IYGB – FP2 PAPER J – QUESTION 1

USING THE STANDARD FORMULA FOR POLAR AREA

$$\text{AREA} = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2 \, d\theta = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2}(a\theta)^2 \, d\theta = \frac{1}{2}a^2 \int_0^{2\pi} \theta^2 \, d\theta$$

$$= \frac{1}{2}a^2 \left[\frac{1}{3}\theta^3 \right]_0^{2\pi} = \frac{1}{6}a^2 \left[8\pi^3 - 0 \right] = \frac{4}{3}\pi^3 a^2$$

IYGB - FP2 PAPER J - QUESTION 2

a) START BY FACTORIZING THE DENOMINATOR FULLY

$$f(x) = \frac{4x}{1-x^4} = \frac{4x}{(1-x^2)(1+x^2)} = \frac{4x}{(1-x)(1+x)(1+x^2)}$$

$$f(x) = \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$$

$$\boxed{4x \equiv A(1+x)(1+x^2) + B(1-x)(1+x^2) + (1-x)(1+x)(Cx+D)}$$

- IF $x=1$

$4 = A(2)(2)$

$A = 1$

- IF $x=-1$

$-4 = (2)(2)B$

$B = -1$

- IF $x=0$

$0 = A + B + D$

$0 = 1 - 1 + D$

$D = 0$

- COMPARING COEFFICIENTS OF x^3 IN BOTH SIDES

$$0 = Ax^3 - Bx^3 - Cx^3$$
$$0 = 1x^3 + 1x^3 - Cx^3$$
$$\therefore C = 2$$

$$\therefore f(x) = \frac{1}{1-x} - \frac{1}{1+x} + \frac{2x}{1+x^2}$$

b) USING PART (a)

$$\int_0^{\frac{1}{2}} f(x)\,dx = \int_0^{\frac{1}{2}} \frac{1}{1-x} - \frac{1}{1+x} + \frac{2x}{1+x^2}\,dx$$

$$= \Big[-\ln|1-x| - \ln|1+x| + \ln|1+x^2|\Big]_0^{\frac{1}{2}}$$

$$= \left(-\ln\tfrac{1}{2} - \ln\tfrac{3}{2} + \ln\tfrac{5}{4}\right) - \left(-\cancel{\ln 1} - \cancel{\ln 1} + \cancel{\ln 1}\right)$$

$$= \ln\tfrac{5}{4} - \ln\tfrac{1}{2} - \ln\tfrac{3}{2}$$

$$= \ln\left(\frac{\frac{5}{4}}{\frac{1}{2}\times\frac{3}{2}}\right)$$

$$= \ln\left(\frac{5}{3}\right)$$

IYGB - FP2 PAPER J - QUESTION 3

a) USING STANDARD SURDS

$$\frac{1}{\sqrt{r+2}+\sqrt{r}} = \frac{\sqrt{r+2}+\sqrt{r}}{(\sqrt{r+2}+\sqrt{r})(\sqrt{r+2}-\sqrt{r})} = \frac{\sqrt{r+2}-\sqrt{r}}{\cancel{r}+2-\cancel{r}}$$

$$= \frac{1}{2}\left(\sqrt{r+2}-\sqrt{r}\right)$$

b) USING PART (a)

$$\boxed{\frac{2}{\sqrt{r+2}+\sqrt{r}} \equiv \sqrt{r+2}-\sqrt{r}}$$

$r=1$: $\dfrac{2}{\sqrt{3}+\sqrt{1}} = \sqrt{3}-\sqrt{1}$

$r=2$: $\dfrac{2}{\sqrt{4}+\sqrt{2}} = \sqrt{4}-\sqrt{2}$

$r=3$: $\dfrac{2}{\sqrt{5}+\sqrt{3}} = \sqrt{5}-\sqrt{3}$

$r=4$: $\dfrac{2}{\sqrt{6}+\sqrt{4}} = \sqrt{6}-\sqrt{4}$

$\vdots \qquad \vdots \qquad \vdots$

$r=n-1$: $\dfrac{2}{\sqrt{n+1}+\sqrt{n-1}} = \sqrt{n+1}-\sqrt{n-1}$

$r=n$: $\dfrac{2}{\sqrt{n+2}+\sqrt{n}} = \sqrt{n+2}-\sqrt{n}$

$$\Longrightarrow \sum_{r=1}^{n} \frac{2}{\sqrt{r+2}+\sqrt{r}} = \sqrt{n+2}+\sqrt{n+1}-\sqrt{2}-1$$

← ADDING BOTH SIDES

$$\Longrightarrow 2\sum \frac{1}{\sqrt{r+2}+\sqrt{r}} = \sqrt{n+2}+\sqrt{n+1}-\sqrt{2}-1$$

$$\Longrightarrow \sum_{r=1}^{n} f(r) = \frac{1}{2}\left[\sqrt{n+2}+\sqrt{n+1}-\sqrt{2}-1\right]$$

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IYGB – FP2 PAPER J – QUESTION 3

c) LET $n = 48$ IN PART (b)

$$\sum_{r=1}^{48} f(r) = \frac{1}{2}\left[\sqrt{50} + \sqrt{49} - \sqrt{2} - 1\right]$$

$$= \frac{1}{2}\left[5\sqrt{2} + 7 - \sqrt{2} - 1\right]$$

$$= \frac{1}{2}\left[4\sqrt{2} + 6\right]$$

$$= 3 + 2\sqrt{2}$$

AS REQUIRED

IYGB – FP2 PAPER J – QUESTION 4

USING THE POLAR TRANSFORMATION EQUATIONS

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Longrightarrow \begin{array}{l} \cos\theta = \frac{x}{r} \\ \sin\theta = \frac{y}{r} \end{array}$$

SUBSTITUTE INTO THE EQUATION

$$\Longrightarrow r = 2(\cos\theta - \sin\theta)$$

$$\Longrightarrow r = 2\left(\frac{x}{r} - \frac{y}{r}\right)$$

$$\Longrightarrow r^2 = 2x - 2y$$

BUT $r^2 = x^2 + y^2$

$$\Longrightarrow x^2 + y^2 = 2x - 2y$$

$$\Longrightarrow x^2 - 2x + y^2 + 2y = 0$$

$$\Longrightarrow (x-1)^2 + (y+1)^2 = 2$$

INDEED A CIRCLE, CENTRE $(1,-1)$, RADIUS $\sqrt{2}$ //

IYGB - FP2 PAPER J - QUESTION 5

BY DIRECT DIFFERENTIATIONS

$$f(x) = \ln(2-e^x)$$

$$f'(x) = \frac{1}{2-e^x} \times (-e^x) = \frac{-e^x}{2-e^x} = \frac{e^x}{e^x-2} = \frac{e^x-2+2}{e^x-2}$$

$$= \frac{e^x-2}{e^x-2} + \frac{2}{e^x-2} = 1 + 2(e^x-2)^{-1}$$

$$f''(x) = -2(e^x-2)^{-2}e^x = -\frac{2e^x}{(e^x-2)^2}$$

$$f'''(x) = -\frac{(e^x-2)^2(2e^x) - 2e^x \times 2(e^x-2)e^x}{(e^x-2)^4}$$

$$= -\frac{2e^x(e^x-2)^2 - 4e^{2x}(e^x-2)}{(e^x-2)^4} = -\frac{2e^x(e^x-2) - 4e^{2x}}{(e^x-2)^3}$$

$$= \frac{4e^{2x} - 2e^{2x} + 4e^x}{(e^x-2)^3} = \frac{2e^{2x}+4e^x}{(e^x-2)^3}$$

NOW EVALUATING AT $x=0$

$$\cancel{f(0) = \ln 1}, \quad f'(0) = -1, \quad f''(0) = -2, \quad f'''(0) = -6$$

BY THE MACLAURIN THEOREM

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + O(x^4)$$

$$\ln(2-e^x) = 0 - x - x^2 - x^3 + O(x^4)$$

IYGB - FP2 PAPER J - QUESTION 6

a) WORKING IN EXPONENTIALS

$$\Rightarrow y = \text{artanh}\, x$$
$$\Rightarrow \tanh y = x$$
$$\Rightarrow \frac{e^{2y}-1}{e^{2y}+1} = x$$
$$\Rightarrow e^{2y}-1 = xe^{2y}+x$$
$$\Rightarrow e^{2y} - xe^{2y} = 1+x$$
$$\Rightarrow e^{2y}(1-x) = 1+x$$
$$\Rightarrow e^{2y} = \frac{1+x}{1-x}$$
$$\Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right)$$
$$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$
$$\Rightarrow \text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

AS REQUIRED

b) USING PART (a)

$$\Rightarrow x = \tanh\left(\ln\sqrt{6x}\right)$$
$$\Rightarrow \text{artanh}\, x = \ln\sqrt{6x}$$
$$\Rightarrow \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \ln(6x)^{\frac{1}{2}}$$
$$\Rightarrow \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\ln(6x)$$
$$\Rightarrow \ln\left(\frac{1+x}{1-x}\right) = \ln(6x)$$
$$\Rightarrow \frac{1+x}{1-x} = 6x$$
$$\Rightarrow 1+x = 6x-6x^2$$
$$\Rightarrow 6x^2-5x+1 = 0$$
$$\Rightarrow (3x-1)(2x-1) = 0$$
$$\Rightarrow x = \begin{cases} 1/2 \\ 1/3 \end{cases}$$

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REWRITE THE O.D.E IN "STANDARD" FORM

$$\Rightarrow x^2 \frac{dy}{dx} + xy(x+3) = 1$$

$$\Rightarrow \frac{dy}{dx} + y\left(\frac{x^2+3x}{x^2}\right) = \frac{1}{x^2}$$

$$\Rightarrow \frac{dy}{dx} + y\left(\frac{x+3}{x}\right) = \frac{1}{x^2}$$

OBTAIN THE INTEGRATING FACTOR

$$\text{I.f.} = e^{\int \frac{x+3}{x}\,dx} = e^{\int 1+\frac{3}{x}\,dx} = e^{x+3\ln x} = e^x \times e^{3\ln x}$$

$$= e^x \times e^{\ln x^3} = x^3 e^x$$

MULTIPLY THROUGH MAKES L.H.S EXACT

$$\Rightarrow \frac{d}{dx}\left[y x^3 e^x\right] = \frac{1}{x^2} x^3 e^x$$

$$\Rightarrow y x^3 e^x = \int x e^x \, dx$$

INTEGRATE THE R.H.S BY PARTS

$$\Rightarrow y x^3 e^x = x e^x - \int e^x \, dx$$

x	1
e^x	e^x

$$\Rightarrow y x^3 e^x = x e^x - e^x + A$$

$$\Rightarrow y = \frac{1}{x^2} - \frac{1}{x^3} + \frac{A}{x^3} e^{-x}$$

IYGB - FP2 PAPER J - QUESTION 7

APPLY THE BOUNDARY CONDITION $(1,1)$

$$\Rightarrow 1 = 1 - 1 + \frac{A}{1}e^{-1}$$

$$\Rightarrow 1 = Ae^{-1}$$

$$\Rightarrow A = \frac{1}{e^{-1}} = e^{1} = e$$

FINALLY LET $x=2$

$$y = \frac{1}{x^2} - \frac{1}{x^3} + \frac{e}{x^3}e^{-x}$$

$$k = \frac{1}{2^2} - \frac{1}{2^3} + \frac{e \times e^{-2}}{2^3}$$

$$k = \frac{1}{4} - \frac{1}{8} + \frac{e^{-1}}{8}$$

$$k = \frac{e^{-1}}{8} + \frac{1}{8}$$

$$k = \frac{1}{8}\left(\frac{1}{e}+1\right)$$

$$k = \frac{1}{8}\left(\frac{e+1}{e}\right)$$

$$k = \frac{e+1}{8e}$$

IYGB - FP2 PAPER J - QUESTION 8

USING HYPERBOLIC IDENTITIES

$$1 + \tan^2 A \equiv \sec^2 A$$
$$1 - \tanh^2 A \equiv \text{sech}^2 A$$
$$1 - \text{sech}^2 A \equiv \tanh^2 A$$

$$\int_0^{\ln 3} \tanh^2 x \, dx = \int_0^{\ln 3} 1 - \text{sech}^2 x \, dx = \Big[x - \tanh x \Big]_0^{\ln 3}$$

$$= \Big[\ln 3 - \tanh(\ln 3)\Big] - \cancel{\Big[0 - 0\Big]}$$

$$= \ln 3 - \frac{e^{2\ln 3} - 1}{e^{2\ln 3} + 1} \quad \text{(OR CALCULATOR)}$$

$$= \ln 3 - \frac{9-1}{9+1}$$

$$= \ln 3 - \frac{4}{5}$$

$$\therefore \text{ MEAN VALUE} = \frac{1}{b-a}\int_a^b f(x)\, dx$$

$$= \frac{1}{\ln 3}\left(\ln 3 - \frac{4}{5}\right)$$

$$= \frac{\ln 3 - \frac{4}{5}}{\ln 3}$$

$$= \frac{5\ln 3 - 4}{5\ln 3} \quad \text{OR} \quad 1 - \frac{4}{5\ln 3}$$

IYGB - FP2 PAPER J - QUESTION 9

a) LET $\cos\theta + i\sin\theta \equiv C + iS$

$$\Rightarrow (\cos\theta + i\sin\theta) = C + iS$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$\Rightarrow (\cos\theta + i\sin\theta)^4 = (C+iS)^4$$

$$\Rightarrow \cos4\theta + i\sin4\theta = C^4 + 4iC^3S - 6C^2S^2 - 4iCS^3 + S^4$$

NOW WE HAVE BY EQUATING REAL & IMAGINARY PARTS

$$\cot4\theta = \frac{\cos4\theta}{\sin4\theta} = \frac{C^4 - 6C^2S^2 + S^4}{4C^3S - 4CS^3} =$$

DIVIDE TOP & BOTTOM OF THE FRACTION BY S^4

$$= \frac{\frac{C^4}{S^4} - \frac{6C^2S^2}{S^4} + \frac{S^4}{S^4}}{\frac{4C^3S}{S^4} - \frac{4CS^3}{S^4}}$$

$$\therefore \cot4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$$

AS REQUIRED

b) START BY THE EQUATION $\cot4\theta = 0$

$$\cot4\theta = 0 \Rightarrow \tan4\theta = \pm\infty$$

$$\Rightarrow 4\theta = \dots \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$$

$$\Rightarrow \theta = \dots \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \dots$$

USING PART (a) WITH $\cot4\theta = 0$

$$\Rightarrow \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta} = 0$$

$$\Rightarrow \cot^4\theta - 6\cot^2\theta + 1 = 0$$

$$\Rightarrow x^2 - 6x + 1 = 0 \qquad [x = \cot^2\theta]$$

$\therefore \cot^2\left(\frac{\pi}{8}\right), \cot^2\left(\frac{3\pi}{8}\right), \cot^2\left(\frac{5\pi}{8}\right), \cot^2\left(\frac{7\pi}{8}\right), \dots$ ARE ROOTS

IYGB - FP2 PAPER J - QUESTION 9

SO SOLUTIONS ARE TWO AS THEY REPEAT

$$\cot^2\left(\frac{\pi}{8}\right) = \cot^2\left(\frac{5\pi}{8}\right) = \cot^2\left(\frac{9\pi}{8}\right) = \ldots$$

$$\& \quad \cot^2\left(\frac{3\pi}{8}\right) = \cot^2\left(\frac{7\pi}{8}\right) = \cot^2\left(\frac{11\pi}{8}\right) = \ldots$$

SO $\cot^2\left(\frac{\pi}{8}\right)$ IS ONE OF THE SOLUTIONS, THE OTHER $\cot^2\left(\frac{3\pi}{8}\right)$

c) USING ROOT RELATIONSHIPS

$$\Rightarrow \alpha + \beta = -\frac{b}{a}$$

$$\Rightarrow \cot^2\frac{\pi}{8} + \cot^2\frac{3\pi}{8} = -\frac{-6}{1}$$

$$\Rightarrow \left(\operatorname{cosec}^2\frac{\pi}{8} - 1\right) + \left(\operatorname{cosec}^2\frac{3\pi}{8}\right) = 6$$

$$\Rightarrow \operatorname{cosec}^2\frac{\pi}{8} + \operatorname{cosec}^2\frac{3\pi}{8} = 8$$

AS REQUIRED

OR IN SIMILAR FASHION

$$\Rightarrow x^2 - 6x + 1 = 0$$

$$\Rightarrow (x-3)^2 - 9 + 1 = 0$$

$$\Rightarrow (x-3)^2 = 8$$

$$\Rightarrow x - 3 = \pm\sqrt{8}$$

$$\Rightarrow \begin{cases} x_1 = 3 + \sqrt{8} \\ x_2 = 3 - \sqrt{8} \end{cases}$$

THUS WE HAVE

$$\cot^2\frac{\pi}{8} + \cot^2\frac{3\pi}{8} = (3+\sqrt{8}) + (3-\sqrt{8})$$

$$\cot^2\frac{\pi}{8} + \cot^2\frac{3\pi}{8} = 6$$

$$\left(\operatorname{cosec}^2\frac{\pi}{8} - 1\right) + \left(\operatorname{cosec}^2\frac{3\pi}{8} - 1\right) = 6$$

$$\operatorname{cosec}^2\frac{\pi}{8} + \operatorname{cosec}^2\frac{3\pi}{8} = 8$$

AS OPPOSITE