IYGB - MP2 PAPER F - QUESTION 1

a) WRITE IN FUNCTION NOTATION

$$f(x) = x^3 + 10x - 4$$

$f(0) = -4 < 0$

$f(1) = +7 > 0$

AS $f(x)$ IS CONTINUOUS IN $(0,1)$ & CHANGES SIGN, THERE IS AT LEAST ONE ROOT IN $(0,1)$

b) "PREPARE THE NEWTON-RAPHSON ITEMS"

$f(x) = x^3 + 10x - 4$

$f'(x) = 3x^2 + 10$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^3 + 10x_n - 4}{3x_n^2 + 10}$$

STARTING WITH $x_1 = 0.5$

$$x_2 = 0.5 - \frac{0.5^3 + 10(0.5) - 4}{3(0.5)^2 + 10} \simeq 0.39534... \left(\frac{17}{43}\right)$$

REAPPLY ONCE MORE

$$x_3 = 0.3938891165...$$

$\therefore \alpha \simeq 0.3939$ 4. d.p.

IYGB - MP2 PAPER F - QUESTION 2

a) SOLVING $y=0$

$$\Rightarrow 0 = 1 - 2\cos\theta$$
$$\Rightarrow 2\cos\theta = 1$$
$$\Rightarrow \cos\theta = \tfrac{1}{2}$$
$$\theta = \begin{cases} \frac{\pi}{3} \\ \frac{5\pi}{3} \leftarrow \left(2\pi - \frac{\pi}{3}\right) \end{cases}$$

$\boxed{\text{CHECK WHICH ONE IS WHICH}}$

$$\theta = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{3} - 4\sin\frac{\pi}{3}$$
$$\Rightarrow x = -2.4690\ldots$$
$$\theta = \frac{5\pi}{3} \Rightarrow x = \frac{5\pi}{3} - 4\sin\frac{5\pi}{3}$$
$$\Rightarrow x = 8.700\ldots$$

$\therefore$ AT P $\theta = \pi/3$

AT Q $\theta = 5\pi/3$

b) SETTING UP AN AREA INTEGRAL IN PARAMETRIC

$$\Rightarrow \text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta$$

$$\Rightarrow \text{AREA} = \int_{\frac{\pi}{3}\,(\text{AT P})}^{\frac{5\pi}{3}\,(\text{AT Q})} \underbrace{(1-2\cos\theta)}_{y(\theta)}\underbrace{(1-4\cos\theta)}_{dx/d\theta}\,d\theta$$

$$\Rightarrow \text{AREA} = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1 - 4\cos\theta - 2\cos\theta + 8\cos^2\theta\,d\theta$$

$$\Rightarrow \text{AREA} = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1 - 6\cos\theta + 8\cos^2\theta\,d\theta$$

AS REQUIRED

c) INTEGRATING USING $\cos2\theta \equiv 2\cos^2\theta - 1 \Rightarrow \cos^2\theta \equiv \frac{1}{2} + \frac{1}{2}\cos2\theta$

$$\text{AREA} = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 1 - 6\cos\theta + 8\left(\tfrac{1}{2} + \tfrac{1}{2}\cos2\theta\right)d\theta = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} 5 - 6\cos\theta + 4\cos2\theta\,d\theta$$

$$= \Big[5\theta - 6\sin\theta + 2\sin2\theta\Big]_{\frac{\pi}{3}}^{\frac{5\pi}{3}} = \left(\tfrac{25}{3}\pi + 3\sqrt{3} - \sqrt{3}\right) - \left(\tfrac{5\pi}{3} - 3\sqrt{3} + \sqrt{3}\right)$$

$$= \frac{20\pi}{3} + 4\sqrt{3}$$

IYGB - MP2 PAPER F - QUESTION 3

REARRANGE INTO A "COMPOUND" ANGLE IDENTITY

$$\Longrightarrow \sin\theta\cos\frac{\pi}{5} = \frac{1}{2} - \cos\theta\sin\frac{\pi}{5}$$

$$\Longrightarrow \sin\theta\cos\frac{\pi}{5} + \cos\theta\sin\frac{\pi}{5} = \frac{1}{2}$$

$$\Longrightarrow \sin\left(\theta+\frac{\pi}{5}\right) = \frac{1}{2}$$

$$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\left(\begin{array}{l} \theta+\frac{\pi}{5} = \frac{\pi}{6} \pm 2n\pi \\ \theta+\frac{\pi}{5} = \frac{5\pi}{6} \pm 2n\pi \end{array}\right. \qquad n = 0,1,2,3,\ldots$$

$$\left(\begin{array}{l} \theta = -\frac{\pi}{30} \pm 2n\pi \\ \theta = \frac{19}{30}\pi \pm 2n\pi \end{array}\right.$$

AND FOR $2\pi < \theta < 4\pi$

$$\theta_1 = \frac{119}{30}\pi$$

$$\theta_2 = \frac{79}{30}\pi$$

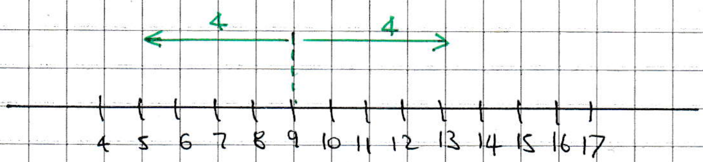
IYGB - MP2 PAPER F - QUESTION 4

a) IF $|y| = 2$ THEN $y = \pm 2$

$$\Rightarrow \quad |3y-1| = \begin{cases} |3\times 2 - 1| = |5| = 5 \\ |3(-2)-1| = |-7| = 7 \end{cases}$$

$\therefore$ EITHER 5 OR 7

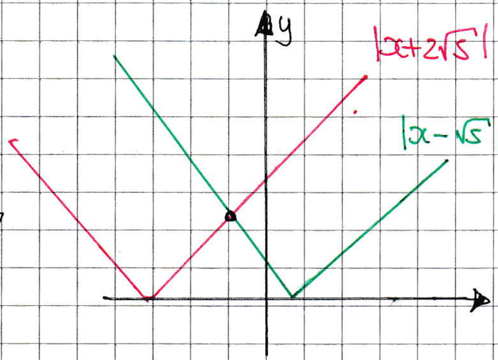
b) IF $5 \leq t \leq 13$, THEN THE DIFFERENCE OF t FROM THE MIDPOINT OF 5 & 13 MUST BE LESS THAN $\frac{13-5}{2} = 4$ (SEE PICTURE)



I.E THE DIFFERENCE (DISTANCE OF t) FROM 9 MUST BE LESS THAN 4

$\therefore \quad |t-9| \leq 4$

c) SOLVING THE EQUATION



$$\Rightarrow |x-\sqrt{2}| = |x+5\sqrt{2}|$$

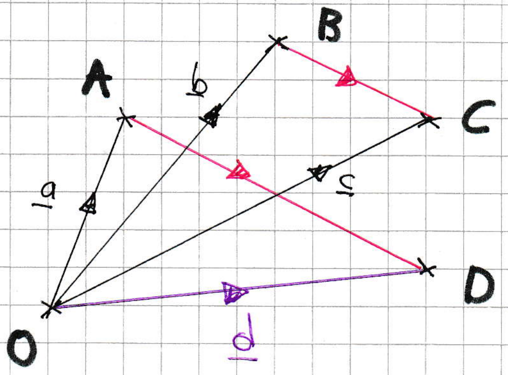
$$\Rightarrow \begin{cases} x-\sqrt{2} = x+5\sqrt{2} & \leftarrow \text{INCONSISTENT} \\ x-\sqrt{2} = -x-5\sqrt{2} & \leftarrow \text{INTERSECTION} \end{cases}$$

$$\Rightarrow 2x = -4\sqrt{2}$$

$$\Rightarrow x = -2\sqrt{2}$$

IYGB - MP2 PAPER F - QUESTION 5

START WITH A DIAGRAM



$A(5,-1,0)$
$B(3,5,-4)$
$C(12,2,8)$
$\overrightarrow{AD} = 2\overrightarrow{BC}$

FORMING A VECTOR EQUATION

$$\Rightarrow \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$$
$$\Rightarrow \overrightarrow{OD} = \overrightarrow{OA} + 2(\overrightarrow{BC})$$
$$\Rightarrow \underline{d} = \underline{a} + 2(\underline{c} - \underline{b})$$
$$\Rightarrow \underline{d} = \underline{a} + 2\underline{c} - 2\underline{b}$$
$$\Rightarrow \underline{d} = (5,-1,0) + 2(12,2,8) - 2(3,5,-4)$$
$$\Rightarrow \underline{d} = (23,-7,24)$$

FINALLY THE DISTANCE CD CAN BE FOUND

$$\Rightarrow |\overrightarrow{CD}| = |\underline{d} - \underline{c}|$$
$$= |(23,-7,24) - (12,2,8)|$$
$$= |11,-9,16|$$
$$= \sqrt{121+81+256}$$
$$= \sqrt{458}$$
$$\simeq 21.40$$

IYGB - MP2 PAPER F - QUESTION 6

PROCEED ACCORDING TO THE HINT

$$\Longrightarrow y = \arctan\left(\tfrac{1}{2}x\right)$$

$$\Longrightarrow \tan y = \tan\left(\arctan\left(\tfrac{1}{2}x\right)\right)$$

$$\Longrightarrow \tan y = \tfrac{1}{2}x$$

$$\Longrightarrow x = 2\tan y$$

$$\Longrightarrow \frac{dx}{dy} = 2\sec^2 y$$

$$\Longrightarrow \frac{dx}{dy} = 2\left(1+\tan^2 y\right)$$

$$\left\{1+\tan^2\theta \equiv \sec^2\theta\right\}$$

$$\Longrightarrow \frac{dx}{dy} = 2 + 2\tan^2 y$$

BY THE INVERSE RULE

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{2+2\tan^2 y}$$

FINALLY WE HAVE

$$\tan y = \tfrac{1}{2}x$$
$$\tan^2 y = \tfrac{1}{4}x^2$$
$$2\tan^2 y = \tfrac{1}{2}x^2$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1}{2+\frac{1}{2}x^2}$$

MULTIPLY TOP & BOTTOM OF THE FRACTION BY 2

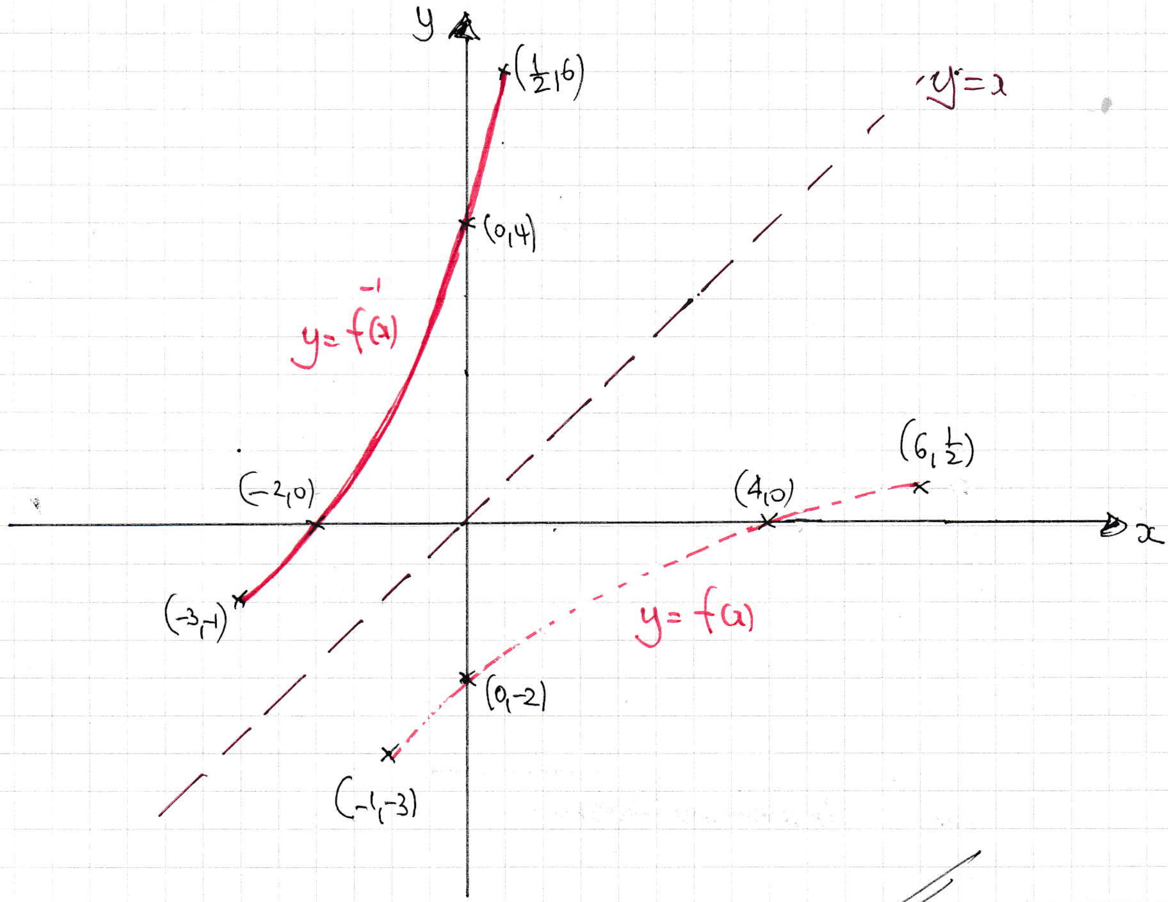
$$\Longrightarrow \frac{dy}{dx} = \frac{2}{4+x^2}$$

AS REQUIRED

IYGB – MP2 PAPER F – QUESTION 7

FOR f & f^{-1} THAT ARE REFLECTIONS OF ONE ANOTHER ABOUT $y=x$,

CO-ORDINATES TRANSPOSE & DOMAINS SWAP WITH RANGES



IYGB – MP2 PAPER F – QUESTION 8

DIFFERENTIATE & MANIPULATE

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{0 + 2\cos\theta}{0 - 2\sin\theta} = \frac{2\cos\theta}{-2\sin\theta}$$

NOW LOOKING AT THE PARAMETRICS

$$2\cos\theta = x - 3 \quad \& \quad -2\sin\theta = -3 - y$$

THUS WE HAVE AT THE END

$$\frac{dy}{dx} = \frac{x-3}{-3-y} = \frac{x-3}{-(3+y)} = \frac{-(3-x)}{-(3+y)} = \frac{3-x}{3+y}$$

// AS REQUIRED

IYGB - MP2 PAPER F - QUESTION 9

a) USING STANDARD FORMULAE

$\Longrightarrow S_\infty = 4 \times u_2$

$\Longrightarrow \frac{a}{1-r} = 4 \times ar^1$

$\Longrightarrow \frac{\cancel{a}}{1-r} = 4\cancel{a}r$

$\Longrightarrow \frac{1}{1-r} = 4r$

$\Longrightarrow 1 = 4r - 4r^2$

$\Longrightarrow 4r^2 - 4r + 1 = 0$

$\Longrightarrow (2r-1) = 0$

$\Longrightarrow r = \frac{1}{2}$

b) USING $S_4 = \frac{a(1-r^4)}{1-r}$

$\Longrightarrow S_4 = 5760$

$\Longrightarrow \frac{a(1-0.5^4)}{1-0.5} = 5760$

$\Longrightarrow a(1-\frac{1}{16}) = 5760 \times \frac{1}{2}$

$\Longrightarrow a \times \frac{15}{16} = 2880$

$\Longrightarrow 15a = 46080$

$\Longrightarrow a = 3072$

c) FINALLY WE HAVE NOTING $S_\infty = 4u_2 = 4ar = 4 \times 3072 \times 0.5 = 6144$

$S_k = S_\infty - 3$

$\frac{3072(1-0.5^k)}{1-0.5} = 6144 - 3$

$3072(1-0.5^k) = 6141 \times 0.5$

$1 - 0.5^k = \frac{2047}{2048}$

$\frac{1}{2048} = \left(\frac{1}{2}\right)^k$

BY INSPECTION OR LOGS

$k = 11$

$\log\left(\frac{1}{2048}\right) = \log\left(\frac{1}{2}\right)^k$

$k\log\left(\frac{1}{2}\right) = \log\left(\frac{1}{2048}\right)$

$k = \frac{\log\left(\frac{1}{2048}\right)}{\log\left(\frac{1}{2}\right)}$

$k = 11$

IYGB - MP2 PAPER F - QUESTION 10

a) FORMING A TABLE OF VALUES - NOTE THAT 5 ORDINATES IS 4 STRIPS

x	1	1.25	1.5	1.75	2
$e^{\frac{1}{10}x^2}$	1.1052	1.1691	1.2523	1.3583	1.4918

USING THE TRAPEZIUM RULE

$$\int_1^2 e^{\frac{1}{10}x^2}\,dx \approx \text{“}\frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2\times\text{REST}\right]\text{”}$$

$$\approx \frac{0.25}{2}\left[1.1052 + 1.4918 + 2\left(1.1691 + 1.2523 + 1.3583\right)\right]$$

$$\approx \underline{1.270}$$

b) PROCEED AS FOLLOWS

$$\int_1^2 e^{1+\frac{1}{10}x^2}\,dx = \int_1^2 e^1 \times e^{\frac{1}{10}x^2}\,dx = e\int_1^2 e^{\frac{1}{10}x^2}\,dx$$

$$\approx e \times 1.270\ldots$$

$$\approx \underline{3.45}$$

IYGB - MP2 PAPER F - QUESTION 11

USING THE SUBSTITUTION GIVEN

$$\Rightarrow u = 1 + x^2 e^{-3x}$$

$$\Rightarrow \frac{du}{dx} = 2xe^{-3x} + x^2(-3e^{-3x})$$

$$\Rightarrow \frac{du}{dx} = 2xe^{-3x} - 3x^2 e^{-3x}$$

$$\Rightarrow \frac{du}{dx} = xe^{-3x}(2-3x)$$

$$\Rightarrow dx = \frac{du}{x(2-3x)e^{-3x}}$$

TRANSFORMING THE INTEGRAL

$$\int \frac{x(2-3x)}{e^{3x}+x^2}\,dx = \int \frac{\cancel{x(2-3x)}}{e^{3x}+x^2}\left(\frac{du}{\cancel{x(2-3x)}e^{-3x}}\right)$$

$$= \int \frac{1}{e^{3x}+x^2} \times \frac{1}{e^{-3x}}\,du$$

$$= \int \frac{1}{1+x^2e^{-3x}}\,du$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln\left(1+x^2e^{-3x}\right) + C$$

IYGB - MP2 PAPER F - QUESTION 12

a) EXPANDING BINOMIALLY UP TO x^2

$$\Rightarrow f(x) = \sqrt{1-x} = (1-x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}(-x)^1 + \frac{\frac{1}{2}(-\frac{1}{2})}{1\times 2}(-x)^2 + \dots$$

$$\Rightarrow f(x) = \sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$$

b) LET $x = (4y - y^2)$ — "CAREFUL WITH THE MINUS ON y^2"

$$\Rightarrow -\frac{1}{2}x = -\frac{1}{2}(4y - y^2) = -2y + \frac{1}{2}y^2$$

$$\Rightarrow -\frac{1}{8}x^2 = -\frac{1}{8}(4y-y^2)^2 = -\frac{1}{8}(16y^2 - 8y^3 + y^4) = -2y^2 + y^3 - \frac{1}{8}y^4$$

HENCE WE HAVE

$$\Rightarrow \sqrt{1-4y+y^2} = \sqrt{1-(4y-y^2)}$$

$$= 1 - 2y + \frac{1}{2}y^2 - 2y^2 + y^3 - \frac{1}{8}y^4 + \dots$$

$$= 1 - 2y - \frac{3}{2}y^2 + O(y^3)$$

IYGB - MP2 PAPER F - QUESTION 13

a) FORMING A DIFFERENTIAL EQUATION

$$\Longrightarrow \frac{dV}{dt} = -kh$$

$$\Longrightarrow \frac{dV}{dh} \times \frac{dh}{dt} = -kh$$

$$\Longrightarrow \left(\frac{1}{16}\pi h^2\right)\frac{dh}{dt} = -kh$$

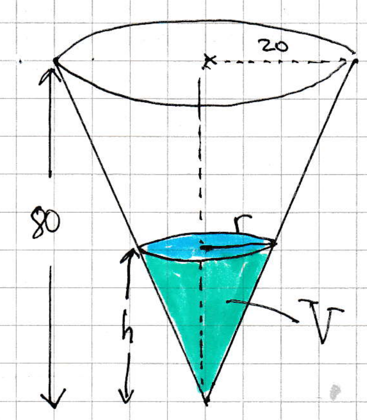
$$\Longrightarrow \pi h^2 \frac{dh}{dt} = -16kh$$

$$\Longrightarrow \frac{dh}{dt} = -\frac{16kh}{\pi h^2}$$

$$\Longrightarrow \frac{dh}{dt} = -\left(\frac{16k}{\pi}\right)\frac{1}{h}$$

$$\Longrightarrow \frac{dh}{dt} = -\frac{A}{h}$$

// AS REQUIRED



- BY SIMILAR SHAPES

$$\frac{r}{h} = \frac{20}{80}$$

$$r = \frac{1}{4}h$$

- VOLUME OF WATER, V

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h$$

$$V = \frac{1}{48}\pi h^3$$

$$\frac{dV}{dh} = \frac{1}{16}\pi h^2$$

b) SOLVE BY SEPARATION OF VARIABLES

$$\Longrightarrow h\,dh = -A\,dt$$

$$\Longrightarrow \int h\,dh = \int -A\,dt$$

$$\Longrightarrow \frac{1}{2}h^2 = -At + B$$

APPLY THE INITIAL CONDITION, IMPLICITLY GIVEN AS $t=0 \quad h=80$

$$\rightarrow \frac{1}{2}\times 80^2 = -A\times 0 + B$$

$$\Longrightarrow B = 3200$$

$$\Longrightarrow \boxed{\frac{1}{2}h^2 = 3200 - At}$$

IYGB − MP2 PAPER F − QUESTION 13

APPLY THE BOUNDARY CONDITION, $t=1 \quad h=78$

$$\Rightarrow \frac{1}{2} \times 78^2 = -A \times 1 + 3200$$
$$\Rightarrow 3042 = -A + 3200$$
$$\Rightarrow A = 158$$

$$\therefore \boxed{\frac{1}{2}h^2 = 3200 - 158t}$$

FINALLY FOR "TOTAL LEAKAGE", $h=0$

$$\Rightarrow 0 = 3200 - 158t$$
$$\Rightarrow 158t = 3200$$
$$\Rightarrow t = \frac{1600}{79} \approx 20.25....$$
$$\approx 20\frac{1}{4} \text{ MINUTES}$$