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RECTANGULAR DISTRIBUTION

Question 1 (***)

An 8 a.m. scheduled bus is known to arrive at a certain bus stop at any random time between 07:57 and 08:13.

The random variable X is used to model the arrival time of the bus after 07:57, where X is measured in minutes.

a) State the name of a suitable distribution that could be used to model X.

b) State the mean arrival time for this bus.

c) Find the value of $\text{Var}(X)$.

The cumulative distribution function of X, is denoted by $F(x)$.

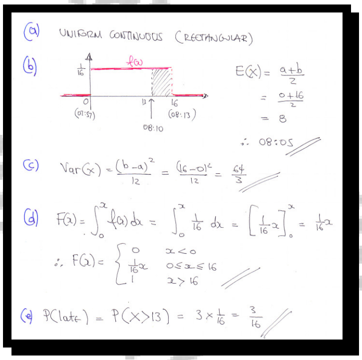
d) Find and specify fully $F(x)$.

Marcus will be late for work if the bus arrives after 08:10.

e) Find the probability that Marcus will be late for work.

$$\boxed{f(x) = \begin{cases} \frac{1}{16} & 0 \le x \le 16 \\ 0 & \text{otherwise} \end{cases}}, \quad \boxed{08:05}, \quad \boxed{\text{Var}(X) = \frac{64}{3}},$$

$$\boxed{F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{16}x & 0 \le x \le 16 \\ 1 & x > 16 \end{cases}}, \quad \boxed{\text{P}(\text{late}) = \frac{3}{16}}$$



(a) UNIFORM CONTINUOUS (RECTANGULAR)

(b) $E(X) = \frac{a+b}{2} = \frac{0+16}{2} = 8$

$\therefore$ 08:05

(c) $\text{Var}(X) = \frac{(b-a)^2}{12} = \frac{(16-0)^2}{12} = \frac{64}{3}$

(d) $F(x) = \int_0^x f(x)\,dx = \int_0^x \frac{1}{16}\,dx = \left[\frac{1}{16}x\right]_0^x = \frac{1}{16}x$

$\therefore F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{16}x & 0 \le x \le 16 \\ 1 & x > 16 \end{cases}$

(e) $P(\text{late}) = P(X > 13) = 3 \times \frac{1}{16} = \frac{3}{16}$

Question 2 (***)

The cumulative distribution function $F(x)$ of a continuous random variable X is given by

$$F(x)=\begin{cases} 0 & x<-3 \\ k(x+3) & -3 \le x \le 5 \\ 1 & x>5 \end{cases}$$

where k is a positive constant.

a) Show that $k=\frac{1}{8}$.

b) Calculate $P(X<1)$.

c) State the value of $P(X=1)$.

d) Find the probability density function of X for all x.

e) State the name of the probability distribution represented by the probability density function of part **(d)**.

f) Determine $E(X)$ and $Var(X)$.

$\boxed{P(X<1)=\frac{1}{2}}$, $\boxed{P(X=1)=0}$, $\boxed{f(x)=\begin{cases} \frac{1}{8} & -3 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}}$, $\boxed{\text{uniform continuous}}$,

$\boxed{E(X)=1}$, $\boxed{Var(X)=\frac{16}{3}}$

(a) $F(5)=1$

$k(5+3)=1$

$8k=1$

$k=\frac{1}{8}$

(b) $P(X<1)=F(1)=\frac{1}{8}(1+3)=\frac{1}{2}$

(c) $P(X=1)=0$

(d) $f(x)=\frac{d}{dx}(F(x))=\frac{d}{dx}\left(\frac{1}{8}(x+3)\right)=\frac{1}{8}$

$f(x)=\begin{cases} \frac{1}{8} & -3<x<5 \\ 0 & \text{OTHERWISE} \end{cases}$

(e) UNIFORM CONTINUOUS (RECTANGULAR)

(f) • $E(X)=\frac{a+b}{2}=\frac{-3+5}{2}=1$

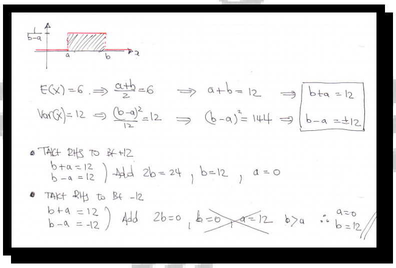
• $Var(X)=\frac{(b-a)^2}{12}=\frac{(5+3)^2}{12}=\frac{64}{12}=\frac{16}{3}$

Question 3 (***)

The continuous random variable X is uniformly distributed in the interval $a \le x \le b$.

Given that $\mathrm{E}(X) = 6$ and $\mathrm{Var}(X) = 12$ determine the value of a and the value of b.

$\boxed{a = 0, \;\; b = 12}$



$\mathrm{E}(X) = 6 \Rightarrow \frac{a+b}{2} = 6 \Rightarrow a + b = 12 \Rightarrow \boxed{b + a = 12}$

$\mathrm{Var}(X) = 12 \Rightarrow \frac{(b-a)^2}{12} = 12 \Rightarrow (b-a)^2 = 144 \Rightarrow \boxed{b - a = \pm 12}$

- TAKE RHS TO BE $+12$
  $b + a = 12$, $b - a = 12$) Add $2b = 24$, $b = 12$, $a = 0$
- TAKE RHS TO BE -12
  $b + a = 12$, $b - a = -12$) Add $2b = 0$, $\cancel{b = 0, a = 12}$ $b > a$ $\therefore a = 0$, $b = 12$

Question 4 (***+)

An automatic coffee dispenser pours coffee into cups. The amount of coffee dispensed into the cup is electronically controlled and is cut off randomly at some point between 240 ml and 260 ml.

The continuous random variable X represents the amount of coffee, in ml, dispensed into a cup.

a) Specify the probability density function of X.

b) Find the value of $\mathrm{P}(X > 248)$.

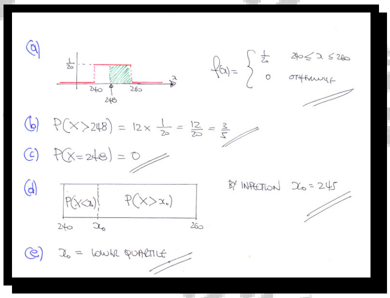
c) State the value of $\mathrm{P}(X = 248)$.

d) Determine the value of x_0 such that

$$P(X > x_0) = 3P(X < x_0).$$

e) State the statistical name of x_0.

$\boxed{f(x) = \begin{cases} \frac{1}{20} & 240 \le x \le 250 \\ 0 & \text{otherwise} \end{cases}}$, $\boxed{0.6}$, $\boxed{0}$, $\boxed{x_0 = 245}$, $\boxed{x_0 = \text{lower quartile}}$



Question 5 (***+)

Olson has a rope which is 180 cm long. The tip of one of the two ends of the rope is dyed red. Olson cuts his rope at a random point so he now has two pieces.

The random variable X represents the length, in cm, of the piece of the rope whose tip is dyed red.

a) Determine the value of …

  i. … $\mathrm{P}(X < 70)$.

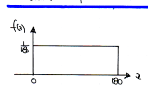
  ii. … $\mathrm{E}(X)$.

  iii. … the standard deviation of X.

b) Calculate the probability that the length of the **shorter** piece of the rope is less than 70 cm.

$\boxed{}$, $\boxed{\mathrm{P}(X<70)=\frac{7}{18}}$, $\boxed{\mathrm{E}(X)=90}$, $\boxed{\sqrt{\mathrm{Var}(X)}=\sqrt{2700}\approx 52.0}$, $\boxed{\frac{7}{9}}$

a) MODELLING WITH A CONTINUOUS UNIFORM DISTRIBUTION (RECTANGULAR)



$$f(x)=\begin{cases}\frac{1}{180} & 0 \leqslant x \leqslant 180\\ 0 & \text{OTHERWISE}\end{cases}$$

I) $P(X<70) = \frac{70}{180} = \frac{7}{18}$

II) USING STANDARD FORMULA $E(X) = \frac{a+b}{2}$

$E(X) = \frac{0+180}{2} = 90$

III) USING STANDARD FORMULA $Var(X) = \frac{(b-a)^2}{12}$

$Var(X) = \frac{(180-0)^2}{12} = 2700$

$\therefore$ STANDARD DEVIATION $= \sqrt{2700} \simeq 52.0$

b) MODEL AS FOLLOWS

$P(\text{shorter piece is AT MOST 70 cm}) = P(X<70) + P(X>110)$

$= \frac{7}{18} + \frac{7}{18}$

$= \frac{7}{9}$

Question 6 (***+)

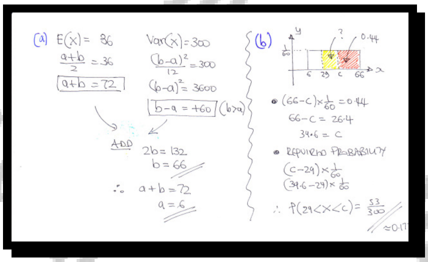
The continuous random variable X is uniformly distributed in the interval $a \le x \le b$.

a) Given that $\mathrm{E}(X) = 36$ and $\mathrm{Var}(X) = 300$ determine the value of a and the value of b.

It is further given that $\mathrm{P}(X > c) = 0.44$, where $a < c < b$.

b) Calculate $\mathrm{P}(29 < X < c)$.

$\boxed{a = 6,\ b = 66}$, $\boxed{\mathrm{P}(29 < X < c) = \frac{53}{300} \approx 0.177}$



Question 7 (****)

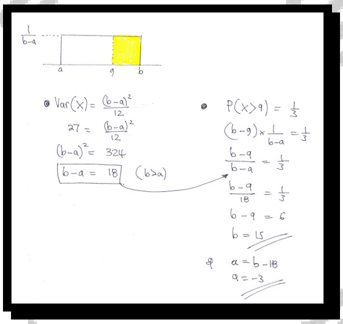
The continuous random variable X is uniformly distributed in the interval $a \le x \le b$.

Given further that

$$\mathrm{Var}(X) = 27 \quad \text{and} \quad \mathrm{P}(X > 9) = \frac{1}{3},$$

determine the value of a and the value of b.

$\boxed{a = -3,\ b = 15}$



Question 8 (****)

The stock control manager in a warehouse, measures the lengths of curtain poles and records their lengths to the nearest cm.

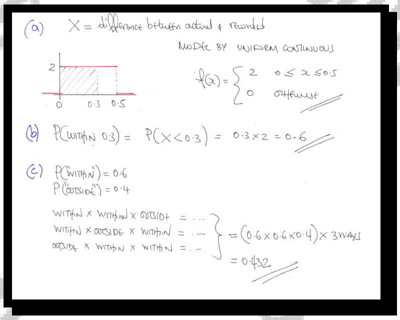
a) Suggest a suitable model to represent the difference between the actual length and the recorded length of a curtain pole. Specify any parameters.

b) Determine the probability that for a randomly chosen curtain pole its actual length will be within 0.3 cm of its recorded length.

Three curtain poles are chosen at random.

c) Find the probability that only two of these poles will be within 0.3 cm of their recorded lengths.

$$\boxed{f(x)=\begin{cases}2 & 0\le x\le 0.5\\ 0 & \text{otherwise}\end{cases}},\ \boxed{0.6},\ \boxed{0.432}$$



(a) $X =$ difference between actual & recorded

MODEL BY UNIFORM CONTINUOUS

$f(x) = \begin{cases} 2 & 0 \le x \le 0.5 \\ 0 & \text{OTHERWISE} \end{cases}$

(b) $P(\text{WITHIN } 0.3) = P(X<0.3) = 0.3 \times 2 = 0.6$

(c) $P(\text{"WITHIN"}) = 0.6$
$P(\text{"OUTSIDE"}) = 0.4$

WITHIN × WITHIN × OUTSIDE = ...
WITHIN × OUTSIDE × WITHIN = ...
OUTSIDE × WITHIN × WITHIN = ...

$= (0.6 \times 0.6 \times 0.4) \times 3 \text{ WAYS}$
$= 0.432$

Question 9 (****)

The radius r of a circle is a continuous random variable R uniformly distributed in the interval $7 \le r \le 11$.

a) State the mean and variance of R.

The circumference of the circle is denoted by C.

b) Determine, in terms of π, the mean and variance of C.

c) Calculate $\text{P}(C > 53)$.

The area of the circle is denoted by A.

d) Find the mean of A.

$\boxed{\text{E}(R) = 9}$, $\boxed{\text{Var}(R) = \frac{4}{3}}$, $\boxed{\text{E}(C) = 18\pi}$, $\boxed{\text{Var}(C) = \frac{16}{3}\pi^2}$, $\boxed{\text{P}(C > 53) \approx 0.6412}$, $\boxed{\text{E}(A) = \frac{247}{3}\pi \approx 259}$

a) $R \sim U(7,11)$

$E(R) = \frac{a+b}{2} = \frac{7+11}{2} = 9$

$Var(R) = \frac{(b-a)^2}{12} = \frac{(11-7)^2}{12} = \frac{4}{3}$

b) $E(C) = E(2\pi R) = 2\pi E(R) = 2\pi \times 9 = 18\pi$

$Var(C) = Var(2\pi R) = (2\pi)^2 Var(R) = 4\pi^2 \times \frac{4}{3} = \frac{16}{3}\pi^2$

c) $P(C>53) = P(2\pi R > 53) = P\left(R > \frac{53}{2\pi}\right) = P(R > 8.4352\ldots)$

$= \frac{11 - 8.4352}{4} = 0.6412$

d) $E(A) = E(\pi R^2) = \pi E(R^2)$

Now $Var(R) = E(R^2) - (E(R))^2$

$\frac{4}{3} = E(R^2) - 9^2$

$\frac{247}{3} = E(R^2)$

$\therefore E(A) = \pi E(R^2) = \pi \times \frac{247}{3} = 258.6577\ldots \simeq 259$

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Question 10 (****)

A piece of string of length a, where a is a positive constant, is cut into 2 pieces at a random point. The continuous random variable X represents the length of the longer piece of the string.

a) Assuming that X is uniformly distributed show by integration that

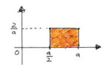
$$\text{Var}(X) = \frac{a^2}{48}.$$

b) Find the probability that the length of the longer piece is more than 4 times the length of the shorter piece.

$\boxed{}$, $\boxed{\frac{2}{5}}$

a) IF X REPRESENTS THE LENGTH OF THE LONGER PIECE THEN X CAN TAKE VALUES BETWEEN $\frac{a}{2}$ & a

$$f(x) = \begin{cases} \frac{2}{a} & \frac{1}{2}a \leqslant x \leqslant a \\ 0 & \text{OTHERWISE} \end{cases}$$



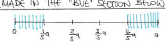
$$E(X) = \int_{\frac{a}{2}}^{a} x f(x)\,dx = \int_{\frac{a}{2}}^{a} \frac{2}{a}x\,dx = \left[\frac{1}{a}x^2\right]_{a/2}^{a} = \frac{1}{a}\left[a^2 - \frac{1}{4}a^2\right]$$
$$= \frac{1}{a} \times \frac{3}{4}a^2 = \frac{3}{4}a$$

$$E(X^2) = \int_{a/2}^{a} x^2 f(x)\,dx = \int_{a/2}^{a} \frac{2}{a}x^2\,dx = \left[\frac{2}{3a}x^3\right]_{a/2}^{a} = \frac{2}{3a}\left[a^3 - \frac{1}{8}a^3\right]$$
$$= \frac{2}{3a} \times \frac{7}{8}a^3 = \frac{7}{12}a^2$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{7}{12}a^2 - \left(\frac{3}{4}a\right)^2 = \frac{1}{48}a^2$$

AS REQUIRED

b) BY INSPECTION – DRAWING THE "ROPE" – THE CONDITION IS SATISFIED IF THE CUT IS MADE IN THE "BLUE" SECTION BELOW



∴ REQUIRED PROBABILITY IS $\frac{2}{5}$

ALTERNATIVE USING PART (a)

$$P(X \geqslant 4(a-X)) = P(X \geqslant 4a - 4X)$$
$$= P(5X \geqslant 4a)$$
$$= P(X \geqslant \tfrac{4}{5}a)$$
$$= \left(a - \tfrac{4}{5}a\right) \times \frac{2}{a}$$
$$= \frac{2}{5}$$

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Question 11 (****)

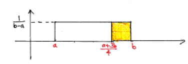
The continuous random variable X is uniformly distributed over the interval $[a,b]$.

a) Determine $P\left(X > \frac{1}{4}a + \frac{3}{4}b\right)$.

b) Given further that $\mathrm{E}(X) = b-8$, show that $\mathrm{E}\left(3X^2 - 256\right) = 3b(b-16)$.

$\boxed{P\left(X > \frac{1}{4}a + \frac{3}{4}b\right) = \frac{1}{4}}$

a) STARTING WITH A SKETCH OF THE P.D.F



BY INSPECTION

$P\left(X > \frac{1}{4}a + \frac{3}{4}b\right) = P\left(X > \frac{a+3b}{4}\right) = \frac{1}{4}$

OR MORE FORMALLY BY LOOKING AT THE AREA

$P\left(X > \frac{1}{4}a + \frac{3}{4}b\right) = P\left(X > \frac{a+3b}{4}\right) = \left(b - \frac{a+3b}{4}\right) \times \frac{1}{b-a}$

$= \frac{4b-a-3b}{4} \times \frac{1}{b-a} = \frac{\cancel{b-a}}{4} \times \frac{1}{\cancel{b-a}} = \frac{1}{4}$

b) STARTING WITH $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$ & $E(X) = b-8$

$\Rightarrow \frac{(b-a)^2}{12} = E(X^2) - (b-8)^2$

$\Rightarrow (b-a)^2 = 12E(X^2) - 12(b-8)^2$

BUT $\boxed{\begin{aligned} E(X) &= b-8 \\ \frac{a+b}{2} &= b-8 \\ a+b &= 2b-16 \\ a+16 &= b \\ a &= b-16 \end{aligned}}$

$\Rightarrow [b-(b-16)]^2 = 12E(X^2) - 12(b-8)^2$

$\Rightarrow 256 = 12E(X^2) - 12(b-8)^2$

$\Rightarrow 64 = 3E(X^2) - 3(b-8)^2$

$\Rightarrow 64 = E(3X^2) - 3(b^2-16b+64)$ $\quad \{E(aX) \equiv aE(X)\}$

$\Rightarrow 64 = E(3X^2) - 3b^2 + 48b - 192$

$\Rightarrow 0 = E(3X^2) - 3b^2 + 48b - 256$

$\Rightarrow 3b^2 - 48b = E(3X^2) - 256$

$\Rightarrow 3b^2 - 48b = E(3X^2 - 256)$

$\Rightarrow E(3X^2 - 256) = 3b(b-16)$

AS REQUIRED

Question 12 (****)

The continuous random variable X is uniformly distributed in the interval $[a,b]$, where a and b are constants.

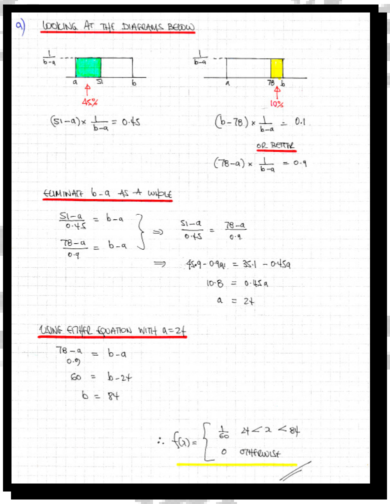
The 45^{th} and 90^{th} percentiles are 51 and 78, respectively.

a) Determine the probability density function of X.

The sum, denoted by S, of 100 random observations of X is obtained.

b) Calculate $\mathrm{P}(S > 5750)$.

$\boxed{}$, $\boxed{f(x) = \begin{cases} \dfrac{1}{60} & 24 < x < 84 \\ 0 & \text{otherwise} \end{cases}}$, $\boxed{\mathrm{P}(S > 5750) \approx 0.217}$



a) LOOKING AT THE DIAGRAMS BELOW

$\frac{1}{b-a}$ a 51 b 45%

$\frac{1}{b-a}$ a 78 b 10%

$(51-a)\times\frac{1}{b-a} = 0.45$

$(b-78)\times\frac{1}{b-a} = 0.1$

OR BETTER

$(78-a)\times\frac{1}{b-a} = 0.9$

ELIMINATE $b-a$ AS A WHOLE

$\left.\begin{array}{l}\frac{51-a}{0.45} = b-a \\ \frac{78-a}{0.9} = b-a\end{array}\right\} \Rightarrow \frac{51-a}{0.45} = \frac{78-a}{0.9}$

$\Rightarrow 45.9 - 0.9a = 35.1 - 0.45a$

$10.8 = 0.45a$

$a = 24$

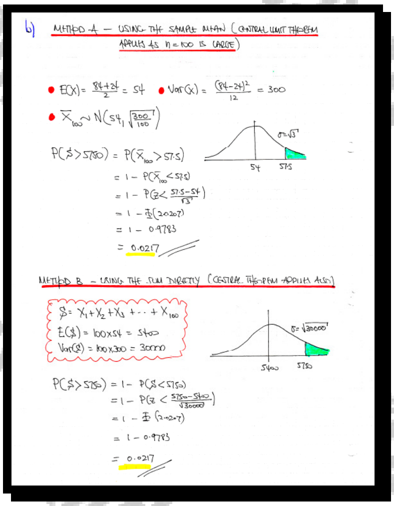
USING EITHER EQUATION WITH $a=24$

$\frac{78-a}{0.9} = b-a$

$60 = b-24$

$b = 84$

$\therefore f(x) = \begin{cases} \frac{1}{60} & 24 < x < 84 \\ 0 & \text{OTHERWISE} \end{cases}$



b) METHOD A – USING THE SAMPLE MEAN (CENTRAL LIMIT THEOREM APPLIES AS $n=100$ IS LARGE)

- $E(X) = \frac{84+24}{2} = 54$ • $\mathrm{Var}(X) = \frac{(84-24)^2}{12} = 300$
- $\bar{X}_{100} \sim N\left(54, \sqrt{\frac{300}{100}}\right)$

$\sigma = \sqrt{3}$ 54 57.5

$P(S > 5750) = P(\bar{X}_{100} > 57.5)$

$= 1 - P(\bar{X}_{100} < 57.5)$

$= 1 - P\left(Z < \frac{57.5-54}{\sqrt{3}}\right)$

$= 1 - \Phi(2.0207)$

$= 1 - 0.9783$

$= 0.0217$

METHOD B – USING THE SUM DIRECTLY (CENTRAL THEOREM APPLIES ALSO)

$S = X_1 + X_2 + X_3 + \dots + X_{100}$

$E(S) = 100 \times 54 = 5400$

$\mathrm{Var}(S) = 100 \times 300 = 30000$

$\sigma = \sqrt{30000}$ 5400 5750

$P(S > 5750) = 1 - P(S < 5750)$

$= 1 - P\left(Z < \frac{5750-5400}{\sqrt{30000}}\right)$

$= 1 - \Phi(2.0207)$

$= 1 - 0.9783$

$= 0.0217$

Question 13 (****+)

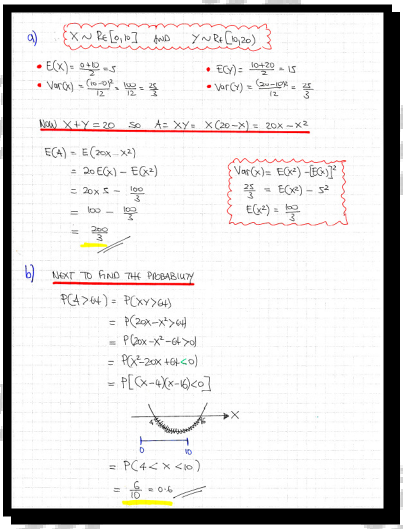
The continuous random variables X and Y represent the sides of a rectangle, and A denotes its area.

X is uniformly distributed in the interval $[0,10]$ and Y is also uniformly distributed in the interval $[10,20]$, with the additional constraint $X+Y=20$.

a) Find the value of $\mathrm{E}(A)$.

b) Determine $\mathrm{P}(A>64)$

$\boxed{}$, $\boxed{\mathrm{E}(A)=\frac{200}{3}}$, $\boxed{\mathrm{P}(A>64)=0.6}$



a) $X \sim R_E[0,10]$ AND $Y \sim R_E[10,20)$

- $E(X) = \frac{0+10}{2} = 5$
- $\mathrm{Var}(X) = \frac{(10-0)^2}{12} = \frac{100}{12} = \frac{25}{3}$
- $E(Y) = \frac{10+20}{2} = 15$
- $\mathrm{Var}(Y) = \frac{(20-10)^2}{12} = \frac{25}{3}$

NOW $X+Y=20$ SO $A = XY = X(20-X) = 20X - X^2$

$E(A) = E(20X - X^2)$
$= 20E(X) - E(X^2)$
$= 20 \times 5 - \frac{100}{3}$
$= 100 - \frac{100}{3}$
$= \frac{200}{3}$

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$
$\frac{25}{3} = E(X^2) - 5^2$
$E(X^2) = \frac{100}{3}$

b) NEXT TO FIND THE PROBABILITY

$P(A>64) = P(XY>64)$
$= P(20X - X^2 > 64)$
$= P(20X - X^2 - 64 > 0)$
$= P(X^2 - 20X + 64 < 0)$
$= P[(X-4)(X-16) < 0]$

$= P(4 < X < 10)$
$= \frac{6}{10} = 0.6$

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Question 14 (****+)

A rectangle $ABCD$ has a fixed perimeter of 20 cm.

The continuous random variable X cm represents the length of AB.

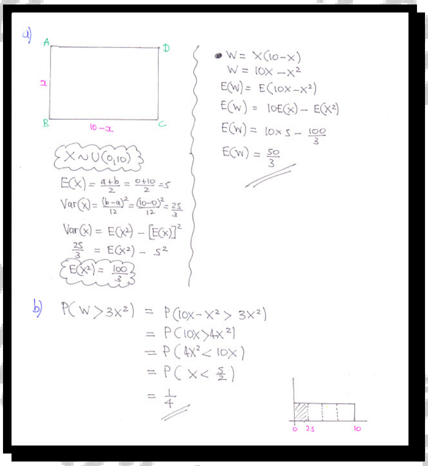
X is uniformly distributed in the interval $(0,10)$.

The continuous random variable W cm^2 represents the area of $ABCD$.

a) Find the value of $\mathrm{E}(W)$.

b) Determine the probability that W is more than three times the area of a square whose side has length X.

$\boxed{\frac{1}{4}}$



a)

$X \sim U(0,10)$

$E(X) = \frac{a+b}{2} = \frac{0+10}{2} = 5$

$Var(X) = \frac{(b-a)^2}{12} = \frac{(10-0)^2}{12} = \frac{25}{3}$

$Var(X) = E(X^2) - [E(X)]^2$

$\frac{25}{3} = E(X^2) - 5^2$

$E(X^2) = \frac{100}{3}$

$W = X(10-X)$

$W = 10X - X^2$

$E(W) = E(10X - X^2)$

$E(W) = 10E(X) - E(X^2)$

$E(W) = 10 \times 5 - \frac{100}{3}$

$E(W) = \frac{50}{3}$

b) $P(W > 3X^2) = P(10X - X^2 > 3X^2)$

$= P(10X > 4X^2)$

$= P(4X^2 < 10X)$

$= P(X < \frac{5}{2})$

$= \frac{1}{4}$

Question 15 (****+)

A wire of length 32 cm is cut into 2 pieces at a random point.

a) Find the probability that the longer piece of the wire will be smaller than 20 cm.

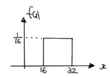
b) Determine the probability that the longer piece of the wire will be 20 cm, correct to the nearest cm.

Each piece of the wire, after it is randomly cut, is bent to form the outline of a square.

c) Find the probability that the difference in the area between these squares will be greater than 32 cm^2.

$\boxed{}$, $\boxed{\frac{1}{4}}$, $\boxed{\frac{1}{16}}$, $\boxed{\frac{1}{2}}$

$\boxed{\begin{array}{l} L = \text{LENGTH OF THE LONGER PIECE} \\ L \sim \text{UNIFORM}(16, 32) \end{array}}$



a) $P(L<20) = (20-16) \times \frac{1}{16}$

$= \frac{1}{4}$

b) $P(L = 20 \text{ NEAREST cm}) = P(19.5 < L < 20.5)$

$= 1 \times \frac{1}{16}$

$= \frac{1}{16}$

c) $\boxed{\begin{array}{ll} L = \text{longer piece} & (L>16) \\ 32-L = \text{shorter piece} & \end{array}}$

$P(\text{DIFFERENCE IN AREAS}) > 32)$

$= P\left[\left(\frac{L}{4}\right)^2 - \left(\frac{32-L}{4}\right)^2 > 32\right]$

$= P\left[L^2 - (32-L)^2 > 16 \times 32\right]$

$= P\left[L^2 - (1024 - 64L + L^2) > 512\right]$

$= P\left[64L - 1024 > 512\right]$

$= P\left[64L > 1536\right]$

$= P(L > 24)$

$= (32-24) \times \frac{1}{16}$

$= 0.5$

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Question 16 (****+)

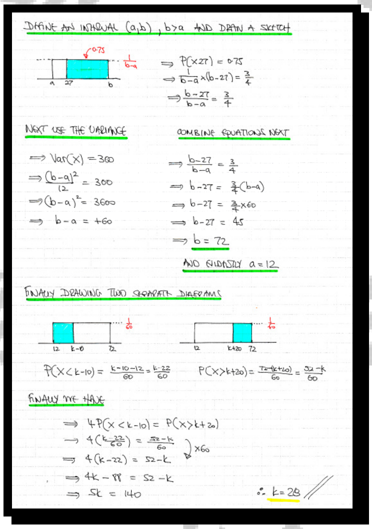
The continuous random variable X is uniformly distributed.

It is further given that

$$\mathrm{P}(X > 27) = 0.75 \quad \blacklozenge \quad \mathrm{Var}(X) = 300 \quad \blacklozenge \quad 4\mathrm{P}(X < k-10) = \mathrm{P}(X > k+20)$$

Determine the value of k.

$\boxed{}$, $\boxed{k = 28}$



DEFINE AN INTERVAL (a,b), $b>a$ AND DRAW A SKETCH

$\Rightarrow \mathrm{P}(X>27) = 0.75$

$\Rightarrow \frac{1}{b-a} \times (b-27) = \frac{3}{4}$

$\Rightarrow \frac{b-27}{b-a} = \frac{3}{4}$

NEXT USE THE VARIANCE

$\Rightarrow \mathrm{Var}(X) = 300$

$\Rightarrow \frac{(b-a)^2}{12} = 300$

$\Rightarrow (b-a)^2 = 3600$

$\Rightarrow b-a = +60$

COMBINE EQUATIONS NEXT

$\Rightarrow \frac{b-27}{b-a} = \frac{3}{4}$

$\Rightarrow b-27 = \frac{3}{4}(b-a)$

$\Rightarrow b-27 = \frac{3}{4} \times 60$

$\Rightarrow b-27 = 45$

$\Rightarrow b = 72$

AND EVIDENTLY $a = 12$

FINALLY DRAWING TWO SEPARATE DIAGRAMS

$\mathrm{P}(X<k-10) = \frac{k-10-12}{60} = \frac{k-22}{60}$ $\qquad$ $\mathrm{P}(X>k+20) = \frac{72-(k+20)}{60} = \frac{52-k}{60}$

FINALLY WE HAVE

$\Rightarrow 4\mathrm{P}(X<k-10) = \mathrm{P}(X>k+20)$

$\Rightarrow 4\left(\frac{k-22}{60}\right) = \frac{52-k}{60}$ $\quad \times 60$

$\Rightarrow 4(k-22) = 52-k$

$\Rightarrow 4k-88 = 52-k$

$\Rightarrow 5k = 140$

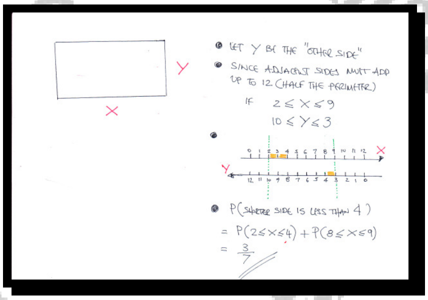
$\therefore k = 28$

Question 17 (****+)

The length, X cm, of one of the sides of a rectangle is uniformly distributed in the interval between 2 cm and 9 cm.

Given that the perimeter of the rectangle is 24 cm, determine the probability that the length of the shorter side of the rectangle is less than 4 cm .

$\boxed{\frac{3}{7}}$

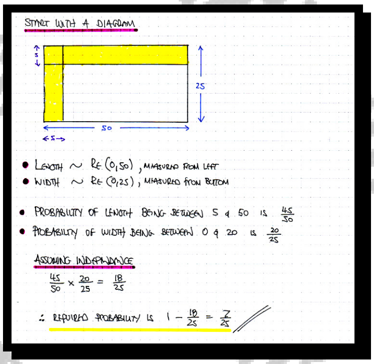


Question 18 (****+)

A dot of negligible dimensions can appear at random anywhere on a rectangular screen which measures 50 cm in width, measured from "left to right" and 25 cm in height, measured from "bottom to top".

Determine the probability that the dot will next appear within 5 cm of the left of the screen or within 5 cm of the top of the screen.

$\boxed{}$, $\boxed{\frac{7}{25}}$



Question 19 (*****)

The continuous random variable X is uniformly distributed in the interval $a \le x \le b$.

Use integration to show that

$$\mathrm{E}(X) = \tfrac{1}{2}(a+b) \quad \text{and} \quad \mathrm{Var}(X) = \tfrac{1}{12}(b-a)^2$$

$\boxed{}$, $\boxed{\text{proof}}$

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \int_a^b x f(x)\,dx = \int_a^b \frac{1}{b-a} x\,dx = \frac{1}{b-a}\int_a^b x\,dx$$

$$= \frac{1}{b-a}\left[\tfrac{1}{2}x^2\right]_a^b = \frac{1}{2} \times \frac{1}{b-a}\left[b^2 - a^2\right] = \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{a+b}{2}$$

$$E(X^2) = \int_a^b x^2 f(x)\,dx = \int_a^b \frac{1}{b-a} x^2\,dx = \frac{1}{b-a}\int_a^b x^2\,dx$$

$$= \frac{1}{b-a}\left[\tfrac{1}{3}x^3\right]_a^b = \frac{1}{3} \times \frac{1}{b-a}\left[b^2 - a^3\right] = \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}} = \frac{b^2+ab+a^2}{3}$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ba+b^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-(3a^2+6ab+3b^2)}{12}$$

$$= \frac{a^2+b^2-2ab}{12} = \frac{(b-a)^2}{12}$$

NOTE $a^3 \pm b^3 \equiv (a \pm b)(a^2 \mp ab + b^2)$