IYGB - MATHEMATICAL METHODS 1 - PAPER A - QUESTION 1

TRANSFER THE SYSTEM INTO MATRIX FORM

$$\left.\begin{array}{l} x+3y+5z=6 \\ 6x-8y+4z=-3 \\ 3x+11y+13z=17 \end{array}\right\} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 6 & -8 & 4 & -3 \\ 3 & 11 & 13 & 17 \end{array}\right]$$

APPLY ROW OPERATIONS

$$\begin{array}{l} r_{12}(-6) \\ r_{13}(-3) \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & -26 & -26 & -39 \\ 0 & 2 & -2 & -1 \end{array}\right] \underset{=}{r_2\left(-\tfrac{1}{26}\right)} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 1 & \frac{3}{2} \\ 0 & 2 & -2 & -1 \end{array}\right] \underset{=}{r_{23}(-2)}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 1 & 3/2 \\ 0 & 0 & -4 & -4 \end{array}\right] \underset{=}{r_3\left(-\tfrac{1}{4}\right)} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 6 \\ 0 & 1 & 1 & \frac{3}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} r_{32}(-1) \\ r_{31}(-5) \end{array} =$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] \quad r_{21}(-3) \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\therefore \quad x=-\tfrac{1}{2}, \quad y=\tfrac{1}{2}, \quad z=1$$

KEY TO ROW OPERATIONS

r_{12} = SWAP ROW 1 & 2

$r_3\left(\frac{1}{2}\right)$ = MULTIPLY ROW 3 BY $\frac{1}{2}$

$r_{23}(-2)$ = MULTIPLY ROW 2 BY -2 AND ADD IT INTO ROW 3

IYGB - MATHEMATICAL METHODS 1 - PAPER A - QUESTION 2

a) WE NOTE THAT $\frac{d}{d\alpha}\left[x^{\alpha}\right] = x^{\alpha}\ln x$.

HENCE WE PROCEED AS FOLLOWS

$$\int_0^1 x^{\alpha}\ln x\,dx = \int_0^1 \frac{\partial}{\partial\alpha}\left(x^{\alpha}\right)dx \qquad \alpha > -1.$$

$$= \frac{\partial}{\partial\alpha}\int_0^1 x^{\alpha}\,dx$$

$$= \frac{\partial}{\partial\alpha}\left[\frac{1}{\alpha+1}x^{\alpha+1}\right]_0^1$$

$$= \frac{\partial}{\partial\alpha}\left[\frac{1}{\alpha+1} - 0\right]$$

$$= -\frac{1}{(\alpha+1)^2}$$

THUS BY LETTING $\alpha = \frac{4}{3}$ WE OBTAIN

$$\int_0^1 x^{\frac{4}{3}}\ln x\,dx = -\frac{1}{\left(\frac{4}{3}+1\right)^2} = -\frac{1}{\left(\frac{7}{3}\right)^2} = -\frac{9}{49}$$

b) VERIFICATION BY PARTS

$$\int_0^1 x^{\frac{4}{3}}\ln x\,dx = \left[\frac{3}{7}x^{\frac{7}{3}}\ln x\right]_0^1 - \frac{3}{7}\int_0^1 x^{\frac{4}{3}}dx$$

$\ln x$	$\frac{1}{x}$
$\frac{3}{7}x^{\frac{7}{3}}$	$x^{\frac{4}{3}}$

$$= \left[\frac{3}{7}x^{\frac{7}{3}}\ln x - \frac{9}{49}x^{\frac{7}{3}}\right]_0^1$$

$$= \left(\cancel{\frac{3}{7}\ln 1} - \frac{9}{49}\right) - \left(\frac{3}{7}\lim_{x\to 0}\left(x^{\frac{7}{3}}\ln x\right) - 0\right)$$

$$= -\frac{9}{49}$$

x TENDS TO ZERO FASTER THAN $\ln x$ TENDS TO $-\infty$

IYGB - MATHEMATICAL METHODS 1 - PAPER A - QUESTION 3

- REWRITE THE O.D.E. IN D OPERATOR FORM

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^{4x}$$

$$(D^2 - 4D + 3)y = e^{4x}$$

$$(D-1)(D-3)y = e^{4x}$$

- COMPLEMENTARY FUNCTION

$$y = Ae^{x} + Be^{3x}$$

- PARTICULAR INTEGRAL

$$y = \frac{1}{D^2-4D+3}\left\{e^{4x}\right\}$$

$$y = \frac{1}{4^2-4\times4+3}\left(e^{4x}\right)$$

$$y = \frac{1}{3}e^{4x}$$

- GENERAL SOLUTION

$$y = Ae^{x} + Be^{3x} + \frac{1}{3}e^{4x}$$

IYGB - MATHEMATICAL METHODS 1 - PAPER A - QUESTION 4

SIMPLIFY DIRECTLY FROM FIRST PRINCIPLES

$$\prod_{m=1}^{3}\prod_{n=1}^{4}\left[\sqrt{mn}\right] = \prod_{m=1}^{3}\left[\sqrt{m\times1}\sqrt{m\times2}\sqrt{m\times3}\sqrt{m\times4}\right]$$

$$= \prod_{m=1}^{3}\left[\sqrt{24m^4}\right]$$

$$= \prod_{m=1}^{3}\left[2\sqrt{6}\,m^2\right]$$

$$= (2\sqrt{6}\times1)\times(2\sqrt{6}\times2^2)\times(2\sqrt{6}\times3^2)$$

$$= (2\sqrt{6})^2(2\sqrt{6})\times2^2\times3^2$$

$$= 4\times6\times2\times2^2\times3^2\times\sqrt{6}$$

$$= 2^2\times2\times3\times2^3\times3^2\times\sqrt{6}$$

$$= 2^6\times3^3\times\sqrt{6}$$

$$= 4^3\times3^3\times\sqrt{6}$$

$$= 12^3\times\sqrt{6}$$

IE $k=3$

IYGB - MATHEMATICAL METHODS 1 - PAPER A - QUESTION 5

$$\lim_{x \to 0}\left[\frac{\cos 7x - 1}{x\sin x}\right] = \ldots$$ USING STANDARD EXPANSIONS

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + O(y^9)$$

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + O(y^8)$$

$$= \lim_{x \to 0}\left[\frac{\overbrace{\left[1 - \frac{(7x)^2}{2!} + O(x^4)\right]}^{\cos 7x} - 1}{x\underbrace{\left[x - \frac{x^3}{3!} + O(x^5)\right]}_{\sin x}}\right]$$

$$= \lim_{x \to 0}\left[\frac{-\frac{49}{2}x^2 + O(x^4)}{x^2 - \frac{1}{6}x^4 + O(x^6)}\right]$$

$$= \lim_{x \to 0}\left[\frac{-\frac{49}{2} + O(x^2)}{1 + O(x^2)}\right]$$

$$= -\frac{49}{2}$$

IYGB - MATHEMATICAL METHODS 1 - PAPER A - QUESTION 6

$$\boxed{f(x,y) = xy(x+2) - y(y+3) = x^2y + 2xy - y^2 - 3y}$$

- OBTAIN THE FIRST AND SECOND DERIVATIVES OF f

$$\frac{\partial f}{\partial x} = 2xy + 2y \qquad \frac{\partial^2 f}{\partial x^2} = 2y$$

$$\frac{\partial f}{\partial y} = x^2 + 2x - 2y - 3 \qquad \frac{\partial^2 f}{\partial y^2} = -2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x + 2$$

- FIND THE STATIONARY POINTS $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

- $2xy + 2y = 0$

  $2y(x+1) = 0$

  $y = 0$ OR $x = -1$

- IF $y = 0$

  $x^2 + 2x - 3 = 0$

  $(x+3)(x-1) = 0$

  $x = \begin{cases} 1 \\ -3 \end{cases}$

  IF $x = -1$

  $1 - 2 - 2y - 3 = 0$

  $-4 = 2y$

  $y = -2$

∴ POSSIBLE STATIONARY POINTS

$y=0$	$x=1$	$z = f(1,0) = 0$
$y=0$	$x=-3$	$z = f(-3,0) = 0$
$x=-1$	$y=-2$	$z = f(-1,-2) = 2(1) + 2(1) = 4$

IYGB - MATHEMATICAL METHODS 1 - PAPER A - QUESTION 6

TO DETERMINE THE NATURE OF EACH OF THESE POINTS

- $(1,0,0)$

$$\frac{\partial^2 f}{\partial x^2} = 0 \ , \quad \frac{\partial^2 f}{\partial y^2} = -2 \ , \quad \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -16 < 0$$

$\therefore (1,0,0)$ IS A SADDLE POINT

- $(-3,0,0)$

$$\frac{\partial^2 f}{\partial x^2} = 0 \ , \quad \frac{\partial^2 f}{\partial y^2} = -2 \ , \quad \frac{\partial^2 f}{\partial x \partial y} = -4$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -16 < 0$$

$\therefore (-3,0,0)$ IS A SADDLE POINT

- $(-1,-2,4)$

$$\frac{\partial^2 f}{\partial x^2} = -4 \ , \quad \frac{\partial^2 f}{\partial y^2} = -2 \ , \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 8 > 0$$

& BOTH $\frac{\partial^2 f}{\partial x^2} < 0$

$\frac{\partial^2 f}{\partial y^2} < 0$

$\therefore (-1,-2,4)$ IS A LOCAL MAXIMUM

IYGB - MATHEMATICAL METHODS 1 - PAPER A - QUESTION 7

a) $$\sum_{n=1}^{\infty}\left[\frac{(n+1)^2}{n^3}\right] = \sum_{n=1}^{\infty}\left[\frac{n^2+2n+1}{n^3}\right] = \sum_{n=1}^{\infty}\left[\frac{1}{n}+\frac{2}{n^2}+\frac{1}{n^3}\right]$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{n}\right] + 2\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right] + \sum_{n=1}^{\infty}\left[\frac{1}{n^3}\right]$$

HARMONIC SERIES WHICH DIVERGES

$$\therefore \sum_{n=1}^{\infty}\left[\frac{(n+1)^2}{n^3}\right] \text{ DIVERGES}$$

b) $$\sum_{r=1}^{\infty}\left[\frac{1}{2r+2^r}\right] < \sum_{r=1}^{\infty}\left[\frac{1}{2^r}\right] = \dots \text{ CONVERGENT G.P WITH } S_{\infty}=1$$

$$< 1$$

AS THIS IS A SERIES OF POSITIVE TERMS, BOUNDED ABOVE THE REQUIRED SERIES CONVERGES

$$\therefore \sum_{r=1}^{\infty}\left[\frac{1}{2r+2^r}\right] \text{ CONVERGES}$$

iYGB - MATHEMATICAL METHODS 1 - PAPER A - QUESTION 8

STARTING WITH A SKETCH IN ORDER TO SET UP THE INTEGRAL

$$\Rightarrow V = \int_{x=2}^{x=4} 1 \, dV$$

$$\Rightarrow V = \int_{x=2}^{x=4} 2\pi x y \, dx$$

$$\Rightarrow V = 2\pi \int_2^4 x(-8+6x-x^2) \, dx$$

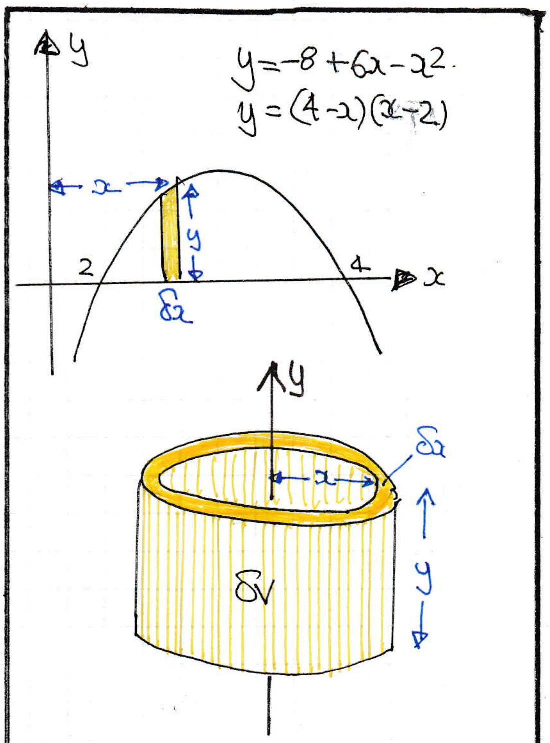
$$\Rightarrow V = 2\pi \int_2^4 -8x+6x^2-x^3 \, dx$$

$$\Rightarrow V = 2\pi \left[-4x^2+2x^3-\tfrac{1}{4}x^4\right]_2^4$$

$$\Rightarrow V = 2\pi\left[\left(\cancel{-64+128-64}\right)-\left(\cancel{-16+16}-4\right)\right]$$

$$\Rightarrow V = 2\pi\,[4]$$

$$\Rightarrow V = 8\pi$$



$$\delta V = \left[\pi(x+\delta x)^2 - \pi x^2\right] y$$

$$\delta V = \pi y\left[(x+\delta x)^2 - x^2\right]$$

$$\delta V = \pi y\left[\cancel{x^2}+2x\delta x+\delta x^2-\cancel{x^2}\right]$$

$$\delta V = 2\pi x y \, \delta x$$

(IGNORING δx^2 TERM)

IYGB - MATHEMATICAL METHODS 1 - PAPER A - QUESTION 9

$$u_{n+2} = u_{n+1} + 6u_n \quad \underline{\text{WITH}} \quad u_1 = 1,\ u_2 = 1$$

REARRANGE THE RELATION

$$u_{n+2} - u_{n+1} - 6u_n = 0$$

"AUXILIARY EQUATION"

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda + 2)(\lambda - 3) = 0$$

$$\lambda = \begin{cases} -2 \\ 3 \end{cases}$$

$\therefore$ GENERAL SOLUTION

$$u_n = A(-2)^n + B(3)^n$$

APPLY CONDITIONS

$$u_1 = 1 \implies 1 = -2A + 3B \quad (n=1)$$

$$u_2 = 1 \implies 1 = 4A + 9B \quad (n=2)$$

SOLVING

$$\left.\begin{aligned} 2 &= -4A + 6B \\ 1 &= 4A + 9B \end{aligned}\right\} \implies 15B = 3$$

$$B = \frac{1}{5}$$

$$\implies 1 = -2A + \frac{3}{5}$$

$$\implies 2A = -\frac{2}{5}$$

$$\implies A = -\frac{1}{5}$$

FINALLY WE OBTAIN

$$u_n = -\frac{1}{5}(-2)^n + \frac{1}{5}(3^n)$$

$$u_n = \frac{1}{5}\left[3^n - (-2)^n\right]$$

IYGB – MATHEMATICAL METHODS 1 – PAPER A – QUESTION 10

- FIRSTLY FIND THE COMPLEMENTARY FUNCTION

$$\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)^2 = 0$$
$$\lambda = 1$$

$$\therefore \quad y = Ae^x + Bxe^x$$

- BY VARIATION OF PARAMETERS

$$\underbrace{1}_{a(x)}\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \underbrace{2xe^x}_{f(x)}$$

$$\boxed{\begin{aligned} a(x) &= 1 \\ e_1 &= e^x \\ e_2 &= xe^x \end{aligned}}$$

- OBTAIN THE WRONSKIAN, $W(x)$

$$W = \begin{vmatrix} e_1 & e_2 \\ e_1' & e_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}} = e^{2x}$$

- HENCE WE CAN FIND THE PARTICULAR INTEGRAL, y_P

$$\Longrightarrow y_P = -e_1 \int \frac{e_2 f(x)}{a(x)\,W(x)}\,dx + e_2 \int \frac{e_1 f(x)}{a(x)\,W(x)}\,dx$$

$$\Longrightarrow y_P = -e^x \int \frac{(xe^x)(2xe^x)}{1 \times e^{2x}}\,dx + xe^x \int \frac{e^x(2xe^x)}{1 \times e^{2x}}\,dx$$

$$\Longrightarrow y_P = -e^x \int \frac{2x^2e^{2x}}{e^{2x}}\,dx + xe^x \int \frac{2xe^{2x}}{e^{2x}}\,dx$$

$$\Rightarrow y_P = -e^{x}\int 2x^2\,dx + xe^{2x}\int 2x\,dx$$

$$\Rightarrow y_P = -e^{x}\left(\tfrac{2}{3}x^3\right) + xe^{2x}\left(x^2\right)$$

$$\Rightarrow y_P = \tfrac{1}{3}x^3e^{x}$$

THUS WE OBTAIN A GENERAL SOLUTION

$$y = Ae^{x} + Bxe^{x} + \tfrac{1}{3}x^3e^{x}$$

(NOTE THAT THE PARTICULAR INTEGRAL CAN ALSO BE FOUND BY INSPECTION)

IYGB - MATHEMATICAL METHODS 1 - PAPER A - QUESTION 11

- REWRITE THE CONIC IN THE STANDARD FORM

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} \overset{x^2}{1} & \overset{xy}{1} \\ \underset{xy}{1} & \underset{y^2}{1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 8 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

- DIAGONALIZE THE SYMMETRIC MATRIX ABOVE

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \implies (1-\lambda)^2 - 1 = 0$$

$$\implies (1-\lambda-1)(1-\lambda+1) = 0$$

$$\implies -\lambda(2-\lambda) = 0$$

$$\implies \lambda = \begin{cases} 0 \\ 2 \end{cases}$$

- FIND EIGENVECTORS FOR EACH EIGENVALUE

IF $\lambda = 0$

$$\left.\begin{array}{l} x+y=0 \\ x+y=0 \end{array}\right\} \quad y=-x \implies \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

IF $\lambda = 2$

$$\left.\begin{array}{l} x+y=2x \\ x+y=2y \end{array}\right\} \quad y=x \implies \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- NORMALIZE THE EIGENVECTORS TO $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ & $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

- THUS IN THE USUAL NOTATION OF DIAGONALIZATION WE HAVE

$$\underline{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \& \quad \underline{D} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

AND USING THE TRANSFORMATION

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\Longrightarrow \boxed{x^2 + 2xy + 8x + y = 0}$$

$$\Longrightarrow (x \ \ y)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (8 \ \ 1)\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Longrightarrow (X \ \ Y)\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + (8 \ \ 1)\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

$$\Longrightarrow 2X^2 + \frac{\sqrt{2}}{2}(9 \ \ 7)\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

$$\Longrightarrow 2X^2 + \frac{\sqrt{2}}{2}(9X + 7Y) = 0$$

$$\Longrightarrow 4X^2 + \sqrt{2}(9X + 7Y) = 0$$

$$\Longrightarrow 4X^2 + 9\sqrt{2}X + 7\sqrt{2}Y = 0$$

IYGB - MATHEMATICAL METHODS 1 - PAPER A - QUESTION 12

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 9x^8 \quad \text{WITH } x=1,\ y=\frac{3}{2},\ \frac{dy}{dx}=2$$

- ASSUME A SOLUTION OF THE FORM $y = x^{\lambda}$

$$y' = \lambda x^{\lambda-1}$$
$$y'' = \lambda(\lambda-1)x^{\lambda-2}$$

- SUBSTITUTE INTO THE L.H.S. OF THE O.D.E (IGNORE R.H.S)

$$\Rightarrow x^2\left[\lambda(\lambda-1)x^{\lambda-2}\right] - 2x\left[\lambda x^{\lambda-1}\right] - 4\left[x^{\lambda}\right] = 0$$
$$\Rightarrow \lambda(\lambda-1)x^{\lambda} - 2\lambda x^{\lambda} - 4x^{\lambda} = 0$$
$$\Rightarrow \left[\lambda(\lambda-1) - 2\lambda - 4\right]x^{\lambda} = 0$$
$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$
$$\Rightarrow (\lambda-4)(\lambda+1) = 0$$
$$\Rightarrow \lambda = \begin{cases} -1 \\ 4 \end{cases}$$

$\therefore$ COMPLEMENTARY FUNCTION $y = Ax^{-1} + Bx^4$

- PARTICULAR INTEGRAL BY INSPECTION

$$\left.\begin{aligned} y &= Px^8 \\ y' &= 8Px^7 \\ y'' &= 56Px^6 \end{aligned}\right\} \Rightarrow x^2\left[56Px^6\right] - 2x\left[8Px^7\right] - 4Px^8 \equiv 9x^8$$

$$\Rightarrow 56Px^8 - 16Px^8 - 4Px^8 \equiv 9x^8$$
$$\Rightarrow 36P = 9$$
$$\Rightarrow P = \frac{1}{4}$$

∴ GENERAL SOLUTION IS

$$y = \frac{A}{x} + Bx^4 + \frac{1}{4}x^8$$

- APPLYING CONDITIONS $x=1$, $y=\frac{3}{2}$, $\frac{dy}{dx}=2$

$$\left.\begin{aligned} y &= \frac{A}{x} + Bx^4 + \frac{1}{4}x^8 \\ \frac{dy}{dx} &= -\frac{A}{x^2} + 4Bx^3 + 2x^7 \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} \frac{3}{2} &= A + B + \frac{1}{4} \\ 2 &= -A + 4B + 2 \end{aligned}\right\} \text{ADDING}$$

$$\Longrightarrow \frac{7}{2} = 5B + \frac{9}{4}$$

$$\Longrightarrow 14 = 20B + 9$$

$$\Longrightarrow 5 = 20B$$

$$\Longrightarrow B = \frac{1}{4}$$

$$\& \quad A = 4B \Longrightarrow A = 1$$

- FINALLY WE HAVE A SOLUTION

$$y = \frac{1}{x} + \frac{1}{4}x^4 + \frac{1}{4}x^8$$

$$y = \frac{1}{x} + \frac{1}{4}x^4(1+x^4)$$

IYGB - MATHEMATICAL METHODS I - PAPER A - QUESTION 13

- START BY CONSIDERING THE GRAPH OF $y = \frac{x(1-x)}{4-x^2}$

$$y = \frac{x-x^2}{4-x^2} = \frac{x(1-x)}{(2-x)(2+x)} = \frac{x(x-1)}{(x-2)(x+2)} = \frac{x^2-x}{x^2-4}$$

$$= \frac{x^2-4}{x^2-4} + \frac{-x+4}{x^2-4} = 1 - \frac{x-4}{x^2-4} = 1 + \frac{4-x}{x^2-4}$$

- FROM THE ABOVE FORMS WE DEDUCE DIRECTLY

  - $x=0$, $y=0$ IE $(0,0)$
  - $y=0$, $x < \begin{matrix} 0 \\ 1 \end{matrix}$ $\quad \begin{matrix} (0,0) \\ (1,0) \end{matrix}$
  - VERTICAL ASYMPTOTES $x = \pm 2$ (DENOMINATOR ZERO)
  - HORIZONTAL ASYMPTOTES $y=1$ (AS $x \to \pm\infty$)

- NEXT LOOK FOR STATIONARY POINTS

$$\frac{d}{dx}\left[1 + \frac{4-x}{x^2-4}\right] = \frac{(x^2-4)(-1) - (4-x)(2x)}{(x^2-4)^2} = \frac{4-x^2-8x+2x^2}{(x^2-4)^2}$$

$$= \frac{x^2-8x+4}{(x^2-4)^2}$$

SOLVING FOR ZERO YIELDS

$$x^2-8x+4=0$$
$$(x-4)^2 - 12 = 0$$
$$(x-4)^2 = 12$$
$$x-4 = \pm 2\sqrt{3}$$
$$x = 4 \pm 2\sqrt{3}$$

AND $x^2 = 16 \pm 16\sqrt{3} + 12$

$$x^2 = 28 \pm 16\sqrt{3}$$

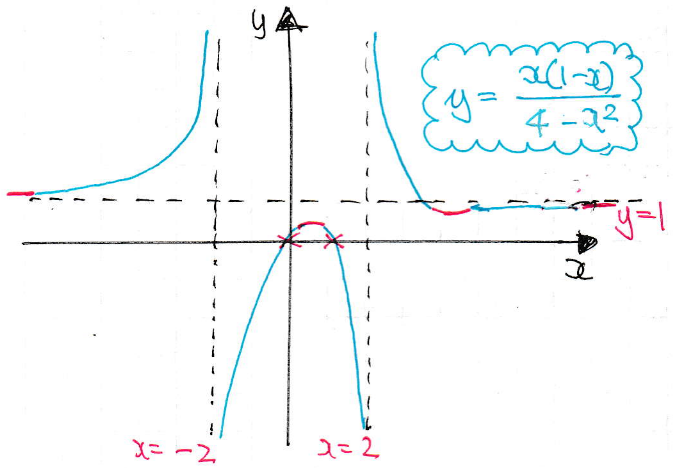
IYGB − MATHEMATICAL METHODS 1 − PAPER A − QUESTION 13

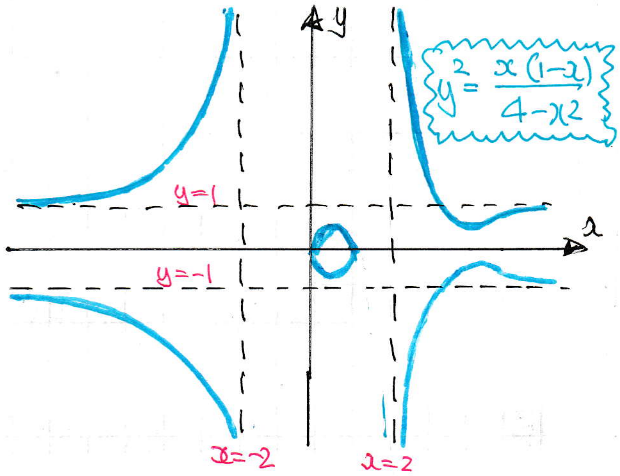
THUS $\dfrac{x^2-x}{x^2-4} = \dfrac{(28 \pm 16\sqrt{3}) - (4 \pm 2\sqrt{3})}{28 \pm 16\sqrt{3} - 4}$

$$= \begin{cases} \dfrac{24+14\sqrt{3}}{24+16\sqrt{3}} = \dfrac{12+7\sqrt{3}}{12+8\sqrt{3}} = \dfrac{1}{4}(2+\sqrt{3}) \\[2ex] \dfrac{24-14\sqrt{3}}{24-16\sqrt{3}} = \dfrac{12-7\sqrt{3}}{12-8\sqrt{3}} = \dfrac{1}{4}(2-\sqrt{3}) \end{cases}$$

HENCE THE STATIONARY POINTS ARE $\left(4 \pm 2\sqrt{3}, \frac{1}{4}(2 \pm \sqrt{3})\right)$

● THUS WE HAVE SO FAR





- y CO-ORDS GET "SQUARE ROOTED"
- PART OF CURVE FOR WHICH $y \geq 0$ REMAINS & PART OF CURVE FOR WHICH $y < 0$ VANISHES
- THE PART OF THE CURVE FOR WHICH $y \geq 0$ IS REFLECTED IN THE y AXIS
- THE CURVE HAS INFINITE GRADIENT AT x INTERCEPTS

NOTE THAT $\frac{1}{4}(2+\sqrt{3}) = \frac{1}{8}(4+2\sqrt{3}) = \frac{1}{8}\left(1^2 + 2 \times 1 \times \sqrt{3} + (\sqrt{3})^2\right)$

$$= \frac{1}{8}(\sqrt{3}+1)^2$$

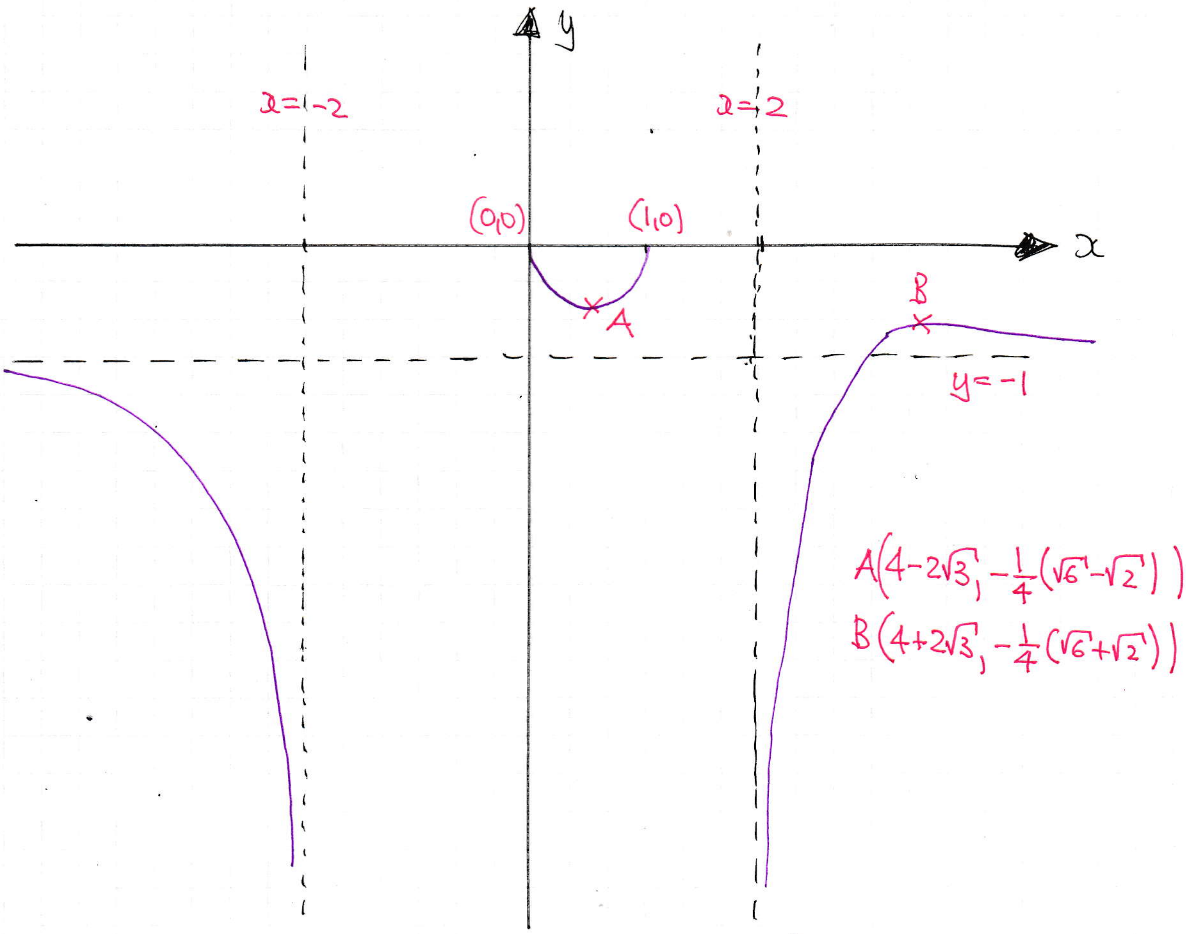
& SIMILARLY $\frac{1}{4}(2-\sqrt{3}) = \frac{1}{8}(\sqrt{3}-1)^2$

IYGB - MATHEMATICAL METHODS 1 - PAPER 4 - QUESTION 13

THUS $\sqrt{\frac{1}{8}(\sqrt{3}+1)^2} = \frac{1}{\sqrt{8}}(\sqrt{3}+1) = \frac{\sqrt{2}}{4}(\sqrt{3}+1) = \frac{1}{4}(\sqrt{6}+\sqrt{2})$

AND SIMILARLY $\sqrt{\frac{1}{8}(\sqrt{3}-1)} = \frac{1}{4}(\sqrt{6}-\sqrt{2})$

- FINALLY THE REQUIRED CURVE IS THE "BOTTOM HALF" OF $y^2 = \frac{x(1-x)}{4-x^2}$



$$\boxed{y = -\sqrt{\frac{x(1-x)}{4-x^2}}}$$

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- THIS IS A FIRST ORDER O.D.E WITH A HOMOGENEOUS R.H.S WHICH IS SOLVED BY THE STANDARD SUBSTITUTION

$$y = x\,v(x)$$

$$\frac{dy}{dx} = v(x) + x\frac{dv}{dx}$$

- TRANSFORMING THE O.D.E. AND SOLVE BY SEPARATING VARIABLES

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+y^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x(xv)}{x^2+x^2v^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-(v+v^3)}{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^3}{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{v^3}{v^2+1}$$

$$\Rightarrow \frac{v^2+1}{v^3}\,dv = -\frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{v} + \frac{1}{v^3}\,dv = \int -\frac{1}{x}\,dx$$

$$\Rightarrow \ln v - \frac{1}{2v^2} = -\ln x + A$$

$$\Rightarrow \ln v + \ln x = A + \frac{1}{2v^2}$$

$$\Rightarrow \ln(xv) = A + \frac{1}{2v^2}$$

$$\Rightarrow \ln y = A + \frac{1}{2\left(\frac{y}{x}\right)^2}$$

$$\Rightarrow \ln y = A + \frac{x^2}{2y^2}$$

$$\Rightarrow y = e^{A+\frac{x^2}{2y^2}}$$

$$\Rightarrow y = e^{A} \times e^{\frac{x^2}{2y^2}}$$

$$\Rightarrow y = Be^{\frac{x^2}{2y^2}}$$

when $x=0$ $y=1$

$\therefore$ $B=1$

$$\therefore y = e^{\frac{x^2}{2y^2}}$$

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$$\frac{dy}{dt} - x = e^{t}$$
$$\frac{dx}{dt} + y = e^{-t}$$
, SUBJECT TO $t=0, x=0, y=0$

- WRITE IN COMPACT NOTATION & TAKE LAPLACE TRANSFORMS IN t

$$\left.\begin{array}{l}\dot{y} - x = e^{t}\\ \dot{x} + y = e^{-t}\end{array}\right\} \Rightarrow \left\{\begin{array}{l} s\bar{y} - y_0 - \bar{x} = \frac{1}{s-1}\\ s\bar{x} - x_0 + \bar{y} = \frac{1}{s+1}\end{array}\right\} \qquad y_0 = x_0 = 0$$

$$\Rightarrow \left\{\begin{array}{l} s\bar{y} - \bar{x} = \frac{1}{s-1}\\ s\bar{x} + \bar{y} = \frac{1}{s+1}.\end{array}\right\} \leftarrow (\times s)$$

$$\Rightarrow \left\{\begin{array}{l} s^2\bar{y} - s\bar{x} = \frac{s}{s-1}\\ s\bar{x} + \bar{y} = \frac{1}{s+1}\end{array}\right\} \quad \text{ADDING}$$

$$\Rightarrow (s^2+1)\bar{y} = \frac{s}{s-1} + \frac{1}{s+1}$$

$$\Rightarrow (s^2+1)\bar{y} = \frac{s^2+s+s-1}{(s-1)(s+1)}$$

$$\Rightarrow \bar{y} = \frac{s^2+2s-1}{(s^2+1)(s-1)(s+1)}$$

- SPLIT BY PARTIAL FRACTIONS IN ORDER TO INVERT

$$\frac{s^2+2s-1}{(s^2+1)(s+1)(s-1)} \equiv \frac{A}{s+1} + \frac{B}{s-1} + \frac{Cs+D}{s^2+1}$$

$$s^2+2s-1 \equiv A(s-1)(s^2+1) + B(s+1)(s^2+1) + (s^2-1)(Cs+D)$$

- If $s=1$, $2 = 4B \Rightarrow B = \frac{1}{2}$
- If $s=-1$, $-2 = -4A \Rightarrow A = \frac{1}{2}$

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- IF $s=0$, $-1 = -A+B-D$

$$D = 1-A+B = 1-\tfrac{1}{2}+\tfrac{1}{2}$$

$$D = 1$$

- IF $s=2$ $\quad 4+4-1 = 5A+15B+3(2C+D)$

$$7 = \tfrac{5}{2}+\tfrac{15}{2}+3(2C+1)$$

$$7 = 10+3(2C+1)$$

$$-3 = 3(2C+1)$$

$$-1 = 2C+1$$

$$-2 = 2C$$

$$C = -1$$

INVERTING THE TRANSFORM, USING STANDARD RESULTS

$$\Rightarrow \bar{y} = \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s+1} - \frac{s-1}{s^2+1}$$

$$\Rightarrow \bar{y} = \tfrac{1}{2}\left(\frac{1}{s-1}\right) + \tfrac{1}{2}\left(\frac{1}{s-1}\right) - \left(\frac{s}{s^2+1}\right) + \left(\frac{1}{s^2+1}\right)$$

$$\Rightarrow y = \tfrac{1}{2}e^{t} + \tfrac{1}{2}e^{-t} - \cos t + \sin t$$

$$\Rightarrow y = \cosh t - \cos t + \sin t$$

TO FIND THE OTHER SOLUTION, USE THE FIRST O.D.E

$$\Rightarrow x = \frac{dy}{dt} - e^{t}$$

$$\Rightarrow x = \sinh t + \sin t + \cos t - e^{t}$$

$$\Rightarrow x = \tfrac{1}{2}e^{t} - \tfrac{1}{2}e^{-t} - e^{t} + \sin t + \cos t$$

$$\Rightarrow x = -\tfrac{1}{2}e^{t} - \tfrac{1}{2}e^{-t} + \sin t + \cos t$$

$$\Rightarrow x = -\cosh t + \cos t + \sin t$$

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a)

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{\cos[(2n-1)x]}{(2n-1)^2}\right]$$

$$\text{Sgn}(x) = \frac{d}{dx}\Big[|x|\Big] = \frac{d}{dx}\left[\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{\cos[(2n-1)x)}{(2n-1)^2}\right]\right]$$

$$= -\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{-(2n-1)\sin[(2n-1)x]}{(2n-1)^2}\right]$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{\sin[(2n-1)x]}{(2n-1)^2}\right]$$

b) NOW FIND THE FOURIER EXPANSION DIRECTLY

$$f(x) = \text{Sign}\,x, \quad -\pi < x < \pi, \quad f(x+2\pi) = f(x)$$

- $$a_0 = \frac{1}{L}\int_a^b f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\text{sign}(x) = 0$$

ODD FUNCTION IN A SYMMETRICAL DOMAIN

- $$a_n = \frac{1}{L}\int_a^b f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\text{sign}(x)}_{\text{ODD}}\,\underbrace{\cos(nx)}_{\text{EVEN}}\,dx = 0$$

ODD INTEGRAND IN A SYMMETRICAL DOMAIN

- $$b_n = \frac{1}{L}\int_a^b f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\text{sign}(x)}_{\text{ODD}}\,\underbrace{\sin(nx)}_{\text{ODD}}\,dx$$

$$= \frac{1}{\pi}\times 2\int_0^{\pi} 1\times\sin(nx)\,dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi n}\Big[\cos nx\Big]_{\pi}^{0} = \frac{2}{\pi n}\Big[1 - \cos n\pi\Big] = \frac{2}{\pi n}\Big[1 - (-1)^n\Big]$$

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$$= \begin{cases} 0 & \text{IF } n \text{ IS EVEN} \\ \frac{4}{\pi n} & \text{IF } n \text{ IS ODD} \end{cases}$$

HENCE WE REWRITE THE COEFFICIENT AS $b_m = \frac{4}{\pi(2m-1)}$, $m = 1,2,3,4,\ldots$

$$\left[\text{OR } b_n = \frac{4}{\pi(2n-1)}, \quad n = 1,2,3,\ldots\right]$$

HENCE WE CAN SUBSTITUTE INTO THE FOURIER FORMULA

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\text{sign}(x) = \sum_{n=1}^{\infty}\left[\frac{4}{\pi(2n-1)} \sin\left[(2n-1)x\right]\right]$$

$$\text{sign}(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{\sin\left[(2n-1)x\right]}{(2n-1)}\right] \quad // \quad \text{AS BEFORE}$$

c) SUBSTITUTING $x = \frac{\pi}{2}$ INTO THE ABOVE FORMULA GIVES

$$\text{sign}\left(\frac{\pi}{2}\right) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)} \sin\left[\frac{\pi}{2}(2n-1)\right]\right]$$

$$1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)}(-1)^{n+1}\right]$$

$$\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{2n-1}\right] = \frac{\pi}{4}$$

$$\text{OR} \quad \sum_{r=1}^{\infty}\left[\frac{(-1)^{r+1}}{2r-1}\right] = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4} \quad //$$

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- START BY PRODUCING DIFFERENTIALS AS FOLLOWS

$$x = 2u + e^{2v}$$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$

$$dx = 2du + 2e^{2v}dv$$

$\times e^{-2u}$

$$e^{-2u}dx = 2e^{-2u}du + 2e^{2v}e^{-2u}dv$$

$$y = 2v + e^{-2u}$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

$$dy = -2e^{-2u}du + 2dv$$

- ADDING THE EXPRESSIONS TO ELIMINATE THE "du" TERMS

$$\Longrightarrow e^{-2u}dx + dy = \left[2 + 2e^{2(v-u)}\right]dv$$

$$\Longrightarrow dv = \frac{e^{-2u}}{2+2e^{2(v-u)}}dx + \frac{1}{2+2e^{2(v-u)}}dy$$

- COMPARING WITH

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\therefore \frac{\partial v}{\partial x} = \frac{e^{-2u}}{2+2e^{2(v-u)}} = \frac{1}{2e^{2u}+2e^{2v}} \quad //$$

$$\frac{\partial v}{\partial y} = \frac{1}{2+2e^{2(v-u)}} = \frac{e^{2u}}{2e^{2u}+2e^{2v}} \quad //$$

- SIMILARLY STARTING FROM THE PREVIOUSLY DERIVED EXPRESSIONS

$$\begin{pmatrix} dx = 2\,du + 2e^{2v}\,dv \\ dy = -2e^{-2u}\,du + 2\,dv \end{pmatrix} \times e^{2v}$$

$$\begin{pmatrix} dx = 2\,du + 2e^{2v}\,dv \\ e^{2v}\,dy = -2e^{2v}e^{-2u}\,du + 2e^{2v}\,dv \end{pmatrix}$$

- SUBTRACTING TO ELIMINATE THE "dv" TERMS

$$\Rightarrow dx - e^{2v}\,dy = \left[2 + 2e^{2(v-u)}\right] du$$

$$\Rightarrow du = \underbrace{\frac{1}{2+2e^{2(v-u)}}}_{\frac{\partial u}{\partial x}}\,dx + \underbrace{\frac{-e^{2v}}{2+2e^{2(v-u)}}}_{\frac{\partial u}{\partial y}}\,dy$$

$$\frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot du$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2+2e^{2(v-u)}} = \frac{e^{2u}}{2e^{2u}+2e^{2v}}$$

$$\frac{\partial u}{\partial y} = \frac{-e^{2v}}{2+2e^{2(v-u)}} = \frac{-e^{2(v+u)}}{2e^{2u}+2e^{2v}}$$

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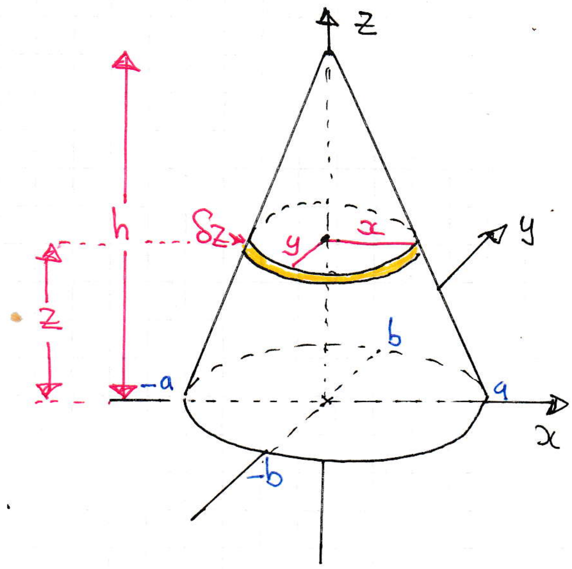
- LOOKING AT THE DIAGRAM OPPOSITE
- THE INFINITESIMAL ELLIPTICAL DISC OF THICKNESS δz HAS VOLUME

$$\delta V = \pi x y \, \delta z$$



- BY SIMILAR TRIANGLES WE HAVE

$$\Rightarrow \frac{x}{a} = \frac{h-z}{h} \quad \& \quad \frac{y}{b} = \frac{h-z}{h}$$

$$\Rightarrow x = \frac{a}{h}(h-z) \quad \& \quad y = \frac{b}{h}(h-z)$$

- SO THE VOLUME IS GIVEN BY

$$\delta V = \pi\left[\frac{a}{h}(h-z)\right]\left[\frac{b}{h}(h-z)\right]\delta z = \frac{\pi ab}{h^2}(h-z)^2\,\delta z$$

$$V = \int_{z=0}^{z=h} 1 \, dV = \int_{z=0}^{z=h} \frac{\pi ab}{h^2}(h-z)^2 \, dz$$

$$= \frac{\pi ab}{h^2}\left[-\frac{1}{3}(h-z)^3\right]_0^h$$

$$= \frac{\pi ab}{3h^2}\left[(h-z)^3\right]_h^0$$

$$= \frac{\pi ab}{3h^2}\left[h^3 - 0\right]$$

$$= \frac{1}{3}\pi abh$$