IYGB

Mathematical Methods 3

Practice Paper E

Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This paper follows the most common syllabi of Mathematical Methods used in the United Kingdom Universities for Mathematics, Physics and Engineering degrees.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.

Full marks may be obtained for answers to ALL questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 10 questions in this question paper.

The total mark for this paper is 100.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear to the Examiner.

Answers without working may not gain full credit.

Non exact answers should be given to an appropriate degree of accuracy.

The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained by attempting all the questions on the paper.

Topics Examined Under Mathematical Methods 3

- Complex Variables including Residues, Laurent Series and Calculus
- Series Solutions of Differential Equations
- Gamma Functions
- Beta Functions
- Laplace Transform
- Fourier Transform
- Special Functions and Polynomials, including Bessel, Legendre, Chebyshev etc.

Question 1

$$f(z) \equiv \frac{2z+1}{z^2-z-2}, \ z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$. **(4)**

Question 2

It is given that the following integral converges

$$\int_0^1 (x \ln x)^n \ dx, \ n \in \mathbb{N}.$$

Evaluate the above integral by a suitable substitution introducing Gamma Functions **(9)**

Question 3

The generating function $g(x,t)$ for the Legendre's polynomials $P_n(x)$, satisfies

$$g(x,t) = \left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\left[t^n P_n(x)\right].$$

Use this relationship to prove that

$$\frac{\partial}{\partial x}\left[g(x,t)\right] + \frac{\partial}{\partial t}\left[g(x,t)\right] = x\left[g(x,t)\right]^3.$$ **(6)**

Question 4

Determine a Laurent series for

$$f(z)=\frac{1}{(z-1)(z+2)},$$

which is valid in ...

a) ... the annulus $1<|z-2|<4$. **(6)**

b) ... in the region for which $|z-2|>4$. **(6)**

Question 5

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$\frac{d^2y}{dx^2}-xy=0.$$

Give the final answer in simplified Sigma notation. **(14)**

Question 6

Find the following inverse Laplace transform

$$\mathcal{L}^{-1}\left[\frac{12}{s^3+8}\right].$$ **(10)**

Question 7

$$I = \int_{-1}^{1} \left(1-x^2\right)^n \, dx \, , \; n \in \mathbb{N} \, .$$

Use techniques involving the Beta function to show that

$$I = \frac{2^{2n+1}(n!)^2}{(2n+1)!} \, .$$ **(9)**

Question 8

The function f is defined by

$$f(x) = \frac{1}{x} \, , \; x \neq 0 \, .$$

a) Determine the Fourier transform of $f(x)$, assuming without proof any standard results about $\displaystyle\int_0^{\infty} \frac{\sin ax}{x} \, dx$. **(5)**

b) By introducing the converging factor $\mathrm{e}^{-\varepsilon|k|}$ and letting $\varepsilon \to 0$, invert the answer of part **(a)** to obtain f. **(8)**

Question 9

$$\int_0^{\infty} \frac{(\ln x)^2}{1+x^2} \, dx .$$

Find the value of the above improper integral, by integrating

$$f(z) = \frac{(\log z)^2}{1+z^2}, \ z \in \mathbb{C} ,$$

over a semicircular contour with a branch cut starting at the origin and oriented in some arbitrary direction in the third or fourth quadrant.

(13)

Question 10

It can be shown that for $n \in \mathbb{N}$

$$\int_{-1}^{1} \left(1-t^2\right)^{n-\frac{1}{2}} e^{ixt} \, dt = \sum_{m=0}^{\infty} \left[\frac{(-1)^m \, \Gamma\left(m+\frac{1}{2}\right) \Gamma\left(n+\frac{1}{2}\right)}{(2m)! \, \Gamma(m+n+1)} \right].$$

Use Legendre's duplication formula for the Gamma Function to show

$$J_n(x) = \frac{x^n}{2^{n-1} \sqrt{\pi} \, \Gamma\left(n+\frac{1}{2}\right)} \int_0^1 \left(1-t^2\right)^{n-\frac{1}{2}} \cos(xt) \, dt .$$

(10)
