IYGB - MATHEMATICAL METHODS 3 - PAPER B - QUESTION 1

FACTORIZING THE FUNCTION

$$f(z) = \frac{z^2+2z+1}{z^2-2z+1} = \frac{(z+1)^2}{(z-1)^2}$$

$f(z)$ HAS A DOUBLE POLE AT $z=1$

$$\lim_{z \to 1}\left[\frac{d}{dz}\left[(z-1)^2 f(z)\right]\right] = \lim_{z \to 1}\left[\frac{d}{dz}\left[\cancel{(z-1)^2}\frac{(z+1)^2}{\cancel{(z-1)^2}}\right]\right]$$

$$= \lim_{z \to 1}\left[\frac{d}{dz}(z+1)^2\right]$$

$$= \lim_{z \to 1}\left[2(z+1)\right]$$

$$= \underline{4}$$

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THIS CAN BE TURNED INTO A GAMMA FUNCTION BY SUBSTITUTION

$$\int_0^\infty x^3 e^{-\frac{1}{2}x^2}\,dx = \int_0^\infty x^3 e^{-u}\left(\frac{du}{x}\right)$$

$$= \int_0^\infty x^2 e^{-u}\,du = \int_0^\infty 2u\,e^{-u}\,du$$

$u = \frac{1}{2}x^2$

$du = x\,dx$

$dx = \frac{du}{x}$

$2u = x^2$

$x = \sqrt{2u}$

LIMITS UNCHANGED

$$\boxed{\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx}$$

$$= 2\int_0^\infty u^{2-1} e^{-u}\,du = 2\,\Gamma(2)$$

$$= 2 \times 1! = \underline{2}$$

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- NEED AN EXPANSION IN POWERS OF $z-1$

$$\Rightarrow \frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{1}{z-1}\left[\frac{1}{z+1}\right] = \frac{1}{z-1}\left[\frac{1}{(z-1)+2}\right]$$

- CREATING A STANDARD BINOMIAL

$$\Rightarrow \frac{1}{z^2-1} = \frac{1}{z-1}\left[\frac{1}{2+(z-1)}\right] = \frac{1}{2(z-1)}\left[\frac{1}{1+\frac{z-1}{2}}\right]$$

- THIS EXPANSION IS VALID FOR

$$0 < \left|\frac{z-1}{2}\right| < 1$$

$$0 < |z-1| < 2 \quad \text{AS REQUIRED}$$

- RETURNING TO THE LAURENT

$$\Rightarrow \frac{1}{z^2-1} = \frac{1}{2(z-1)}\left[1 - \frac{z-1}{2} + \frac{(z-1)^2}{2^2} - \frac{(z-1)^3}{2^3} + \frac{(z-1)^4}{2^4} \dots\right]$$

$$\Rightarrow \frac{1}{z^2-1} = \frac{1}{2(z-1)} - \frac{1}{2^2} + \frac{(z-1)}{2^3} - \frac{(z-1)^2}{2^4} + \frac{(z-1)^4}{2^5} \dots$$

$$\Rightarrow \frac{1}{z^2-1} = \sum_{r=-1}^{\infty}\left[\frac{(z-1)^r(-1)^{r+1}}{2^{r+2}}\right]$$

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NOTING AN ALTERNATIVE DEFINITION OF BETA

$$B(m,n) \equiv \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

PROCEED BY A SUBSTITUTION AIMING FOR THE ABOVE FORM

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx = \dots \text{ EVEN INTEGRAND } = 2\int_0^\infty \frac{x^2}{1+x^4}\, dx$$

$u = x^4$

$x = u^{\frac{1}{4}}$

$dx = \frac{1}{4}u^{-\frac{3}{4}}\, du$

LIMITS UNCHANGED

$$= 2\int_0^\infty \frac{u^{\frac{1}{2}}}{1+u}\left(\frac{1}{4}u^{-\frac{3}{4}}\, du\right) = \frac{1}{2}\int_0^\infty \frac{u^{-\frac{1}{4}}}{(1+u)^1}\, du$$

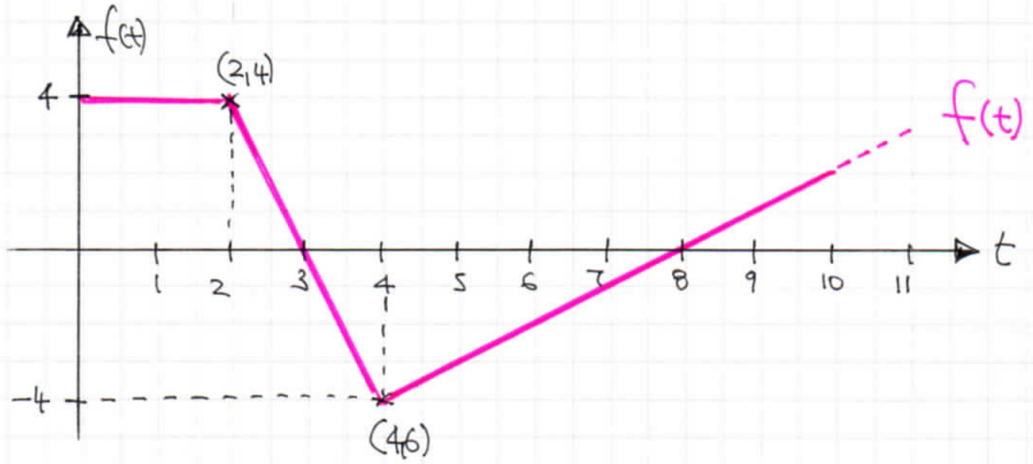
$$= \frac{1}{2}\int_0^\infty \frac{u^{\frac{3}{4}-1}}{(1+u)^{\frac{3}{4}+\frac{1}{4}}}\, du = \frac{1}{2} B\left(\frac{3}{4},\frac{1}{4}\right) = \frac{1}{2}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

FINALLY USING $\Gamma(x)\Gamma(1-x) \equiv \frac{\pi}{\sin \pi x}$

$$\dots = \frac{1}{2} \times \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{2\times\frac{\sqrt{2}}{2}} = \frac{\pi}{\sqrt{2}}$$

IYGB - MATHEMATICAL METHODS 3 - PAPER B - QUESTION 5

a) $f(t) = \begin{cases} 4 & 0 \le t \le 2 \\ 12-4t & 2 < t \le 4 \\ t-8 & t > 4 \end{cases}$



b) EXPRESSING THE GRAPH IN TERMS OF "HEAVISIDES"

$$\Rightarrow f(t) = 4H(t) - 4H(t-2) + (12-4t)H(t-2) - (12-4t)H(t-4) + (t-8)H(t-4)$$

"ON"
"OFF"

$$\Rightarrow f(t) = 4H(t) + (8-4t)H(t-2) + (5t-20)H(t-4)$$

$$\Rightarrow f(t) = 4H(t) - 4(t-2)H(t-2) + 5(t-4)H(t-4)$$

$$\Rightarrow \bar{f}(s) = \frac{4e^{0s}}{s} - 4\left(\frac{e^{-2s}}{s^2}\right) + 5\left(\frac{e^{-4s}}{s^2}\right)$$

$$\Rightarrow \bar{f}(s) = \frac{4}{s} - \frac{4}{s^2}e^{-2s} + \frac{5}{s^2}e^{-4s}$$

NOTE THAT $\mathcal{L}[f(t-a)H(t-a)] = e^{-as}\mathcal{L}[f(t)]$

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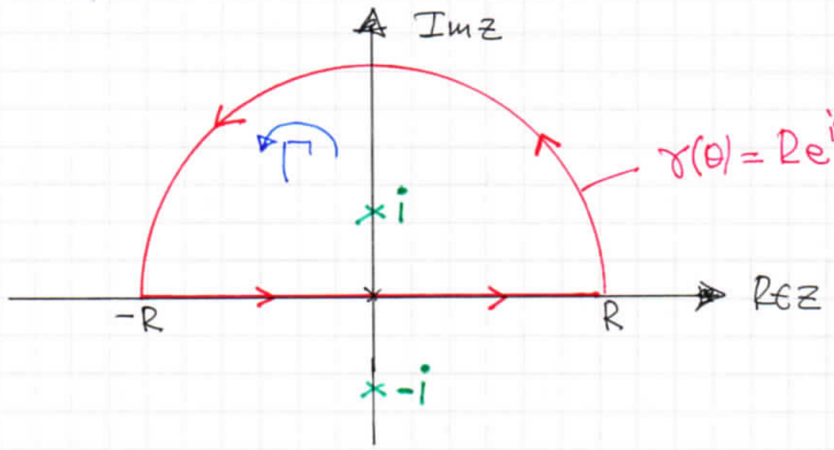
- CONSIDER $\oint_{\Gamma} f(z)\,dz$, WHERE $f(z) = \frac{e^{iz}}{1+z^2}$ AND Γ IS THE "STANDARD" SEMICIRCULAR CONTOUR, SHOWN BELOW



- $f(z)$ HAS SIMPLE POLES AT $\pm i$, OF WHICH ONLY THE ONE AT i IS INSIDE Γ

- CALCULATE THE RESIDUE OF THIS POLE

$$\lim_{z \to i}\left[(z-i)f(z)\right] = \lim_{z \to i}\left[(z-i)\frac{e^{iz}}{z^2+1}\right]$$

$$= \lim_{z \to i}\left[\cancel{(z-i)}\frac{e^{iz}}{\cancel{(z-i)}(z+i)}\right]$$

$$= \frac{e^{-1}}{2i}$$

- BY THE RESIDUE THEOREM

$$\Rightarrow \oint_{\Gamma} f(z)\,dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$

$$\Rightarrow \oint_{\Gamma} \frac{e^{iz}}{1+z^2}\,dz = 2\pi i \times \frac{e^{-1}}{2i}$$

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$$\Longrightarrow \int_{-R}^{R} \frac{e^{ix}}{1+x^2}\,dx + \int_{\gamma} \frac{e^{iz}}{1+z^2}\,dz = \frac{\pi}{e}$$

$z = x + 0i$ ALONG THE STRAIGHT LINE

ALONG THE ARC

- NOW $g(z) = \dfrac{e^{iz}}{1+z^2}$ SATISFIES JORDAN'S LEMMA, SO AS $R \to \infty$ THE INTEGRAL AROUND γ VANISHES.

$$\Longrightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \frac{\pi}{e}$$

ODD

$$\Longrightarrow \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} + i\,\cancel{\frac{\sin x}{1+x^2}}\,dx = \frac{\pi}{e}$$

$$\Longrightarrow \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{e}$$

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STARTING FROM THE GENERATING FUNCTION

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = e^{\frac{1}{2}xt} \times e^{-\frac{1}{2}\frac{x}{t}} = \left[\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}xt\right)^k}{k!}\right]\left[\sum_{m=0}^{\infty} \frac{\left(-\frac{x}{2t}\right)^m}{m!}\right]$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}xt\right)^k\left(-\frac{x}{2t}\right)^m}{k!\,m!}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m\left(\frac{1}{2}\right)^{k+m} x^{k+m}\, t^k\, t^{-m}}{k!\,m!}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \left[(-1)^m\left(\frac{1}{2}\right)^{k+m} \frac{x^{k+m}\, t^{k-m}}{k!\,m!}\right]$$

NOW WE SPLIT INTO TWO CASES - POWER OF t IS POSITIVE, SAY $n \geqslant 0$

$$\boxed{\begin{aligned} &k-m = n \geqslant 0 \\ &k = m+n \quad \& \quad k+m = 2m+n \end{aligned}}$$

$$\ldots = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left[(-1)^m\left(\frac{1}{2}\right)^{2m+n} \frac{x^{2m+n}\, t^n}{(n+m)!\,m!}\right]$$

$$= \sum_{n=0}^{\infty}\left[t^n\left[\sum_{m=0}^{\infty} \frac{(-1)^m}{(n+m)!\,m!}\left(\frac{x}{2}\right)^{2m+n}\right]\right]$$

$$= \sum_{n=0}^{\infty} t^n J_n(x)$$

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CASE B — THE POWER OF t IS A NEGATIVE INTEGER, SAY $-n<0$

LET $k-m=-n \implies k=m-n$
$\implies k+m=2m-n$

$$\ldots = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[(-1)^m\left(\frac{1}{2}\right)^{k+m}\frac{x^{k+m}\,t^{k-m}}{k!\,m!}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[(-1)^m\left(\frac{1}{2}\right)^{2m-n}\frac{x^{2m-n}\,t^{-n}}{(m-n)!\,m!}\right]$$

$$= \sum_{n=0}^{\infty}\left[t^{-n}\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{(m-n)!\,m!}\left(\frac{x}{2}\right)^{2m-n}\right]\right]$$

$$= \sum_{n=0}^{\infty}\left[t^{-n}\,J_{-n}(x)\right]$$

$$\therefore\ e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

IYGB - MATHEMATICAL METHODS 3 - PAPER B - QUESTION 8

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = e^{2x}$$

AS x, -1 & e^{2x} ARE ANALYTIC EVERYWHERE, WE MAY ASSUME A SOLUTION OF THE FORM

$$y = \sum_{r=0}^{\infty} a_r x^r \ ; \quad \frac{dy}{dx} = \sum_{r=1}^{\infty} a_r r x^{r-1} \ ; \quad \frac{d^2y}{dx^2} = \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}$$

SUBSTITUTE INTO THE O.D.E

$$\Longrightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} + x\sum_{r=1}^{\infty} a_r r x^{r-1} - \sum_{r=0}^{\infty} a_r x^r = e^{2x}$$

$$\Longrightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} + \sum_{r=1}^{\infty} a_r r x^r - \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} \frac{(2x)^r}{r!}$$

PULL OUT OF THE SUMMATIONS x^0, THE LOWEST POWER OF x

- $2 \times 1 \times a_2 x^0 - a_0 x^0 = x^0 \quad \Longrightarrow \quad 2a_2 - a_0 = 1$

$$\Longrightarrow \quad a_2 = \frac{1}{2} + \frac{1}{2}a_0$$

- $$\sum_{r=3}^{\infty} a_r r(r-1)x^{r-2} + \sum_{r=1}^{\infty} a_r r x^r - \sum_{r=1}^{\infty} a_r x^r = \sum_{r=1}^{\infty}\left(\frac{2^r}{r!}\right)x^r$$

ADJUST r, SO ALL THE SUMMATIONS START FROM $r=1$

$$\Longrightarrow \sum_{r=1}^{\infty} a_{r+2}(r+2)(r+1)x^r + \sum_{r=1}^{\infty} a_r r x^r - \sum_{r=1}^{\infty} a_r x^r = \sum_{r=1}^{\infty}\left(\frac{2^r}{r!}\right)x^r$$

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HENCE A RECURRENCE RELATION IS OBTAINED

$$\Longrightarrow a_{r+2}(r+2)(r+1) + a_r r - a_r = \frac{2^r}{r!}$$

$$\Longrightarrow \boxed{a_{r+2}(r+2)(r+1) + (r-1)a_r = \frac{2^r}{r!}}$$

NOW GENERATE TERMS

- IF $r=0$: $2a_2 - a_0 = 1 \Longrightarrow a_2 = \frac{1}{2} + \frac{1}{2}a_0$ (ALREADY KNOWN)
- IF $r=1$: $6a_3 = 2 \Longrightarrow a_3 = \frac{1}{3}$
- IF $r=2$: $12a_4 + a_2 = 2$

$$12a_4 + \frac{1}{2} + \frac{1}{2}a_0 = 2$$
$$24a_4 + 1 + a_0 = 4$$
$$24a_4 = 3 - a_0 \Longrightarrow a_4 = \frac{1}{8} - \frac{1}{24}a_0$$

- IF $r=3$: $20a_5 + 2a_3 = \frac{4}{3}$

$$20a_5 + \frac{2}{3} = \frac{4}{3}$$
$$20a_5 = \frac{2}{3} \Longrightarrow a_5 = \frac{1}{30}$$

- IF $r=4$: $30a_6 + 3a_4 = \frac{2}{3}$

$$30a_6 + 3\times\left(\frac{1}{8} - \frac{1}{24}a_0\right) = \frac{2}{3}$$
$$720a_6 + 72\left(\frac{1}{8} - \frac{1}{24}a_0\right) = 16$$
$$720a_6 + 9 - 3a_0 = 16$$
$$720a_6 = 7 + 3a_0 \Longrightarrow a_6 = \frac{7}{720} + \frac{1}{240}a_0$$

- IF $r=5$: $42a_7 + 4a_5 = \frac{4}{15}$

$$42a_7 + \frac{2}{15} = \frac{4}{15}$$
$$42a_7 = \frac{2}{15} \Longrightarrow a_7 = \frac{1}{315}$$

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- IF $r=6$:
$$56a_8 + 5a_6 = \frac{4}{45}$$
$$56a_8 + 5\left(\frac{7}{720} + \frac{1}{240}a_0\right) = \frac{4}{45}$$
$$56a_8 + \frac{7}{144} + \frac{1}{48}a_0 = \frac{4}{45}$$
$$56a_8 = \frac{29}{720} - \frac{1}{48}a_0 \implies a_8 = \frac{29}{40320} - \frac{1}{2688}a_0$$

TIDYING UP WE OBTAIN

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

$$y = a_0 + a_1x + \frac{1}{2}a_0x^2 - \frac{1}{24}a_0x^4 + \frac{1}{240}a_0x^6 - \frac{1}{2688}a_0x^8$$
$$+ \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{30}x^5 + \frac{7}{720}x^6 + \frac{1}{315}x^7 + \frac{29}{40320}x^8$$

$$y = a_1x + a_0\left[1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{240}x^6 - \frac{1}{2688}x^8 + \dots\right]$$
$$+ x^2\left[\frac{1}{2} + \frac{1}{3}x + \frac{1}{8}x^2 + \frac{1}{30}x^3 + \frac{7}{720}x^4 + \frac{1}{315}x^5 + \frac{29}{40320}x^6\right]$$

$$\therefore \boxed{y = Ax + B\left[1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{240}x^6 - \frac{1}{2688}x^8 + \dots\right] + x^2\left[\frac{1}{2} + \frac{1}{3}x + \frac{1}{8}x^2 + \frac{1}{30}x^3 + \frac{7}{720}x^4 + \frac{1}{315}x^5 + \frac{29}{40320}x^6 + \dots\right]}$$

IYGB - MATHEMATICAL METHODS 3 - PAPER B - QUESTION 9

a) STARTING BY THE DEFINITION OF A LAPLACE TRANSFORM

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = \int_0^\infty \frac{d^2y}{dt^2} e^{-st}\,dt.$$

PROCEED BY INTEGRATION BY PARTS

e^{-st}	$-se^{-st}$
$\frac{dy}{dt}$	$\frac{d^2y}{dt^2}$

$$\dots = \left[\frac{dy}{dt}e^{-st}\right]_0^\infty - \int_0^\infty -s\frac{dy}{dt}e^{-st}\,dt$$

$$= 0 - \left.\frac{dy}{dt}\right|_{t=0} + s\int_0^\infty \frac{dy}{dt}e^{-st}\,dt$$

INTEGRATION BY PARTS AGAIN

e^{-st}	$-se^{-st}$
y	$\frac{dy}{dt}$

$$= -\left.\frac{dy}{dt}\right|_{t=0} + s\left[\left[ye^{-st}\right]_0^\infty - \int -sye^{-st}\,dt\right]$$

$$= -\left.\frac{dy}{dt}\right|_{t=0} + s\left[0 - y\Big|_{t=0} + s\int_0^\infty ye^{-st}\,dt\right]$$

$$= -\left.\frac{dy}{dt}\right|_{t=0} - sy\Big|_{t=0} + s^2\int_0^\infty ye^{-st}$$

$$= -\left.\frac{dy}{dt}\right|_{t=0} - sy\Big|_{t=0} + s^2\mathcal{L}[y]$$

$$= s^2\mathcal{L}[y(t)] - sy(0) - \frac{dy}{dt}(0)$$

AS REQUIRED

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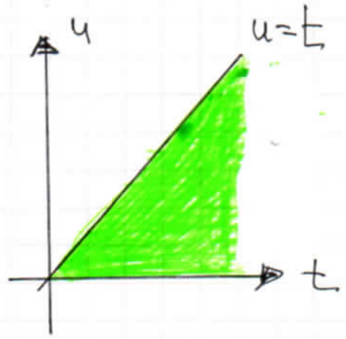
b) AGAIN STARTING BY THE DEFINITION

$$\mathcal{L}\left[\int_0^t f(t-u)\,g(u)\,du\right] = \int_{t=0}^{\infty} e^{-st}\left[\int_{u=0}^{t} f(t-u)\,g(u)\,du\right]dt$$

REVERSING THE ORDER OF INTEGRATION



FROM $u=0$ TO $u=t$
& FROM $t=0$ TO $t=\infty$

OR

FROM $t=u$ TO $t=\infty$
FROM $u=0$ TO $u=\infty$

$$\dots = \int_{u=0}^{\infty}\int_{t=u}^{t=\infty} e^{-st} f(t-u)\,g(u)\,dt\,du$$

$$= \int_{u=0}^{\infty} g(u)\left[\int_{t=u}^{t=\infty} e^{-st} f(t-u)\,dt\right]du$$

USING A SUBSTITUTION IN THE "INNER" INTEGRAL

$v = t-u \iff t = v+u$

$dv = dt$ (u IS CONSTANT IN THIS INTEGRAL)

$t=u \longmapsto v=0$

$t=\infty \longmapsto v=\infty$

$$\dots = \int_{u=0}^{\infty} g(u)\left[\int_{v=0}^{\infty} e^{-s(u+v)} f(v)\,dv\right]du$$

$$= \int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-su} g(u)\, e^{-sv} f(v)\,dv\,du$$

$$= \left[\int_{u=0}^{\infty} e^{-su} g(u)\,du\right]\left[\int_{v=0}^{\infty} e^{-sv} f(v)\,dv\right]$$

$$= \left[\int_{t=0}^{\infty} e^{-st} g(t)\,dt\right]\left[\int_{t=0}^{\infty} e^{-st} f(t)\,dt\right] = \mathcal{L}[g(t)]\,\mathcal{L}[f(t)]$$

IYGB - MATHEMATICAL METHODS 3 - PAPER B - QUESTION 10

STARTING WITH THE GENERATING FUNCTION

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right]$$

DIFFERENTIATE FIRST WITH RESPECT TO x & NEXT WITH RESPECT TO t ONLY (SEPARATELY)

$$\left.\begin{aligned} -\tfrac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2x+2t) &= \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right] \\ -\tfrac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2t) &= \sum_{n=0}^{\infty}\left[t^n P_n'(x)\right] \end{aligned}\right\} \Longrightarrow$$

$$\left.\begin{aligned} (x-t)(1-2xt+t^2)^{-\frac{3}{2}} &= \sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right] \\ t(1-2xt+t^2)^{-\frac{3}{2}} &= \sum_{n=0}^{\infty}\left[t^n P_n'(x)\right] \end{aligned}\right\} \Longrightarrow$$

$$\left.\begin{aligned} t(x-t)(1-2xt+t^2)^{-\frac{3}{2}} &= t\sum_{n=0}^{\infty}\left[nt^{n-1}P_n(x)\right] \\ (x-t)t(1-2xt+t^2)^{-\frac{3}{2}} &= (x-t)\sum_{n=0}^{\infty}\left[t^n P_n'(x)\right] \end{aligned}\right\} \Longrightarrow$$

EQUATING THE RHS OF THE ABOVE EQUATIONS

$$\Longrightarrow \sum_{n=0}^{\infty}\left[nt^n P_n(x)\right] = \sum_{n=0}^{\infty}\left[xt^n P_n'(x) - t^{n+1}P_n'(x)\right]$$

FINALLY "EQUATE POWERS OF t" IN THE ABOVE EQUATION, SAY $[t^n]$

$$\Longrightarrow nP_n(x) = xP_n'(x) - P_{n-1}'(x)$$

AS REQUIRED

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AS $\hat{g}(k) = -i \operatorname{sign} k$ IS NOT ABSOLUTELY INTEGRABLE, WE INTRODUCE A CONVERGENCE FACTOR $e^{-\varepsilon|k|}$, AND LET $\varepsilon \to 0$ AT THE END

$$\Rightarrow g(x) = \lim_{\varepsilon \to 0}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\underbrace{-i \operatorname{sign} k}_{\text{(ODD)}}\ \underbrace{e^{-\varepsilon|k|}}_{\text{(EVEN)}}\right)e^{ikx}\,dk\right]$$

ONLY THE ODD PART SURVIVES

$$\Rightarrow g(x) = \lim_{\varepsilon \to 0}\left[-\frac{2i}{\sqrt{2\pi}}\int_{0}^{\infty}\operatorname{sign} k\ e^{-\varepsilon|k|}\,(i\sin kx)\,dk\right]$$

$$\Rightarrow g(x) = \lim_{\varepsilon \to 0}\left[-\frac{2i^2}{\sqrt{2\pi}}\int_{0}^{\infty}1 \times e^{-\varepsilon k}\sin kx\,dk\right]$$

$$\Rightarrow g(x) = \lim_{\varepsilon \to 0}\left[\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}e^{-\varepsilon k}\sin kx\,dk\right]$$

CARRY THE INTEGRATION BY COMPLEX NUMBERS (OR TWICE BY PARTS)

$$\Rightarrow g(x) = \lim_{\varepsilon \to 0}\left[\sqrt{\frac{2}{\pi}}\,\mathrm{Im}\left[\int_{0}^{\infty}e^{-\varepsilon k}e^{ikx}\,dk\right]\right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\mathrm{Im}\left[\int_{0}^{\infty}e^{(-\varepsilon+ix)k}\,dk\right]\right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\mathrm{Im}\left[\frac{1}{-\varepsilon+ix}e^{(-\varepsilon+ix)k}\right]_{k=0}^{k=\infty}\right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}}\lim_{\varepsilon \to 0}\left[\mathrm{Im}\left[\frac{-\varepsilon-ix}{\varepsilon^2+x^2}e^{-\varepsilon k}(\cos kx + i\sin kx)\right]_{k=0}^{\infty}\right]$$

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$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\left[\frac{-e^{-\varepsilon k}}{\varepsilon^2 + x^2} (\varepsilon \sin kx + x \cos kx) \right]_{k=0}^{k=\infty} \right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\left[\frac{e^{-\varepsilon k}}{\varepsilon^2 + x^2} (\varepsilon \sin kx + x \cos kx) \right]_{k=\infty}^{k=0} \right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon^2 + x^2} (0 + x) - 0 \right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0} \left[\frac{x}{\varepsilon^2 + x^2} \right]$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}} \times \frac{x}{x^2}$$

$$\Rightarrow g(x) = \sqrt{\frac{2}{\pi}} \frac{1}{x}$$