IYGB

Mathematical Methods 3

Practice Paper A

Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This paper follows the most common syllabi of Mathematical Methods used in the United Kingdom Universities for Mathematics, Physics and Engineering degrees.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.

Full marks may be obtained for answers to ALL questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 11 questions in this question paper.

The total mark for this paper is 100.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear to the Examiner.

Answers without working may not gain full credit.

Non exact answers should be given to an appropriate degree of accuracy.

The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained by attempting all the questions on the paper.

Topics Examined Under Mathematical Methods 3

- Complex Variables including Residues, Laurent Series and Calculus
- Series Solutions of Differential Equations
- Gamma Functions
- Beta Functions
- Laplace Transform
- Fourier Transform
- Special Functions and Polynomials, including Bessel, Legendre, Chebyshev etc.

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Question 1

$$f(z) \equiv \frac{\sin z}{z^2}, \ z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$. **(4)**

Question 2

By using techniques involving the Gamma function, find the exact value of

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right) \right]^{a-1} \, dx,$$

where $a \neq 1, \ 0, -1, -2, -3, \ldots$ **(4)**

Give the answer in terms of a Gamma function.

Question 3

Determine a Laurent series for

$$f(z) = \frac{1}{z},$$

which is valid in the infinite annulus $|z-1| > 1$. **(6)**

Question 4

By using techniques involving the Beta function, find the exact value of

$$\int_0^\infty \frac{x^3}{\left(1+8x^3\right)^2} \, dx.$$ **(8)**

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Question 5

The generating function of the Bessel function of the first kind is

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right],\ n \in \mathbb{Z}.$$

a) By differentiating the generating function relation with respect to x, show that

$$\frac{1}{2}J_{n-1}(x) - \frac{1}{2}J_{n+1}(x) = J_n'(x).$$ **(4)**

b) By differentiating the generating function relation with respect to t, show that

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right].$$ **(4)**

c) Hence find a simplified expression for

$$\frac{d}{dx}\left[\left(x^n + x^{-n}\right)J_n(x)\right].$$ **(4)**

Question 6

The piecewise continuous function $f(t)$ is defined as

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\left(2 - 2e^{-4s} - e^{-6s}\right)\right].$$

Sketch the graph of $f(t)$. **(10)**

Question 7

Find the two independent solutions of the following differential equation

$$\left(x^2-1\right)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0 \ , \quad |x|<1 .$$

Give the final answer in simplified form without involving infinite sums. **(10)**

Question 8

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{2\pi} \cos^6\theta \sin^6\theta \ d\theta . \qquad \textbf{(11)}$$

Question 9

The generating function for the Legendre's polynomials $P_n(x)$, satisfies

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

a) Use this result to show that

$$P_n(-1)=(-1)^n . \qquad \textbf{(4)}$$

b) By using the result of part **(a)** and Legendre's equation, deduce that

$$P_n'(-1)=\frac{1}{2}n(n+1)(-1)^{n+1} . \qquad \textbf{(5)}$$

Question 10

Use the differential equation

$$\frac{d^2x}{dt^2} = a^2 x \,,\ t \ge 0 \,,$$

with appropriate initial conditions to show that

$$\mathcal{L}(\cosh at) = \frac{s}{s^2 - a^2} \qquad \text{and} \qquad \mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2} \,. \qquad (10)$$

You may not use integration in this question.

Question 11

The Gaussian function $f(x)$ is defined by

$$f(x) = A\mathrm{e}^{-\alpha x^2} \,,$$

where A and α are positive constants.

Find the Fourier transform of $f(x)$. **(16)**
