PROBABILITY DENSITY FUNCTIONS

P.D.F. CALCULATIONS

Question 1 (***)

The lifetime of a certain brand of battery, in tens of hours, is modelled by the continuous random variable X with probability density function $f(x)$ given by

$$f(x) = \begin{cases} \dfrac{2}{75}x & 0 \le x \le 5 \\ \dfrac{2}{15} & 5 < x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

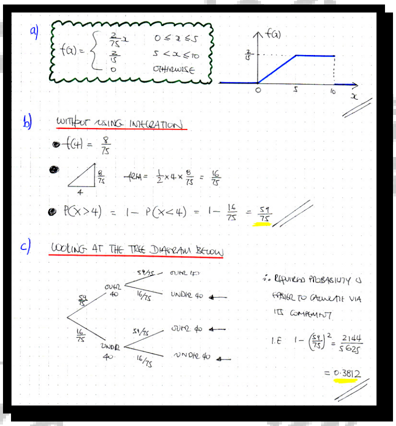
a) Sketch $f(x)$ for all x.

b) Determine $\mathrm{P}(X > 4)$.

Two such batteries are needed by a piece of electronic equipment. This equipment will only operate if **both** batteries are still functional.

c) If two new batteries are fitted to this equipment, determine the probability that this equipment will stop working within the next 40 hours.

$\boxed{}$, $\boxed{\mathrm{P}(X > 4) = \dfrac{59}{75}}$, $\boxed{\dfrac{2144}{5625} \approx 0.381}$



Question 2 (***)

The lengths of telephone conversations, in minutes, by sales reps of a certain company are modelled by the continuous random variable T.

The probability density function of T is denoted by $f(t)$, and is given by

$$f(t) = \begin{cases} kt & 0 \le t \le 12 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $k = \dfrac{1}{72}$.

b) Determine $\mathrm{P}(T > 5)$.

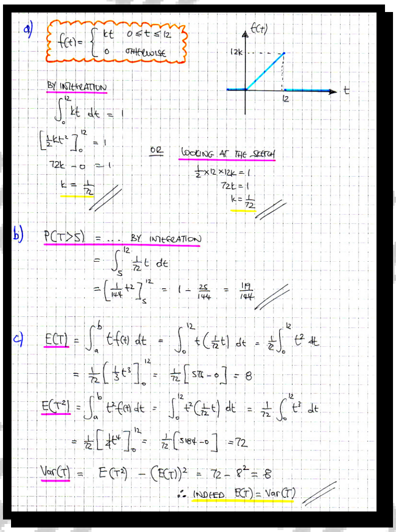
c) Show by calculation that $\mathrm{E}(T) = \mathrm{Var}(T)$.

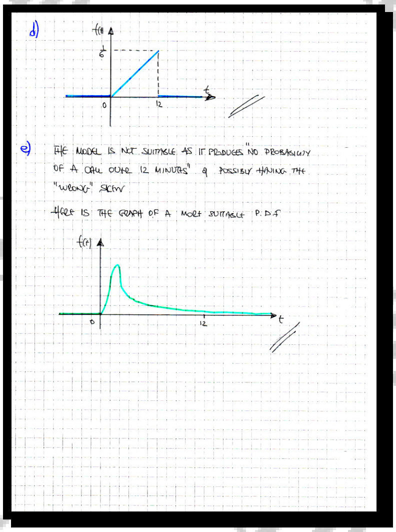
d) Sketch $f(t)$ for all t.

A statistician suggests that the probability density function $f(t)$ as defined above, might not provide a good model for T.

e) Give a reason for his suggestion.

$\boxed{}$, $\boxed{\mathrm{P}(T > 5) = \dfrac{119}{144}}$, $\boxed{\mathrm{E}(T) = \mathrm{Var}(T) = 8}$





Question 3 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases} 2x+k & 3 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $k=-6$.

b) Sketch $f(x)$ for all x.

c) State the mode of X.

d) Calculate, showing detailed workings, the value of …

i. … $\mathrm{E}(X)$.

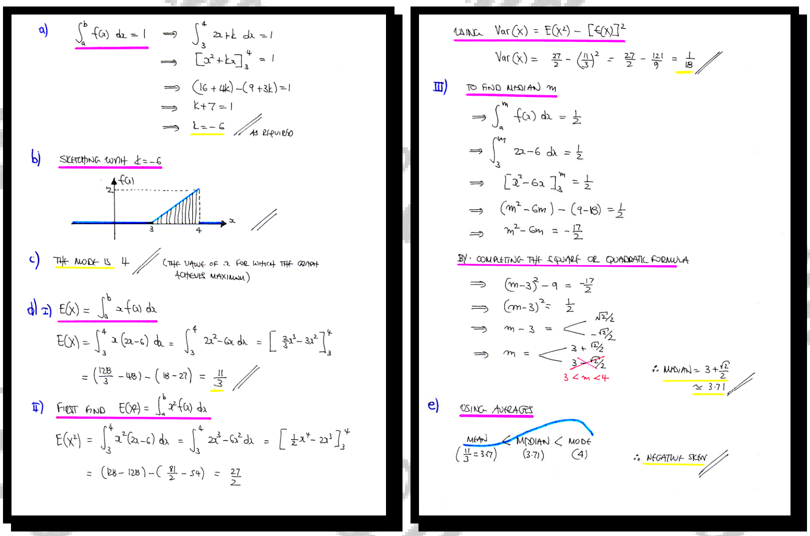
ii. … $\mathrm{Var}(X)$.

iii. … the median of X.

e) Determine with justification the skewness of the distribution.

$\boxed{\quad}$, $\boxed{\text{mode}=4}$, $\boxed{\mathrm{E}(X)=\frac{11}{3}\approx 3.67}$, $\boxed{\mathrm{Var}(X)=\frac{1}{18}\approx 0.0556}$,

$\boxed{\text{median}=3+\frac{\sqrt{2}}{2}\approx 3.71}$, $\boxed{\text{mean} < \text{median} < \text{mode} \Rightarrow \text{negative skew}}$



a) $\int_a^b f(x)\,dx = 1 \implies \int_3^4 2x+k\,dx = 1$

$\implies \left[x^2+kx\right]_3^4 = 1$

$\implies (16+4k)-(9+3k)=1$

$\implies k+7=1$

$\implies k=-6$ // AS REQUIRED

b) SKETCHING WITH $k=-6$

c) THE MODE IS 4 // (THE VALUE OF x FOR WHICH THE GRAPH ACHIEVES MAXIMUM)

d) I) $E(X)=\int_a^b x f(x)\,dx$

$E(X)=\int_3^4 x(2x-6)\,dx = \int_3^4 2x^2-6x\,dx = \left[\frac{2}{3}x^3-3x^2\right]_3^4$

$=\left(\frac{128}{3}-48\right)-(18-27)=\frac{11}{3}$ //

II) FIRST FIND $E(X^2)=\int_a^b x^2 f(x)\,dx$

$E(X^2)=\int_3^4 x^2(2x-6)\,dx = \int_3^4 2x^3-6x^2\,dx = \left[\frac{1}{2}x^4-2x^3\right]_3^4$

$=(128-128)-\left(\frac{81}{2}-54\right)=\frac{27}{2}$

USING $\mathrm{Var}(X)=E(X^2)-[E(X)]^2$

$\mathrm{Var}(X)=\frac{27}{2}-\left(\frac{11}{3}\right)^2=\frac{27}{2}-\frac{121}{9}=\frac{1}{18}$ //

III) TO FIND MEDIAN m

$\implies \int_a^m f(x)\,dx=\frac{1}{2}$

$\implies \int_3^m 2x-6\,dx=\frac{1}{2}$

$\implies \left[x^2-6x\right]_3^m=\frac{1}{2}$

$\implies (m^2-6m)-(9-18)=\frac{1}{2}$

$\implies m^2-6m=-\frac{17}{2}$

BY COMPLETING THE SQUARE OR QUADRATIC FORMULA

$\implies (m-3)^2-9=-\frac{17}{2}$

$\implies (m-3)^2=\frac{1}{2}$

$\implies m-3 = \sqrt{2}/2$ or $-\sqrt{2}/2$

$\implies m = 3+\sqrt{2}/2$ or ~~$3-\sqrt{2}/2$~~

$3<m<4$

$\therefore$ MEDIAN $=3+\frac{\sqrt{2}}{2} \simeq 3.71$ //

e) USING AVERAGES

MEAN $\left(\frac{11}{3}=3.67\right)$ < MEDIAN (3.71) < MODE (4)

$\therefore$ NEGATIVE SKEW //

Question 4 (****)

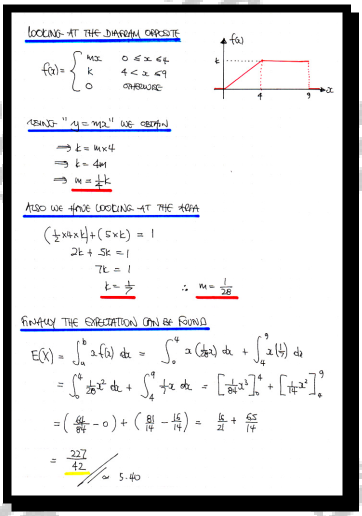
A continuous random variable X has probability density function $f(x)$ given by

$$f(x) \equiv \begin{cases} mx & 0 \le x \le 4 \\ k & 4 \le x \le 9 \\ 0 & \text{otherwise} \end{cases}$$

where m and k are positive constants.

Find as an exact simplified fraction the value of $\mathrm{E}(X)$.

$\boxed{\mathrm{E}(X) = \frac{227}{42}}$



LOOKING AT THE DIAGRAM OPPOSITE

$$f(x) = \begin{cases} mx & 0 \le x \le 4 \\ k & 4 < x \le 9 \\ 0 & \text{OTHERWISE} \end{cases}$$

USING "$y = mx$" WE OBTAIN

$\Rightarrow k = m \times 4$

$\Rightarrow k = 4m$

$\Rightarrow m = \frac{1}{4}k$

ALSO WE HAVE LOOKING AT THE AREA

$\left(\frac{1}{2} \times 4 \times k\right) + (5 \times k) = 1$

$2k + 5k = 1$

$7k = 1$

$k = \frac{1}{7}$ $\quad \therefore m = \frac{1}{28}$

FINALLY THE EXPECTATION CAN BE FOUND

$$\mathrm{E}(X) = \int_a^b x f(x)\, dx = \int_0^4 x\left(\tfrac{1}{28}x\right) dx + \int_4^9 x\left(\tfrac{1}{7}\right) dx$$

$$= \int_0^4 \tfrac{1}{28}x^2\, dx + \int_4^9 \tfrac{1}{7}x\, dx = \left[\tfrac{1}{84}x^3\right]_0^4 + \left[\tfrac{1}{14}x^2\right]_4^9$$

$$= \left(\tfrac{64}{84} - 0\right) + \left(\tfrac{81}{14} - \tfrac{16}{14}\right) = \tfrac{16}{21} + \tfrac{65}{14}$$

$$= \tfrac{227}{42} \approx 5.40$$

Question 5 (****+)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x) = \begin{cases} kx\left(16 - x^2\right) & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $k = \dfrac{1}{64}$.

b) Calculate, showing detailed workings, the value of …

i. … $\mathrm{E}(X)$.

ii. … $\mathrm{Var}(X)$.

c) Show by calculation, that the median is 2.165, correct to 3 decimal places.

d) Use calculus to find the mode of X.

e) Sketch the graph of $f(x)$ for all x.

f) Determine with justification the skewness of the distribution.

$\boxed{\quad}$, $\boxed{\mathrm{E}(X) = \frac{32}{15} \approx 2.13}$, $\boxed{\mathrm{Var}(X) = \frac{176}{225} \approx 0.782}$, $\boxed{\text{mode} \approx 2.31}$,

$\boxed{\text{mean} < \text{median} < \text{mode} \Rightarrow \text{negative skew}}$

a) USING $\int_a^b f(x)\,dx = 1$

$\Rightarrow \int_0^4 kx(16-x^2)\,dx = 1$

$\Rightarrow k\int_0^4 16x - x^3\,dx = 1$

$\Rightarrow k\left[8x^2 - \frac{1}{4}x^4\right]_0^4 = 1$

$\Rightarrow k\left[(128-64) - 0\right] = 1$

$\Rightarrow 64k = 1$

$\Rightarrow k = \frac{1}{64}$ AS REQUIRED

b) I) USING $E(X) = \int_a^b x f(x)\,dx$

$E(X) = \int_0^4 x \times \frac{1}{64}x(16-x^2)\,dx = \frac{1}{64}\int_0^4 16x^2 - x^4\,dx$

$= \frac{1}{64}\left[\frac{16}{3}x^3 - \frac{1}{5}x^5\right]_0^4 = \frac{1}{64}\left[\left(\frac{1024}{3} - \frac{1024}{5}\right) - 0\right] = \frac{32}{15}$ ≈ 2.13

II) FIRSTLY COMPUTE $E(X^2) = \int_a^b x^2 f(x)\,dx$

$E(X^2) = \int_0^4 x^2 \times \frac{1}{64}x(16-x^2)\,dx = \frac{1}{64}\int_0^4 16x^3 - x^5\,dx$

$= \frac{1}{64}\left[4x^4 - \frac{1}{6}x^6\right]_0^4 = \frac{1}{64}\left[\left(1024 - \frac{2048}{3}\right) - 0\right] = \frac{16}{3}$

USING $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$\mathrm{Var}(X) = \frac{16}{3} - \left(\frac{32}{15}\right)^2 = \frac{176}{225} \approx 0.782$

c) THE MEDIAN (m) SATISFIES THE EQUATION $\int_a^m f(x)\,dx = \frac{1}{2}$

$\Rightarrow \int_0^m \frac{1}{64}x(16-x^2)\,dx = \frac{1}{2}$

$\Rightarrow \int_0^m 16x - x^3\,dx = 32$

$\Rightarrow \left[8x^2 - \frac{1}{4}x^4\right]_0^m = 32$

$\Rightarrow \left(8m^2 - \frac{1}{4}m^4\right) - 0 = 32$

$\Rightarrow 32m^2 - m^4 = 128$

$\Rightarrow 0 = m^4 - 32m^2 + 128$

QUADRATIC FORMULA OR COMPLETING THE SQUARE

$\Rightarrow (m^2 - 16)^2 - 16^2 + 128 = 0$

$\Rightarrow (m^2 - 16)^2 = 128$

$\Rightarrow m^2 - 16 = \pm\sqrt{128}$

$\Rightarrow m^2 = \cancel{16 + \sqrt{128}}$ ← GREATER THAN 4 WHEN SQUARE ROOTED, $16 - \sqrt{128}$

$\Rightarrow m = +\sqrt{16 - \sqrt{128}} \approx 2.165$ AS REQUIRED

d) BY DIFFERENTIATION

$f(x) = \frac{1}{64}x(16-x^2) = \frac{1}{64}(16x - x^3)$

$f'(x) = \frac{1}{64}(16 - 3x^2)$

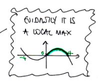
SOLVING FOR ZERO YIELDS

$\frac{1}{64}(16 - 3x^2) = 0$

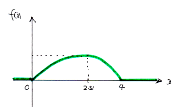
$16 = 3x^2$

$x^2 = \frac{16}{3}$

$x = +\frac{4}{\sqrt{3}} \approx 2.31$



e) SKETCHING THE CUBIC $f(x) = \frac{1}{64}x(16-x^2) = \frac{1}{64}x(4-x)(4+x)$



f) USING THE AVERAGES

MEAN < MEDIAN < MODE
2.13 2.165 2.31

∴ NEGATIVE SKEW

Question 6 (**+)**

The continuous random variable X has probability density function $f(x)$, given by

$$f(x) = \begin{cases} kx(a-x) & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

where k and a are positive constants.

A statistician claims that $a \ge 4$.

a) Justify the statistician's claim.

b) Show clearly that

$$k = \frac{3}{8(3a-8)}.$$

It is further given that $\mathrm{E}(X) = 2.4$.

c) Show further that

$$k = \frac{9}{80(a-3)}.$$

d) Hence determine the value of a and the value of k.

e) Sketch the graph of $f(x)$ for all x and hence state the mode of X.

$\boxed{\text{FS}}$, $\boxed{a=6}$, $\boxed{k=\frac{3}{80}=0.0375}$, $\boxed{\text{mode}=3}$

a) "ARGUING" AS FOLLOWS SINCE $f(x) \geqslant 0$ FOR $0 \leqslant x \leqslant 4$

$k > 0$, $0 \leqslant x \leqslant 4$, & $a - x \geqslant 0$

$\therefore$ a HAS TO BE AT LEAST 4, OTHERWISE $f(4) = 4k(a-4) < 0$

b) USING THE FACT THAT $\int_a^b f(x)\,dx = 1$

$\Rightarrow \int_0^4 kx(a-x)\,dx = 1$

$\Rightarrow k\int_0^4 ax - x^2\,dx = 1$

$\Rightarrow k\left[\frac{1}{2}ax^2 - \frac{1}{3}x^3\right]_0^4 = 1$

$\Rightarrow k\left[\left(8a - \frac{64}{3}\right) - 0\right] = 1$ $\times 3$

$\Rightarrow k\left[24a - 64\right] = 3$

$\Rightarrow k = \frac{3}{24a-64}$

$\Rightarrow k = \frac{3}{8(3a-8)}$

c) USING $\mathrm{E}(X) = \int_a^b x f(x)\,dx$

$\Rightarrow \int_0^4 x \times kx(a-x)\,dx = 2.4$

$\Rightarrow k\int_0^4 ax^2 - x^3\,dx = \frac{12}{5}$

$\Rightarrow k\left[\frac{1}{3}ax^3 - \frac{1}{4}x^4\right]_0^4 = \frac{12}{5}$

$\Rightarrow k\left[\left(\frac{64}{3}a - 64\right) - 0\right] = \frac{12}{5}$

$\Rightarrow k\left(\frac{64}{3}a - 64\right) = \frac{12}{5}$ $\times 3$

$\Rightarrow k\left(64a - 192\right) = \frac{36}{5}$

$\Rightarrow 64k(a-3) = \frac{36}{5}$ $\div 4$

$\Rightarrow 16k(a-3) = \frac{9}{5}$

$\Rightarrow k = \frac{9}{80(a-3)}$ AS REQUIRED

d) SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l} k = \frac{3}{8(3a-8)} \\ k = \frac{9}{80(a-3)} \end{array}\right\}$$

$\Rightarrow \frac{3}{8(3a-8)} = \frac{9}{80(a-3)}$ $\div 3, \times 8$

$\Rightarrow \frac{1}{3a-8} = \frac{3}{10(a-3)}$

$\Rightarrow 10(a-3) = 3(3a-8)$

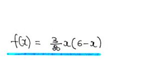
$\Rightarrow 10a - 30 = 9a - 24$

$\Rightarrow a = 6$

& $k = \frac{9}{80(6-3)}$

$k = \frac{3}{80}$

e) $f(x) = \frac{3}{80}x(6-x)$



$\therefore$ MODE IS 3

Question 7 (****+)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x) = \begin{cases} \frac{2}{21}x & 0 \le x \le k \\ \frac{2}{15}(6-x) & k < x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

Determine the value of the positive constant k, and hence or otherwise find

$$\text{P}\left[X < \tfrac{1}{3}k \,\middle|\, X < k\right].$$

$\boxed{k = 3.5}$, $\boxed{\text{P}\left[X < \tfrac{1}{3}k \,\middle|\, X < k\right] = \frac{1}{9}}$

● USING THE FACT $\int_a^b f(x)\,dx = 1$

$\Rightarrow \int_0^k \frac{2}{21}x\,dx + \int_k^6 \frac{2}{15}(6-x)\,dx = 1$

MULTIPLY THE EQUATION BY 105, THE LCM OF 21 & 15

$\Rightarrow \int_0^k 10x\,dx + \int_k^6 14(6-x)\,dx = 105$

$\Rightarrow \int_0^k 10x\,dx + \int_k^6 84 - 14x\,dx = 105$

$\Rightarrow \left[5x^2\right]_0^k + \left[84x - 7x^2\right]_k^6 = 105$

$\Rightarrow (5k^2 - 0) + \left[(504 - 252) - (84k - 7k^2)\right] = 105$

$\Rightarrow 5k^2 + 252 - 84k + 7k^2 = 105$

$\Rightarrow 12k^2 - 84k + 147 = 0$ $\div 3$

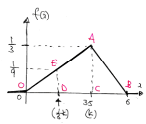
$\Rightarrow 4k^2 - 28k + 49 = 0$

NOTICING THE PERFECT SQUARE OR USE THE QUADRATIC FORMULA

$\Rightarrow (2k-7)^2 = 0$

$\Rightarrow k = \frac{7}{2}$

● SKETCHING THE P.D.F NEXT



WITHOUT ACTUAL EVALUATIONS OF PROBABILITIES (SIMILAR TRIANGLES)

AREA OF $\hat{OAC}$ = 9 × AREA $\hat{OED}$

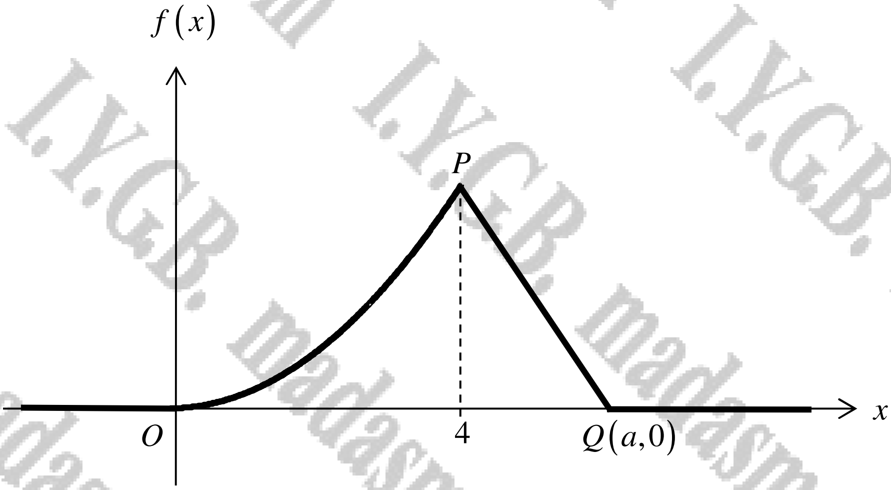
(SCALE FACTOR 3)

$\Rightarrow P(X < \frac{1}{3}k \mid X < k) = \frac{1}{9}$

ALTERNATIVE BY CALCULATION (INTEGRATION AND NO DIAGRAM)

- $P(X<k) = P(X<3.5) = \int_0^{3.5} \frac{2}{21}x\,dx = \left[\frac{1}{21}x^2\right]_0^{3.5}$
  $= \frac{1}{21}(3.5)^2 - 0 = \frac{7}{12}$
- $P(X < \frac{1}{3}k) = P(X < \frac{3.5}{3}) = P(X < \frac{7}{6}) = \int_0^{\frac{7}{6}} \frac{2}{21}x\,dx$
  $= \left[\frac{1}{21}x^2\right]_0^{\frac{7}{6}} = \frac{1}{21}\left(\frac{7}{6}\right)^2 - 0 = \frac{7}{108}$
- $P(X < \frac{1}{3}k \mid X<k) = \frac{\frac{7}{108}}{\frac{7}{12}} = \frac{7 \times 12}{7 \times 108} = \frac{12}{108} = \frac{1}{9}$ AS BEFORE

Question 8 (****+)



The figure above shows the graph of the probability density function $f(x)$ of a continuous random variable X.

The graph consists of the curved segment OP with equation

$$f(x) = kx^2, \quad 0 \le x \le 4,$$

where k is a positive constant.

The graph of $f(x)$ further consists ofm a straight line segment from P to $Q(a,0)$, for $4 < x \le a$, where a is a positive constant.

For all other values of x, $f(x) = 0$.

a) State the mode of X.

It is given that the mode of X is equal to the median of X.

b) Show that $k = \dfrac{3}{128}$, and find the value of a.

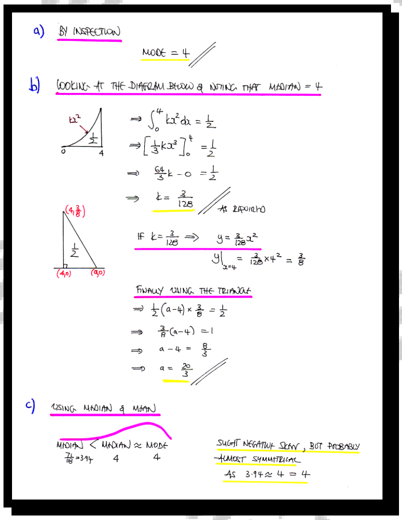
[continues overleaf]

[continued from overleaf]

It is further given that $\mathrm{E}(X)=\dfrac{71}{18}$.

c) Determine with justification the skewness of the distribution.

$\boxed{\quad\quad}$, $\boxed{\text{mode}=4}$, $\boxed{a=\dfrac{20}{3}}$, $\boxed{\text{mean} < \text{median} = \text{mode} \Rightarrow \text{negative skew}}$



a) BY INSPECTION

MODE $= 4$

b) LOOKING AT THE DIAGRAM BELOW & NOTING THAT MEDIAN $= 4$

$\Rightarrow \int_0^4 kx^2\,dx = \frac{1}{2}$

$\Rightarrow \left[\frac{1}{3}kx^3\right]_0^4 = \frac{1}{2}$

$\Rightarrow \frac{64}{3}k - 0 = \frac{1}{2}$

$\Rightarrow k = \frac{3}{128}$ AS REQUIRED

IF $k=\frac{3}{128} \Rightarrow y=\frac{3}{128}x^2$

$y\big|_{x=4} = \frac{3}{128}\times 4^2 = \frac{3}{8}$

FINALLY USING THE TRIANGLE

$\Rightarrow \frac{1}{2}(a-4)\times\frac{3}{8} = \frac{1}{2}$

$\Rightarrow \frac{3}{8}(a-4) = 1$

$\Rightarrow a-4 = \frac{8}{3}$

$\Rightarrow a = \frac{20}{3}$

c) USING MEDIAN & MEAN

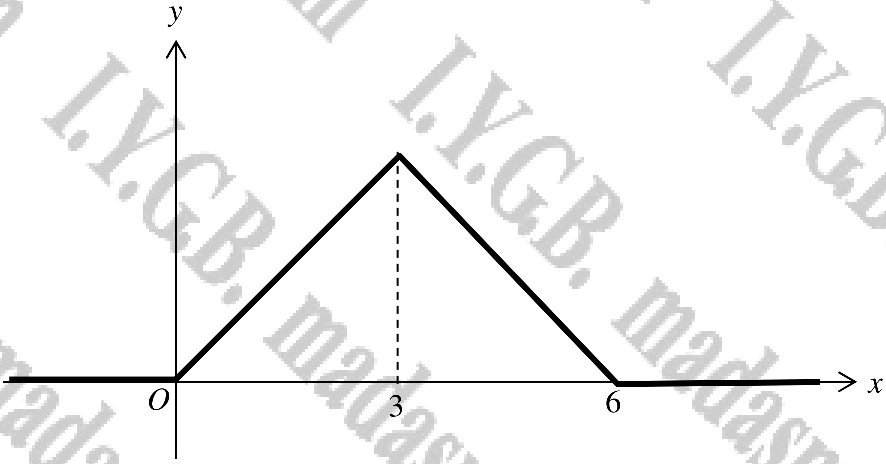
MEAN < MEDIAN ≈ MODE

$\frac{71}{18}=3.94$ 4   4

SLIGHT NEGATIVE SKEW, BUT PROBABLY ALMOST SYMMETRICAL

AS $3.94 \approx 4 = 4$

Question 9 (**+)**



The figure above shows the graph of a probability density function $f(x)$ of a continuous random variable X.

The graph consists of two straight line segments of **equal** length joined up at the point where $x = 3$.

The probability density function $f(x)$ is fully specified as

$$f(x) = \begin{cases} ax & 0 \le x \le 3 \\ b + cx & 3 < x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

where a, b and c are non zero constants.

a) Show that $b = \frac{2}{3}$, $c = -\frac{1}{9}$ and find the value of a.

b) State the value of $\mathrm{E}(X)$.

c) Show that $\mathrm{Var}(X) = 1.5$.

[continues overleaf]

[continued from overleaf]

d) Determine the upper and lower quartile of X.

A statistician claims that $P(|X-\mu|<\sigma)>0.5$.

e) Show that the statistician's claim is correct.

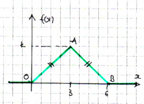
$\boxed{}$, $\boxed{a=\frac{1}{9}}$, $\boxed{\mathrm{E}(X)=3}$, $\boxed{Q_1 \approx 2.121}$, $\boxed{Q_3 \approx 3.879}$

a) PROCEED AS FOLLOWS

$\frac{1}{2}\times 6\times k = 1$

$3k = 1$

$k = \frac{1}{3}$



GRADIENT OF OA $= \frac{1/3}{3} = \frac{1}{9} \implies a = \frac{1}{9}$

GRADIENT OF AB $= -\frac{1}{9}$ BY SYMMETRY $\implies c = -\frac{1}{9}$

EQUATION OF AB : "$y = mx + c$" $\implies y = b - \frac{1}{9}x$

$\implies 0 = b - \frac{1}{9}\times 6 \quad (6,0)$

$\implies b = \frac{2}{3}$

b) $E(X) = 3$ BY SYMMETRY

c) $E(X^2) = \int_{x_1}^{x_2} x^2 f(x)\,dx$

$E(X^2) = \int_0^3 x^2\left(\frac{1}{9}x\right)dx + \int_3^6 x^2\left(\frac{2}{3}-\frac{1}{9}x\right)dx$

$= \frac{1}{9}\int_0^3 x^3\,dx + \int_3^6 \frac{2}{3}x^2 - \frac{1}{9}x^3\,dx$

$= \frac{1}{36}\left[x^4\right]_0^3 + \left[\frac{2}{9}x^3 - \frac{1}{36}x^4\right]_3^6$

$= \left(\frac{81}{36}-0\right) + \left[(48-36)-\left(6-\frac{81}{36}\right)\right]$

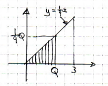
$= \frac{21}{2}$

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$\mathrm{Var}(X) = 10.5 - 3^2$

$\mathrm{Var}(X) = 1.5$ AS REQUIRED

d) LOOKING AT THE DIAGRAM BELOW



$\frac{1}{2}\times Q \times \frac{1}{9}Q = \frac{1}{4}$

$\frac{1}{18}Q^2 = \frac{1}{4}$

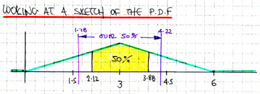
$Q^2 = \frac{9}{2}$

$Q_1 = \frac{3}{2}\sqrt{2} \approx 2.12$

& $Q_3 \approx 3.88$

BY SYMMETRY $6 - 2.12$

e) LOOKING AT A SKETCH OF THE P.D.F



$\sigma = \sqrt{1.5}\,,\ \mu = 3$

$\Rightarrow P(|X-\mu|<\sigma) = P(\mu-\sigma < X < \mu+\sigma)$

$= P(3-\sqrt{1.5} < X < 3+\sqrt{1.5})$

$= P(1.78 < X < 4.22) > 0.5$

THE STATISTICIAN IS CORRECT

Question 10 (****+)

The continuous random variable X has probability density function $f(x)$, given by

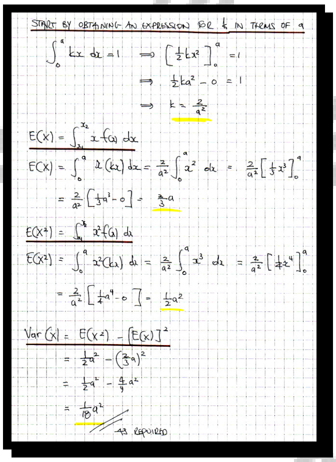
$$f(x) = \begin{cases} kx & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

where a and k are positive constants.

Show, by a detailed method, that

$$\text{Var}(X) = \frac{1}{18}a^2 .$$

[ ] , [proof]



START BY OBTAINING AN EXPRESSION FOR k IN TERMS OF a

$$\int_0^a kx\,dx = 1 \implies \left[\tfrac{1}{2}kx^2\right]_0^a = 1$$
$$\implies \tfrac{1}{2}ka^2 - 0 = 1$$
$$\implies k = \frac{2}{a^2}$$

$$E(X) = \int_{x_1}^{x_2} x f(x)\,dx$$

$$E(X) = \int_0^a x(kx)\,dx = \frac{2}{a^2}\int_0^a x^2\,dx = \frac{2}{a^2}\left[\tfrac{1}{3}x^3\right]_0^a$$
$$= \frac{2}{a^2}\left[\tfrac{1}{3}a^3 - 0\right] = \frac{2}{3}a$$

$$E(X^2) = \int_{x_1}^{x_2} x^2 f(x)\,dx$$

$$E(X^2) = \int_0^a x^2(kx)\,dx = \frac{2}{a^2}\int_0^a x^3\,dx = \frac{2}{a^2}\left[\tfrac{1}{4}x^4\right]_0^a$$
$$= \frac{2}{a^2}\left[\tfrac{1}{4}a^4 - 0\right] = \frac{1}{2}a^2$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$
$$= \tfrac{1}{2}a^2 - \left(\tfrac{2}{3}a\right)^2$$
$$= \tfrac{1}{2}a^2 - \tfrac{4}{9}a^2$$
$$= \tfrac{1}{18}a^2$$

AS REQUIRED

P.D.F. to C.D.F CALCULATIONS

Question 1 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x) = \begin{cases} \frac{2}{9}(5-x) & 2 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

The cumulative distribution function of X, is denoted by $F(x)$.

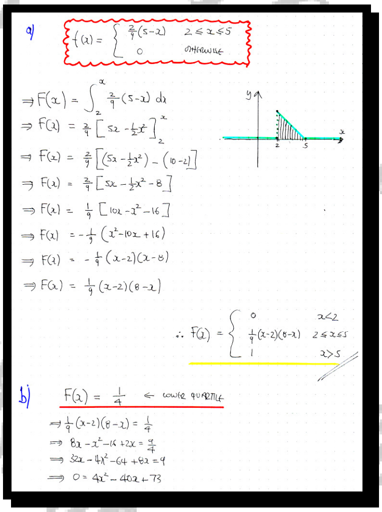
a) Find and specify fully $F(x)$.

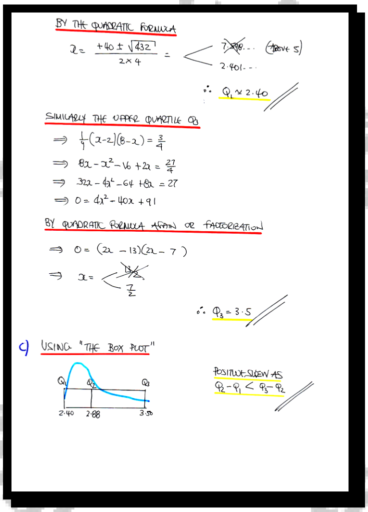
b) Use $F(x)$, to show that the lower quartile of X is approximately 2.40, and find the value of the upper quartile.

c) Given that the median of X is 2.88, comment on the skewness of X.

$$\boxed{F(x) = \begin{cases} 0 & x < 2 \\ -\frac{1}{9}(x-8)(x-2) & 2 \le x \le 5 \\ 1 & x > 5 \end{cases}}, \quad \boxed{Q_3 = 3.5},$$

$$\boxed{Q_2 - Q_1 < Q_3 - Q_2 \Rightarrow \text{+ve skew}}$$





Question 2 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x) = \begin{cases} kx(x-3) & 3 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Use integration to show that $k = \dfrac{6}{11}$.

b) Calculate the value of $\mathrm{E}(X)$.

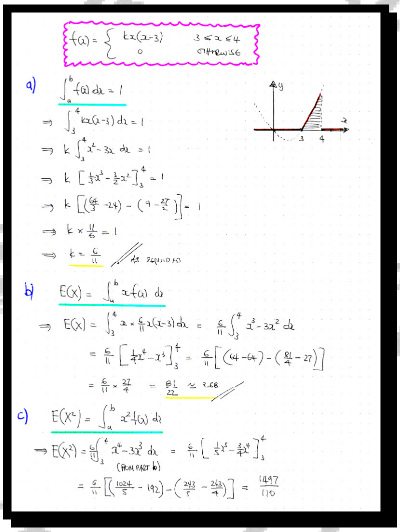
c) Show that $\mathrm{Var}(X) = 0.053$, correct to three decimal places.

The cumulative distribution function of X, is denoted by $F(x)$.

d) Find and specify fully $F(x)$.

e) Show that the median of X lies between 3.70 and 3.75.

$\boxed{}$, $\boxed{\mathrm{E}(X) = \dfrac{81}{22} \approx 3.68}$, $\boxed{F(x) = \begin{cases} 0 & x < 3 \\ \dfrac{1}{11}\left(2x^3 - 9x^2 + 27\right) & 3 \le x \le 4 \\ 1 & x > 4 \end{cases}}$



$f(x) = \begin{cases} kx(x-3) & 3 \le x \le 4 \\ 0 & \text{OTHERWISE} \end{cases}$

a) $\int_a^b f(x)\,dx = 1$

$\Rightarrow \int_3^4 kx(x-3)\,dx = 1$

$\Rightarrow k\int_3^4 x^2 - 3x\,dx = 1$

$\Rightarrow k\left[\frac{1}{3}x^3 - \frac{3}{2}x^2\right]_3^4 = 1$

$\Rightarrow k\left[\left(\frac{64}{3} - 24\right) - \left(9 - \frac{27}{2}\right)\right] = 1$

$\Rightarrow k \times \frac{11}{6} = 1$

$\Rightarrow k = \frac{6}{11}$ // AS REQUIRED

b) $E(X) = \int_a^b x f(x)\,dx$

$\Rightarrow E(X) = \int_3^4 x \times \frac{6}{11}x(x-3)\,dx = \frac{6}{11}\int_3^4 x^3 - 3x^2\,dx$

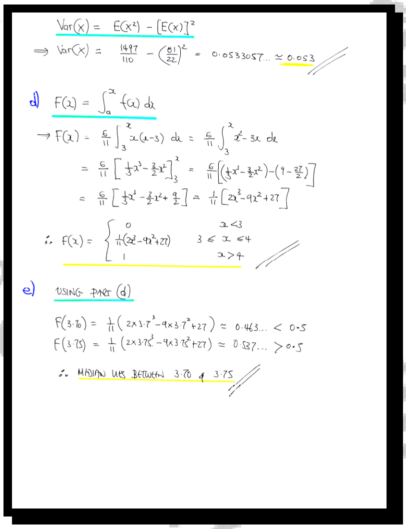
$= \frac{6}{11}\left[\frac{1}{4}x^4 - x^3\right]_3^4 = \frac{6}{11}\left[(64-64) - \left(\frac{81}{4} - 27\right)\right]$

$= \frac{6}{11} \times \frac{27}{4} = \frac{81}{22} \approx 3.68$

c) $E(X^2) = \int_a^b x^2 f(x)\,dx$

$\rightarrow E(X^2) = \frac{6}{11}\int_3^4 x^4 - 3x^3\,dx = \frac{6}{11}\left[\frac{1}{5}x^5 - \frac{3}{4}x^4\right]_3^4$ (FROM PART b)

$= \frac{6}{11}\left[\left(\frac{1024}{5} - 192\right) - \left(\frac{243}{5} - \frac{243}{4}\right)\right] = \frac{1497}{110}$



$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$\Rightarrow \mathrm{Var}(X) = \frac{1497}{110} - \left(\frac{81}{22}\right)^2 = 0.0533057\ldots \approx 0.053$

d) $F(x) = \int_a^x f(t)\,dt$

$\rightarrow F(x) = \frac{6}{11}\int_3^x t(t-3)\,dt = \frac{6}{11}\int_3^x t^2 - 3t\,dt$

$= \frac{6}{11}\left[\frac{1}{3}t^3 - \frac{3}{2}t^2\right]_3^x = \frac{6}{11}\left[\left(\frac{1}{3}x^3 - \frac{3}{2}x^2\right) - \left(9 - \frac{27}{2}\right)\right]$

$= \frac{6}{11}\left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{9}{2}\right] = \frac{1}{11}\left[2x^3 - 9x^2 + 27\right]$

$\therefore F(x) = \begin{cases} 0 & x < 3 \\ \frac{1}{11}(2x^3 - 9x^2 + 27) & 3 \le x \le 4 \\ 1 & x > 4 \end{cases}$

e) USING PART (d)

$F(3.70) = \frac{1}{11}(2 \times 3.7^3 - 9 \times 3.7^2 + 27) = 0.463\ldots < 0.5$

$F(3.75) = \frac{1}{11}(2 \times 3.75^3 - 9 \times 3.75^2 + 27) = 0.537\ldots > 0.5$

$\therefore$ MEDIAN LIES BETWEEN 3.70 & 3.75

Question 3 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x) = \begin{cases} k(x+2)^2 & -2 \le x \le 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Show clearly that $k = \frac{3}{8}$.

b) Find the value of $\mathrm{E}(X)$.

c) Show that $\mathrm{Var}(X) = 0.15$.

The cumulative distribution function of X, is denoted by $F(x)$.

d) Find and specify fully $F(x)$.

e) Determine $P(-1 \le X \le 1)$.

$\boxed{}$, $\boxed{\mathrm{E}(X) = -0.5}$, $\boxed{F(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{8}x^3 + \frac{3}{4}x^2 + \frac{3}{2}x + 1 & -2 \le x \le 0 \\ 1 & x > 0 \end{cases}}$,

$\boxed{P(-1 \le X \le 1) = 0.875}$

a) USING $\int_a^b f(x)\,dx = 1$

$\int_{-2}^{0} k(x+2)^2\,dx = 1 \Rightarrow \left[\frac{1}{3}k(x+2)^3\right]_{-2}^{0} = 1$

$\Rightarrow \frac{8}{3}k - 0 = 1$

$\Rightarrow k = \frac{3}{8}$

b) $E(X) = \int_a^b x f(x)\,dx$

$E(X) = \int_{-2}^{0} x \times \frac{3}{8}(x+2)^2\,dx = \frac{3}{8}\int_{-2}^{0} x^3 + 4x^2 + 4x\,dx$

$= \frac{3}{8}\left[\frac{1}{4}x^4 + \frac{4}{3}x^3 + 2x^2\right]_{-2}^{0} = 0 - \frac{3}{8}\left[4 - \frac{32}{3} + 8\right] = -\frac{1}{2}$

c) FIRST FIND $E(X^2) = \int_a^b x^2 f(x)\,dx$

$E(X^2) = \int_{-2}^{0} x^2 \times \frac{3}{8}(x+2)^2\,dx = \frac{3}{8}\int_{-2}^{0} x^4 + 4x^3 + 4x^2\,dx$

$= \frac{3}{8}\left[\frac{1}{5}x^5 + x^4 + \frac{4}{3}x^3\right]_{-2}^{0} = 0 - \frac{3}{8}\left[-\frac{32}{5} + 16 - \frac{32}{3}\right] = \frac{2}{5}$

USING $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$\mathrm{Var}(X) = \frac{2}{5} - \left(-\frac{1}{2}\right)^2 = \frac{3}{20}$ AS REQUIRED

d) $F(x) = \int_a^x f(x)\,dx$

$F(x) = \int_{-2}^{x} \frac{3}{8}(x+2)^2\,dx = \left[\frac{1}{8}(x+2)^3\right]_{-2}^{x} = \frac{1}{8}(x+2)^3 - 0 = \frac{1}{8}(x+2)^3$

$\therefore F(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{8}(x+2)^3 & -2 \le x \le 0 \\ 1 & x > 0 \end{cases}$

e) USING PART (d)

$P(-1 \le X \le 1) = P(-1 \le X \le 0)$

$= 1 - P(X \le -1)$

$= 1 - F(-1)$

$= 1 - \frac{1}{8}(-1+2)^3$

$= 1 - \frac{1}{8}$

$= \frac{7}{8} = 0.875$

Question 4 (****)

The continuous random variable X has probability density function $f(x)$, given by:

$$f(x) = \begin{cases} \frac{1}{60}x^3 & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of $\mathrm{E}(X)$.

b) Show that the standard deviation of X is 0.516, correct to 3 decimal places.

The cumulative distribution function of X, is denoted by $F(x)$.

c) Find and specify fully $F(x)$.

d) Determine $P(X > 3.5)$.

e) Calculate the median of X, correct to two decimal places.

$\boxed{\mathrm{E}(X) = \frac{248}{75} \approx 3.31}$, $\boxed{F(x) = \begin{cases} 0 & x < 2 \\ \frac{1}{240}\left(x^4 - 16\right) & 2 \le x \le 4 \\ 1 & x > 4 \end{cases}}$,

$\boxed{P(X > 3.5) = \frac{113}{256}}$, $\boxed{\text{median} \approx 3.41}$

a) $\mathrm{E}(X) = \int_a^b x f(x)\,dx$

$\mathrm{E}(X) = \int_2^4 x\left(\frac{1}{60}x^3\right)dx = \int_2^4 \frac{1}{60}x^4 = \left[\frac{1}{300}x^5\right]_2^4$

$= \frac{1}{300}\left[1024 - 32\right] = \frac{248}{75} \approx 3.31$

b) START BY FINDING $\mathrm{E}(X^2) = \int_a^b x^2 f(x)\,dx$

$\mathrm{E}(X^2) = \int_2^4 x^2\left(\frac{1}{60}x^3\right)dx = \int_2^4 \frac{1}{60}x^5\,dx = \frac{1}{360}\left[x^6\right]_2^4$

$= \frac{1}{360}\left[4096 - 64\right] = \frac{56}{5}$

NOW USING $\mathrm{Var}(X) = \mathrm{E}(X^2) - \left[\mathrm{E}(X)\right]^2$

$\mathrm{Var}(X) = \frac{56}{5} - \left(\frac{248}{75}\right)^2$

$\mathrm{Var}(X) = 0.2659555\ldots$

$\therefore$ STANDARD DEVIATION $= \sqrt{0.2659555\ldots}$

≈ 0.516 // 3 d.p

c) $F(x) = \int_a^x f(x)\,dx$

$F(x) = \int_2^x \frac{1}{60}x^3\,dx = \left[\frac{1}{240}x^4\right]_2^x = \frac{1}{240}\left(x^4 - 16\right)$

$\therefore F(x) = \begin{cases} 0 & x < 2 \\ \frac{1}{240}(x^4 - 16) & 2 \le x \le 4 \\ 1 & x > 4 \end{cases}$

d) $P(X > 3.5) = 1 - P(X < 3.5)$

$= 1 - F(3.5)$

$= 1 - \frac{1}{240}\left(3.5^4 - 16\right)$

$= \frac{113}{256}$ //

e) SOLVING $F(x) = \frac{1}{2}$

$\frac{1}{240}\left(x^4 - 16\right) = \frac{1}{2}$

$x^4 - 16 = 120$

$x^4 = 136$

$x = +\sqrt[4]{136}$

$x \approx 3.41$ // 2 d.p.

Question 5 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases} kx(6-x) & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

a) Show clearly that $k=\dfrac{3}{100}$.

b) Calculate the value of $\mathrm{E}(X)$.

c) Find the mode of X.

The cumulative distribution function of X, is denoted by $F(x)$.

d) Find and specify fully $F(x)$.

e) Verify that the median of X lies between 2.85 and 2.9.

f) Determine with justification the skewness of the distribution.

$\boxed{\mathrm{E}(X)=\dfrac{45}{16}}$, $\boxed{\text{mode}=3}$, $\boxed{F(x)=\begin{cases} 0 & x<0 \\ \dfrac{1}{100}x^2(9-x) & 0 \le x \le 5 \\ 1 & x>5 \end{cases}}$,

$\boxed{\text{mean} < \text{median} < \text{mode} \Rightarrow \text{negative skew}}$

(a) $\int_a^b f(x)\,dx = 1$

$\Rightarrow \int_0^5 kx(6-x)\,dx = 1$

$\Rightarrow k\int_0^5 6x - x^2\,dx = 1$

$\Rightarrow k\left[3x^2 - \frac{1}{3}x^3\right]_0^5 = 1$

$\Rightarrow k\left[75 - \frac{5}{3}\right] = 1$

$\Rightarrow k \times \frac{100}{3} = 1$

$\Rightarrow k = \frac{3}{100}$ AS REQUIRED

(b) $E(X) = \int_a^b x f(x)\,dx$

$= \int_0^5 x \times \frac{3}{100}x(6-x)\,dx$

$= \frac{3}{100}\int_0^5 6x^2 - x^3\,dx$

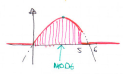
$= \frac{3}{100}\left[2x^3 - \frac{1}{4}x^4\right]_0^5$

$= \frac{3}{100}\left[\left(250 - \frac{625}{4}\right) - 0\right]$

$= \frac{45}{16}$ $(= 2.8125)$

(c) BY DIFFERENTIATION OR USING GRAPH

BY SYMMETRY, MODE IS 3



(d) $F(x) = \int_a^x f(x)\,dx = \int_0^x \frac{3}{100}x(6-x) = \frac{3}{100}\int_0^x 6x - x^2\,dx$

$= \frac{3}{100}\left[3x^2 - \frac{1}{3}x^3\right]_0^x = \frac{3}{100}\left[\left(3x^2 - \frac{1}{3}x^3\right) - 0\right] = \frac{3}{100}\left(3x^2 - \frac{1}{3}x^3\right)$

$= \frac{1}{100}\left[9x^2 - x^3\right] = \frac{1}{100}x^2(9-x)$

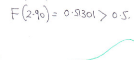
$\therefore F(x) = \begin{cases} 0 & x<0 \\ \frac{1}{100}x^2(9-x) & 0 \le x \le 5 \\ 1 & x>5 \end{cases}$

(e) $F(2.85) = 0.49953375 < 0.5$

$F(2.90) = 0.51301 > 0.5$

HENCE MEDIAN LIES BETWEEN 2.85 & 2.90

(f)



MEAN < MEDIAN < MODE $\Longrightarrow$ NEGATIVE SKEW

2.8125   2.85–2.90   3

Question 6 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x) = \begin{cases} k(x-1)(x-4) & 1 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Show clearly that $k = -\frac{2}{9}$

b) Sketch the graph of $f(x)$, for all x.

c) State the value of $\mathrm{E}(X)$.

d) Calculate the $\mathrm{Var}(X)$.

The cumulative distribution function of X, is denoted by $F(x)$.

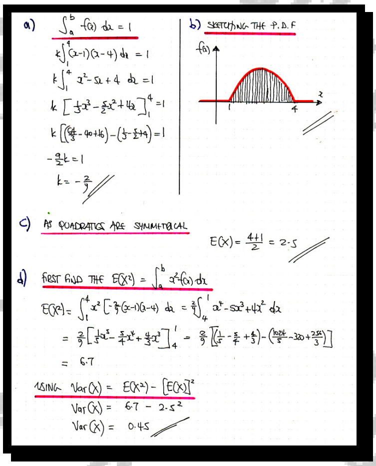
e) Find and specify fully $F(x)$.

f) Determine with justification the skewness of the distribution.

$\boxed{\mathrm{E}(X) = 2.5}$, $\boxed{\mathrm{Var}(X) = 0.45}$

$$\boxed{F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{27}\left(11 - 24x + 15x^2 - 2x^3\right) & 1 \le x \le 4 \\ 1 & x > 4 \end{cases}},$$

$\boxed{\text{mean} = \text{median} = \text{mode} = 2.5 \Rightarrow \text{zero skew}}$



a) $\int_a^b f(x)\,dx = 1$

$k\int_1^4 (x-1)(x-4)\,dx = 1$

$k\int_1^4 x^2 - 5x + 4\,dx = 1$

$k\left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x\right]_1^4 = 1$

$k\left[\left(\frac{64}{3} - 40 + 16\right) - \left(\frac{1}{3} - \frac{5}{2} + 4\right)\right] = 1$

$-\frac{9}{2}k = 1$

$k = -\frac{2}{9}$

b) SKETCHING THE P.D.F

c) AS QUADRATICS ARE SYMMETRICAL

$E(X) = \frac{4+1}{2} = 2.5$

d) FIRST FIND THE $E(X^2) = \int_a^b x^2 f(x)\,dx$

$E(X^2) = \int_1^4 x^2\left[-\frac{2}{9}(x-1)(x-4)\right]dx = \frac{2}{9}\int_4^1 x^4 - 5x^3 + 4x^2\,dx$

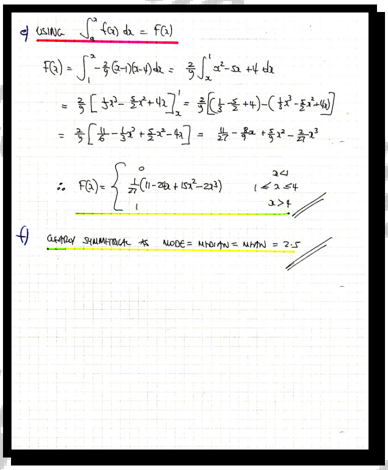
$= \frac{2}{9}\left[\frac{1}{5}x^5 - \frac{5}{4}x^4 + \frac{4}{3}x^3\right]_4^1 = \frac{2}{9}\left[\left(\frac{1}{5} - \frac{5}{4} + \frac{4}{3}\right) - \left(\frac{1024}{5} - 320 + \frac{256}{3}\right)\right]$

$= 6.7$

USING $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$\mathrm{Var}(X) = 6.7 - 2.5^2$

$\mathrm{Var}(X) = 0.45$



e) USING $\int_a^x f(x)\,dx = F(x)$

$F(x) = \int_1^x -\frac{2}{9}(x-1)(x-4)\,dx = \frac{2}{9}\int_x^1 x^2 - 5x + 4\,dx$

$= \frac{2}{9}\left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x\right]_x^1 = \frac{2}{9}\left[\left(\frac{1}{3} - \frac{5}{2} + 4\right) - \left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x\right)\right]$

$= \frac{2}{9}\left[\frac{11}{6} - \frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x\right] = \frac{11}{27} - \frac{8}{9}x + \frac{5}{9}x^2 - \frac{2}{27}x^3$

$\therefore F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{27}(11 - 24x + 15x^2 - 2x^3) & 1 \le x \le 4 \\ 1 & x > 4 \end{cases}$

f) CLEARLY SYMMETRICAL AS MODE = MEDIAN = MEAN = 2.5

Question 7 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases}\dfrac{2+x}{k} & 2 \le x \le 5 \\ 0 & \text{otherwise}\end{cases}$$

a) Show clearly that $k = \dfrac{33}{2}$.

b) Find the value of $\mathrm{E}(X)$.

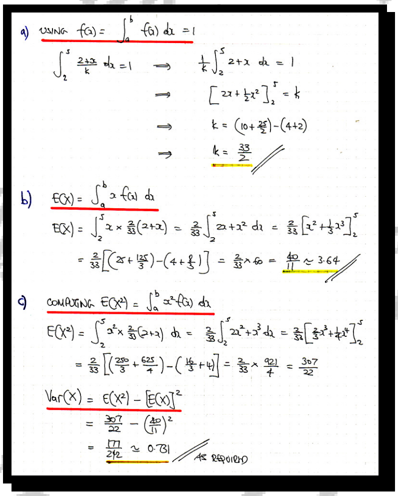
c) Show that $\mathrm{Var}(X) = 0.731$, correct to three decimal places.

The cumulative distribution function of X, is denoted by $F(x)$.

d) Find and specify fully $F(x)$.

e) Determine the median of X.

$\boxed{}$, $\boxed{\mathrm{E}(X)=\dfrac{40}{11}}$, $\boxed{F(x)=\begin{cases}0 & x<2\\ \dfrac{1}{33}(x+6)(x-2) & 2\le x\le 5\\ 1 & x>5\end{cases}}$, $\boxed{\text{median} \approx 3.70}$



a) USING $f(x) = \int_a^b f(x)\,dx = 1$

$\int_2^5 \frac{2+x}{k}\,dx = 1 \Rightarrow \frac{1}{k}\int_2^5 2+x\,dx = 1$

$\Rightarrow \left[2x+\frac{1}{2}x^2\right]_2^5 = k$

$\Rightarrow k = \left(10+\frac{25}{2}\right)-(4+2)$

$\Rightarrow k = \frac{33}{2}$

b) $E(X) = \int_a^b x f(x)\,dx$

$E(X) = \int_2^5 x \times \frac{2}{33}(2+x) = \frac{2}{33}\int_2^5 2x+x^2\,dx = \frac{2}{33}\left[x^2+\frac{1}{3}x^3\right]_2^5$

$= \frac{2}{33}\left[\left(25+\frac{125}{3}\right)-\left(4+\frac{8}{3}\right)\right] = \frac{2}{33}\times 60 = \frac{40}{11} \simeq 3.64$

c) COMPUTING $E(X^2) = \int_a^b x^2 f(x)\,dx$

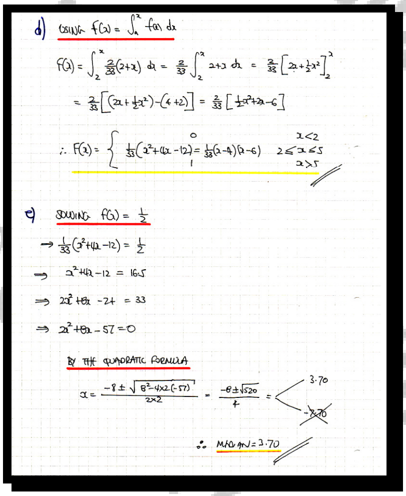
$E(X^2) = \int_2^5 x^2 \times \frac{2}{33}(2+x)\,dx = \frac{2}{33}\int_2^5 2x^2+x^3\,dx = \frac{2}{33}\left[\frac{2}{3}x^3+\frac{1}{4}x^4\right]_2^5$

$= \frac{2}{33}\left[\left(\frac{250}{3}+\frac{625}{4}\right)-\left(\frac{16}{3}+4\right)\right] = \frac{2}{33}\times\frac{921}{4} = \frac{307}{22}$

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$= \frac{307}{22} - \left(\frac{40}{11}\right)^2$

$= \frac{177}{242} \simeq 0.731$ AS REQUIRED



d) USING $F(x) = \int_a^x f(x)\,dx$

$F(x) = \int_2^x \frac{2}{33}(2+x)\,dx = \frac{2}{33}\int_2^x 2+x\,dx = \frac{2}{33}\left[2x+\frac{1}{2}x^2\right]_2^x$

$= \frac{2}{33}\left[\left(2x+\frac{1}{2}x^2\right)-(4+2)\right] = \frac{2}{33}\left[\frac{1}{2}x^2+2x-6\right]$

$\therefore F(x) = \begin{cases} 0 & x<2 \\ \frac{1}{33}(x^2+4x-12) = \frac{1}{33}(x-4)(x-6) & 2\le x\le 5 \\ 1 & x>5\end{cases}$

e) SOLVING $F(x) = \frac{1}{2}$

$\Rightarrow \frac{1}{33}(x^2+4x-12) = \frac{1}{2}$

$\Rightarrow x^2+4x-12 = 16.5$

$\Rightarrow 2x^2+8x-24 = 33$

$\Rightarrow 2x^2+8x-57=0$

BY THE QUADRATIC FORMULA

$x = \frac{-8\pm\sqrt{8^2-4\times2(-57)}}{2\times2} = \frac{-8\pm\sqrt{520}}{4} = \begin{cases}3.70 \\ \cancel{-7.70}\end{cases}$

$\therefore$ MEDIAN $= 3.70$

Question 8 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x)=\begin{cases} \frac{1}{3} & 0 \le x \le 2 \\ \frac{4}{195}x^3 & 2 < x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

a) Show that $\mathrm{E}(X)=1.532$, correct to three decimal places.

The cumulative distribution function of X, is denoted by $F(x)$.

b) Find and specify fully $F(x)$.

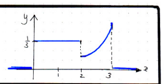
c) Calculate the median of X.

d) Determine with justification the skewness of the distribution.

$$\boxed{F(x)=\begin{cases} 0 & x<0 \\ \frac{1}{3}x & 0 \le x \le 2 \\ \frac{1}{195}x^4+\frac{38}{65} & 2<x\le 3 \\ 1 & x>3 \end{cases}}, \quad \boxed{\text{median} = 1.5},$$

$$\boxed{(\text{mode}) < \text{median} < \text{mean} \Rightarrow \text{slight positive skew}}$$

$$f(x)=\begin{cases} \frac{1}{3} & 0 \le x \le 2 \\ \frac{4}{195}x^3 & 2 \le x \le 3 \\ 0 & \text{OTHERWISE} \end{cases}$$



a) $\mathrm{E}(X) = \int_a^b x f(x)\,dx$

$\mathrm{E}(X) = \int_0^3 x f(x)\,dx = \int_0^2 x\left(\frac{1}{3}\right)dx + \int_2^3 x\left(\frac{4}{195}x^3\right)dx$

$= \int_0^2 \frac{1}{3}x\,dx + \int_2^3 \frac{4}{195}x^4\,dx = \left[\frac{1}{6}x^2\right]_0^2 + \left[\frac{4}{975}x^5\right]_2^3$

$= \left(\frac{2}{3}-0\right) + \left(\frac{324}{325} - \frac{128}{975}\right) = \frac{498}{325} \simeq 1.532$

b) $F(x) = \int_a^x f(x)\,dx$

- FOR $0 \le x \le 2$

$F_1(x) = \int_0^x \frac{1}{3}\,dx = \left[\frac{1}{3}x\right]_0^x = \frac{1}{3}x - 0 = \frac{1}{3}x$

- $F_1(2) = \frac{2}{3}$

- $F_2(x) = \frac{2}{3} + \int_2^x \frac{4}{195}x^3\,dx = \frac{2}{3} + \left[\frac{1}{195}x^4\right]_2^x$

$= \frac{2}{3} + \left[\frac{1}{195}x^4 - \frac{16}{195}\right] = \frac{1}{195}x^4 - \frac{38}{65} = \frac{1}{195}\left[x^4 - 114\right]$

$$F(x)=\begin{cases} 0 & x<0 \\ \frac{1}{3}x & 0 \le x \le 2 \\ \frac{1}{195}(x^4-114) & 2<x\le 3 \\ 1 & x>3 \end{cases}$$

c) FOR MEDIAN $F(x) = \frac{1}{2}$

$\Rightarrow$ MEDIAN LIES IN THE FIRST SECTION, SINCE $F(2) = \frac{2}{3}$

$\Rightarrow \frac{1}{3}x = \frac{1}{2}$

$\Rightarrow x = \frac{3}{2}$

$\Rightarrow Q_2 = 1.5$

d) USING THE AVERAGES

(MODE) < MEDIAN < MEAN

1.5 1.532

$\therefore$ (SLIGHT) POSITIVE SKEW

Question 9 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x) = \begin{cases} \frac{1}{4}x & 0 \le x \le 2 \\ kx^3 & 2 < x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Show clearly that $k = \frac{1}{120}$.

The cumulative distribution function of X, is denoted by $F(x)$.

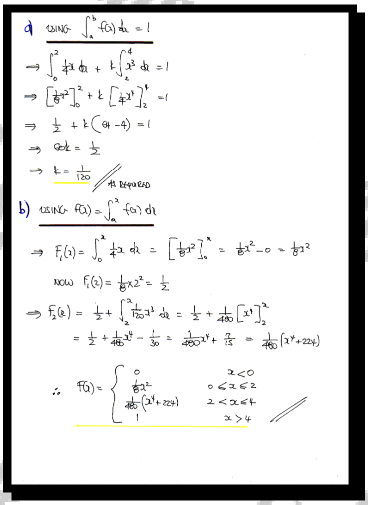
b) Find and specify fully $F(x)$.

c) State the median of X.

d) Show that $\mathrm{E}(X) = \frac{58}{25}$.

e) Calculate the interquartile range of X.

$\boxed{}$, $\boxed{F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{8}x^2 & 0 \le x \le 2 \\ \frac{x^4 + 224}{480} & 2 < x \le 4 \\ 1 & x > 4 \end{cases}}$, $\boxed{\text{median} = 2}$, $\boxed{\text{IQR} \approx 2.00}$



a) USING $\int_a^b f(x)\,dx = 1$

$\Rightarrow \int_0^2 \frac{1}{4}x\,dx + k\int_2^4 x^3\,dx = 1$

$\Rightarrow \left[\frac{1}{8}x^2\right]_0^2 + k\left[\frac{1}{4}x^4\right]_2^4 = 1$

$\Rightarrow \frac{1}{2} + k(64 - 4) = 1$

$\Rightarrow 60k = \frac{1}{2}$

$\Rightarrow k = \frac{1}{120}$ AS REQUIRED

b) USING $F(x) = \int_a^x f(x)\,dx$

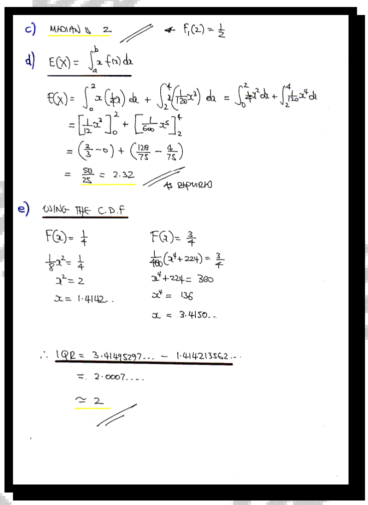
$\Rightarrow F_1(x) = \int_0^x \frac{1}{4}x\,dx = \left[\frac{1}{8}x^2\right]_0^x = \frac{1}{8}x^2 - 0 = \frac{1}{8}x^2$

NOW $F_1(2) = \frac{1}{8} \times 2^2 = \frac{1}{2}$

$\Rightarrow F_2(x) = \frac{1}{2} + \int_2^x \frac{1}{120}x^3\,dx = \frac{1}{2} + \frac{1}{480}\left[x^4\right]_2^x$

$= \frac{1}{2} + \frac{1}{480}x^4 - \frac{1}{30} = \frac{1}{480}x^4 + \frac{7}{15} = \frac{1}{480}(x^4 + 224)$

$\therefore F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{8}x^2 & 0 \le x \le 2 \\ \frac{1}{480}(x^4 + 224) & 2 < x \le 4 \\ 1 & x > 4 \end{cases}$



c) MEDIAN IS 2 $\leftarrow F_1(2) = \frac{1}{2}$

d) $E(X) = \int_a^b x f(x)\,dx$

$E(X) = \int_0^2 x\left(\frac{1}{4}x\right)dx + \int_2^4 x\left(\frac{1}{120}x^3\right)dx = \int_0^2 \frac{1}{4}x^2\,dx + \int_2^4 \frac{1}{120}x^4\,dx$

$= \left[\frac{1}{12}x^3\right]_0^2 + \left[\frac{1}{600}x^5\right]_2^4$

$= \left(\frac{2}{3} - 0\right) + \left(\frac{128}{75} - \frac{4}{75}\right)$

$= \frac{58}{25} = 2.32$ AS REQUIRED

e) USING THE C.D.F

$F(x) = \frac{1}{4}$

$\frac{1}{8}x^2 = \frac{1}{4}$

$x^2 = 2$

$x = 1.4142\ldots$

$F(x) = \frac{3}{4}$

$\frac{1}{480}(x^4 + 224) = \frac{3}{4}$

$x^4 + 224 = 360$

$x^4 = 136$

$x = 3.4150\ldots$

$\therefore$ IQR $= 3.41495297\ldots - 1.414213562\ldots$

$= 2.0007\ldots$

$\simeq 2$

Question 10 (****)

The continuous random variable X has probability density function $f(x)$, given by

$$f(x) = \begin{cases} k\left(x^2 - 2x + 3\right) & 0 \le x < 2 \\ \frac{1}{3}k & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

a) Show clearly that $k = \frac{1}{5}$.

The cumulative distribution function of X, is denoted by $F(x)$.

b) Find and specify fully $F(x)$.

c) Show that $\mathrm{E}(X) = \frac{11}{10}$.

d) Show that the median of X lies between 1.05 and 1.1.

$\boxed{}$, $$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{15}x\left(x^2 - 3x + 9\right) & 0 \le x < 2 \\ \frac{1}{15}(x+12) & 2 \le x \le 3 \\ 1 & x > 4 \end{cases}$$

a) USING $\int_a^b f(x) = 1$

$\int_0^2 k(x^2-2x+3)\,dx + \int_2^3 \frac{1}{3}k\,dx = 1$

$k\left[\frac{1}{3}x^3 - x^2 + 3x\right]_0^2 + \frac{1}{3}k\left[x\right]_2^3 = 1$

$k\left[\left(\frac{8}{3} - 4 + 6\right) - 0\right] + \frac{1}{3}k(3-2) = 1$

$\frac{14}{3}k + \frac{1}{3}k = 1$

$5k = 1$

$k = \frac{1}{5}$ //

b) USING $F(x) = \int_a^x f(x)\,dx$

$F_1(x) = \int_0^x \frac{1}{5}(x^2 - 2x + 3)\,dx = \frac{1}{5}\left[\frac{1}{3}x^3 - x^2 + 3x\right]_0^x$

$= \frac{1}{5}\left[\left(\frac{1}{3}x^3 - x^2 + 3x\right) - 0\right] = \frac{1}{5}\left[\frac{1}{3}x^3 - x^2 + 3x\right]$

$F_1(2) = \frac{1}{5}\left[\frac{8}{3} - 4 + 6\right] = \frac{14}{15}$

$F_2(x) = \frac{14}{15} + \int_2^x \frac{1}{3}k\,dx = \frac{14}{15} + \int_2^x \frac{1}{15}\,dx$

$= \frac{14}{15} + \left[\frac{1}{15}x\right]_2^x = \frac{14}{15} + \left[\frac{1}{15}x - \frac{2}{15}\right]$

$= \frac{1}{15}x + \frac{4}{5} = \frac{1}{15}\left[x + 12\right]$

$$\therefore\ F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{15}\left[x^3 - 3x^2 + 9x\right) & 0 \le x < 2 \\ \frac{1}{15}\left[x + 12\right] & 2 \le x \le 3 \\ 1 & x > 3 \end{cases}$$ //

c) $E(X) = \int_a^b x f(x)\,dx$

$E(X) = \frac{1}{5}\int_0^2 x(x^2 - 2x + 3)\,dx + \frac{1}{5}\int_2^3 \frac{1}{3}x\,dx$

k k

$E(X) = \frac{1}{5}\int_0^2 x^3 - 2x^2 + 3x\,dx + \frac{1}{5}\int_2^3 \frac{1}{3}x\,dx$

$E(X) = \frac{1}{5}\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2\right]_0^2 + \frac{1}{5}\left[\frac{1}{6}x^2\right]_2^3$

$E(X) = \frac{1}{5}\left[\left(4 - \frac{16}{3} + 6\right) - 0\right] + \frac{1}{5}\left[\frac{3}{2} - \frac{2}{3}\right]$

$E(X) = \frac{14}{15} + \frac{1}{6} = \frac{11}{10}$ // AS REQUIRED

d) USING THE C.D.F & NOTING THAT THE MEDIAN MUST LIE BETWEEN 0 & 2 SINCE $F_1(2) = \frac{14}{15} > 0.5$

$F_1(1.05) = \frac{1}{15}\left(1.05^3 - 3\times1.05^2 + 9\times1.05\right) = 0.486675 < 0.5$

$F_1(1.1) = \frac{1}{15}\left(1.1^3 - 3\times1.1^2 + 9\times1.1\right) = 0.506733\ldots > 0.5$

INDEED THE MEDIAN LIES BETWEEN 1.05 & 1.1 //

Question 11 (****+)

The continuous random variable X has probability density function $f(x)$, given by:

$$f(x) = \begin{cases} \frac{1}{10}x & 0 \le x < 4 \\ 2 - \frac{2}{5}x & 4 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch the graph of $f(x)$ for all x.

b) State the mode of X.

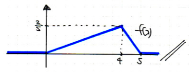
c) Show clearly that $\mathrm{E}(X) = 3$.

d) Calculate the value of $\mathrm{Var}(X)$.

e) Find and specify fully the cumulative distribution function of X, $F(x)$.

$\boxed{}$, $\boxed{\text{mode} = 4}$, $\boxed{\mathrm{Var}(X) = \frac{7}{6}}$, $\boxed{F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{20}x^2 & 0 \le x < 4 \\ -\frac{1}{5}\left(x^2 - 10x + 20\right) & 4 \le x \le 5 \\ 1 & x > 5 \end{cases}}$

a) SKETCHING THE P.D.F CONSISTING OF TWO LINES



b) MODE IS 4

c) $E(X) = \int_a^b x f(x)\,dx$

$E(X) = \int_0^4 x\left(\frac{1}{10}x\right)dx + \int_4^5 x\left(2 - \frac{2}{5}x\right)dx = \int_0^4 \frac{1}{10}x^2\,dx + \int_4^5 2x - \frac{2}{5}x^2\,dx$

$= \left[\frac{1}{30}x^3\right]_0^4 + \left[x^2 - \frac{2}{15}x^3\right]_4^5 = \left(\frac{64}{30} - 0\right) + \left(25 - \frac{250}{15}\right) - \left(16 - \frac{128}{15}\right)$

$= \frac{32}{15} + 25 - \frac{50}{3} - 16 + \frac{128}{15} = 3$ AS REQUIRED

d) FIRST COMPUTE $E(X^2) = \int_a^b x^2 f(x)\,dx$

$E(X^2) = \int_0^4 x^2\left(\frac{1}{10}x\right)dx + \int_4^5 x^2\left(2 - \frac{2}{5}x\right)dx = \int_0^4 \frac{1}{10}x^3\,dx + \int_4^5 2x^2 - \frac{2}{5}x^3\,dx$

$= \left[\frac{1}{40}x^4\right]_0^4 + \left[\frac{2}{3}x^3 - \frac{1}{10}x^4\right]_4^5$

$= \left(\frac{32}{5} - 0\right) + \left(\frac{250}{3} - \frac{125}{2}\right) - \left(\frac{128}{3} - \frac{128}{5}\right) = \frac{61}{6}$

USING $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$\mathrm{Var}(X) = \frac{61}{6} - 3^2$

$\mathrm{Var}(X) = \frac{7}{6}$

$F(x) = \int_a^x f(t)\,dt$

$F_1(x) = \int_0^x \frac{1}{10}t\,dt = \left[\frac{1}{20}t^2\right]_0^x = \frac{1}{20}x^2 - 0 = \frac{1}{20}x^2$

$F_1(4) = \frac{4}{5}$

$F_2(x) = \frac{4}{5} + \int_4^x 2 - \frac{2}{5}t\,dt = \frac{4}{5} + \left[2t - \frac{1}{5}t^2\right]_4^x$

$= \frac{4}{5} + \left[2x - \frac{1}{5}x^2\right] - \left[8 - \frac{16}{5}\right]$

$= \frac{4}{5} + 2x - \frac{1}{5}x^2 - 8 + \frac{16}{5}$

$= -\frac{1}{5}x^2 + 2x - 4$

SPECIFYING

$$\begin{cases} 0 & x < 0 \\ \frac{1}{20}x^2 & 0 \le x < 4 \\ -\frac{1}{5}(x^2 - 10x + 20) & 4 \le x < 5 \\ 1 & x > 5 \end{cases}$$

Question 12 (****+)

The continuous random variable X has probability density function $f(x)$, defined by the piecewise continuous function

$$f(x) = \begin{cases} \frac{1}{12}(x-1) & 1 \le x \le 3 \\ \frac{1}{6} & 3 < x \le 6 \\ \frac{5}{12} - \frac{1}{24}x & 6 < x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

a) Sketch the graph of $f(x)$, for all values of x.

The cumulative distribution function of X, is denoted by $F(x)$.

b) Show by detailed calculations that

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{24}x^2 - \frac{1}{12}x + \frac{1}{24} & 1 \le x \le 3 \\ \frac{1}{6}x - \frac{1}{3} & 3 < x \le 6 \\ -\frac{1}{48}x^2 + \frac{5}{12}x - \frac{13}{12} & 6 < x \le 10 \\ 1 & x > 10 \end{cases}$$

[continues overleaf]

[continued from overleaf]

c) Calculate the median of X.

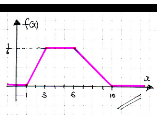
d) Determine the value of $P(2 < X < 4) \cup P(5 < X < 9)$.

e) Find the value for $\mathrm{E}(X)$.

$\boxed{}$, $\boxed{\text{median} = 5}$, $\boxed{P(2 < X < 4) \cup P(5 < X < 9) = \frac{37}{48}}$, $\boxed{\mathrm{E}(X) = \frac{61}{12}}$

a)

$$f(x) = \begin{cases} \frac{1}{12}(x-1) & 1 \leqslant x \leqslant 3 \\ \frac{1}{6} & 3 \leqslant x \leqslant 6 \\ \frac{5}{12} - \frac{1}{24}x & 6 < x \leqslant 10 \\ 0 & \text{OTHERWISE} \end{cases}$$



b) PROCEED AS FOLLOWS

IF $1 \leqslant x \leqslant 3$

$F_1(x) = \int_1^x \frac{1}{12}(x-1)\,dx = \int_1^x \frac{1}{12}x - \frac{1}{12}\,dx = \left[\frac{1}{24}x^2 - \frac{1}{12}x\right]_1^x$

$= \left(\frac{1}{24}x^2 - \frac{1}{12}x\right) - \left(\frac{1}{24} - \frac{1}{12}\right) = \underline{\frac{1}{24}x^2 - \frac{1}{12}x + \frac{1}{24}}$

IF $3 < x \leqslant 6$

$F_1(3) = \frac{1}{24}\times 3^2 - \frac{1}{12}\times 3 + \frac{1}{24} = \frac{1}{6}$

$F_2(x) = \frac{1}{6} + \int_3^x \frac{1}{6}\,dx = \frac{1}{6} + \left[\frac{1}{6}x\right]_3^x = \frac{1}{6} + \left(\frac{1}{6}x - \frac{1}{2}\right)$

$= \underline{\frac{1}{6}x - \frac{1}{3}}$

IF $6 < x \leqslant 10$

$F_2(6) = \frac{1}{6}\times 6 - \frac{1}{3} = \frac{2}{3}$

$F_3(x) = \frac{2}{3} + \int_6^x \frac{5}{12} - \frac{1}{24}x\,dx = \frac{2}{3} + \left[\frac{5}{12}x - \frac{1}{48}x^2\right]_6^x$

$= \frac{2}{3} + \left[\left(\frac{5}{12}x - \frac{1}{48}x^2\right) - \left(\frac{5}{2} - \frac{3}{4}\right)\right]$

$= \underline{\frac{5}{12}x - \frac{1}{48}x^2 - \frac{13}{12}}$

$$\therefore\ F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{24}x^2 - \frac{1}{12}x + \frac{1}{24} & 1 \leqslant x \leqslant 3 \\ \frac{1}{6}x - \frac{1}{3} & 3 < x \leqslant 6 \\ -\frac{1}{48}x^2 + \frac{5}{12}x - \frac{13}{12} & 6 < x \leqslant 10 \\ 1 & x > 10 \end{cases}$$

c) FROM THE CONSTRUCTION OF $F(x)$ IT IS EVIDENT THAT THE MEDIAN LIES IN THE "MIDDLE SECTION"

$F(x) = \frac{1}{2} \Rightarrow \frac{1}{6}x - \frac{1}{3} = \frac{1}{2}$

$\Rightarrow \frac{1}{6}x = \frac{5}{6}$

$\Rightarrow \underline{x = 5}$

d) USING THE C.D.F

$P(2 < X < 4) \cup P(5 < X < 9)$

$= [P(X<4) - P(X<2)] + [P(X<9) - P(X<5)]$

$= F(4) - F(2) + F(9) - F(5)$

$= [F(4) + F(9)] - [F(2) + F(5)]$

$= \left[\left(\frac{1}{6}\times 4 - \frac{1}{3}\right) + \left(-\frac{1}{48}\times 9^2 + \frac{5}{12}\times 9 - \frac{13}{12}\right)\right] - \left[\left(\frac{1}{24}\times 2^2 - \frac{1}{12}\times 2 + \frac{1}{24}\right) + \left(\frac{1}{6}\times 5 - \frac{1}{3}\right)\right]$

$= \left(\frac{1}{3} + \frac{47}{48}\right) - \left(\frac{1}{24} + \frac{1}{2}\right)$

$= \underline{\frac{37}{48}}$

e) $\mathrm{E}(X) = \int_a^b x f(x)\,dx$

$\mathrm{E}(X) = \int_1^3 x \times \frac{1}{12}(x-1)\,dx + \int_3^6 x \times \frac{1}{6}\,dx + \int_6^{10} \left(\frac{5}{12} - \frac{1}{24}x\right)x\,dx$

$\mathrm{E}(X) = \int_1^3 \frac{1}{12}x^2 - \frac{1}{12}x\,dx + \int_3^6 \frac{1}{6}x\,dx + \int_6^{10} \frac{5}{12}x - \frac{1}{24}x^2\,dx$

$\mathrm{E}(X) = \left[\frac{1}{36}x^3 - \frac{1}{24}x^2\right]_1^3 + \left[\frac{1}{12}x^2\right]_3^6 + \left[\frac{5}{24}x^2 - \frac{1}{72}x^3\right]_6^{10}$

$\mathrm{E}(X) = \left(\frac{3}{4} - \frac{3}{8}\right) - \left(\frac{1}{36} - \frac{1}{24}\right) + \left(3 - \frac{3}{4}\right) + \left(\frac{125}{6} - \frac{125}{9}\right) - \left(\frac{15}{2} - 3\right)$

$\mathrm{E}(X) = \frac{3}{8} + \frac{1}{72} + \frac{9}{4} + \frac{125}{18} - \frac{9}{2}$

$\underline{\mathrm{E}(X) = \frac{61}{12}}$

C.D.F. to P.D.F CALCULATIONS

Question 1 (***)

The cumulative distribution function, $F(x)$, of a continuous random variable X is given by the following expression

$$F(x) = \begin{cases} 0 & x < 1 \\ k\left(x^4 + 2x^2 - 3\right) & 1 \le x \le 3 \\ 1 & x > 3 \end{cases}$$

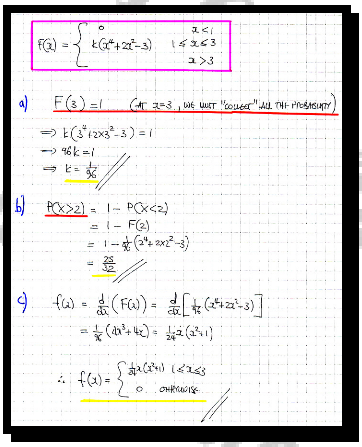
where k is a positive constant.

a) Show clearly that $k = \dfrac{1}{96}$.

b) Find $\mathrm{P}(X > 2)$.

c) Determine the probability density function of X, for all values of x.

$\boxed{}$, $\boxed{\mathrm{P}(X>2) = \dfrac{25}{32}}$, $\boxed{f(x) = \begin{cases} \dfrac{1}{24}\left(x^3 + x\right) & 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}}$



$F(x) = \begin{cases} 0 & x<1 \\ k(x^4+2x^2-3) & 1 \le x \le 3 \\ 1 & x>3 \end{cases}$

a) $F(3) = 1$ (AT $x=3$, WE MUST "COVER" ALL THE PROBABILITY)

$\Rightarrow k(3^4+2\times3^2-3) = 1$

$\Rightarrow 96k = 1$

$\Rightarrow k = \frac{1}{96}$

b) $P(X>2) = 1 - P(X<2)$

$= 1 - F(2)$

$= 1 - \frac{1}{96}(2^4+2\times2^2-3)$

$= \frac{25}{32}$

c) $f(x) = \frac{d}{dx}(F(x)) = \frac{d}{dx}\left[\frac{1}{96}(x^4+2x^2-3)\right]$

$= \frac{1}{96}(4x^3+4x) = \frac{1}{24}x(x^2+1)$

$\therefore f(x) = \begin{cases} \frac{1}{24}x(x^2+1) & 1 \le x \le 3 \\ 0 & \text{OTHERWISE} \end{cases}$

Question 2 (***)

The cumulative distribution function, $F(x)$, of a continuous random variable X is given by

$$F(x) = \begin{cases} 0 & x < -3 \\ k(x+3) & -3 \le x \le 8 \\ 1 & x > 8 \end{cases}$$

where k is a positive constant.

a) Show clearly that $k = \frac{1}{11}$.

b) Calculate $\mathrm{P}(X < 1)$.

c) Find $\mathrm{P}(X = 1)$.

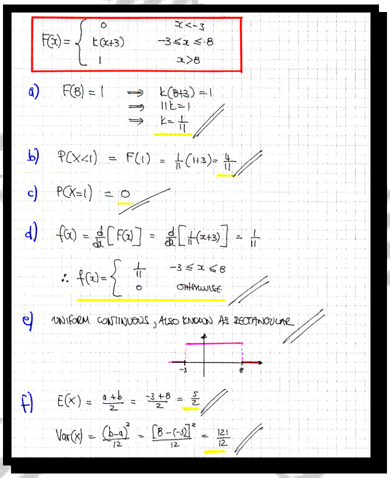
d) Define the probability density function of X, for all values of x.

e) State the name of this probability distribution.

f) Determine the value of $\mathrm{E}(X)$ and the value of $\mathrm{Var}(X)$.

$\boxed{}$, $\boxed{\mathrm{P}(X<1) = \frac{4}{11}}$, $\boxed{\mathrm{P}(X=1)=0}$, $\boxed{f(x) = \begin{cases} \frac{1}{11} & -3 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}}$,

$\boxed{\text{uniform continuous}}$, $\boxed{\mathrm{E}(X) = \frac{5}{2}}$, $\boxed{\mathrm{Var}(X) = \frac{121}{12}}$



$$F(x) = \begin{cases} 0 & x < -3 \\ k(x+3) & -3 \le x \le 8 \\ 1 & x > 8 \end{cases}$$

a) $F(8) = 1 \Rightarrow k(8+3) = 1 \Rightarrow 11k = 1 \Rightarrow k = \frac{1}{11}$

b) $P(X<1) = F(1) = \frac{1}{11}(1+3) = \frac{4}{11}$

c) $P(X=1) = 0$

d) $f(x) = \frac{d}{dx}[F(x)] = \frac{d}{dx}\left[\frac{1}{11}(x+3)\right] = \frac{1}{11}$

$\therefore f(x) = \begin{cases} \frac{1}{11} & -3 \le x \le 8 \\ 0 & \text{OTHERWISE} \end{cases}$

e) UNIFORM CONTINUOUS, ALSO KNOWN AS RECTANGULAR

f) $E(X) = \frac{a+b}{2} = \frac{-3+8}{2} = \frac{5}{2}$

$\mathrm{Var}(X) = \frac{(b-a)^2}{12} = \frac{[8-(-3)]^2}{12} = \frac{121}{12}$

Question 3 (***)

The continuous random variable T represents the lifetime, in tens of hours, for a certain brand of battery.

The cumulative distribution function $F(t)$ of the variable T is given by

$$F(t)=\begin{cases} 0 & t<2 \\ \frac{1}{20}(t+6)(t-2) & 2 \le t \le 4 \\ 1 & t>4 \end{cases}$$

a) Calculate the value of …

i. … $\text{P}(T>3.5)$.

ii. … $\text{P}(T=3)$.

b) Show that the median of T is 3.10, correct to three significant figures.

c) Define the probability density function of T, for all values of t.

Four new batteries, whose lifetime is modelled by T, are fitted to a road hazard lantern. This lantern will only remain operational if all four batteries are working.

d) Determine the probability that the lantern will operate for more than 35 hours.

$\boxed{}$, $\boxed{\text{P}(T>3.5)=\frac{23}{80}}$, $\boxed{\text{P}(T=3)=0}$, $\boxed{f(t)=\begin{cases} \frac{1}{10}(t+2) & 2 \le t \le 4 \\ 0 & \text{otherwise} \end{cases}}$, $\boxed{0.00683}$

a) I) $P(T>3.5) = 1 - P(T<3.5) = 1 - F(3.5) = 1 - \frac{57}{80}$
$= \frac{23}{80} = 0.2875$

II) $P(T=3) = 0$ (CONTINUOUS VARIABLE)

b) BY DIRECT CALCULATION

$F(t) = \frac{1}{2}$

$\frac{1}{20}(t+6)(t-2) = \frac{1}{2}$

$(t+6)(t-2) = 10$

$t^2+4t-12 = 10$

$t^2+4t-22 = 0$

$(t+2)^2 - 26 = 0$

$(t+2)^2 = 26$

$t+2 = \pm\sqrt{26}$

$t = -2+\sqrt{26} \approx 3.10$

OR BY VERIFICATION

$F(3.095) = \frac{1}{20}(3.095+6)(3.095-2)$
$= 0.4979\ldots$
< 0.5

$F(3.105) = \frac{1}{20}(3.105+6)(3.105-2)$
$= 0.50305\ldots$
> 0.5

INDEED THE MEDIAN IS 3.10 CORRECT TO 3 s.f.

c) USING $f(t) = \frac{d}{dt}[F(t)]$

$\frac{d}{dt}\left[\frac{1}{20}(t+6)(t-2)\right] = \frac{d}{dt}\left[\frac{1}{20}(t^2+4t-12)\right] = \frac{1}{20}(2t+4) = \frac{1}{10}(t+2)$

$\therefore f(t) = \begin{cases} \frac{1}{10}(t+2) & 2 \le t \le 4 \\ 0 & \text{OTHERWISE} \end{cases}$

d) USING PART (a) & ASSUMING INDEPENDENCE

WE REQUIRE $[P(T>3.5)]^4 = \left(\frac{23}{80}\right)^4 \approx 0.00683$

Question 4 **(***)**

The cumulative distribution function, $F(y)$, of a continuous random variable Y is given by

$$F(y) = \begin{cases} 0 & y < 0 \\ ky^3 & 0 \le y \le b \\ 1 & y > b \end{cases}$$

It is further given that $\mathrm{P}(1 < Y < 3) = \frac{13}{32}$.

a) Find the value of k and hence determine the probability density function of Y for all y.

b) Calculate the median of Y.

c) **State** the mode of Y and hence comment on the skewness of the distribution.

$\boxed{\quad}$, $\boxed{f(y) = \begin{cases} \frac{3}{64}y^2 & 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}}$, $\boxed{\text{median} \approx 3.17}$, $\boxed{\text{mode} = 4}$,

$\boxed{\text{median} < \text{mode} \Rightarrow \text{negative skew}}$

a) $P(1<Y<3) = \frac{13}{32}$

$\Rightarrow P(Y<3) - P(Y<1) = \frac{13}{32}$

$\Rightarrow F(3) - F(1) = \frac{13}{32}$

$\Rightarrow k \times 3^3 - k \times 1^3 = \frac{13}{32}$

$\Rightarrow 27k - k = \frac{13}{32}$

$\Rightarrow 26k = \frac{13}{32}$

$\Rightarrow k = \frac{1}{64}$

NOW USING $F(b) = 1$

$\frac{1}{64}b^3 = 1$

$b^3 = 64$

$b = 4$

$\uparrow$ UPPER LIMIT

NOW $f(y) = \frac{d}{dy}[F(y)]$

$f(y) = \frac{d}{dy}\left[\frac{1}{64}y^3\right] = \frac{3}{64}y^2$

$\therefore f(y) = \begin{cases} \frac{3}{64}y^2 & 0 \le y \le 4 \\ 0 & \text{OTHERWISE} \end{cases}$

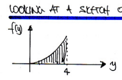
b) THE MEDIAN SATISFIES $F(y) = \frac{1}{2}$

$\frac{1}{64}y^3 = \frac{1}{2}$

$y^3 = 32$

$y = \sqrt[3]{32} \approx 3.17$

c) LOOKING AT A SKETCH OF THE P.D.F



MODE IS 4

(MEAN) < MEDIAN < MODE
3.17 4

$\therefore$ NEGATIVE SKEW

Question 5 (***)

The cumulative distribution function $F(x)$ of a continuous random variable X is given by the following expression

$$F(x) = \begin{cases} 0 & x < 0 \\ kx(x+2) & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

where k is a positive constant.

a) Find the value of k.

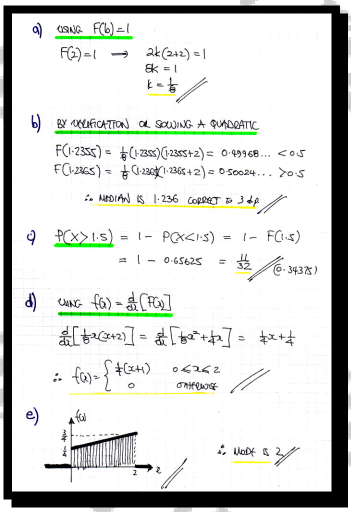
b) Show that the median of X is 1.236, correct to 3 decimal places.

c) Calculate $\text{P}(X > 1.5)$.

d) Define the probability density function of X, $f(x)$, for all x.

e) Sketch the graph of $f(x)$ for all x, and hence state the mode of X.

$\boxed{}$, $\boxed{k = \frac{1}{8}}$, $\boxed{\text{P}(X > 1.5) = \frac{11}{32}}$, $\boxed{f(x) = \begin{cases} \frac{1}{4}(x+1) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}}$, $\boxed{\text{mode} = 2}$



a) USING $F(b) = 1$

$F(2) = 1 \implies 2k(2+2) = 1$

$8k = 1$

$k = \frac{1}{8}$

b) BY VERIFICATION OR SOLVING A QUADRATIC

$F(1.2355) = \frac{1}{8}(1.2355)(1.2355+2) = 0.49968\ldots < 0.5$

$F(1.2365) = \frac{1}{8}(1.2365)(1.2365+2) = 0.50024\ldots > 0.5$

$\therefore$ MEDIAN IS 1.236 CORRECT TO 3 d.p.

c) $P(X > 1.5) = 1 - P(X < 1.5) = 1 - F(1.5)$

$= 1 - 0.65625 = \frac{11}{32}$ (0.34375)

d) USING $f(x) = \frac{d}{dx}[F(x)]$

$\frac{d}{dx}\left[\frac{1}{8}x(x+2)\right] = \frac{d}{dx}\left[\frac{1}{8}x^2 + \frac{1}{4}x\right] = \frac{1}{4}x + \frac{1}{4}$

$\therefore f(x) = \begin{cases} \frac{1}{4}(x+1) & 0 \le x \le 2 \\ 0 & \text{OTHERWISE} \end{cases}$

e)

$\therefore$ MODE IS 2

Question 6 (***+)

The time, in weeks, a patient has to wait for an appointment to see a psychiatrist in a certain hospital, is modelled by the continuous random variable T.

The cumulative distribution function of T is given by

$$F(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{1728}t^3 & 0 \le t \le 12 \\ 1 & t > 12 \end{cases}$$

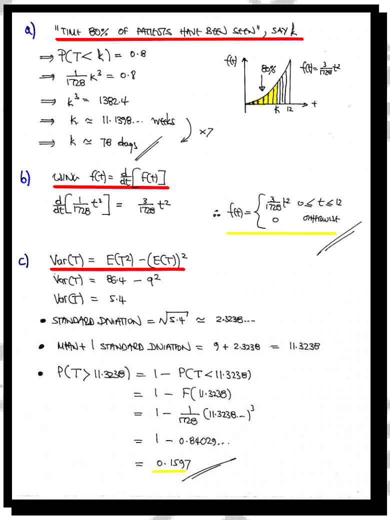
a) Find, to the nearest day, the time within which 80% of the patients have been given an appointment.

b) Define the probability density function of T, for all values of t.

It is given that $\mathrm{E}(T) = 9$ and $\mathrm{E}\left(T^2\right) = 86.4$.

c) Determine the probability that the time a patient has to wait for an appointment, is more than one standard deviation above the mean.

$\boxed{\approx 78 \text{ days}}$, $\boxed{f(t) = \begin{cases} \frac{1}{576}t^2 & 0 \le t \le 12 \\ 0 & \text{otherwise} \end{cases}}$, $\boxed{\approx 0.1597}$



a) "TIME 80% OF PATIENTS HAVE BEEN SEEN", SAY k

$\Rightarrow P(T < k) = 0.8$

$\Rightarrow \frac{1}{1728}k^3 = 0.8$

$\Rightarrow k^3 = 1382.4$

$\Rightarrow k \simeq 11.1398\ldots$ weeks

$\Rightarrow k \simeq 78$ days

$\times 7$

b) USING $f(t) = \frac{d}{dt}\left[F(t)\right]$

$\frac{d}{dt}\left[\frac{1}{1728}t^3\right] = \frac{3}{1728}t^2$

$\therefore f(t) = \begin{cases} \frac{3}{1728}t^2 & 0 \le t \le 12 \\ 0 & \text{otherwise} \end{cases}$

c) $\mathrm{Var}(T) = \mathrm{E}(T^2) - (\mathrm{E}(T))^2$

$\mathrm{Var}(T) = 86.4 - 9^2$

$\mathrm{Var}(T) = 5.4$

- STANDARD DEVIATION $= \sqrt{5.4} \simeq 2.3238\ldots$
- MEAN + 1 STANDARD DEVIATION $= 9 + 2.3238 = 11.3238$
- $P(T > 11.3238) = 1 - P(T < 11.3238)$

$= 1 - F(11.3238)$

$= 1 - \frac{1}{1728}(11.3238\ldots)^3$

$= 1 - 0.84029\ldots$

$= 0.1597$

Question 7 (***+)

The time, in hours, spent on a piece of homework by a group of students is modelled by the continuous random variable T.

The cumulative distribution function of T is given by

$$F(t) = \begin{cases} 0 & t < 0 \\ k\left(2t^3 - t^4\right) & 0 \le t \le b \\ 1 & t > b \end{cases}$$

where k is a positive constant.

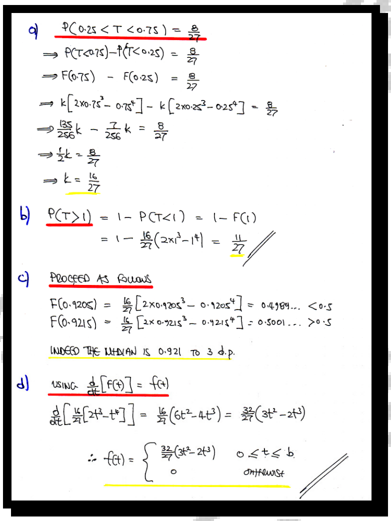
a) If the probability that a student from this group took between one quarter and three quarters of an hour to complete the homework is $\frac{8}{27}$, show that $k = \frac{16}{27}$.

b) Find the proportion of students who spent over an hour in their homework.

c) Verify that the median is 0.921, correct to 3 decimal places.

d) Define the probability density function of T, for all values of t.

$\boxed{}$, $\boxed{\mathrm{P}(T > 1) = \frac{11}{27}}$, $\boxed{f(t) = \begin{cases} \frac{32}{27}\left(3t^2 - 2t^3\right) & 0 \le t \le b \\ 0 & \text{otherwise} \end{cases}}$



a) $P(0.25 < T < 0.75) = \frac{8}{27}$

$\Rightarrow P(T<0.75) - P(T<0.25) = \frac{8}{27}$

$\Rightarrow F(0.75) - F(0.25) = \frac{8}{27}$

$\Rightarrow k\left[2\times0.75^3 - 0.75^4\right] - k\left[2\times0.25^3 - 0.25^4\right] = \frac{8}{27}$

$\Rightarrow \frac{135}{256}k - \frac{7}{256}k = \frac{8}{27}$

$\Rightarrow \frac{1}{2}k = \frac{8}{27}$

$\Rightarrow k = \frac{16}{27}$

b) $P(T>1) = 1 - P(T<1) = 1 - F(1)$

$= 1 - \frac{16}{27}\left(2\times1^3 - 1^4\right) = \frac{11}{27}$

c) PROCEED AS FOLLOWS

$F(0.9205) = \frac{16}{27}\left[2\times0.9205^3 - 0.9205^4\right] = 0.4984\ldots < 0.5$

$F(0.9215) = \frac{16}{27}\left[2\times0.9215^3 - 0.9215^4\right] = 0.5001\ldots > 0.5$

INDEED THE MEDIAN IS 0.921 TO 3 d.p.

d) USING $\frac{d}{dt}\left[F(t)\right] = f(t)$

$\frac{d}{dt}\left[\frac{16}{27}\left[2t^3 - t^4\right]\right] = \frac{16}{27}\left(6t^2 - 4t^3\right) = \frac{32}{27}\left(3t^2 - 2t^3\right)$

$\therefore f(t) = \begin{cases} \frac{32}{27}\left(3t^2 - 2t^3\right) & 0 \le t \le b \\ 0 & \text{otherwise} \end{cases}$

Question 8 (***+)

A continuous random variable X has the following cumulative distribution function $F(x)$, defined by

$$F(x)=\begin{cases} 0 & x<0 \\ \frac{1}{5}x^2\left(6-x^2\right) & 0\le x\le 1 \\ 1 & x>1 \end{cases}$$

a) Find $\mathrm{P}(X>0.5)$.

b) Define $f(x)$, the probability density function of X, for all values of x.

A skewness coefficient can be calculated by the formula

$$\frac{\text{mean} - \text{mode}}{\text{standard deviation}}.$$

c) Given that $\mathrm{E}(X)=\frac{16}{25}$ and $\mathrm{E}\left(X^2\right)=\frac{7}{15}$, evaluate the skewness coefficient for this distribution.

$\boxed{}$, $\boxed{\mathrm{P}(X>0.5)=\frac{57}{80}}$, $\boxed{f(x)=\begin{cases} \frac{4}{5}x\left(3-x^2\right) & 0\le x\le 1 \\ 0 & \text{otherwise} \end{cases}}$, $\boxed{-1.507}$

a) $P(X>0.5) = 1 - P(X<0.5) = 1 - F(0.5)$

$= 1 - \frac{1}{5}(0.5^2)(6-0.5^2) = 1 - \frac{23}{80}$

$= \frac{57}{80}$

b) USING $f(x) = \frac{d}{dx}\left[F(x)\right]$

$\frac{d}{dx}\left[\frac{1}{5}x^2(6-x^2)\right] = \frac{d}{dx}\left(\frac{6}{5}x^2 - \frac{1}{5}x^4\right) = \frac{12}{5}x - \frac{4}{5}x^3$

$= \frac{4}{5}x(3-x^2)$

$\therefore f(x) = \begin{cases} \frac{4}{5}x(3-x^2) & 1 \le 0 \le 1 \\ 0 & \text{OTHERWISE} \end{cases}$

c) $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$\mathrm{Var}(X) = \frac{7}{15} - \left(\frac{16}{25}\right)^2 = \frac{107}{1875} = 0.057066\ldots$

$\therefore$ STANDARD DEVIATION $= \sqrt{0.057066\ldots} \simeq 0.238886\ldots$

$\therefore$ MODE BY DIFFERENTIATION

$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\frac{12}{5}x - \frac{4}{5}x^3\right) = \frac{12}{5} - \frac{12}{5}x^2$

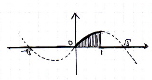
SET TO ZERO

$\frac{12}{5} - \frac{12}{5}x^2 = 0$

$\frac{12}{5} = \frac{12}{5}x^2$

$x^2 = 1$

$x = +1$



$\therefore$ MODE IS 1

FINALLY

$\frac{\text{MEAN} - \text{MODE}}{\text{STANDARD DEVIATION}} = \frac{0.64 - 1}{0.238886\ldots} \approx -1.507$

Question 9 **(****)**

The cumulative distribution function, $F(x)$, of a continuous random variable X is given by the following expression

$$F(x) = \begin{cases} 0 & x < 1 \\ k\left(-x^3 + 6x^2 - 5\right) & 1 \le x \le 4 \\ 1 & x > 4 \end{cases}$$

where k is a positive constant.

a) Show clearly that $k = \frac{1}{27}$.

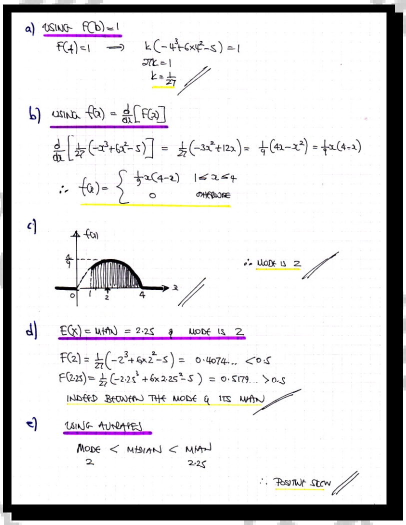
b) Define $f(x)$, the probability density function of X, for all values of x.

c) Sketch the graph of $f(x)$ for all x, and hence state the mode of X.

d) Given that $\mathrm{E}(X) = 2.25$, show that the median of X lies between its mode and its mean.

e) State the skewness of the distribution.

$\boxed{}$, $\boxed{f(x) = \begin{cases} \frac{1}{9}x(4-x) & 1 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}}$, $\boxed{\text{mode} = 2}$, $\boxed{\text{positive skew}}$



a) USING $F(b)=1$

$F(4)=1 \implies k(-4^3+6\times4^2-5)=1$

$27k=1$

$k=\frac{1}{27}$

b) USING $f(x) = \frac{d}{dx}[F(x)]$

$\frac{d}{dx}\left[\frac{1}{27}(-x^3+6x^2-5)\right] = \frac{1}{27}(-3x^2+12x) = \frac{1}{9}(4x-x^2) = \frac{1}{9}x(4-x)$

$\therefore f(x) = \begin{cases} \frac{1}{9}x(4-x) & 1 \le x \le 4 \\ 0 & \text{OTHERWISE} \end{cases}$

c)

$\therefore$ MODE IS 2

d) $E(X)$ = MEAN = 2.25 & MODE IS 2

$F(2) = \frac{1}{27}(-2^3+6\times2^2-5) = 0.4074... < 0.5$

$F(2.25) = \frac{1}{27}(-2.25^3+6\times2.25^2-5) = 0.5179... > 0.5$

INDEED BETWEEN THE MODE & ITS MEAN

e) USING AVERAGES

MODE < MEDIAN < MEAN

2 2.25

$\therefore$ POSITIVE SKEW

Question 10 (****)

The cumulative distribution function, $F(x)$, of a continuous random variable X is given by the following expression

$$F(x)=\begin{cases} 0 & x<1 \\ (3-x)(x-1)^2 & 1\le x\le 2 \\ 1 & x>2 \end{cases}$$

a) Find $\mathrm{P}(X>1.2)$.

b) Show that the median of X lies between 1.59 and 1.60.

c) Show that for $1\le x\le 2$, the probability density function of X, $f(x)$, is

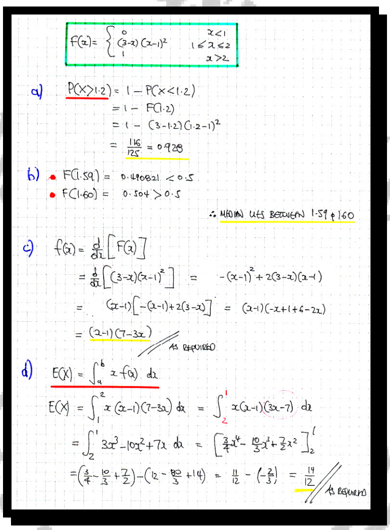
$$f(x)=(1-x)(3x-7).$$

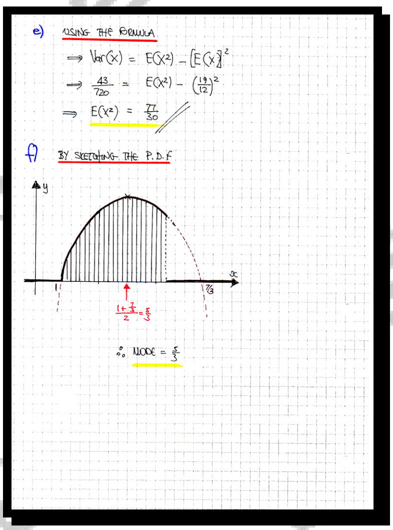
d) Show further that the mean of X is $\frac{19}{12}$.

e) Given that the variance of X is $\frac{43}{720}$, find the exact value of $\mathrm{E}(X^2)$.

f) Find the mode of X.

$\boxed{\mathrm{P}(X>1.2)=0.928}$, $\boxed{\mathrm{E}(X^2)=\frac{77}{30}}$, $\boxed{\text{mode}=\frac{5}{3}}$





Question 11 (****+)

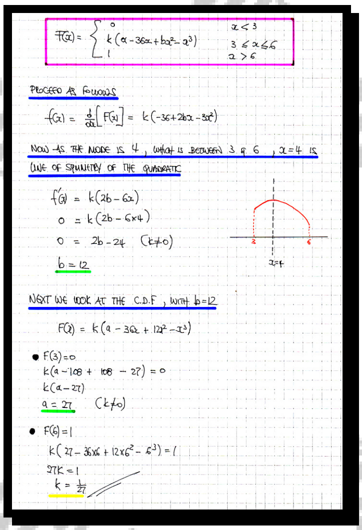
The cumulative distribution function, $F(x)$, of a continuous random variable X is given by

$$F(x)=\begin{cases} 0 & x<3 \\ k\left(a-36x+bx^2-x^3\right) & 3 \le x \le 6 \\ 1 & x>6 \end{cases}$$

where k, a and b are non zero constants.

Determine the value of k, given that the mode of X is 4.

$\boxed{\text{F31}}$, $\boxed{k=\frac{1}{27}}$



$$F(x)=\begin{cases} 0 & x<3 \\ k(a-36x+bx^2-x^3) & 3 \le x \le 6 \\ 1 & x>6 \end{cases}$$

PROCEED AS FOLLOWS

$f(x)=\frac{d}{dx}\left[F(x)\right]=k(-36+2bx-3x^2)$

NOW AS THE MODE IS 4, WHICH IS BETWEEN 3 & 6, $x=4$ IS LINE OF SYMMETRY OF THE QUADRATIC

$f'(x)=k(2b-6x)$

$0=k(2b-6\times4)$

$0=2b-24 \quad (k\neq0)$

$b=12$

NEXT WE LOOK AT THE C.D.F, WITH $b=12$

$F(x)=k(a-36x+12x^2-x^3)$

- $F(3)=0$

  $k(a-108+108-27)=0$

  $k(a-27)$

  $a=27 \quad (k\neq0)$

- $F(6)=1$

  $k(27-36\times6+12\times6^2-6^3)=1$

  $27k=1$

  $k=\frac{1}{27}$

Question 12 (*****)

The continuous random variable X has the following cumulative distribution function

$$F(x)=\begin{cases} 0 & x<0 \\ ax-bx^2 & 0 \le x \le k \\ 1 & x>k \end{cases}$$

where vehicles a, b and k are positive constants.

The variable Y is related to X by

$$Y=3X-2.$$

Determine the value of a, b and k given further that $\mathrm{E}(Y)=2$ and $\mathrm{Var}(Y)=8$.

$\boxed{}$, $\boxed{a=\frac{1}{2}}$, $\boxed{b=\frac{1}{16}}$, $\boxed{k=4}$

START BY REVERSING THE TRANSFORMATIONS

$\Rightarrow Y=3X-2$
$\Rightarrow E(Y)=E(3X-2)$
$\Rightarrow 2=3E(X)-2$
$\Rightarrow E(X)=\frac{4}{3}$

$\Rightarrow Y=3X-2$
$\Rightarrow \mathrm{Var}(Y)=\mathrm{Var}(3X-2)$
$\Rightarrow 8=3^2\mathrm{Var}(X)$
$\Rightarrow \mathrm{Var}(X)=\frac{8}{9}$

OBTAIN A RELATIONSHIP FROM THE C.D.F

$F(k)=1 \Rightarrow \boxed{ak-bk^2=1}$

OBTAIN THE P.D.F NEXT

$f(x)=\frac{d}{dx}[F(x)]=a-2bx$

NEXT THE EXPRESSION FOR E(X) YIELDS

$\Rightarrow E(X)=\int_0^k x(a-2bx)\,dx=\int_0^k ax-2bx^2\,dx$
$\Rightarrow \frac{4}{3}=\left[\frac{1}{2}ax^2-\frac{2}{3}bx^3\right]_0^k$
$\Rightarrow \frac{4}{3}=\frac{1}{2}ak^2-\frac{2}{3}bk^3$
$\Rightarrow \boxed{8=3ak^2-4bk^3}$

FINALLY AN EXPRESSION FOR THE VARIANCE

$\Rightarrow \mathrm{Var}(X)=\int_0^k x^2(a-2bx)\,dx-[E(X)]^2$
$\Rightarrow \frac{8}{9}=\int_0^k ax^2-2bx^3\,dx-\left(\frac{4}{3}\right)^2$
$\Rightarrow \frac{8}{9}=\left[\frac{1}{3}ax^3-\frac{1}{2}bx^4\right]_0^k-\frac{16}{9}$
$\Rightarrow \frac{1}{3}ak^3-\frac{1}{2}bk^4=\frac{8}{3}$
$\Rightarrow \boxed{2ak^3-3bk^4=16}$

SOLVING THE EQUATIONS AS FOLLOWS

(I) $ak-bk^2=1 \longrightarrow ak=bk^2+1$
(II) $3ak^2-4bk^3=8$
(III) $2ak^3-3bk^4=16$

SUB INTO THE OTHER 2

(I) $3k(bk^2+1)-4bk^3=8$
(II) $2k^2(bk^2+1)-3bk^4=16$

$\Rightarrow \begin{cases} 3bk^3+3k-4bk^3=8 \\ 2bk^4+2k^2-3bk^4=16 \end{cases}$
$\Rightarrow \begin{cases} 3k-bk^3=8 \\ 2k^2-bk^4=16 \end{cases}$
$\Rightarrow \begin{cases} bk^3=3k-8 \\ bk^4=2k^2-16 \end{cases}$ DIVIDING
$\Rightarrow \frac{1}{k}=\frac{3k-8}{2k^2-16}$
$\Rightarrow 2k^2-16=3k^2-8k$
$\Rightarrow 0=k^2-8k+16$
$\Rightarrow (k-4)^2=0$

HENCE $k=4$

$\Rightarrow 3k-bk^3=8$ & $ak=bk^2+1$
$\Rightarrow 12-64b=8$; $4a=\frac{1}{16}\times16+1$
$\Rightarrow 4=64b$; $4a=2$
$\Rightarrow b=1/16$; $a=\frac{1}{2}$

$\therefore a=\frac{1}{2}$ & $b=\frac{1}{16}$