0.4$

$\Rightarrow \; -0.4 \leqslant -k \leqslant 0$

$\Rightarrow \; -1.6 \leqslant -4k \leqslant 0$

$\Rightarrow \; -1.6 + 4 \leqslant -4k + 4 \leqslant 0 + 4$

$\Rightarrow \; 2.4 \leqslant E(X) \leqslant 4$

b) $E(X^2) = \sum x^2 P(X=x)$

$= (1^2\times k) + (2^2\times 0.1) + (3^2\times 0.2) + (4^2\times 0.3) + [5^2\times(0.4-k)]$

$= k + 0.4 + 1.8 + 4.8 + 25(0.4-k)$

$= k + 7 + 10 - 25k$

$= 17 - 24k$

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$\Rightarrow \; 1.36 = 17 - 24k - (4-4k)^2$

$\Rightarrow \; 1.36 = 17 - 24k - (16 - 32k + 16k^2)$

$\Rightarrow \; 1.36 = 17 - 24k - 16 + 32k - 16k^2$

$\Rightarrow \; 16k^2 - 8k + 0.36 = 0$

BY THE QUADRATIC FORMULA

$k = \text{"}\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}\text{"}$

$k = \dfrac{8 \pm \sqrt{40.96}}{2\times 16}$

$k = \cancel{0.45}$ or 0.05 $0 \leqslant k \leqslant 0.4$

Question 35 (****+)

Luke has 6 chocolates of which 2 have a hazelnut at their centre.

Luke eats his chocolates one after the other.

The random variable X represents the number of chocolates Luke eats, up and including the first chocolate with a hazelnut at its centre.

Show, with detailed workings, that $\text{Var}(X) = \frac{14}{9}$.

$\boxed{}$, $\boxed{\text{proof}}$

X = NO OF CHOCOLATES EATEN, UNTIL THE FIRST CHOCOLATE WITH A HAZELNUT IS EATEN

- $P(X=1) = \frac{2}{6} = \frac{1}{3}$
- $P(X=2) = \frac{4}{6} \times \frac{2}{5} = \frac{8}{30} = \frac{4}{15}$
- $P(X=3) = \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} = \frac{24}{120} = \frac{1}{5}$
- $P(X=4) = \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} = \frac{48}{360} = \frac{2}{15}$
- $P(X=5) = \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} \times \frac{2}{2} = \frac{48}{720} = \frac{1}{15}$

FORMING THE PROBABILITY DISTRIBUTION OF X

x	1	2	3	4	5
$P(X=x)$	$\frac{1}{3}$	$\frac{4}{15}$	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{15}$

$E(X) = \sum x P(X=x)$

$$E(X) = \left(1 \times \tfrac{1}{3}\right) + \left(2 \times \tfrac{4}{15}\right) + \left(3 \times \tfrac{1}{5}\right) + \left(4 \times \tfrac{2}{15}\right) + 5 \times \tfrac{1}{15}$$
$$= \tfrac{1}{3} + \tfrac{8}{15} + \tfrac{3}{5} + \tfrac{8}{15} + \tfrac{1}{3}$$
$$= \tfrac{7}{3}$$

$E(X^2) = \sum x^2 P(X=x)$

$$E(X^2) = \left(1^2 \times \tfrac{1}{3}\right) + \left(2^2 \times \tfrac{4}{15}\right) + \left(3^2 \times \tfrac{1}{5}\right) + \left(4^2 \times \tfrac{2}{15}\right) + \left(5^2 \times \tfrac{1}{15}\right)$$
$$= \tfrac{1}{3} + \tfrac{16}{15} + \tfrac{9}{5} + \tfrac{32}{15} + \tfrac{25}{15}$$
$$= 7$$

$\text{Var}(X) = E(X^2) - [E(X)]^2$

$$\text{Var}(X) = 7 - \left(\tfrac{7}{3}\right)^2$$
$$= 7 - \tfrac{49}{9}$$
$$= \tfrac{14}{9}$$

Question 36 (****+)

The probability of a biased coin landing on "*tails*" is 0.3.

An experiment consists of tossing the coin until "*tails*" is shown for the first time, up to a maximum of 4 tosses.

If "*tails*" is shown before the 4^{th} toss the experiment stops.

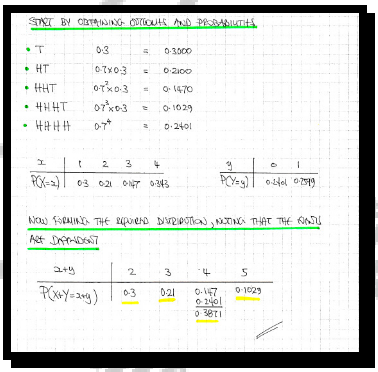
If after the 4^{th} toss no "*tails*" has been obtained, then no more tosses are made.

- The discrete random variable X is defined as the number of tosses in this experiment.

- The discrete random variable Y is defined as the number of "*tails*" in this experiment.

Determine the probability distribution of $X+Y$.

$x+y$	2	3	4	5
probability	0.3	0.21	0.3871	0.1029



START BY OBTAINING OUTCOMES AND PROBABILITIES

- T $\quad 0.3 = 0.3000$
- HT $\quad 0.7 \times 0.3 = 0.2100$
- HHT $\quad 0.7^2 \times 0.3 = 0.1470$
- HHHT $\quad 0.7^3 \times 0.3 = 0.1029$
- HHHH $\quad 0.7^4 = 0.2401$

x	1	2	3	4
$P(X=x)$	0.3	0.21	0.147	0.343

y	0	1
$P(Y=y)$	0.2401	0.7599

NOW FORMING THE REQUIRED DISTRIBUTION, NOTING THAT THE EVENTS ARE DEPENDENT

$x+y$	2	3	4	5
$P(X+Y=x+y)$	0.3	0.21	$0.147 + 0.2401 = 0.3871$	0.1029

Question 37 (****+)

A box contains three blue discs and two red discs.

An experiment is conducted where three discs are selected at random from the box **without** replacement.

The variable X represents the number of blue discs selected.

a) Show that the probability distribution of X is given by

x	1	2	3
$\mathrm{P}(X = x)$	$\frac{3}{10}$	$\frac{6}{10}$	$\frac{1}{10}$

Four independent observations of X are recorded, labelled as X_1, X_2, X_3 and X_4.

b) Determine $\mathrm{P}(X_1 + X_2 + X_3 + X_4 \geq 10)$ and $\mathrm{Var}(X)$.

$\boxed{}$, $\boxed{\mathrm{P}(X_1 + X_2 + X_3 + X_4 \geq 10) = 0.0253}$

a) START WITH A TREE DIAGRAM OR SOME KIND OF SORTER

$X =$ NO OF BLUE DISCS PICKED

$P(X=1) = P(\text{Red}-\text{Red}-\text{blue}) \times 3 \text{ WAYS} = \left(\frac{2}{5} \times \frac{1}{4} \times \frac{3}{3}\right) \times 3 = \frac{3}{10}$

$P(X=2) = P(\text{Red}-\text{Blue}-\text{Blue}) \times 3 \text{ WAYS} = \left(\frac{2}{5} \times \frac{3}{4} \times \frac{2}{3}\right) \times 3 = \frac{6}{10}$

$P(X=3) = P(\text{blue}-\text{blue}-\text{blue}) \times 1 \text{ WAY} = \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} = \dots = \frac{1}{10}$

b) ORGANISING OUTCOMES

$3,3,3,3 \Longrightarrow \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = 0.0001$

$\left.\begin{matrix} 3,3,3,2 \\ 3,3,2,3 \\ 3,2,3,3 \\ 2,3,3,3 \end{matrix}\right\} \Longrightarrow \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{6}{10} \times 4 \text{ WAYS} = 0.0024$

$\left.\begin{matrix} 3322 \\ 3232 \\ 3223 \\ 2332 \\ 2323 \\ 2233 \end{matrix}\right\} \Longrightarrow \frac{1}{10} \times \frac{1}{10} \times \frac{6}{10} \times \frac{6}{10} \times 6 \text{ WAYS} = 0.0216$

$\left.\begin{matrix} 3331 \\ 3313 \\ 3133 \\ 1333 \end{matrix}\right\} \Longrightarrow \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{3}{10} \times 4 \text{ WAYS} = 0.0012$

ADDING GIVES 0.0253

Question 38 (*****)

The probability distribution of a discrete random variable X is given by

$$\mathrm{P}(X = x) = \begin{cases} \dfrac{1}{7} & x = 1, 2, 3, \ldots 7 \\ 0 & \text{otherwise} \end{cases}$$

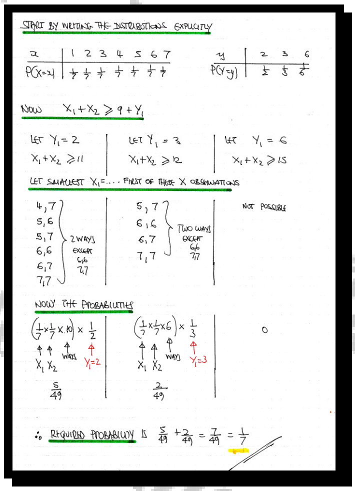
The probability distribution of another discrete random variable Y is given by

$$\mathrm{P}(Y = y) = \begin{cases} \dfrac{1}{y} & x = 2, 3, 6 \\ 0 & \text{otherwise} \end{cases}$$

Two observations of X are made, denoted by X_1 and X_2, and one observation of Y, denoted by Y_1 are considered.

Assuming these three observations are independent, calculate $P(X_1 + X_2 \geq 9 + Y_1)$.

$$\boxed{P(X_1 + X_2 \geq 9 + Y_1) = \frac{1}{7}}$$



START BY WRITING THE DISTRIBUTIONS EXPLICITLY

x	1	2	3	4	5	6	7
$P(X=x)$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

y	2	3	6
$P(Y=y)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

NOW $X_1 + X_2 \geqslant 9 + Y_1$

LET $Y_1 = 2$: $X_1 + X_2 \geqslant 11$ | LET $Y_1 = 3$: $X_1 + X_2 \geqslant 12$ | LET $Y_1 = 6$: $X_1 + X_2 \geqslant 15$

LET SMALLEST $X_1 = \ldots$ FIRST OF THESE X OBSERVATIONS

4,7; 5,6; 5,7; 6,6; 6,7; 7,7 — 2 WAYS EXCEPT 6,6 7,7 | 5,7; 6,6; 6,7; 7,7 — TWO WAYS EXCEPT 6,6 7,7 | NOT POSSIBLE

NOW THE PROBABILITIES

$\left(\frac{1}{7} \times \frac{1}{7} \times 10\right) \times \frac{1}{2}$ (X_1, X_2, WAYS, $Y_1 = 2$) $= \frac{5}{49}$ | $\left(\frac{1}{7} \times \frac{1}{7} \times 6\right) \times \frac{1}{3}$ (X_1, X_2, WAYS, $Y_1 = 3$) $= \frac{2}{49}$ | 0

$\therefore$ REQUIRED PROBABILITY IS $\frac{5}{49} + \frac{2}{49} = \frac{7}{49} = \frac{1}{7}$

Question 39 (*****)

A biased six sided die has the following probability distribution

x	1	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$

where the random variable X represents the number shown on its uppermost face when it comes to rest after it is rolled.

The die is rolled twice and the two independent observations of X, X_1 and X_2, produce the score Y defined as

$$Y = \begin{cases} 6 & \text{if } X_1 = 6 \\ X_1 + X_2 & \text{if } X_1 \neq 6 \end{cases}$$

a) Find the value of $\mathrm{P}(Y=6)$.

b) Determine the probability distribution of Y and hence calculate the $\mathrm{E}(Y)$.

c) Find the value of $\mathrm{P}(Y<7 \mid Y>4)$.

$\boxed{}$, $\boxed{\mathrm{P}(Y=6)=0.55}$, $\boxed{\mathrm{E}(Y)=6.75}$, $\boxed{\mathrm{P}(Y<7 \mid Y>4)=\frac{59}{94}}$

a)

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$

$P(Y=6) = \text{"6"} + \text{"1,5"} + \text{"2,4"} + \text{"3,3"} + \text{"5,1"} + \text{"4,2"}$

$= \frac{1}{2} + 5 \times \frac{1}{10} \times \frac{1}{10}$

$= 0.55$

b) WRITING ALL THE OUTCOMES

6 6

1 1 2   2 1 3   3 1 4   4 1 5   5 1 6
1 2 3   2 2 4   3 2 5   4 2 6   5 2 7
1 3 4   2 3 5   3 3 6   4 3 7   5 3 8
1 4 5   2 4 6   3 4 7   4 4 8   5 4 9
1 5 6   2 5 7   3 5 8   4 5 9   5 5 10
1 6 7   2 7 8   3 6 9   4 6 10   5 6 11

2 3 4 5 6 7 8 9 10 11

NOTE THAT ANY "DOUBLE OUTCOME WHICH INCLUDES A 6 HAS PROBABILITY $\frac{1}{2} \times \frac{1}{10} = \frac{1}{20}$ AND ALL OTHERS HAVE PROBABILITY $\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$

y	2	3	4	5	6	7	8	9	10	11
$P(Y=y)$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{3}{100}$	$\frac{4}{100}$	$\frac{55}{100}$	$\frac{9}{100}$	$\frac{8}{100}$	$\frac{7}{100}$	$\frac{6}{100}$	$\frac{5}{100}$
					Part(a)	$\frac{1}{20}+\frac{4}{100}$	$\frac{1}{20}+\frac{3}{100}$	$\frac{1}{20}+\frac{2}{100}$	$\frac{1}{20}+\frac{1}{100}$	$\frac{1}{20}$

NOW THE EXPECTATION OF Y CAN BE FOUND

$E(Y) = \sum y P(Y=y)$

$= \left(2 \times \frac{1}{100}\right) + \left(3 \times \frac{2}{100}\right) + \left(4 \times \frac{3}{100}\right) + \dots + \left(11 \times \frac{5}{100}\right)$

$= \frac{2+6+12+20+330+63+64+63+60+55}{100} \quad \frac{675}{100}$

$= 6.75$

c) $P(Y<7 \mid Y>4) = \frac{P(Y<7 \cap Y>4)}{P(Y>4)}$

$= \frac{P(4<Y<7)}{P(Y>4)}$

$= \frac{P(Y=5,6)}{P(Y=5,6,7,\dots 11)}$

$= \frac{P(Y=5,6)}{1 - P(Y=2,3,4)}$

$= \frac{\frac{4}{100} + \frac{55}{100}}{1 - \left[\frac{1}{100} + \frac{2}{100} + \frac{3}{100}\right]}$

$= \frac{59/100}{94/100}$

$= \frac{59}{94} \approx 0.6277$

Question 40 (*****)

The probability distribution of a discrete random variable X is given by

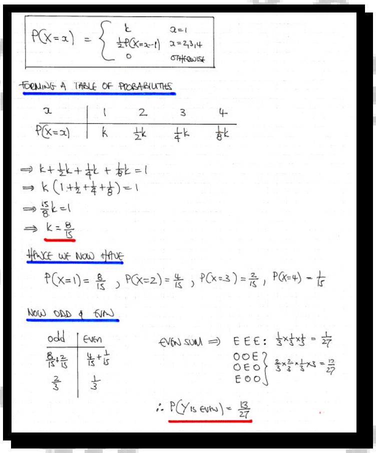
$$\mathrm{P}(X=x)=\begin{cases} k & x=1 \\ \frac{1}{2}\mathrm{P}(X=x-1) & x=2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

where k is a positive constant.

Three independent observations of X are made, denoted by X_1, X_2 and X_3, and the variable Y is defined as $Y=X_1+X_2+X_3$.

If Y is an even number, determine the probability that Y is greater than 9.

$\boxed{\text{MMS-Z}}$, $\boxed{\frac{1}{65}}$



$$\mathrm{P}(X=x)=\begin{cases} k & x=1 \\ \frac{1}{2}\mathrm{P}(X=x-1) & x=2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

FORMING A TABLE OF PROBABILITIES

x	1	2	3	4
$\mathrm{P}(X=x)$	k	$\frac{1}{2}k$	$\frac{1}{4}k$	$\frac{1}{8}k$

$\Rightarrow k+\frac{1}{2}k+\frac{1}{4}k+\frac{1}{8}k=1$

$\Rightarrow k\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)=1$

$\Rightarrow \frac{15}{8}k=1$

$\Rightarrow k=\frac{8}{15}$

HENCE WE NOW HAVE

$\mathrm{P}(X=1)=\frac{8}{15}$, $\mathrm{P}(X=2)=\frac{4}{15}$, $\mathrm{P}(X=3)=\frac{2}{15}$, $\mathrm{P}(X=4)=\frac{1}{15}$

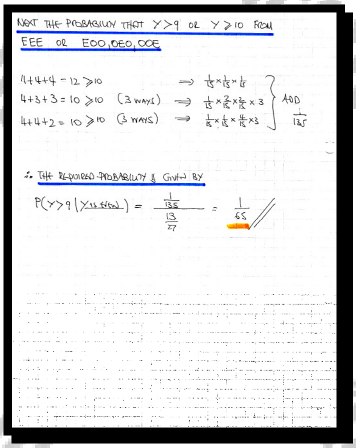
NOW ODD & EVEN

odd	Even
$\frac{8}{15}+\frac{2}{15}$	$\frac{4}{15}+\frac{1}{15}$
$\frac{2}{3}$	$\frac{1}{3}$

EVEN SUM $\Rightarrow$ EEE: $\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}=\frac{1}{27}$

$\left.\begin{matrix}\text{OOE}\\ \text{OEO}\\ \text{EOO}\end{matrix}\right\}\ \frac{2}{3}\times\frac{2}{3}\times\frac{1}{3}\times 3=\frac{12}{27}$

$\therefore \mathrm{P}(Y \text{ is even})=\frac{13}{27}$



NEXT THE PROBABILITY THAT $Y>9$ OR $Y\geqslant 10$ FROM EEE OR EOO, OEO, OOE

$\left.\begin{matrix} 4+4+4=12\geqslant 10 & & \Rightarrow \frac{1}{15}\times\frac{1}{15}\times\frac{1}{15} \\ 4+3+3=10\geqslant 10 & (3 \text{ WAYS}) & \Rightarrow \frac{1}{15}\times\frac{2}{15}\times\frac{2}{15}\times 3 \\ 4+4+2=10\geqslant 10 & (3 \text{ WAYS}) & \Rightarrow \frac{1}{15}\times\frac{1}{15}\times\frac{4}{15}\times 3 \end{matrix}\right\}$ ADD $\frac{1}{135}$

$\therefore$ THE REQUIRED PROBABILITY IS GIVEN BY

$\mathrm{P}(Y>9 \mid Y \text{ is even})=\dfrac{\frac{1}{135}}{\frac{13}{27}}=\frac{1}{65}$

Question 41 (*****)

The probability distribution of a discrete random variable X is given by

$$\mathrm{P}(X=r+1)=\begin{cases}\frac{2}{3}\mathrm{P}(X=r) & r=1,2,3,4,5,\ldots \\ 0 & \text{otherwise}\end{cases}$$

Determine $\mathrm{P}(2 \le X \le 4)$.

M.M., $\boxed{\mathrm{P}(2 \le X \le 4)=\frac{38}{31}}$

$$\boxed{\mathrm{P}(X=r+1)=\begin{cases}\frac{2}{3}\mathrm{P}(X=r) & r=1,2,3,4,5,\ldots \\ 0 & \end{cases}}$$

USE A CONDITIONAL PROBABILITY TABLE

r	1	2	3	4	5	6	...
$\mathrm{P}(X=r)$	k	$\frac{2}{3}k$	$\frac{4}{9}k$	$\frac{8}{27}k$	$\frac{16}{81}k$	$\frac{32}{243}k$	..

$\sum \mathrm{P}(X=r)=1$

$\Rightarrow k\left(1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\frac{32}{243}+\ldots\right)=1$

THIS IS A CONVERGENT G.P. WITH $a=1$ & $r=\frac{2}{3}$ & $S_\infty=\frac{a}{1-r}$

$\Rightarrow k\left(\frac{1}{1-\frac{2}{3}}\right)=1$

$\Rightarrow k \times 3=1$

$\Rightarrow k=\frac{1}{3}$

$\therefore \mathrm{P}(2 \le X \le 4)=\mathrm{P}(X=2)+\mathrm{P}(X=3)+\mathrm{P}(X=4)$

$=\frac{2}{3}k+\frac{4}{9}k+\frac{8}{27}k$

$=\frac{2}{9}+\frac{4}{27}+\frac{8}{81}$

$=\frac{38}{81}$