IYGB - M456 PAPER A - QUESTION 1

$$\frac{d^2\underline{r}}{dt^2} - \frac{d\underline{r}}{dt} = 6(\underline{r} + t\underline{i} - 2\underline{j}) \qquad t=0 \qquad \underline{r} = \underline{i} + 2\underline{j}$$
$$\underline{v} = 3\underline{i} - \underline{j}$$

START BY REWRITING THE O.D.E IN ITS "USUAL FORM"

$$\frac{d^2\underline{r}}{dt^2} - \frac{d\underline{r}}{dt} - 6\underline{r} = 6t\underline{i} - 12\underline{j}$$

THIS SPLITS INTO TWO O.D.Es FOR $\underline{r} = \underline{r}(x(t), y(t))$

$$\boxed{\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 6t} \qquad \boxed{\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = -12}$$

THE AUXILIARY EQUATION FOR EITHER O.D.E IS

$$\Rightarrow \lambda^2 - \lambda - 6 = 0$$
$$\Rightarrow (\lambda + 2)(\lambda - 3) = 0$$
$$\Rightarrow \lambda = \begin{cases} 3 \\ -2 \end{cases}$$

$$\therefore \quad x = Ae^{3t} + Be^{-2t} \quad \& \quad y = Ce^{3t} + De^{-2t}$$

PARTICULAR INTEGRAL FOR THE "FIRST" EQUATION

- $x = Pt + Q$
- $\frac{dx}{dx} = P$
- $\frac{d^2x}{dt} = 0$

SUB INTO THE O.D.E.

$$\Rightarrow -P - 6(Pt + Q) \equiv 6t$$
$$\Rightarrow -6Pt - P - 6Q \equiv 6t$$

$$\therefore \quad P = -1 \qquad Q = \frac{1}{6}$$

IYGB - M456 PAPER A - QUESTION 1

PARTICULAR INTEGRAL FOR THE "SECOND" EQUATION

BY INSPECTION $y=2$

HENCE WE HAVE THE INDIVIDUAL GENERAL SOLUTIONS BELOW TO APPLY CONDITIONS

$x = Ae^{3t} + Be^{-2t} - t + \frac{1}{6}$

$\frac{dx}{dt} = 3Ae^{3t} - 2Be^{-2t} - 1$

- $t=0, x=1$

$\Rightarrow 1 = A + B + \frac{1}{6}$

$\Rightarrow A + B = \frac{5}{6}$

- $t=0, \frac{dx}{dt} = 3$

$\Rightarrow 3 = 3A - 2B - 1$

$\Rightarrow 3A - 2B = 4$

$\left[A = \frac{5}{6} - B\right]$

$\Rightarrow 3\left(\frac{5}{6} - B\right) - 2B = 4$

$\Rightarrow \frac{5}{2} - 3B - 2B = 4$

$\Rightarrow -5B = \frac{3}{2}$

$\Rightarrow B = -\frac{3}{10}, \quad A = \frac{17}{15}$

$y = Ce^{3t} + De^{-2t} + 2$

$\frac{dy}{dt} = 3Ce^{3t} - 2De^{-2t}$

- $t=0 \quad y=2$

$\Rightarrow 2 = C + D + 2$

$\Rightarrow C + D = 0$

- $t=0 \quad \frac{dy}{dt} = -1$

$\Rightarrow -1 = 3C - 2D$

$[D = -C]$

$-1 = 3C - 2(-C)$

$-1 = 5C$

$C = -\frac{1}{5}$

$D = \frac{1}{5}$

IYGB - MP456 PAPER A - QUESTION 1

HENCE WE FINALLY OBTAIN

$$x = \frac{17}{15}e^{3t} - \frac{3}{10}e^{-2t} - t + \frac{1}{6} =$$

$$y = \frac{1}{5}e^{-2t} - \frac{1}{5}e^{3t} + 2$$

$$\underline{r} = \left[\frac{17}{15}e^{3t} - \frac{3}{10}e^{-2t} - t + \frac{1}{6}\right]\underline{i} + \left[-\frac{1}{5}e^{3t} + \frac{1}{5}e^{-2t} + 2\right]\underline{j}$$

OR

$$\underline{r} = \frac{1}{15}e^{3t}\left[17\underline{i} - 3\underline{j}\right] - \frac{1}{10}e^{-2t}\left[3\underline{i} - 2\underline{j}\right] + \frac{1}{6}(\underline{i} + 12\underline{j}) - t\underline{i}$$

IYGB – M456 PAPER A – QUESTION 2

a)

FORCE	$4\underline{i} + b\underline{j}$	$3a\underline{i} + 2b\underline{j}$	$10b\underline{i} + 3\underline{j}$
POINT	$(1,2)$	$(4,-2)$	$(-3,-5)$

FIRSTLY TOTAL FORCE IS ZERO

$$(4\underline{i} + b\underline{j}) + (3a\underline{i} + 2b\underline{j}) + (10b\underline{i} + 3\underline{j}) = \underline{0}$$

$$(4 + 3a + 10b)\underline{i} + (3b + 3)\underline{j} = \underline{0}$$

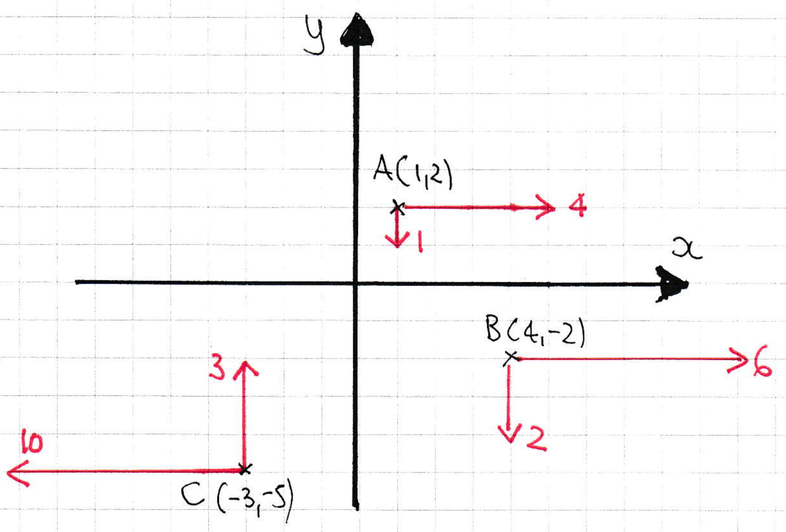
$$3b + 3 = 0 \qquad\qquad 4 + 3a + 10b = 0$$

$$3b = -3 \qquad\qquad 4 + 3a - 10 = 0$$

$$b = -1 \qquad\qquad 3a = 6$$

$$a = 2$$

NEXT DRAW A DIAGRAM – TAKE MOMENTS ABOUT O



$$\left.\begin{array}{l} -(4\times 2) \\ -(1\times 1) \end{array}\right\} \underline{F}_1 \text{ AT } A$$

$$\left.\begin{array}{l} +(6\times 2) \\ -(2\times 4) \end{array}\right\} \underline{F}_2 \text{ AT } B$$

$$\left.\begin{array}{l} -(3\times 3) \\ -(10\times 5) \end{array}\right\} \underline{F}_3 \text{ AT } C$$

$\therefore$ TOTAL MOMENT IS

$$= -8 - 1 + 12 - 8 - 9 - 50$$

$$= -64$$

$$= 64 \text{ Nm CLOCKWISE}$$

IYGB - M456 PAPER A - QUESTION 2

b) MOMENT ABOUT C NOW

$$\underbrace{-(1\times4)-(4\times7)}_{F_1}-\underbrace{(6\times3)-(2\times7)}_{F_2} = -4-28-18-14$$

$$= -64$$

$$= 64 \text{ Nm clockwise}$$

ALTERNATIVE BY CROSS PRODUCTS

a) $$\underline{G}_0 = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 4 & -1 & 0 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 4 & -2 & 0 \\ 6 & -2 & 0 \end{vmatrix} + \begin{vmatrix} i & j & k \\ -3 & -5 & 0 \\ -10 & 3 & 0 \end{vmatrix}$$

$$\underline{G}_0 = (0,0,-9)+(0,0,4)+(0,0,-59)$$

$$\underline{G}_0 = (0,0,-64)$$

I.E $|\underline{G}_0| = 64$ clockwise

b) $$\overrightarrow{CA} = \underline{a}-\underline{c} = (1,2)-(-3,-5) = (4,7)$$

$$\overrightarrow{CB} = \underline{b}-\underline{c} = (4,-2)-(-3,-5) = (7,3)$$

$$\underline{G}_c = \begin{vmatrix} i & j & k \\ 4 & 7 & 0 \\ 4 & -1 & 0 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 7 & 3 & 0 \\ 6 & -2 & 0 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & 0 \\ -10 & 3 & 0 \end{vmatrix}$$

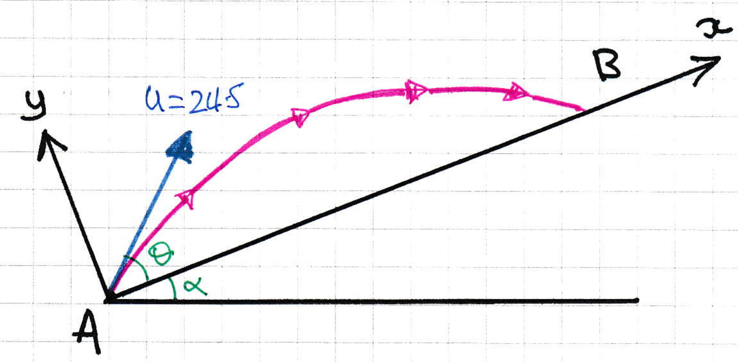
$$\underline{G}_c = (0,0,-32)+(0,0,-32)+(0,0,0)$$

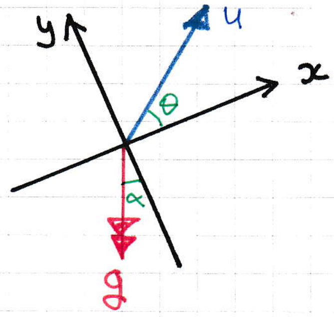
$$\underline{G}_c = (0,0,-64)$$

I.E $\underline{G}_c = 64$ clockwise

IYGB - M456 PAPER A - QUESTION 3

- START WITH A DIAGRAM





$\tan\alpha = \frac{5}{12}$
$\sin\alpha = \frac{5}{13}$
$\cos\alpha = \frac{12}{13}$
$\tan\theta = \frac{3}{4}$
$\sin\theta = \frac{3}{5}$
$\cos\theta = \frac{4}{5}$

- DERIVE THE EQUATIONS OF MOTION IN THE ROTATED SET OF AXES (LOOKING AT DIAGRAMS)

- $\ddot{x} = -g\sin\alpha$
- $\dot{x} = -gt\sin\alpha + u\cos\theta$
- $x = ut\cos\theta - \frac{1}{2}gt^2\sin\alpha$

- $\ddot{y} = -g\cos\alpha$
- $\dot{y} = -gt\cos\alpha + u\sin\theta$
- $y = ut\sin\theta - \frac{1}{2}gt^2\cos\alpha$

- FIND THE FLIGHT TIME FROM A TO B BY SOLVING $y=0$

$$\Rightarrow 0 = ut\sin\theta - \frac{1}{2}gt^2\cos\alpha$$

$$\Rightarrow \frac{1}{2}t\left[2u\sin\theta - gt\cos\alpha\right]$$

$$\Rightarrow t = \frac{2u\sin\theta}{g\cos\alpha} \quad (t \neq 0)$$

$$\Rightarrow t = \frac{2 \times 24.5 \times 3/5}{9.8 \times \frac{12}{13}}$$

$$\Rightarrow t = \frac{13}{4} = 3.25$$

IYGB - M456 PAPER A - QUESTION 3

- NEXT WE FIND THE COMPONENTS OF THE VELOCITY PARALLEL AND PERPENDICULAR TO THE PLANE AS THE PARTICLE HITS B

$$\Rightarrow \dot{x} = u\cos\theta - gt\sin\alpha$$

$$\Rightarrow \dot{x} = 24.5 \times \frac{4}{5} - 9.8 \times \frac{13}{4} \times \frac{5}{13}$$

$$\Rightarrow \dot{x} = 7.35$$

AND

$$\Rightarrow \dot{y} = u\sin\theta - gt\cos\alpha$$

$$\Rightarrow \dot{y} = 24.5 \times \frac{3}{5} - 9.8 \times \frac{13}{4} \times \frac{12}{13}$$

$$\Rightarrow \dot{y} = -14.7$$

$\therefore$ THE SPEEDS AFTER THE IMPACT WILL BE $\dot{x} = 7.35$ (UNCHANGED)

$$e|\dot{y}| = \frac{\sqrt{3}}{2} \times 14.7 = 7.35\sqrt{3}$$

- FINALLY THE REBOUND SPEED WILL BE

$$\Rightarrow \text{REBOUND SPEED} = \sqrt{(7.35)^2 + (7.35\sqrt{3})^2}$$

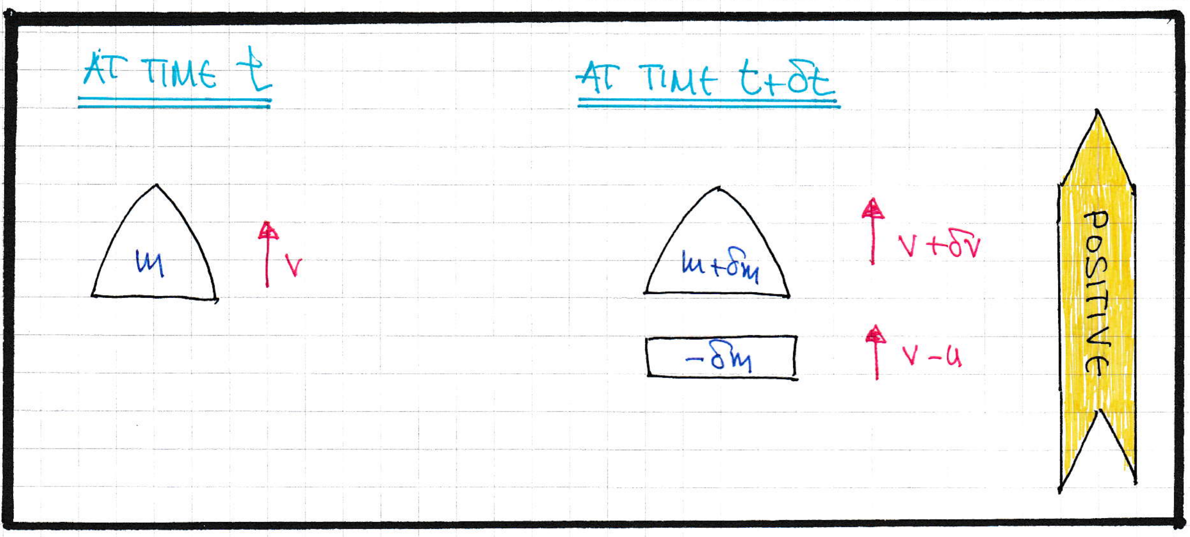
$$= 7.35\sqrt{1^2 + \sqrt{3}^2}$$

$$= 7.35 \times 2$$

$$= 14.7 \text{ ms}^{-1}$$

AS REQUIRED

IYGB - M456 - PAPER A - QUESTION 4



BY THE IMPULSE-MOMENTUM PRINCIPLE, NOTING FURTHER THAT THERE ARE NO EXTERNAL FORCES

$$\Longrightarrow \quad 0 = \left[(m+\delta m)(v+\delta v) - \delta m(v-u)\right] - mv$$

$$\Longrightarrow \quad 0 = \cancel{mv} + m\delta v + \cancel{v\delta m} + \delta m\delta v - \cancel{v\delta m} + u\delta m - \cancel{mv}$$

$$\Longrightarrow \quad 0 = m\frac{\delta v}{\delta m} + \frac{\delta m\delta v}{\delta m} + u\frac{\delta m}{\delta m}$$

TAKING LIMITS, WE OBTAIN

$$\Longrightarrow \quad m\frac{dv}{dm} + u = 0$$

SOLVE THE O.D.E, SUBJECT TO THE INITIAL CONDITIONS

$$\Longrightarrow \quad m\frac{dv}{dm} = -u$$

$$\Longrightarrow \quad 1\,dv = -\frac{u}{m}\,dm$$

IYGB - M456 - PAPER A - QUESTION 4

$$\Rightarrow \int_{v=U}^{v} 1 \, dv = \int_{m=M}^{\frac{1}{4}M} -\frac{u}{m} \, dm$$

$$\Rightarrow \Big[v \Big]_{U}^{v} = \Big[-u\ln m \Big]_{M}^{\frac{1}{4}M}$$

$$\Rightarrow v - U = \Big[u\ln m \Big]_{\frac{1}{4}M}^{M}$$

$$\Rightarrow v - U = u\left[\ln M - \ln \tfrac{1}{4}M\right]$$

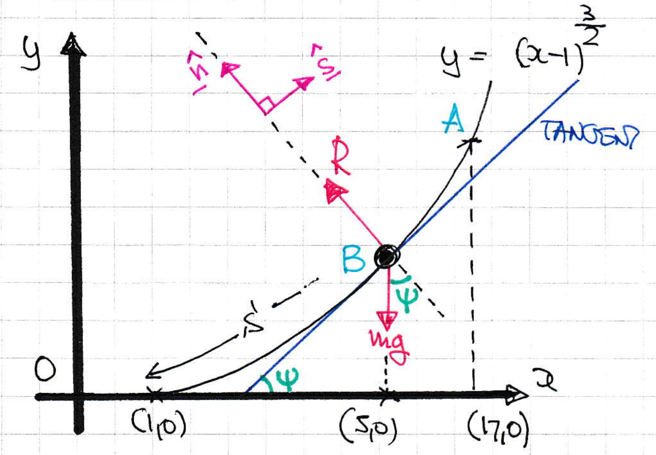
$$\Rightarrow v - U = u\ln\left[\frac{\cancel{M}}{\frac{1}{4}\cancel{M}}\right]$$

$$\Rightarrow v - U = u\ln 4$$

$$\Rightarrow v = U + u\ln 4$$

IYGB - M456 - PAPER A - QUESTION 5

a) STARTING WITH A DIAGRAM AND PREPARING SOME AUXILIARY RESULTS

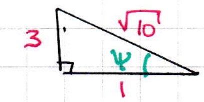


$$\frac{dy}{dx} = \frac{3}{2}(x-1)^{\frac{1}{2}}$$

$$\tan\psi = \frac{3}{2}(x-1)^{\frac{1}{2}}$$

AT POINT B, $x = 5$

$$\tan\psi = 3$$



BY ENERGIES, TAKING THE LEVEL OF THE x AXIS AS THE ZERO POTENTIAL LEVEL

$$y_A = (17-1)^{\frac{3}{2}} = 64$$

$$y_B = (5-1)^{\frac{3}{2}} = 8$$

$$\cancel{KE_A} + P.E_A = KE_B + P.E_B$$

$$\cancel{m}g\,y_A = \frac{1}{2}\cancel{m}v^2 + \cancel{m}g\,y_B$$

$$2g \times 64 = v^2 + 2g \times 8$$

$$v^2 = 128g - 16g$$

$$v^2 = 112g$$

IYGB - M456 - PAPER A - QUESTION 5

NEXT THE RADIUS OF CURVATURE ρ, IN CARTESIAN

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left[\frac{3}{2}(x-1)^{\frac{1}{2}}\right]^2\right]^{\frac{3}{2}}}{\frac{3}{4}(x-1)^{-\frac{1}{2}}}$$

$$\rho_B = \frac{\left[1+\left(\frac{3}{2}\times 4^{\frac{1}{2}}\right)^2\right]^{\frac{3}{2}}}{\frac{3}{4}\times 4^{-\frac{1}{2}}} = \frac{10\sqrt{10}}{\frac{3}{8}} = \frac{80}{3}\sqrt{10}$$

ACCELERATION IN INTRINSICS $\underline{a} = \ddot{s}\,\hat{\underline{s}} + \frac{\dot{s}^2}{\rho}\hat{\underline{n}}$

- TANGENTIALLY

$$m\ddot{s} = -mg\sin\psi$$

$$\ddot{s} = -g\sin\psi$$

$$\ddot{s}_B = -g\left(\frac{3}{\sqrt{10}}\right)$$

$$\left|\ddot{s}_B\right| = \frac{3g}{\sqrt{10}}$$

- NORMALLY

$$\frac{\dot{s}^2}{\rho} = \frac{v^2}{\rho}$$

$$\left[\frac{\dot{s}^2}{\rho}\right]_B = \frac{112g}{\frac{80}{3}\sqrt{10}}$$

$$\left[\frac{\dot{s}^2}{\rho}\right]_B = \frac{21g}{5\sqrt{10}}$$

MAGNITUDE OF ACCELERATION AT B

$$\sqrt{\left|\ddot{s}\right|_B^2 + \left[\frac{\dot{s}^2}{\rho}\right]_B^2} = \frac{3g}{\sqrt{10}}\sqrt{1^2+\left(\frac{7}{5}\right)^2}$$

$$= \frac{3g}{5}\sqrt{\frac{74}{10}}$$

$$\approx 16.0 \text{ ms}^{-2}$$

IYGB - M456 - PAPER A - QUESTION 5

b) LOOKING AT THE MOTION, NORMALLY

$$\Longrightarrow m\left(\frac{v^2}{\rho}\right) = R - mg\cos\psi$$

$$\Longrightarrow R = \frac{mv^2}{\rho} + mg\cos\psi$$

AT B WE HAVE $\rho = \frac{80}{3}\sqrt{10}$, $v^2 = 112g$, $\cos\psi = \frac{1}{\sqrt{10}}$

$$\Longrightarrow R = \frac{200 \times 112g}{\frac{80}{3}\sqrt{10}} + 200g \times \frac{1}{\sqrt{10}}$$

$$\Longrightarrow R = \frac{840g}{\sqrt{10}} + \frac{200g}{\sqrt{10}}$$

$$\Longrightarrow R = \frac{1040g}{\sqrt{10}}$$

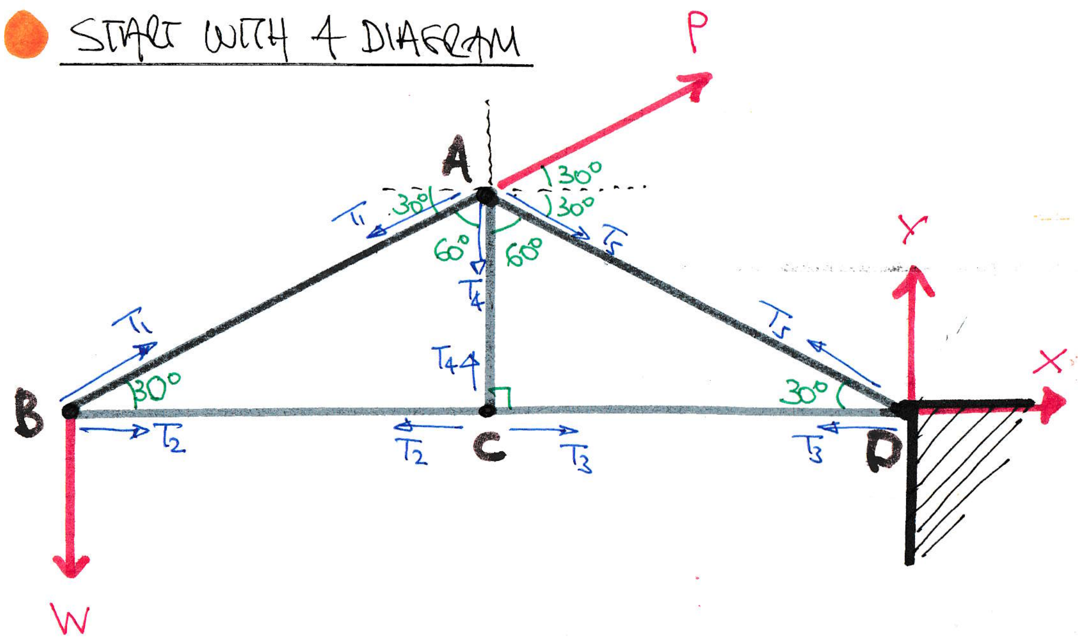
$$\Longrightarrow R = 104\sqrt{10}\,g$$

$$\Longrightarrow R = 3222.993391\ldots$$

$$\Longrightarrow R \simeq 3223 \text{ N}$$

IYGB - MMS6 PAPER A - QUESTION 6

a) START WITH A DIAGRAM



NEXT SOME LENGTHS

LET $|AB| = |AD| = 2a$

$|AC| = |AB|\sin 30° = a$

$|BC| = |CD| = |AB|\cos 30° = \sqrt{3}a$

MOMENTS ABOUT D

$W|BD| = P\cos 30|AC| + P\sin 30|CD|$

$2\sqrt{3}aW = \frac{\sqrt{3}}{2}Pa + \frac{1}{2}P\sqrt{3}a$

$2W = \frac{P}{2} + \frac{P}{2}$

$P = W$

RESOLVING EXTERNAL FORCES

- $X + P\cos 30° = 0$

  $X = -P\cos 30$

  $X = -W\frac{\sqrt{3}}{2}$

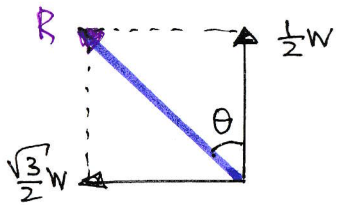
  $X = \frac{\sqrt{3}}{2}W$ (TO THE "LEFT")

- $Y + P\sin 30 = W$

  $Y + \frac{1}{2}W = W$

  $Y = \frac{1}{2}W$

IYGB – M456 PAPER A – QUESTION 6



$$R = \sqrt{\left(\tfrac{1}{2}W\right)^2 + \left(\tfrac{\sqrt{3}}{2}W\right)^2} = \sqrt{\tfrac{1}{4}W^2 + \tfrac{3}{4}W^2} = W$$

$$\tan\theta = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}\ , \quad \theta = 60°$$

---

b) • LOOKING AT B VERTICALLY

$T_1 \sin 30 = W$

$\frac{1}{2}T_1 = W$

$T_1 = 2W$ (TENSION)

• LOOKING AT B HORIZONTALLY

$T_2 + T_1 \cos 30 = 0$

$T_2 = -T_1 \cos 30$

$T_2 = -2W\left(\frac{\sqrt{3}}{2}\right)$

$T_2 = \sqrt{3}W$ (THRUST)

• LOOKING AT C, HORIZONTALLY

$T_2 = T_3$

$T_3 = \sqrt{3}W$ (THRUST)

• LOOKING AT C, VERTICALLY

$T_4 = 0$

• LOOKING AT A, HORIZONTALLY

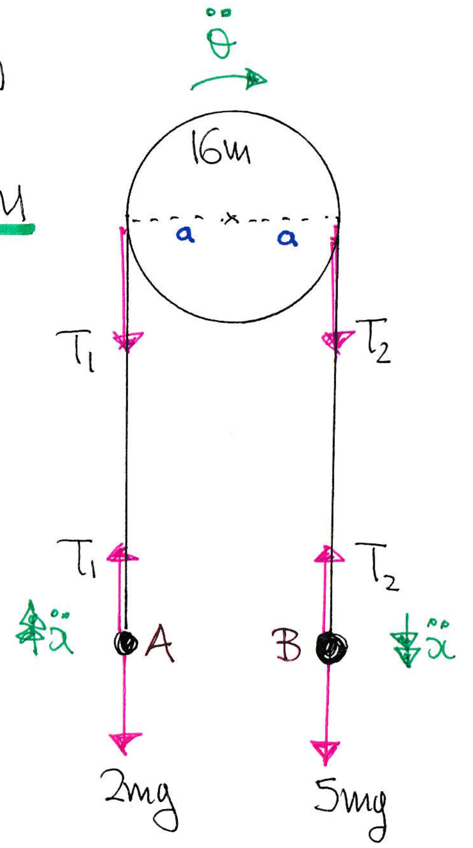
$T_1 \cancel{\cos 30°} = P\cancel{\cos 30} + T_5 \cancel{\cos 30°}$

$2W = W + T_5$

$T_5 = W$ (TENSION)

IYGB - M456 PAPER A - QUESTION 7

- **FORMING THE EQUATION OF MOTION FOR EACH COMPONENT OF THE SYSTEM**



LOOKING AT A

$T_1 - 2mg = 2m\ddot{x}$

LOOKING AT B

$5mg - T_2 = 5m\ddot{x}$

LOOKING AT PULLEY

$T_2 a - T_1 a - mga = I\ddot{\theta}$

$(T_2 - T_1)a - mga = 8ma^2\ddot{\theta}$

$T_2 - T_1 - mg = 8ma\ddot{\theta}$

- MOMENT OF INERTIA OF PULLEY IS $\frac{1}{2}(16m)a^2$ $8ma^2$
- FRICTIONAL COUPLE ON PULLEY mga
- NO SLIP $\Rightarrow$ $x = a\theta$
  $\dot{x} = a\dot{\theta}$
  $\ddot{x} = \ddot{\theta}$

- **"SUBTRACTING" THE FIRST TWO EQUATIONS GIVES**

$\Rightarrow T_1 - T_2 + 3mg = 7m\ddot{x}$

$\Rightarrow 3mg - 7m\ddot{x} = T_2 - T_1$

- **SUBSTITUTING INTO THE THIRD EQUATION**

$\Rightarrow (T_2 - T_1) - mg = 8ma\ddot{\theta}$

$\Rightarrow (3mg - 7m\ddot{x}) - mg = 8ma\ddot{\theta}$

$\Rightarrow 3g - 7\ddot{x} - g = 8a\ddot{\theta}$

$\Rightarrow 2g - 7\ddot{x} = 8\ddot{x}$

$\Rightarrow 15\ddot{x} = 2g$

$\Rightarrow \ddot{x} = \frac{2g}{15}$

- **HENCE WE OBTAIN**

$T_1 - 2mg = 2m\ddot{x}$ & $5mg - T_2 = 5m\ddot{x}$

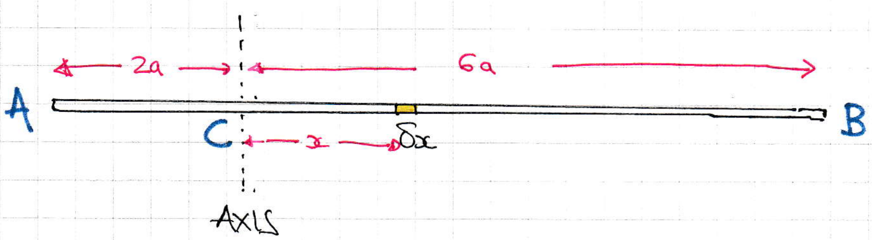
$T_1 - 2mg = 2m\left(\frac{2g}{15}\right)$ $5mg - T_2 = 5m\left(\frac{2}{15}g\right)$

$T_1 - 2mg = \frac{4}{15}mg$ $5mg - T_2 = \frac{2}{3}mg$

$T_1 = \frac{34}{15}mg$ $T_2 = \frac{13}{3}mg$

IYGB - M456 PAPER A - QUESTION 8

a)

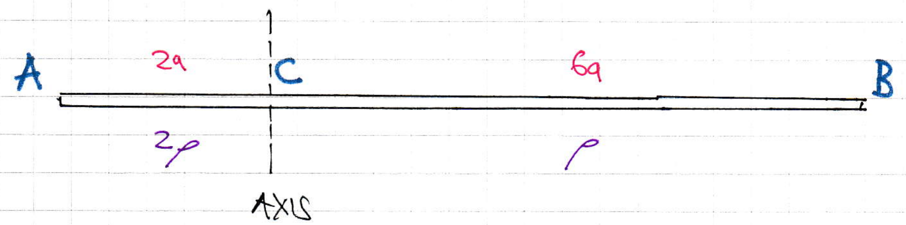


- $\rho = \frac{m}{8a}$ = MASS PER UNIT LENGTH (DENSITY)
- $\delta I = (\rho \delta x)x^2 = \rho x^2 \delta x$
- $I = \int_{x=-2a}^{x=6a} \rho x^2 \, dx = \rho \int_{-2a}^{6a} x^2 \, dx = \frac{1}{3}\rho \left[x^3 \right]_{-2a}^{6a}$

$$= \frac{1}{3}\left(\frac{m}{8a}\right)\left[(6a)^3 - (-2a)^3\right] = \frac{m}{24a} \times 224a^3 = \frac{28}{3}ma^2$$

$\uparrow \lambda$

b)



"4aρ PARTS" "6aρ PARTS" IE RATIO 2:3

USING THE STANDARD RESULT FOR THE MOMENT OF INERTIA OF A ROD ABOUT ITS ENDPOINT ("$\frac{4}{3}M\ell^2$") & THE ADDITION RULE

$$\Longrightarrow I = \underbrace{\frac{4}{3}\left(\frac{2}{5}m\right)a^2}_{AC} + \underbrace{\frac{4}{3}\left(\frac{3}{5}m\right)(3a)^2}_{CB}$$

$$\Longrightarrow I = \frac{8}{15}ma^2 + \frac{36}{5}ma^2$$

$$\Longrightarrow I = \frac{116}{15}ma^2$$

$\uparrow \mu$

IYGB - M456 PAPER A - QUESTION 9

FORMING A DIFFERENTIAL EQUATION

$$\frac{dv}{dt} = 10 - kv$$

DECREASE AT A RATE PROPORTIONAL TO ITS VELOCITY

CONSTANT RATE OF VELOCITY INCREASE ($g = 10$ HERE)

RATE OF CHANGE OF VELOCITY

NEXT WE ARE GIVEN THE TERMINAL VELOCITY AS 100

$\Rightarrow$ WHEN $V = 100$, $\frac{dv}{dt} = 0$

$\Rightarrow 0 = 10 - k \times 100$

$\Rightarrow 100k = 10$

$\Rightarrow k = \frac{1}{10}$

$$\therefore \frac{dv}{dt} = 10 - \frac{1}{10}V$$

SOLVING THE O.D.E BY SEPARATION OF VARIABLES

$\Rightarrow 10\frac{dv}{dt} = 100 - V$

$\Rightarrow \frac{10}{100-V}\,dv = 1\,dt$

INTEGRATE SUBJECT TO THE CONDITION, $t=0, V=0$

$$\Rightarrow \int_0^V \frac{10}{100-V}\,dV = \int_0^t 1\,dt$$

IYGB – M456 PAPER A – QUESTION 9

$$\Rightarrow \left[-10\ln(100-v)\right]_0^v = \left[t\right]_0^t$$

$$\Rightarrow \left[\ln(100-v)\right]_0^v = \left[-\tfrac{1}{10}t\right]_0^t$$

$$\Rightarrow \ln(100-v) - \ln 100 = -\tfrac{1}{10}t$$

$$\Rightarrow \ln\left(\frac{100-v}{100}\right) = -\tfrac{1}{10}t$$

$$\Rightarrow \frac{100-v}{100} = e^{-\frac{1}{10}t}$$

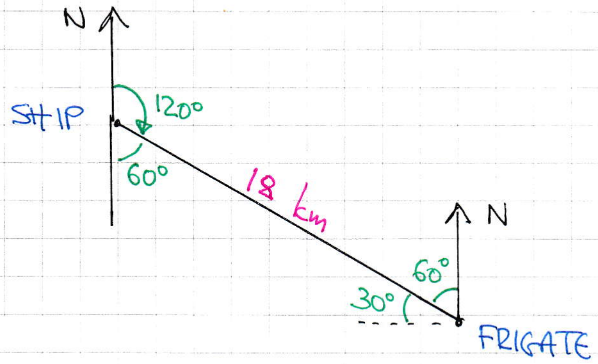
$$\Rightarrow 100 - v = 100e^{-\frac{1}{10}t}$$

$$\Rightarrow 100 - 100e^{-\frac{1}{10}t} = v$$

$$\Rightarrow v = 100\left(1 - e^{-\frac{1}{10}t}\right)$$

IYGB - M456 PAPER A - QUESTION 10

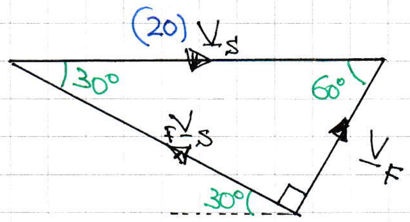
a) START WITH THE INITIAL CONFIGURATION



$$_F\underline{V}_S = \underline{V}_F - \underline{V}_S$$
$$\underline{V}_F = {}_F\underline{V}_S + \underline{V}_S$$

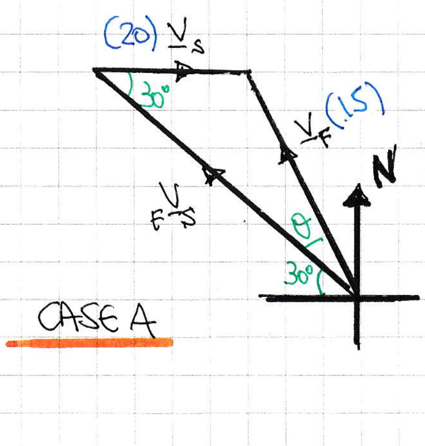
FOR INTERCEPTION THE VELOCITY OF THE FRIGATE MUST BE PERPENDICULAR TO THE RELATIVE VELOCITY BETWEEN THE TWO VESSELS

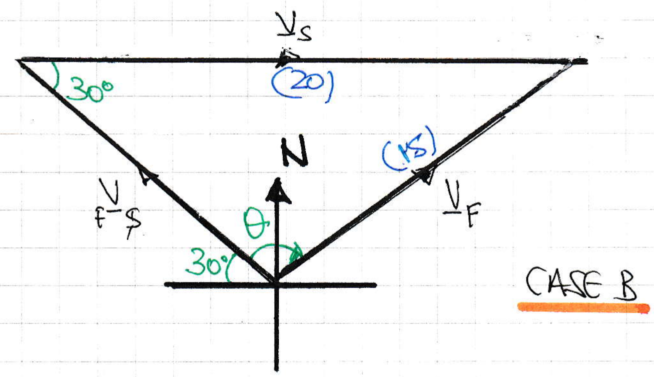
LET THE SHIP BE "FIXED" – THEN AN OBSERVER ON THE SHIP WILL BE SEEING THE FRIGATE HEADING DIRECTLY TOWARDS THEM, IF ALONG THE LINE JOINING THE TWO VESSELS – HENCE WE HAVE



$$|\underline{V}_F| = 20\sin 30° = 10 \text{ km h}^{-1}$$

b) FOR INTERCEPTION AGAIN AN OBSERVER ON THE SHIP WILL BE SEEING THE FRIGATE HEADING DIRECTLY TOWARDS THE SHIP ALONG THE LINE JOINING THEM BUT $\underline{V}_F$ IS NOT PERPENDICULAR TO $_F\underline{V}_S$





IYGB – M456 PAPER A – QUESTION 10

BY THE SINE RULE IN EACH OF THE TWO CASES

$$\Longrightarrow \frac{\sin 30^\circ}{15} = \frac{\sin\theta}{20}$$

$$\Longrightarrow \sin\theta = \frac{2}{3}$$

$$\Longrightarrow \theta = \begin{cases} 41.81^\circ & \leftarrow \text{CASE A ("SHORTEST" INTERCEPTION)} \\ 138.19^\circ & \leftarrow \text{CASE B ("LONGEST INTERCEPTION)} \end{cases}$$

$$\Longrightarrow \text{BEARING} \begin{cases} 270^\circ + 30^\circ + 41.81^\circ \simeq 342^\circ \\ 30^\circ + \theta - 90^\circ \simeq 78^\circ \end{cases}$$

NOW USING THE SINE RULE IN "CASE A" TO FIND $|{}_F\underline{V}_S|$

$$\frac{|{}_F\underline{V}_S|}{\sin(180-30-\theta)} = \frac{15}{\sin 30^\circ} \qquad \text{CASE A, } \theta = 41.81^\circ$$

$$|{}_F\underline{V}_S| = \frac{15\sin(108.19)}{\sin 30}$$

$$|{}_F\underline{V}_S| = 28.50084\ldots \text{ km h}^{-1}$$

NOW USING THIS SPEED, THE FRIGATE WILL HAVE TO COVER THE INITIAL DISTANCE OF 18 km

$$T = \frac{18}{28.50084\ldots} = 0.63156\ldots \text{ hours}$$

$$\xrightarrow{\times 60} \simeq 38 \text{ minutes}$$

IYGB – M456 PAPER A – QUESTION 11

GIVEN IN THE PROBLEM

$\dot{\theta} = \omega = \text{CONSTANT}$

$\hat{\underline{r}}: \quad (\ddot{r} - r\dot{\theta}^2) = -2\omega^2 r$

$t = 0$
$\theta = 0$
$r = a$
$\dot{r} = \sqrt{3}\,a\omega$

LOOKING AT THE ACCELERATION RADIALLY $(\hat{\underline{r}})$

$$\Rightarrow \ddot{r} - r\dot{\theta}^2 = -2\omega^2 r$$

$$\Rightarrow \ddot{r} - r\omega^2 = -2\omega^2 r$$

$$\Rightarrow \ddot{r} = -\omega^2 r$$

SOLVING THE O.D.E, WHICH IS A STANDARD S.H.M SOLUTION

$$\Rightarrow \frac{d^2 r}{dt^2} = -\omega^2 r$$

$$\Rightarrow r(t) = A\cos\omega t + B\sin\omega t$$

APPLY THE CONDITION $t=0$, $r=a$ YIELDS $\quad a = A$

$$\Rightarrow r(t) = a\cos\omega t + B\sin\omega t$$

DIFFERENTIATE TO APPLY THE OTHER CONDITION

$$\Rightarrow \dot{r}(t) = -a\omega\sin\omega t + B\omega\cos\omega t$$

$t=0$
$\dot{r} = \sqrt{3}\,a\omega$

$$\sqrt{3}\,a\omega = B\omega$$

$$B = \sqrt{3}\,a$$

IYGB - M456 PAPER A - QUESTION 11

MANIPULATE AS FOLLOWS

$\Rightarrow r = a\cos\omega t + \sqrt{3}a\sin\omega t$

$\Rightarrow r = 2a\left[\frac{1}{2}\cos\omega t + \frac{\sqrt{3}}{2}\sin\omega t\right]$

$\Rightarrow r = 2a\left[\sin\frac{\pi}{6}\cos\omega t + \cos\frac{\pi}{6}\sin\omega t\right]$

$\Rightarrow r = 2a\sin\left(\omega t + \frac{\pi}{6}\right)$

FINALLY TO "LOSE" t

$$\dot{\theta} = \omega$$

$$\frac{d\theta}{dt} = \omega$$

$$1\,d\theta = \omega\,dt$$

$$\int_{\theta=0}^{\theta} 1\,d\theta = \int_{t=0}^{t} \omega\,dt$$

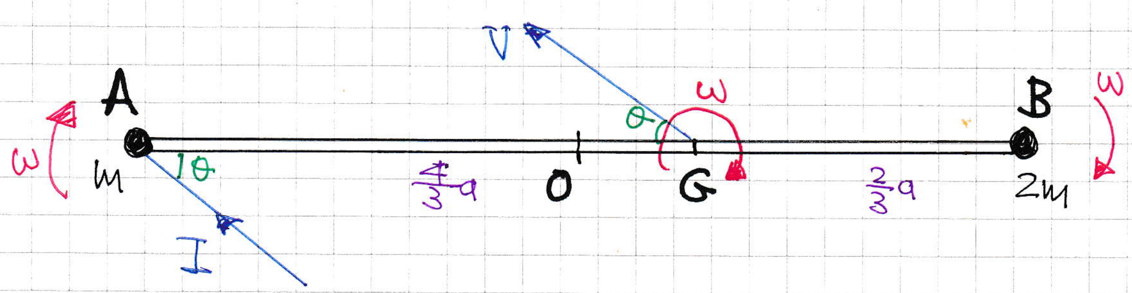
$$\left[\theta\right]_0^{\theta} = \left[\omega t\right]_0^{t}$$

$$\theta = \omega t$$

$$\therefore\ r(\theta) = 2a\sin\left(\theta + \frac{\pi}{6}\right)$$

IYGB – M456 PAPER A – QUESTION 12

a) STARTING WITH A DIAGRAM



AS THE ROD IS LIGHT, BY INSPECTION, THE CENTRE OF MASS OF THE SYSTEM G, IS SUCH SO THAT $|AG| = \frac{4}{3}a$, $|BG| = \frac{2}{3}a$

SINCE THE SYSTEM IS NOT CONSTRAINED, ITS CENTRE OF MASS G, WILL MOVE WITH SPEED V IN THE SAME DIRECTION AS I

$\Rightarrow I = 3m(v-u)$

$\Rightarrow I = 3m(V-0)$

$\Rightarrow I = 3mV$

$$\boxed{V = \frac{I}{3m}}$$

THE SYSTEM WILL ALSO ACQUIRE ANGULAR VELOCITY ω, ABOUT ITS CENTRE OF MASS G

$\Rightarrow$ MOMENT OF IMPULSE ABOUT G = CHANGE IN ANG MOMENTUM ABOUT G

$\Rightarrow I\sin\theta \times |AG| = \underbrace{\left[m \times |AG|^2\right]}_{\text{MOMENT OF INERTIA OF A}} \times \omega + \underbrace{\left[2m \times |BG|^2\right]}_{\text{MOMENT OF INERTIA OF B}} \times \omega$

$\Rightarrow 3\cancel{m}V\sin\theta \times \frac{4}{3}\cancel{a} = \cancel{m}\left(\frac{16}{9}a^{\cancel{2}}\right)\omega + 2\cancel{m}\left(\frac{4}{9}a^{\cancel{2}}\right)\omega$

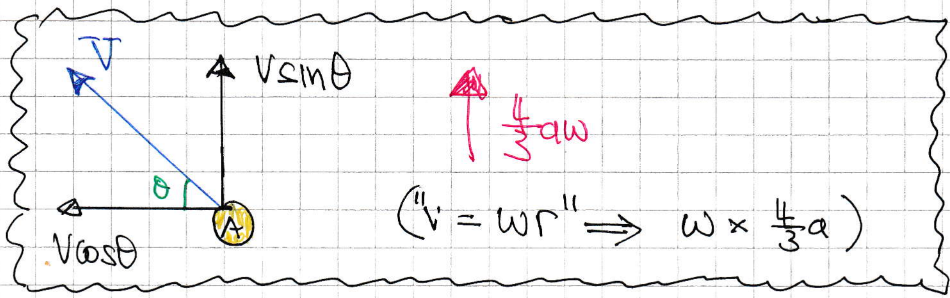
IYGB - M456 PAPER A - QUESTION 12

$$\Rightarrow 4V\sin\theta = \frac{8}{3}a\omega$$

$$\Rightarrow V\sin\theta = \frac{2}{3}a\omega$$

$$\Rightarrow \boxed{\omega = \frac{3V\sin\theta}{2a}}$$

NOW WE CAN OBTAIN THE SPEED OF EACH PARTICLE BY REFERRING TO THE DIAGRAM BELOW



$$\Rightarrow (\text{SPEED})^2 = \left(V\sin\theta + \tfrac{4}{3}a\omega\right)^2 + (V\cos\theta)^2$$

$$\Rightarrow (\text{SPEED})^2 = \left(V\sin\theta + \tfrac{4}{3}a\,\frac{3V\sin\theta}{2a}\right)^2 + V^2\cos^2\theta$$

$$\Rightarrow (\text{SPEED})^2 = (V\sin\theta + 2V\sin\theta)^2 + V^2\cos^2\theta$$

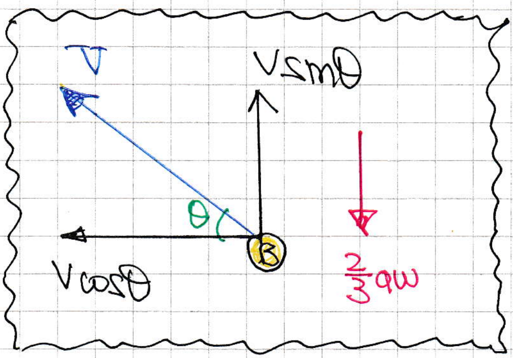
$$\Rightarrow (\text{SPEED})^2 = 9V^2\sin^2\theta + V^2\cos^2\theta$$

$$\Rightarrow (\text{SPEED})^2 = V^2(9\sin^2\theta + \cos^2\theta)$$

$$\Rightarrow (\text{SPEED})^2 = \left(\frac{I}{3m}\right)^2(8\sin^2\theta + 1)$$

$$\Rightarrow \text{SPEED OF } A = \frac{I}{3m}\sqrt{1+8\sin^2\theta}$$

IYGB - M456 PAPER A - QUESTION 12



$$\Rightarrow (\text{SPEED})^2 = \left(V\sin\theta - \tfrac{2}{3}a\omega\right)^2 + (V\cos\theta)^2$$

$$\Rightarrow (\text{SPEED})^2 = \left(V\sin\theta - \tfrac{2}{3}a\,\frac{3V\sin\theta}{2a}\right)^2 + V^2\cos^2\theta$$

$$\Rightarrow (\text{SPEED})^2 = \cancel{\left(V\sin\theta - V\sin\theta\right)^2} + V^2\cos^2\theta$$

$$\Rightarrow \text{SPEED OF } B = \frac{I}{3m}\cos\theta$$

b) THE GAIN IN KINETIC ENERGY IS GIVEN BY

$$\frac{1}{2}m\frac{I^2}{9m^2}\left(1+8\sin^2\theta\right) + \frac{1}{2}(2m)\frac{I^2}{9m^2}\cos^2\theta$$

$$= \frac{I^2}{18m}\left(1+8\sin^2\theta\right) + \frac{I^2}{9m}\cos^2\theta$$

$$= \frac{I^2}{18m}\left[1+8\sin^2\theta + 2\cos^2\theta\right]$$

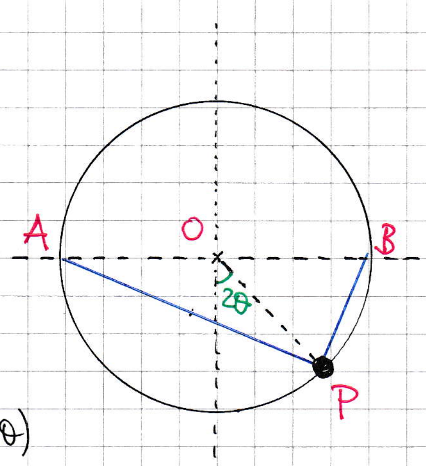
$$= \frac{I^2}{18m}\left[3+6\sin^2\theta\right]$$

$$= \frac{I^2}{6m}\left(1+2\sin^2\theta\right)$$

NOTE THAT $\frac{1}{2}(3m)v^2 + \frac{1}{2}I_A\omega^2 + \frac{1}{2}I_B\omega^2$ YIELDS THE SAME ANSWER WHERE I_A & I_B ARE THE RESPECTIVE MOMENTS OF INERTIA OF A & B ABOUT G

IYGB - M456 PAPER A - QUESTION 13

LOOKING AT THE DIAGRAM



- $|AP| = 2|AO|\sin\left(\frac{90+2\theta}{2}\right) = 2a\sin(45+\theta)$
- $|PB| = 2|OB|\sin\left(\frac{90-2\theta}{2}\right) = 2a\sin(45-\theta)$

LENGTH OF THE STRING

$$|AP| + |PB| = 2a\sin(45+\theta) + 2a\sin(45-\theta)$$
$$= 2a\left[\sin45\cos\theta + \cancel{\cos45\sin\theta} + \sin45\cos\theta - \cancel{\cos45\sin\theta}\right]$$
$$= 4a\sin45\cos\theta$$
$$= 2\sqrt{2}\,a\cos\theta$$

THE EXTENSION OF THE STRING

$$2\sqrt{2}a\cos\theta - 2a = 2a\left[\sqrt{2}\cos\theta - 1\right)$$

ELASTIC ENERGY

$$\text{“}\frac{\lambda}{2l}x^2\text{”} = \frac{kmg}{2(2a)}\left[2a\left(\sqrt{2}\cos\theta - 1\right)\right]^2$$
$$= \frac{kmg}{4a} \times 4a^2\left(\sqrt{2}\cos\theta - 1\right)^2$$
$$= kmga\left(\sqrt{2}\cos\theta - 1\right)^2$$

POTENTIAL ENERGY TAKING THE LEVEL OF AB AS THE ZERO POTENTIAL LEVEL

$$= -mga\cos2\theta$$

IYGB - M456 PAPER A - QUESTION 13

TOTAL ENERGY FOR THE SYSTEM

$$\Rightarrow V(\theta) = kmga\left(\sqrt{2}\cos\theta - 1\right)^2 - mga\cos 2\theta + C$$

$$\Rightarrow V(\theta) = mga\left[k\left(\sqrt{2}\cos\theta - 1\right)^2 - \cos 2\theta\right] + C$$

$$\Rightarrow V'(\theta) = mga\left[2k\left(\sqrt{2}\cos\theta - 1\right)\left(-\sqrt{2}\sin\theta\right) + 2\sin 2\theta\right]$$

$$\Rightarrow V'(\theta) = 2mga\left[\sin 2\theta - k\sqrt{2}\sin\theta\left(\sqrt{2}\cos\theta - 1\right)\right]$$

$$\Rightarrow V'(\theta) = 2mga\left[\sin 2\theta - 2k\sin\theta\cos\theta + k\sqrt{2}\sin\theta\right]$$

$$\Rightarrow V'(\theta) = 2mga\left[k\sqrt{2}\sin\theta + \sin 2\theta - k\sin 2\theta\right]$$

$$\Rightarrow V'(\theta) = 2mga\left[k\sqrt{2}\sin\theta + (1-k)\sin 2\theta\right]$$

SOLVING FOR ZERO WE OBTAIN

$$\Rightarrow k\sqrt{2}\sin\theta + 2(1-k)\sin\theta\cos\theta = 0$$

$$\Rightarrow \sin\theta\left[k\sqrt{2} + 2(1-k)\cos\theta\right] = 0$$

EITHER $\sin\theta = 0$ OR $\cos\theta = \dfrac{k\sqrt{2}}{2(k-1)}$

$$\Rightarrow \cos\frac{\pi}{6} = \frac{k\sqrt{2}}{2(k-1)}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{k\sqrt{2}}{2(k-1)}$$

IYGB - M456 PAPER A - QUESTION 13

$$\Longrightarrow \sqrt{3} = \frac{\sqrt{2}\,k}{k-1}$$

$$\Longrightarrow \sqrt{3}k - \sqrt{3} = \sqrt{2}k$$

$$\Longrightarrow (\sqrt{3} - \sqrt{2})k = \sqrt{3}$$

$$\Longrightarrow k = \frac{\sqrt{3}}{\sqrt{3} - \sqrt{2}}$$

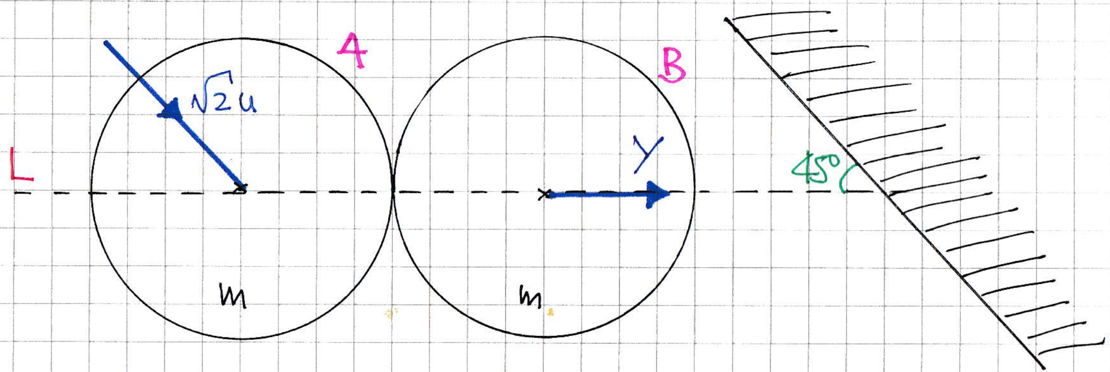
$$\Longrightarrow k = \frac{\sqrt{3}(\sqrt{3} + \sqrt{2})}{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})}$$

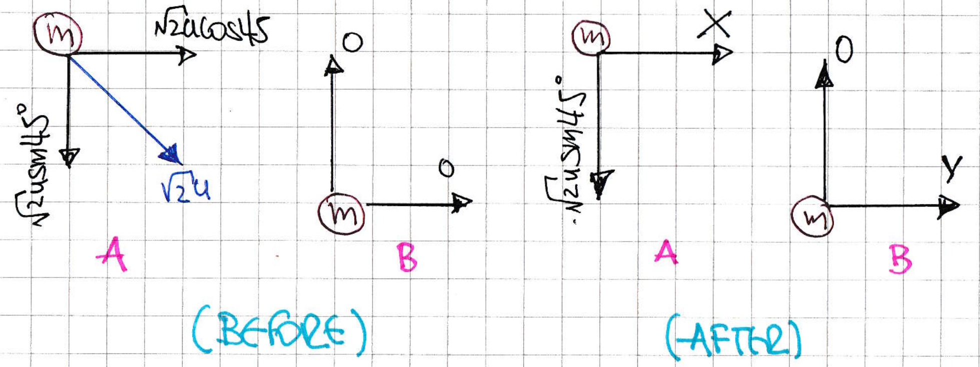
$$\Longrightarrow k = \frac{3 + \sqrt{6}}{3 - 2}$$

$$\Longrightarrow k = 3 + \sqrt{6}$$

IYGB – M456 PAPER A – QUESTION 14

START BY A DIAGRAM





BY CONSERVATION OF MOMENTUM ALONG L

$$m\sqrt{2}u\cos45 + 0 = mX + mY$$

$$X + Y = u$$

BY RESTITUTION ALONG L

$$\frac{Y - X}{\sqrt{2}u\cos45} = e$$

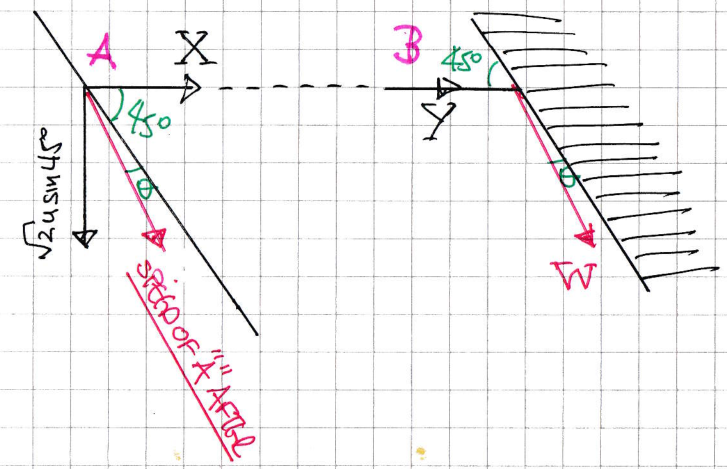
$$-X + Y = eu$$

SOLVING THE EQUATIONS

$2Y = u + eu \qquad 2X = u - eu$

$Y = \frac{1}{2}u(1+e) \qquad X = \frac{1}{2}u(1-e)$

LOOKING AT ANOTHER DIAGRAM



IYGB - M456 PAPER A - QUESTION 14

NEXT WE LOOK AT THE COLLISION WITH THE WALL

- $Y\cos 45 = W\cos\theta$ (NO MOMENTUM EXCHANGE)
- $\dfrac{W\sin\theta}{Y\sin 45} = E$ ← RESTITUTION WITH THE WALL

$$\left.\begin{aligned} W\sin\theta &= EY\sin 45 \\ W\cos\theta &= Y\cos 45 \end{aligned}\right)$$ DIVIDE, NOTING THAT $\sin 45 = \cos 45$, GIVES $\tan\theta = E$

FINALLY LOOKING AT THE SPEED OF OF A, AFTER THE COLLISION

$$\Rightarrow \tan(45+\theta) = \frac{\sqrt{2}u\sin 45}{x} = \frac{\sqrt{2}u \times \frac{1}{\sqrt{2}}}{\frac{1}{2}u(1-e)} = \frac{2}{1-e}$$

$$\Rightarrow \frac{\tan 45 + \tan\theta}{1-\tan 45\tan\theta} = \frac{2}{1-e}$$

$$\Rightarrow \frac{1+E}{1-E} = \frac{2}{1-e}$$

$$\Rightarrow (1-e) + E(1-e) = 2 - 2E$$

$$\Rightarrow (3-e)E = 2-(1-e)$$

$$\Rightarrow E = \frac{1+e}{3-e}$$

AS REQUIRED