IYGB – FP2 PAPER W – QUESTION 1

USING THE "STANDARD TRANSFORMATION" EQUATIONS

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2 = r^2$

$\Rightarrow r = \cos\theta + \sin\theta$

$\Rightarrow r = \frac{x}{r} + \frac{y}{r}$

$\Rightarrow r = \frac{x+y}{r}$

$\Rightarrow r^2 = x+y$

$\Rightarrow x^2+y^2-x-y=0$

$\Rightarrow \left(x-\frac{1}{2}\right)^2 - \frac{1}{4} + \left(y-\frac{1}{2}\right)^2 - \frac{1}{4} = 0$

$\Rightarrow \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{2}$

$\therefore$ INDEED A CIRCLE

CENTRE AT $\left(\frac{1}{2}, \frac{1}{2}\right)$

RADIUS $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

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PROCEED AS FOLLOWS

$\cosh(4x-3y)=1$

$4x-3y=0$

$y = \frac{1}{x}e^{\text{arsinh}\frac{4}{3}}$

$xy = e^{\ln\left[\frac{4}{3}+\sqrt{\frac{16}{9}+1}\right]}$

$xy = e^{\ln\left(\frac{4}{3}+\sqrt{\frac{25}{9}}\right)}$

$xy = \frac{4}{3}+\frac{5}{3}$

$xy = 3$

NOW ANY SENSIBLE APPROACH – MULTIPLY THE FIRST EQUATION BY y

$\Rightarrow 4x-3y=0$

$\Rightarrow 4xy-3y^2=0$

$\Rightarrow 4\times3-3y^2=0$

$\Rightarrow 12=3y^2$

$\Rightarrow y^2=4$

$\Rightarrow y = \left\langle \begin{matrix} 2 \\ \cancel{-2} \end{matrix}\right.$ $y>0$

FINALLY WE CAN GET x

$\Rightarrow 4x-3y=0$

$\Rightarrow 4x=3y$

$\Rightarrow 4x=6$

$\Rightarrow x=\frac{3}{2}$

$\therefore (x,y)=\left(\frac{3}{2},2\right)$

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WORKING AS FOLLOWS

$$\ln(x^2+4x+4) = \ln\left[(x+2)^2\right] = 2\ln(x+2) = 2\ln(2+x)$$

$$= 2\ln\left[2\left(1+\tfrac{1}{2}x\right)\right]$$

$$= 2\ln 2 + 2\ln\left(1+\tfrac{1}{2}x\right)$$

NOW USING STANDARD EXPANSIONS

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots$$

$$\ln\left(1+\tfrac{1}{2}x\right) = \tfrac{1}{2}x - \tfrac{1}{2}\left(\tfrac{1}{2}x\right)^2 + \tfrac{1}{3}\left(\tfrac{1}{2}x\right)^3 - \tfrac{1}{4}\left(\tfrac{1}{2}x\right)^4 + \dots$$

$$= \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{24}x^3 - \tfrac{1}{64}x^4 + \dots$$

$$\therefore \ \ln(x^2+4x+4) = 2\ln 2 + 2\left[\tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{24}x^3 - \tfrac{1}{64}x^4 + \dots\right]$$

$$= 2\ln 2 + x - \tfrac{1}{4}x^2 + \tfrac{1}{12}x^3 - \tfrac{1}{32}x^4 + O(x^5)$$

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DIFFERENTIATE THE FIRST EQUATION WITH RESPECT TO x

$$\frac{dx}{dt} = x - 2y$$

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - 2\frac{dy}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - 2(5x - y)$$

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - 10x + 2y$$

BUT $2y = x - \frac{dx}{dt}$

$$\frac{d^2x}{dt^2} = \cancel{\frac{dx}{dt}} - 10x + \left(x - \cancel{\frac{dx}{dt}}\right)$$

$$\frac{d^2x}{dt^2} = -9x$$

$$\frac{d^2x}{dt^2} + 9x = 0$$

AUXILIARY EQUATION

$$\lambda^2 + 9 = 0$$

$$\lambda = \pm 3i$$

$$\therefore \underline{\underline{x(t) = A\cos 3t + B\sin 3t}}$$

DIFFERENTIATE $x(t)$ WITH RESPECT TO t

$$\frac{dx}{dt} = -3A\sin 3t + 3B\cos 3t$$

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SUBSTITUTE x & $\dot{x}$ INTO $2y = x - dx/dt$

$$\Rightarrow 2y = x - \frac{dx}{dt}$$

$$\Rightarrow 2y = A\cos 3t + B\sin 3t - (-3A\sin 3t + 3B\cos 3t)$$

$$\Rightarrow 2y = A\cos 3t + B\sin 3t + 3A\sin 3t - 3B\cos 3t$$

$$\Rightarrow y = \frac{A-3B}{2}\cos 3t + \frac{B+3A}{2}\sin 3t$$

APPLY CONDITIONS FIRST INTO $x(t)$

$t=0 \quad x=-1 \Rightarrow -1 = A$

APPLY CONDITION TO $y(t)$

$$t=0 \quad y=2 \Rightarrow 2 = \frac{A-3B}{2}$$

$$4 = A - 3B$$

$$3B = A - 4$$

$$3B = -5$$

$$B = -\frac{5}{3}$$

FINALLY WE HAVE IF $A=-1$ & $B = 5/3$

$$x(t) = -\cos 3t - \frac{5}{3}\sin 3t$$

$$y(t) = 2\cos 3t - \frac{7}{3}\sin 3t$$

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a) SETTING UP A STANDARD VOLUME INTEGRAL IN PARAMETRIC, STARTING BY FINDING THE VALUE OF t AT POINT "A"

AT A $y=0$, $x>0$

$4\sin 2t = 0$

$\sin 2t = 0$

$2t = 0, \pi, 2\pi, 3\pi, \dots$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots$

$$V = \pi \int_{x_1}^{x_2} [y(x)]^2 \, dx = \pi \int_{t_1}^{t_2} (y(t))^2 \frac{dx}{dt} \, dt$$

$$V = \pi \int_0^{\pi/2} (4\sin 2t)^2 (2t) \, dt = \pi \int_0^{\pi/2} 32t \sin^2 2t \, dt$$

(BY INSPECTION)

$$V = \pi \int_0^{\pi/2} 32t \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 4t\right) dt = \pi \int_0^{\frac{\pi}{2}} 16t - 16t\cos 4t \, dt$$

AS REQUIRED

b) EVALUATING THE ABOVE

$$V = \pi \int_0^{\pi/2} 16t - 16t\cos 4t \, dt$$

BY PARTS

$4t$	4
$\sin 4t$	$4\cos 4t$

$$V = \pi \left[\left[8t^2\right]_0^{\pi/2} - \left[\left[4t\sin 4t\right]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} 4\sin 4t \, dt\right]\right]$$

$$V = \pi \left[\left[8t^2\right]_0^{\frac{\pi}{2}} - \left[4t\sin 4t\right]_0^{\pi/2} + \int_0^{\pi/2} 4\sin 4t \, dt\right]$$

$$V = \pi \left[8t^2 - 4t\sin 4t - \cos 4t \right]_0^{\pi/2}$$

$$V = \pi \left[(2\pi^2 - 0 + 1) - (0 - 0 + 1) \right]$$

$$V = \pi (2\pi^2)$$

$$\therefore V = 2\pi^3$$

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AS THE INTEGRAND IS EVEN WE MAY REWRITE AS FOLLOWS

$$\int_{-1.5}^{1.5} 8x \arcsin\left(\tfrac{1}{3}x\right) dx = 2\int_{0}^{1.5} 8x \arcsin\left(\tfrac{1}{3}x\right) dx$$

$$= 16\int_{0}^{1.5} x\arcsin\left(\tfrac{1}{3}x\right) dx$$

USING A SUBSTITUTION

$$= 16\int_{0}^{\frac{\pi}{6}} (3\sin\theta)\,\theta\,(3\cos\theta\,d\theta)$$

$$= 144\int_{0}^{\pi/6} \theta\cos\theta\sin\theta\,d\theta$$

$$= 72\int_{0}^{\frac{\pi}{6}} \theta\sin 2\theta\,d\theta$$

$\theta = \arcsin\left(\frac{1}{3}x\right)$

$\sin\theta = \frac{1}{3}x$

$x = 3\sin\theta$

$\frac{dx}{d\theta} = 3\cos\theta$

$dx = 3\cos\theta\,d\theta$

$x = 0 \longmapsto \theta = 0$

$x = 1.5 \longmapsto \theta = \frac{\pi}{6}$

PROCEED BY INTEGRATION BY PARTS

$$= 72\left[\left[-\tfrac{1}{2}\theta\cos 2\theta\right]_{0}^{\frac{\pi}{6}} - \int_{0}^{\frac{\pi}{6}} -\tfrac{1}{2}\cos 2\theta\,d\theta\right]$$

$$= 72\left[\left[-\tfrac{1}{2}\theta\cos 2\theta\right]_{0}^{\frac{\pi}{6}} + \int_{0}^{\pi/6} \tfrac{1}{2}\cos 2\theta\,d\theta\right]$$

$$= 72\left[-\tfrac{1}{2}\theta\cos 2\theta + \tfrac{1}{4}\sin 2\theta\right]_{0}^{\frac{\pi}{6}}$$

$$= 72\left[\left(-\tfrac{1}{2}\times\tfrac{\pi}{6}\times\tfrac{1}{2} + \tfrac{1}{4}\times\tfrac{\sqrt{3}}{2}\right) - (0)\right]$$

$$= 72\left[\frac{\sqrt{3}}{8} - \frac{\pi}{24}\right]$$

$$= 9\sqrt{3} - 3\pi$$

θ	1
$-\frac{1}{2}\cos 2\theta$	$\sin\theta$

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a) USING DE MOIVRE'S THEOREM

$$\cos\theta + i\sin\theta \equiv C + iS$$
$$(\cos\theta + i\sin\theta)^7 \equiv (C+iS)^7$$
$$(\cos\theta + i\sin\theta)^7 \equiv C^7 + 7iC^6S - 21C^5S^2 - 35iC^4S^3 + 35C^3S^4 + 21iC^2S^5 - 7CS^6 - iS^7$$
$$\cos7\theta + i\sin7\theta \equiv [C^7 - 21C^5S^2 + 35C^3S^4 - 7CS^6] + [7C^6S - 35C^4S^3 + 21C^2S^5 - S^7]i$$

EQUATE IMAGINARY PARTS

$$\sin7\theta = 7C^6S - 35C^4S^3 + 21C^2S^5 - S^7$$
$$= S(7C^6 - 35S^2C^4 + 21S^4C^2 - S^6)$$
$$= S(7(1-S^2)^3 - 35S^2(1-S^2)^2 + 21S^4(1-S^2) - S^6)$$
$$= S(7(1-3S^2+3S^4-S^6) - 35S^2(1-2S^2+S^4) + 21S^4 - 21S^6 - S^6)$$
$$= S(7 - 21S^2 + 21S^4 - 7S^6 - 35S^2 + 70S^4 - 35S^6 + 21S^4 - 21S^6 - S^6)$$
$$= S(7 - 56S^2 + 112S^4 - 64S^6)$$
$$\sin7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

// AS REQUIRED

b) SOLVING THE EQUATION $\sin7\theta = 0$

- $\theta = 0, \pi, 2\pi, 3\pi, \ldots$
- $\theta = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \ldots$

$$7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta = 0$$
$$-\sin\theta\underbrace{(64\sin^6\theta - 112\sin^4\theta + 56\sin^2\theta - 7)}_{\theta \neq 0 \quad \theta = \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \ldots} = 0$$

$\uparrow$
$\theta = 0$

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LET $z = \sin^2\theta \implies 64z^3 - 112z^2 + 56z - 7 = 0$

$$\alpha + \beta + \gamma = \frac{112}{64} = \frac{7}{4}$$

$$\alpha\beta + \gamma\beta + \gamma\alpha = \frac{56}{64} = \frac{7}{8}$$

$$\alpha\beta\gamma = \frac{7}{64}$$

NOW NOTE THAT $\sin\frac{\pi}{7} = \sin\frac{6\pi}{7}$, $\sin\frac{2\pi}{7} = \sin\frac{5\pi}{7}$, $\sin\frac{3\pi}{7} = \sin\frac{4\pi}{7}$

$$\operatorname{cosec}^2\frac{\pi}{7} + \operatorname{cosec}^2\frac{2\pi}{7} + \operatorname{cosec}^2\frac{3\pi}{7} = \frac{1}{\sin^2\frac{\pi}{7}} + \frac{1}{\sin^2\frac{2\pi}{7}} + \frac{1}{\sin^2\frac{3\pi}{7}}$$

$$= \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$$

$$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{\frac{7}{8}}{\frac{7}{64}}$$

$$= \frac{7 \times 64}{7 \times 8}$$

$$= 8$$

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MANIPULATE $f(r)$ AS FOLLOWS

$$f(r) = \frac{6r^4+6r^3-ar^2-ar+1}{r(r+1)} = \frac{6r^3(r+1)-ar(r+1)+1}{r(r+1)}$$

$$= 6r^2 - a + \frac{1}{r(r+1)}$$

PARTIAL FRACTIONS BY INSPECTION

$$= 6r^2 - a + \frac{1}{r} - \frac{1}{r+1}$$

NOW PROCEED BY THE METHOD OF DIFFERENCES

$$\boxed{f(r) \equiv 6r^2 - a + \frac{1}{r} - \frac{1}{r+1}}$$

$r=1 \qquad f(1) = 6\times1^2 - a + \frac{1}{1} - \cancel{\frac{1}{2}}$

$r=2 \qquad f(2) = 6\times2^2 - a + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$

$r=3 \qquad f(3) = 6\times3^2 - a + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$

$r=4 \qquad f(4) = 6\times4^2 - a + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$r=n \qquad f(n) = 6\times n^2 - a + \cancel{\frac{1}{n}} - \frac{1}{n+1}$

ADD $\quad \sum_{r=1}^{n} f(r) = 6\sum_{r=1}^{n} r^2 - na + 1 - \frac{1}{n+1} \quad$ ADD

$$= 6\times\tfrac{1}{6}n(n+1)(2n+1) - an + \frac{n+1-1}{n+1}$$

$$= n(n+1)(2n+1) - an + \frac{n}{n+1}$$

$$= \frac{n(n+1)^2(2n+1) - an(n+1) + n}{n+1}$$

IYGB - FP2 PAPER W - QUESTION 8

COMPARING NUMERATORS

$$\frac{n^2(n+2)(2n+1)}{n+1} \equiv \frac{n(n+1)^2(2n+1) - an(n+1) + n}{n+1}$$

$$n^2(2n^2+5n+2) \equiv n(2n+1)(n^2+2n+1) - an(n+1) + n$$

$$n(2n^2+5n+2) \equiv (2n+1)(n^2+2n+1) - a(n+1) + 1$$

$$2n^3 + 5n^2 + 2n \equiv \begin{array}{l} 2n^3 + 4n^2 + 2n \\ \quad\;\; n^2 + 2n + 1 \\ \qquad\quad -an - a \\ \qquad\qquad\;\; +1 \end{array}$$

$$2n^3 + 5n + 2n \equiv 2n^3 + 5n^2 + (4-a)n + (2-a)$$

$\therefore\ 4 - a = 2$ & $2 - a = 0$

$a = 2$ $\qquad$ $a = 2$

$\therefore\ a = 2$