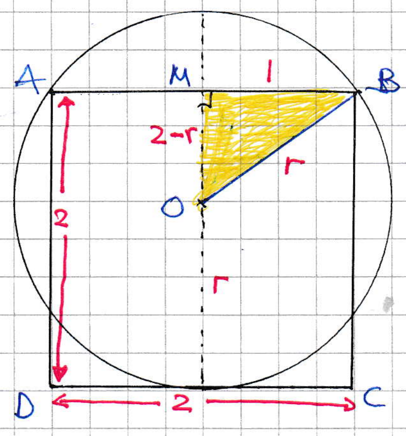
IYGB - MP2 PAPER S - QUESTION 1

LOOKING AT THE DIAGRAM OPPOSITE

BY PYTHAGORAS ON $\hat{MOB}$



$$|MO|^2 + |MB|^2 = |OB|^2$$

$$(2-r)^2 + 1^2 = r^2$$

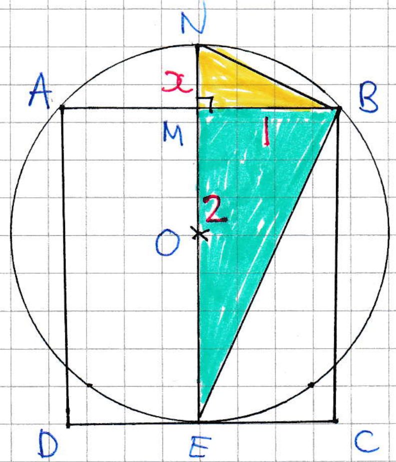
$$4 - 4r + \cancel{r^2} + 1 = \cancel{r^2}$$

$$5 = 4r$$

$$r = \frac{5}{4} = 1.25$$

ALTERNATIVE

BY SIMILAR TRIANGLES



$$\frac{|MB|}{|ME|} = \frac{|MN|}{|MB|}$$

$$\frac{1}{2} = \frac{x}{1}$$

$$x = \frac{1}{2}$$

& THE RADIUS IS GIVEN BY

$$r = \frac{1}{2}|EN| = \frac{1}{2}(x+2)$$

$$= \frac{1}{2}\left(\frac{1}{2}+2\right)$$

$$= \frac{1}{2} \times \frac{5}{2}$$

$$= \frac{5}{4}$$

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PROOF BY CONTRADICTION

SUPPOSE THAT $\log_{10}5$ IS RATIONAL

$$\Rightarrow \log_{10}5 = \frac{a}{b}$$ WHERE a & b ARE POSITIVE INTEGERS, WITHOUT COMMON FACTORS

$$\Rightarrow 10^{\log_{10}5} = 10^{\frac{a}{b}}$$

$$\Rightarrow 5 = 10^{\frac{a}{b}}$$

$$\Rightarrow (5)^b = \left(10^{\frac{a}{b}}\right)^b$$

$$\Rightarrow 5^b = 10^a$$

BUT THIS IS A CONTRADICTION AS POWERS OF 5 ARE ODD (5, 25, 125, 625, 3125...) AND THE POWERS OF 10 ARE EVEN (10, 100, 1000, 10000, ...)

$\Rightarrow$ THE ASSERTION THAT $\log_{10}5$ IS RATIONAL IS FALSE

$\Rightarrow$ $\log_{10}5$ IS IRRATIONAL

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- START BY DIRECT DIFFERENTIATION

$$y = \ln(1+\cos x)$$

$$\frac{dy}{dx} = \frac{-\sin x}{1+\cos x} = -\frac{\sin x}{1+\cos x}$$

$$\frac{d^2y}{dx^2} = -\frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= -\frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = -\frac{1+\cos x}{(1+\cos x)} = -\frac{1}{1+\cos x}$$

- NOW PROCEED AS FOLLOWS

$$\Rightarrow y = \ln(1+\cos x) \quad \text{AND} \quad \frac{d^2y}{dx^2} = -\frac{1}{1+\cos x}$$

$$\Rightarrow e^y = 1+\cos x$$

$$\Rightarrow e^{-y} = \frac{1}{1+\cos x}$$

$$\Rightarrow -e^{-y} = -\frac{1}{1+\cos x}$$

$$\Rightarrow -e^{-y} = \frac{d^2y}{dx^2}$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} + e^{-y} = 0}$$

- DIFFERENTIATE THE ABOVE EXPRESSION WITH RESPECT TO x GIVES

$$\Rightarrow \frac{d^3y}{dx^3} - e^{-y}\frac{dy}{dx} = 0$$

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- FINALLY, ONE MORE DIFFERENTIATION W.R.T x AND TIDY

$$\Rightarrow \frac{d^4y}{dx^4} - \frac{d}{dx}\left[e^{-y}\frac{dy}{dx}\right] = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} - \left[-e^{-y}\frac{dy}{dx} \times \frac{dy}{dx} + e^{-y}\frac{d^2y}{dx^2}\right] = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} + e^{-y}\left(\frac{dy}{dx}\right)^2 - e^{-y}\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} + e^{-y}\left(\frac{dy}{dx}\right)^2 - e^{-y}\left(-e^{-y}\right) = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} + e^{-y}\left(\frac{dy}{dx}\right)^2 + e^{-2y} = 0$$

// AS REQUIRED

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WORK AS FOLLOWS

$x = t^2 - p^2$ $\qquad\qquad$ $y = 2tp$

$\qquad\qquad\qquad\qquad$ $y^2 = 4t^2p^2$

$p^2 = 2t^2 - 1$

$x = t^2 - (2t^2 - 1)$ $\qquad\qquad$ $y^2 = 4t^2(2t^2 - 1)$

$x = 1 - t^2$ $\qquad\qquad$ $y^2 = 4t^2(2t^2 - 1)$

FINALLY BY SUBSTITUTION $t^2 = 1 - x$

$$y^2 = 4(1-x)[2(1-x) - 1]$$

$$y^2 = 4(1-x)(2-2x-1)$$

$$y^2 = 4(1-x)(1-2x)$$

$$y^2 = 4(-1)(x-1)(-1)(2x-1)$$

$$y^2 = 4(x-1)(2x-1)$$

AS REQUIRED

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$$f(x) \equiv \frac{e^{\sin x \cos x}+1}{e^{\sin x \cos x}-1} \qquad x \in \mathbb{R}$$

LET US FIRST NOTE THAT

$$\sin(-x) \equiv -\sin x$$
$$\cos(-x) \equiv \cos x$$

THUS WE NOW HAVE

$$f(-x) = \frac{e^{\sin(-x)\cos(-x)}+1}{e^{\sin(-x)\cos(-x)}-1} = \frac{e^{-\sin x \cos x}+1}{e^{-\sin x \cos x}-1}$$

$$= \frac{e^{-\sin x \cos x} \times e^{\sin x \cos x} + 1 \times e^{\sin x \cos x}}{e^{-\sin x \cos x} \times e^{\sin x \cos x} - 1 \times e^{\sin x \cos x}}$$

$$= \frac{e^{0}+e^{\sin x \cos x}}{e^{0}-e^{\sin x \cos x}} = \frac{1+e^{\sin x \cos x}}{1-e^{\sin x \cos x}}$$

$$= \frac{1+e^{\sin x \cos x}}{-(e^{\sin x \cos x}-1)} = -\frac{e^{\sin x \cos x}+1}{e^{\sin x \cos x}-1}$$

$$= -f(x)$$

AS $f(-x) = -f(x)$, f IS ODD

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PROCEED AS FOLLOWS

$$u_{n+2} = u_{n+1} + u_n$$

$$\frac{u_{n+2}}{u_{n+1}} = \frac{u_{n+1}}{u_{n+1}} + \frac{u_n}{u_{n+1}}$$

AS ALL THE TERMS ARE POSITIVE WE MAY DIVIDE THROUGH BY u_{n+1}

NOW WE ARE GIVEN THAT AS $r \to \infty$

$$\frac{u_{r+1}}{u_r} = \phi \implies \frac{u_{n+2}}{u_{n+1}} \to \phi \text{ AS } n \to \infty$$

$$\frac{u_n}{u_{n+1}} \to \frac{1}{\phi} \text{ AS } n \to \infty$$

HENCE WE OBTAIN

$$\implies \phi = 1 + \frac{1}{\phi}$$

$$\implies \phi^2 = \phi + 1$$

$$\implies \phi^2 - \phi - 1 = 0$$

$$\implies \left(\phi - \tfrac{1}{2}\right)^2 - \frac{5}{4} = 0$$

$$\implies \left(\phi - \tfrac{1}{2}\right)^2 = \frac{5}{4}$$

$$\implies \phi - \frac{1}{2} = \frac{\pm\sqrt{5}}{2}$$

$$\implies \phi = \frac{1 \pm \sqrt{5}}{2}$$

$$\implies \phi = \begin{cases} \frac{1}{2}(1+\sqrt{5}) \\ \cancel{\frac{1}{2}(1-\sqrt{5})} < 0 \end{cases}$$

AND RATIO OF POSITIVE TERMS CANNOT BE NEGATIVE

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- DEFINE SOME VARIABLES IN ORDER TO FORM SOME EQUATIONS

a = 1ST TERM OF A.P.
d = COMMON DIFFERENCE OF A.P.
A = 1ST TERM OF G.P.
r = COMMON RATIO OF G.P.

NTH TERM OF A.P
$u_n = a + (n-1)d$

NTH TERM OF A G.P.
$u_n = ar^{n-1}$

- HENCE WE NOW HAVE

$A = a + 6d$ — I
$Ar = a + 3d$ — II
$Ar^2 = a + d$ — III

- SOLVE III FOR d: $d = Ar^2 - a$

- SUBSTITUTE INTO I & II

$$\left.\begin{aligned} A &= a + 6Ar^2 - 6a \\ Ar &= a + 3Ar^2 - 3a \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} 5a &= 6Ar^2 - A \\ 2a &= 3Ar^2 - Ar \end{aligned}\right\} \Rightarrow$$

- DIVIDE EQUATIONS

$$\Rightarrow \frac{5a}{2a} = \frac{6Ar^2 - A}{3Ar^2 - Ar}$$

$$\Rightarrow \frac{5}{2} = \frac{6r^2 - 1}{3r^2 - r}$$

$$\Rightarrow 15r^2 - 5r = 12r^2 - 2$$

$$\Rightarrow 3r^2 - 5r + 2 = 0$$

$$\Rightarrow (3r-2)(r-1) = 0$$

$$\Rightarrow r = \cancel{1} \quad (r \neq \pm 1, 0)$$

$$r = \frac{2}{3}$$

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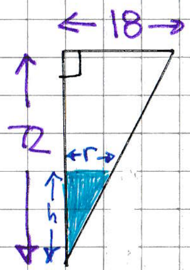
START BY CONNECTING DERIVATIVES

$$\Rightarrow \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt},$$

WHERE h IS THE HEIGHT OF THE WATER IN THE CONE AND V ITS VOLUME

$$\Rightarrow \frac{dh}{dt} = \frac{dV}{dh} \times k$$

WE NEED $V = f(h)$ TO FIND $\frac{dV}{dh}$



$$\frac{r}{h} = \frac{18}{72}$$

$$r = \frac{1}{4}h$$

$$\Rightarrow V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h$$

$$\Rightarrow V = \frac{1}{48}\pi h^3$$

$$\Rightarrow \frac{dV}{dh} = \frac{1}{16}\pi h^2$$

$$\boxed{\frac{dh}{dV} = \frac{16}{\pi h^2}}$$

RETURNING TO THE "MAIN" LINE

$$\Rightarrow \frac{dh}{dt} = \frac{16}{\pi h^2} \times k$$

$$\Rightarrow \frac{dh}{dt} = \frac{16k}{\pi h^2}$$

NEXT CONVERT THE TIME INTO HEIGHT h

... CONSTANT RATE OF k cm^3 PER SECOND ...

... $12\frac{1}{2}$ MINUTES = 750 SECONDS

... $V = 750k \text{ cm}^3$

... $\frac{1}{48}\pi h^3 = 750k$

... $\pi h^3 = 36000k$

FINALLY WE ARE GIVEN THAT $\left.\frac{dh}{dV}\right|_{t=60} = \frac{2}{75}$

... $\frac{2}{75} = \frac{16k}{\pi h^2}$

... $\pi h^2 = 600k$

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SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l}\pi h^3 = 36000k \\ \pi h^2 = 600k\end{array}\right) \Longrightarrow \text{DIVIDING} \quad h = 60$$

$$\Longrightarrow \pi h^2 = 600k$$

$$\Longrightarrow \pi \times 60^2 = 600k$$

$$\Longrightarrow \pi \times 60 = 10k$$

$$\Longrightarrow k = 6\pi$$

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SOLVE THE O.D.E. BY SEPARATING VARIABLES

$$\Rightarrow \frac{dy}{dx} = \frac{2xy(y+1)}{\sin^2\left(x+\frac{\pi}{6}\right)}$$

$$\Rightarrow \frac{1}{y(y+1)}\,dy = \frac{2x}{\sin^2\left(x+\frac{\pi}{6}\right)}\,dx.$$

$$\Rightarrow \int \frac{1}{y(y+1)}\,dy = \int 2x\,\mathrm{cosec}^2\left(x+\frac{\pi}{6}\right)\,dx$$

THE L.H.S REQUIRES PARTIAL FRACTIONS (BY INSPECTION) AND THE R.H.S INTEGRATION BY PARTS

$2x$	2
$-\cot\left(x+\frac{\pi}{6}\right)$	$\mathrm{cosec}^2\left(x+\frac{\pi}{6}\right)$

$$\Longrightarrow \int \frac{1}{y} - \frac{1}{y+1}\,dy = -2x\cot\left(x+\frac{\pi}{6}\right) - \int -2\cot\left(x+\frac{\pi}{6}\right)dx$$

$$\Longrightarrow \ln|y| - \ln|y+1| = -2x\cot\left(x+\frac{\pi}{6}\right) + \int 2\cot\left(x+\frac{\pi}{6}\right)\,dx$$

$$\Longrightarrow \ln|y| - \ln|y+1| = -2x\cot\left(x+\frac{\pi}{6}\right) + 2\ln\left|\sin\left(x+\frac{\pi}{6}\right)\right| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

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APPLY CONDITION $x=0,\ y=1$

$$\cancel{\ln 1} - \ln 2 = 0 + 2\ln\left(\sin\frac{\pi}{6}\right) + C$$

$$-\ln 2 = 2\ln\tfrac{1}{2} + C$$

$$-\ln 2 = -2\ln 2 + C$$

$$C = \ln 2$$

THUS WE NOW HAVE

$$\ln|y| - \ln|y+1| = \ln 2 - 2x\cot\left(x+\frac{\pi}{6}\right) + 2\ln\left|\sin\left(x+\frac{\pi}{6}\right)\right|$$

WHEN $x=\frac{\pi}{12}$

$$\Longrightarrow \ln\left|\frac{y}{y+1}\right| = \ln 2 - 2\times\left(\frac{\pi}{12}\right)\cot\frac{\pi}{4} + 2\ln\left(\sin\frac{\pi}{4}\right)$$

$$\Longrightarrow \ln\left|\frac{y}{y+1}\right| = \ln 2 - \frac{\pi}{6} + 2\ln\left(\frac{1}{\sqrt{2}}\right)$$

$$\Longrightarrow \ln\left|\frac{y}{y+1}\right| = \ln 2 - \frac{\pi}{6} + 2\ln 2^{-\frac{1}{2}}$$

$$\Longrightarrow \ln\left|\frac{y}{y+1}\right| = \cancel{\ln 2} - \frac{\pi}{6} - \cancel{\ln 2}$$

$$\Longrightarrow \frac{y}{y+1} = e^{-\frac{\pi}{6}}$$

$$\Longrightarrow \frac{y+1}{y} = e^{\frac{\pi}{6}}$$

$$\Longrightarrow 1 + \frac{1}{y} = e^{\frac{\pi}{6}}$$

$$\Longrightarrow \frac{1}{y} = e^{\frac{\pi}{6}} - 1$$

$$\Longrightarrow y = \frac{1}{e^{\frac{\pi}{6}} - 1}$$

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MANIPULATE THE SERIES STEP BY STEP

$$\Rightarrow S = 1 + \frac{2}{4} + \frac{2\times3}{4\times8} + \frac{2\times3\times4}{4\times8\times12} + \frac{2\times3\times4\times5}{4\times8\times12\times16} + \dots$$

$$\Rightarrow S = 1 + \frac{2}{4(1)} + \frac{2\times3}{4^2(1\times2)} + \frac{2\times3\times4}{4^3(1\times2\times3)} + \frac{2\times3\times4\times5}{4^4(1\times2\times3\times4)} + \dots$$

$$\Rightarrow S = 1 + \frac{2}{1!}\left(\frac{1}{4}\right) + \frac{2\times3}{2!}\left(\frac{1}{4}\right)^2 + \frac{2\times3\times4}{3!}\left(\frac{1}{4}\right)^3 + \frac{2\times3\times4\times5}{4!}\left(\frac{1}{4}\right)^4 + \dots$$

FINALLY WE NEED TO "TAKE CARE" OF THE SIGNS, IN ORDER TO FORM A CONVERGENT BINOMIAL EXPANSION

$$\Rightarrow S = 1 + \frac{-2}{1!}\left(-\frac{1}{4}\right)^1 + \frac{(-2)(-3)}{2!}\left(-\frac{1}{4}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(-\frac{1}{4}\right)^3 + \frac{(-2)(-3)(-4)(-5)}{4!}\left(-\frac{1}{4}\right)^4 + \dots$$

$$\Rightarrow S = \left(1 - \frac{1}{4}\right)^{-2}$$

$$\Rightarrow S = \left(\frac{3}{4}\right)^{-2}$$

$$\Rightarrow S = \left(\frac{9}{16}\right)^{-1}$$

$$\Rightarrow S = \frac{16}{9}$$

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- START BY REARRANGING THE SECOND EQUATION FOR x

$$\Rightarrow 4y - 2x = \pi$$
$$\Rightarrow 4y - \pi = 2x$$
$$\Rightarrow x = 2y - \frac{\pi}{2}$$

- SUBSTITUTE INTO THE OTHER

$$\Rightarrow 4\cos y = 3 - 2\sin x$$
$$\Rightarrow 4\cos y = 3 - 2\sin\left(2y - \frac{\pi}{2}\right)$$
$$\Rightarrow 4\cos y = 3 + 2\sin\left(\frac{\pi}{2} - 2y\right)$$
$$\Rightarrow 4\cos y = 3 + 2\cos(2y)$$
$$\Rightarrow 4\cos y = 3 + 2\left[2\cos^2 y - 1\right]$$
$$\Rightarrow 4\cos y = 3 + 4\cos^2 y - 2$$
$$\Rightarrow 0 = 4\cos^2 y - 4\cos y + 1$$
$$\Rightarrow (2\cos y - 1)^2 = 0$$
$$\Rightarrow \cos y = \frac{1}{2}$$

- HENCE WE HAVE

$$y = \dots -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \dots$$

$$x = \dots -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \dots$$

$\therefore$ ONLY SOLUTION IS $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ //

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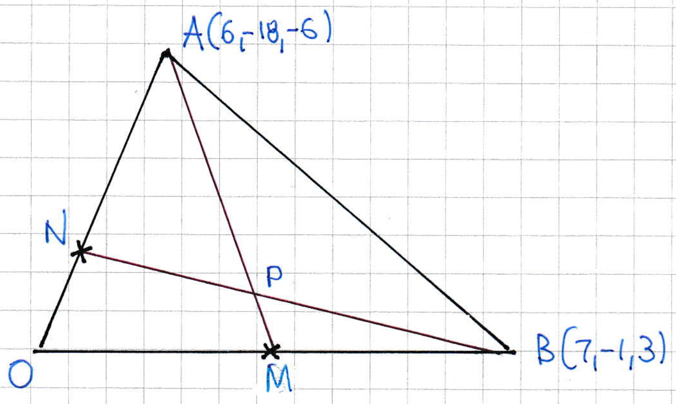
STARTING WITH A DIAGRAM



- BY INSPECTION

$N(2,-6,-2)$

$M\left(\frac{7}{2},-\frac{1}{2},\frac{3}{2}\right)$

WORK AS FOLLOWS

$\overrightarrow{NP} = k\overrightarrow{NB}$, $0 < k < 1$

$\overrightarrow{NP} = k(\underline{b} - \underline{n}) = k[(7,-1,3)-(2,-6,-2)] = k(5,5,5)$

$$\overrightarrow{NP} = (5k,5k,5k)$$

NEXT WE WORK AN EXPRESSION FOR $\overrightarrow{MP}$

$\overrightarrow{MP} = \overrightarrow{MO} + \overrightarrow{ON} + \overrightarrow{NP}$

$\overrightarrow{MP} = -\underline{m} + \underline{n} + (5k,5k,5k)$

$\overrightarrow{MP} = -\left(\frac{7}{2},-\frac{1}{2},\frac{3}{2}\right) + (2,-6,-2) + (5k,5k,5k)$

$\overrightarrow{MP} = \left(5k-\frac{3}{2}, 5k-\frac{11}{2}, 5k-\frac{7}{2}\right)$

NEXT A SIMILAR EXPRESSION FOR $\overrightarrow{PA}$

$\overrightarrow{PA} = \overrightarrow{PN} + \overrightarrow{NA}$

$\overrightarrow{PA} = -\overrightarrow{NP} + 2\overrightarrow{ON}$

$\overrightarrow{PA} = (-5k,-5k,-5k) + (4,-12,-4)$

$\overrightarrow{PA} = (-5k+4,-5k-12,-5k-4)$

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BUT P, M & A ARE COLLINEAR

$\Rightarrow \overrightarrow{MP} = \lambda \overrightarrow{MA}$ FOR SOME SCALAR λ

$\Rightarrow \left(5k-\frac{3}{2}, 5k-\frac{11}{2}, 5k-\frac{7}{2}\right) = \lambda(-5k+4, -5k-12, -5k-4)$

EQUATE ANY TWO COMPONENTS

$$\left.\begin{array}{l} 5k-\frac{3}{2} = -5\lambda k + 4\lambda \\ 5k-\frac{11}{2} = -5\lambda k - 12\lambda \end{array}\right\} \Rightarrow \left.\begin{array}{l} 5k+5\lambda k = 4\lambda + \frac{3}{2} \\ 5k+5\lambda k = \frac{11}{2} - 12\lambda \end{array}\right\} \Rightarrow$$

$\Rightarrow 4\lambda + \frac{3}{2} = \frac{11}{2} - 12\lambda$

$\Rightarrow 16\lambda = 4$

$\Rightarrow \lambda = \frac{1}{4}$

USING $5k - \frac{3}{2} = -5\lambda k + 4\lambda$

$5k - \frac{3}{2} = -\frac{5}{4}k + 1$

$\frac{25}{4}k = \frac{5}{2}$

$k = \frac{2}{5}$

CHECKING FOR CONSISTENCY THE THIRD COMPONENT (NOT USED ABOVE)

$5k - \frac{7}{2} = 5 \times \frac{2}{5} - \frac{7}{2} = 2 - \frac{7}{2} = -\frac{3}{2}$.

$-5\lambda k - 4\lambda = -5 \times \frac{1}{4} \times \frac{2}{5} - 4 \times \frac{1}{4} = -\frac{1}{2} - 1 = -\frac{3}{2}$

(ALL FINE!)

FINALLY WE HAVE

$\overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NP}$

$= (2,-6,2) + (5k,5k,5k)$

$= (2,-6,2) + (2,2,2)$

$= (4,-4,0)$

$\therefore P(4,-4,0)$

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$$y = |x^2 - 16| + 2(x-4), \quad x \in \mathbb{R}$$

THE CRITICAL VALUES OF THIS GRAPH (DUE TO THE MODULUS) ARE DETERMINED BY THE INEQUALITY

$$\Longrightarrow \quad x^2 - 16 \geqslant 0$$
$$\Longrightarrow \quad x^2 \geqslant 16$$
$$\Longrightarrow \quad x \leqslant -4 \quad \text{OR} \quad x \geqslant 4$$

HENCE WE HAVE TWO CASES TO CONSIDER

IF $x^2 - 16 \geqslant 0$

$x \leqslant 4$ OR $x \geqslant 4$

THE GRAPH REDUCES TO

$$\Longrightarrow y = +(x^2-16) + 2(x-4)$$
$$\Longrightarrow y = (x-4)(x+4) + 2(x-4)$$
$$\Longrightarrow y = (x-4)(x+4+2)$$
$$\Longrightarrow y = (x-4)(x+6)$$

$x \leqslant -4$ OR $x \geqslant 4$

IF $x^2 - 16 \leqslant 0$

$-4 \leqslant x \leqslant 4$

THE GRAPH REDUCES TO

$$\Longrightarrow y = -(x^2-16) + 2(x-4)$$
$$\Longrightarrow y = -(x-4)(x+4) + 2(x-4)$$
$$\Longrightarrow y = (x-4)[-x-4+2]$$
$$\Longrightarrow y = (x-4)(-x-2)$$
$$\Longrightarrow y = (x+2)(4-x)$$

$-4 \leqslant x \leqslant 4$

"SOLVING SIMULTANEOUSLY" TO LOCATE CUSPS

$x = 4, \quad y = 0 \quad \Longrightarrow \quad (4, 0)$

$x = -4, \quad y = -16 \quad \Longrightarrow \quad (-4, -16)$

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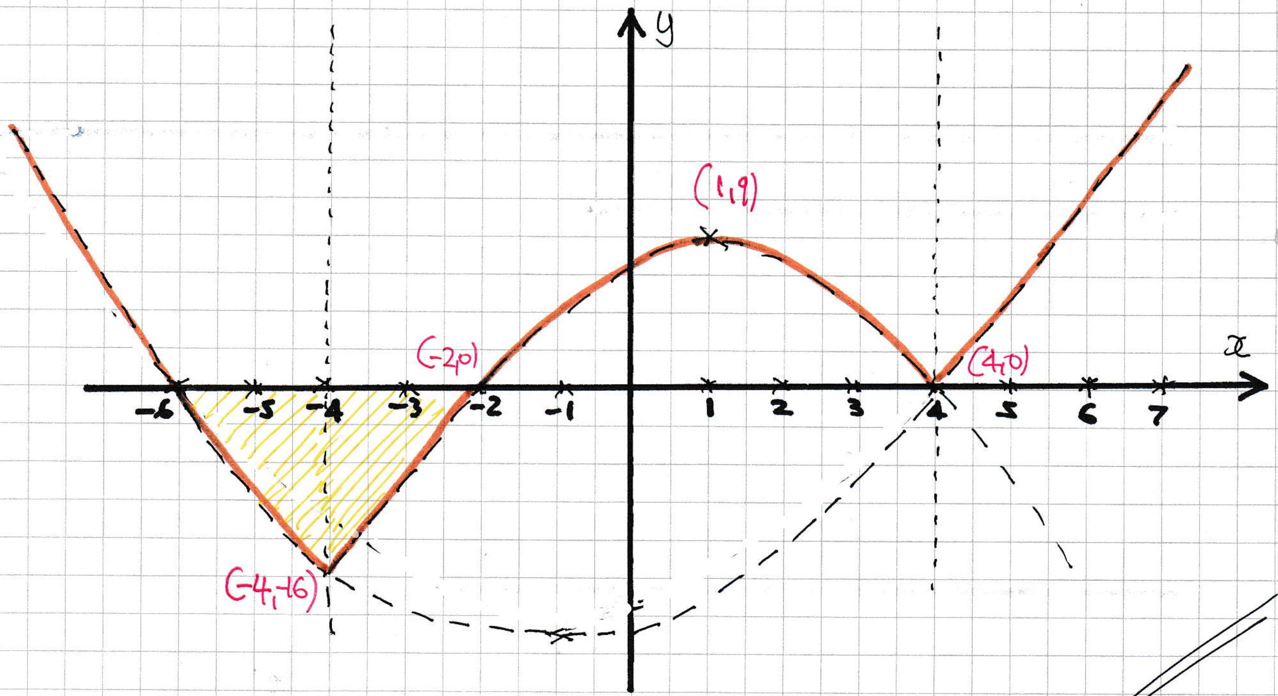
THE SKETCH CAN NOW BE PRODUCED



THE REQUIRED AREA CAN BE FOUND AS

$$\text{AREA} = \left| \int_{-6}^{-4} (x+6)(x-4)\,dx + \int_{-4}^{-2} (x+2)(4-x)\,dx \right|$$

$$= \int_{-4}^{-6} x^2+2x-24\,dx + \int_{-2}^{-4} -x^2+2x+8\,dx$$

$$= \left[\tfrac{1}{3}x^3+x^2-24x\right]_{-4}^{-6} + \left[-\tfrac{1}{3}x^3+x^2+8x\right]_{-2}^{-4}$$

$$= (-72+36+144)-\left(-\tfrac{64}{3}+16+96\right)+\left(\tfrac{64}{3}+16-32\right)-\left(\tfrac{8}{3}+4-16\right)$$

$$= 108-\frac{272}{3}+\frac{16}{3}+\frac{28}{3}$$

$$= 32$$

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USING THE SUBSTITUTION GIVEN

$$u = \sec x + \sqrt{\tan x} = \sec x + (\tan x)^{\frac{1}{2}}$$

$$\frac{du}{dx} = \sec x \tan x + \frac{1}{2}(\tan x)^{-\frac{1}{2}}\sec^2 x$$

$$\frac{du}{dx} = \frac{1}{\cos x}\,\frac{\sin x}{\cos x} + \frac{(\cos x)^{\frac{1}{2}}}{2(\sin x)^{\frac{1}{2}}\cos^2 x}$$

$$\frac{du}{dx} = \frac{\sin x}{\cos^2 x} + \frac{(\cos x)^{\frac{1}{2}}}{2(\sin x)^{\frac{1}{2}}\cos^2 x}$$

$$\frac{du}{dx} = \frac{1}{\cos^2 x}\left[\sin x + \frac{(\cos x)^{\frac{1}{2}}}{2(\sin x)^{\frac{1}{2}}}\right]$$

$$\frac{du}{dx} = \frac{1}{\cos^2 x}\left[\frac{2(\sin x)^{3/2} + (\cos x)^{\frac{1}{2}}}{2(\sin x)^{\frac{1}{2}}}\right]$$

$$\frac{du}{dx} = \frac{2\sin^{\frac{3}{2}}x + \cos^{\frac{1}{2}}x}{2\cos^2 x\,\sin^{\frac{1}{2}}x}$$

$$\boxed{dx = \frac{2\cos^2 x\,\sin^{\frac{1}{2}}x}{2\sin^{\frac{3}{2}}x + \cos^{\frac{1}{2}}x}\,du}$$

TRANSFORMING THE INTEGRAL

$$\int \frac{2\sin x\sqrt{\tan x} + 1}{2\cos x\sqrt{\tan x}\left[\cos x\sqrt{\tan x} + 1\right]}\,dx$$

$$= \int \frac{2\sin x\left(\frac{\sin^{\frac{1}{2}}x}{\cos^{\frac{1}{2}}x}\right) + 1}{2\cos x\left(\frac{\sin^{\frac{1}{2}}x}{\cos^{\frac{1}{2}}x}\right)\left[\cos x\left(\frac{\sin^{\frac{1}{2}}x}{\cos^{\frac{1}{2}}x}\right) + 1\right]} \times \frac{2\cos^2 x\,\sin^{\frac{1}{2}}x}{2\sin^{\frac{3}{2}}x + \cos^{\frac{1}{2}}x}\,du$$

$$= \int \frac{2\sin^{\frac{3}{2}}x\,\sec^{\frac{1}{2}}x + 1}{\cancel{2\cos^{\frac{1}{2}}x\,\sin^{\frac{1}{2}}x}\left[\cos^{\frac{1}{2}}x\,\sin^{\frac{1}{2}}x + 1\right]} \times \frac{\cancel{2}\cos^{\cancel{2}\,\frac{3}{2}}x\,\cancel{\sin^{\frac{1}{2}}x}}{2\sin^{\frac{3}{2}}x + \cos^{\frac{1}{2}}x}\,du$$

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$$= \int \frac{\left[2\sin^{\frac{3}{2}}x \sec^{\frac{1}{2}}x + 1\right]\left[\cos^{\frac{3}{2}}x\right]}{\left[\cos^{\frac{1}{2}}x \sin^{\frac{1}{2}}x + 1\right]\left[2\sin^{\frac{3}{2}}x + \cos^{\frac{1}{2}}x\right]} \, du$$

$$= \int \frac{2\sin^{\frac{3}{2}}x \cos x + \cos^{\frac{3}{2}}x}{\left[\cos^{\frac{1}{2}}x \sin^{\frac{1}{2}}x + 1\right]\left[2\sin^{\frac{3}{2}}x + \cos^{\frac{1}{2}}x\right]} \, du$$

$$= \int \frac{\cos x \cancel{\left[2\sin^{\frac{3}{2}}x + \cos^{\frac{1}{2}}x\right]}}{\left[\cos^{\frac{1}{2}}x \sin^{\frac{1}{2}}x + 1\right]\cancel{\left[2\sin^{\frac{3}{2}}x + \cos^{\frac{1}{2}}x\right]}} \, du$$

$$= \int \frac{\cos x}{\cos^{\frac{1}{2}}x \sin^{\frac{1}{2}}x + 1} \, du$$

MULTIPLY "TOP" & BOTTOM BY $\sec x$

$$= \int \frac{\sec x \cos x}{\sec x \cos^{\frac{1}{2}}x \sin^{\frac{1}{2}}x + \sec x} \, du$$

$$= \int \frac{1}{\frac{\sin^{\frac{1}{2}}x}{\cos^{\frac{1}{2}}x} + \sec x} \, du$$

$$= \int \frac{1}{\sqrt{\tan x} + \sec x} \, du$$

$$= \int \frac{1}{u} \, du$$

$$= \ln|u| + C$$

$$= \ln\left|\sqrt{\tan x} + \sec x\right| + C$$

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a) START BY DIFFERENTIATION TO LOCATE THE STATIONARY POINT

$$f(x) = 3x^4 + 8x^3 + 3x^2 - 12x - 6$$

$$f'(x) = 12x^3 + 24x^2 + 6x - 12$$

$$f'(x) = 6(2x^3 + 4x^2 + x - 2)$$

BY INSPECTION

$$f'(0) = -12 < 0$$

$$f'(1) = 30 > 0$$

AS $f'(x)$ IS CONTINUOUS AND CHANGES SIGN IN THE INTERVAL $[0,1]$ THERE IS AT LEAST A ROOT OF $f'(x) = 0$ IN THIS INTERVAL

b) INVESTIGATE EACH OF THE THREE RELATIONS USING THE MIDPOINT OF THE INTERVAL $x = 0.5$

- $x_{n+1} = \dfrac{2}{2x_n^2 + 4x_n + 1}$, $\dfrac{d}{dx}\left[2(2x^2+4x+1)^{-1}\right] = \dfrac{-2(4x+4)}{(2x^2+4x+1)^2}$

  EVALUATE AT $x = 0.5$ GIVES $-\dfrac{48}{49}$

- $x_{n+1} = \dfrac{1}{x_n^2} - \dfrac{1}{2x_n} - 2$, $\dfrac{d}{dx}\left[\dfrac{1}{x^2} - \dfrac{1}{2x} - 2\right] = -\dfrac{2}{x^3} + \dfrac{1}{2x^2}$

  EVALUATE AT $x = 0.5$ GIVES -14

- $x_{n+1} = \sqrt{\dfrac{2 - x_n}{4 + 2x_n}}$, $\dfrac{d}{dx}\left[(2-x)^{\frac{1}{2}}(4+2x)^{-\frac{1}{2}}\right] = \ldots$

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$$\ldots = -\frac{1}{2}(2-x)^{-\frac{1}{2}}(4+2x)^{-\frac{1}{2}} - (2-x)^{\frac{1}{2}}(4+2x)^{-\frac{3}{2}}$$

EVALUATE AT $x=0.5$

$$= -\frac{1}{2}\sqrt{\frac{2}{3}}\,\frac{1}{\sqrt{5}} - \sqrt{\frac{3}{2}} \times \frac{1}{5\sqrt{5}}$$

$$= -\sqrt{\frac{1}{20}} - \sqrt{\frac{3}{250}}$$

$$= -0.29211$$

HENCE WE DEDUCE

- $x_{n+1} = \frac{1}{x_n^2} - \frac{1}{2x_n} - 2$ — DIVERGES BY OSCILLATION (COBWEB) AS $|-14| > 1$ & NEGATIVE. RATE OF DIVERGENCE WILL BE RAPID $(14 >> 1)$

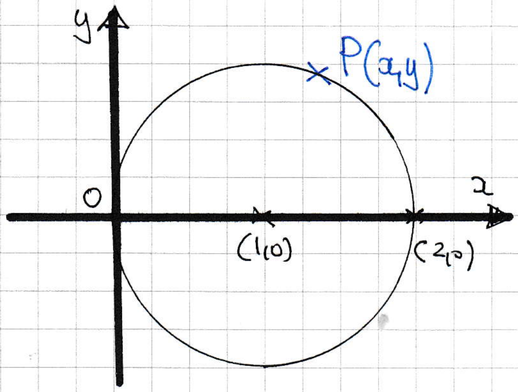
- $x_{n+1} = \frac{2}{2x_n^2 + 4x_n + 1}$ — CONVERGES BY OSCILLATION (COBWEB) AS $\left|-\frac{48}{49}\right| < 1$ & NEGATIVE. SLOW CONVERGING AS IT IS ALMOST 1

- $x_{n+1} = \sqrt{\frac{2-x_n}{4+2x_n}}$ — CONVERGES BY OSCILLATION (COBWEB) AS $|-0.292..| < 1$ & NEGATIVE. CONVERGES FAST AS 0.292 IS CLOSER TO ZERO THAN TO 1

IYGB – MP2 PAPER S – QUESTION 16

THE EQUATION OF A CIRCLE WITH CENTRE (1,0) AND RADIUS 1 IS

$$(x-1)^2 + y^2 = 1$$



EVIDENTLY THE GREATEST VALUE OF $x+y$ CANNOT BE IN THE LOWEST PORTION OF THE CIRCLE WHERE y WILL BE NEGATIVE.

$$\Rightarrow y^2 = 1-(x-1)^2$$
$$\Rightarrow y^2 = 1-x^2+2x-1$$
$$\Rightarrow y^2 = 2x-x^2$$
$$\Rightarrow y = +(2x-x^2)^{\frac{1}{2}} \quad \leftarrow \text{CONSTRAINT}$$

NOW LOOKING AT THE EXPRESSION TO BE MAXIMIZED

$$\Rightarrow f(x,y) = x+y$$
$$\Rightarrow f(x) = x+(2x-x^2)^{\frac{1}{2}}$$

DIFFERENTIATE AND SOLVE FOR ZERO

$$\Rightarrow f'(x) = 1+\tfrac{1}{2}(2-2x)(2x-x^2)^{-\frac{1}{2}}$$
$$\Rightarrow 0 = 1+(1-x)(2x-x^2)^{-\frac{1}{2}}$$
$$\Rightarrow 0 = 1+\frac{1-x}{(2x-x^2)^{\frac{1}{2}}}$$
$$\Rightarrow 0 = (2x-x^2)^{\frac{1}{2}}+1-x$$
$$\Rightarrow x-1 = (2x-x^2)^{\frac{1}{2}}$$

SQUARING, SO SOLUTIONS MUST BE CHECKED

$$\Rightarrow x^2-2x+1 = 2x-x^2$$

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$$\Longrightarrow \quad 2x^2 - 4x + 1 = 0$$

$$x = \frac{4 \pm \sqrt{16 - 4\times2\times1}}{2\times2} = \frac{4 \pm \sqrt{8}}{4} = \frac{4 \pm 2\sqrt{2}}{4} = 1 \pm \frac{1}{2}\sqrt{2}$$

THE "NEGATIVE" VERSION DOES NOT SATISFY THE EQUATION

$x - 1 = (2x - x^2)^{\frac{1}{2}}$ AS IT GIVES $-\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$

$$\Longrightarrow \quad x = 1 + \frac{1}{2}\sqrt{2}$$

HENCE WE CAN FIND f_{MAX} (DEFINITELY A MAX AS IT WILL BE POSITIVE AND $x+y=0$ AT THE ORIGIN)

$$f\left(1+\frac{1}{2}\sqrt{2}\right) = \left(1+\frac{1}{2}\sqrt{2}\right) + \left(2\left(1+\frac{1}{2}\sqrt{2}\right) - \left(1+\frac{1}{2}\sqrt{2}\right)^2\right)^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2}\sqrt{2} + \left(2 + \cancel{\sqrt{2}} - 1 - \cancel{\sqrt{2}} - \frac{1}{2}\right)^{\frac{1}{2}}$$

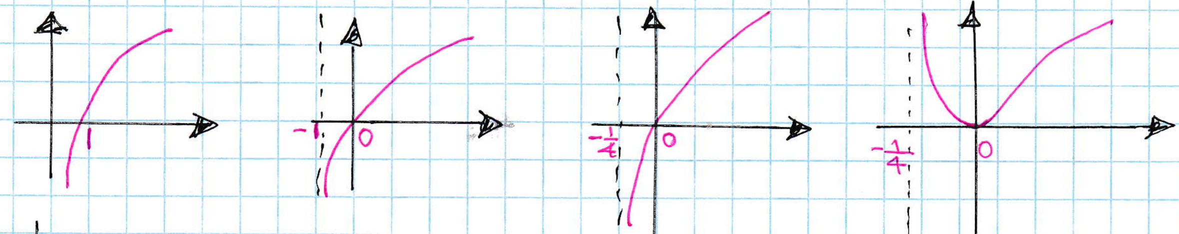
$$= 1 + \frac{1}{2}\sqrt{2} + \left(\frac{1}{2}\right)^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2}\sqrt{2} + \frac{\sqrt{2}}{2}$$

$$= 1 + \sqrt{2}$$

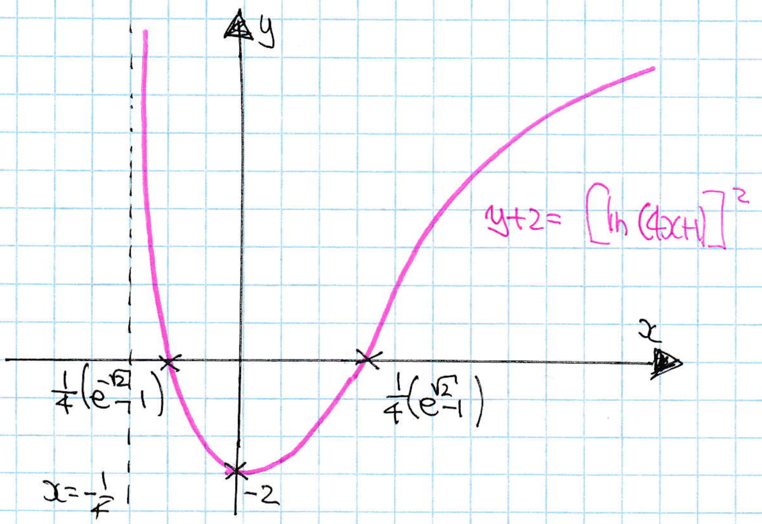
IYGB – MP2 PAPER S – QUESTION 17

- START WE A SKETCH OF THE CURVE STARTING WITH $\left[\ln(4x+1)\right]^2$



$y = \ln x$ $\qquad y = \ln(x+1)$ $\qquad y = \ln(4x+1)$ $\qquad y = \left[\ln(4x+1)\right]^2$

- HENCE WE HAVE THE GRAPH OF $\quad y+2 = \left[\ln(4x+1)\right]^2$



$$y+2 = \left[\ln(4x+1)\right]^2$$
$$2 = \left[\ln(4x+1)\right]^2$$
$$\ln(4x+1) = \begin{cases} \sqrt{2} \\ -\sqrt{2} \end{cases}$$
$$4x+1 = \begin{cases} e^{\sqrt{2}} \\ e^{-\sqrt{2}} \end{cases}$$
$$x = \begin{cases} \frac{1}{4}(e^{\sqrt{2}}-1) \\ \frac{1}{4}(e^{-\sqrt{2}}-1) \end{cases}$$

- NOW TO FIND THE REQUIRED AREA BY INTEGRATION IN x

$$\Longrightarrow \text{AREA} = \left| \int_0^{\frac{1}{4}(e^{\sqrt{2}}-1)} \left[\ln(4x+1)\right]^2 - 2 \, dx \right|$$

$$\Longrightarrow \text{AREA} = \int_{\frac{1}{4}(e^{\sqrt{2}}-1)}^{0} \left[\ln(4x+1)\right]^2 - 2 \, dx$$

... BY SUBSTITUTION BEFORE ATTEMPTING INTEGRATION BY PARTS ...

$$u = 4x+1 \qquad\qquad x = 0 \longmapsto u = 1$$
$$\frac{du}{dx} = 4 \qquad\qquad x = \tfrac{1}{4}(e^{\sqrt{2}}-1) \longmapsto u = e^{\sqrt{2}}$$

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$$\Longrightarrow \text{AREA} = \int_{e^{\sqrt{2}}}^{1} [\ln u]^2 \left(\frac{1}{4}\,du\right) - \int_{\frac{1}{4}(e^{\sqrt{2}}-1)}^{0} 2\,dx$$

$$\Longrightarrow \text{AREA} = \frac{1}{2}\left(e^{\sqrt{2}}-1\right) + \frac{1}{4}\int_{e^{\sqrt{2}}}^{1} [\ln u]^2\,du$$

.... BY PARTS ....

$(\ln u)^2$	$\frac{2}{u}\ln u$
u	1

$$\Longrightarrow \text{AREA} = \frac{1}{2}\left(e^{\sqrt{2}}-1\right) + \frac{1}{4}\left\{\left[u[\ln u]^2\right]_{e^{\sqrt{2}}}^{1} - \int_{e^{\sqrt{2}}}^{1} 2\ln u\,du\right\}$$

$$\Longrightarrow \text{AREA} = \frac{1}{2}\left(e^{\sqrt{2}}-1\right) + \frac{1}{4}\left\{0 - 2e^{\sqrt{2}} - 2\left[\underbrace{[u\ln u - u]}\right]_{e^{\sqrt{2}}}^{1}\right\}$$

BY PARTS OR NOTING THAT

$$\int \ln x\,dx = x\ln x - x + C$$

$$\Longrightarrow \text{AREA} = \frac{1}{2}\left[e^{\sqrt{2}}-1\right] - \frac{1}{2}e^{\sqrt{2}} - \frac{1}{2}\left[(0-1) - \left(\sqrt{2}e^{\sqrt{2}} - e^{\sqrt{2}}\right)\right]$$

$$\Longrightarrow \text{AREA} = \cancel{\frac{1}{2}e^{\sqrt{2}}} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}e^{\sqrt{2}}} - \frac{1}{2}\left[\cancel{-1} - \sqrt{2}e^{\sqrt{2}} + e^{\sqrt{2}}\right]$$

$$\Longrightarrow \text{AREA} = \frac{1}{2}\sqrt{2}e^{\sqrt{2}} - \frac{1}{2}e^{\sqrt{2}}$$

$$\Longrightarrow \text{AREA} = \frac{1}{2}e^{\sqrt{2}}\left(\sqrt{2}-1\right)$$

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ALTERNATIVE INTEGRATION PARALLEL TO THE y AXIS

- START BY REARRANGING THE EQUATION

$$y+2 = \left[\ln(4x+1)\right]^2$$

$$\left[\ln(4x+1)\right] = \begin{cases} +\sqrt{y+2} \\ -\sqrt{y+2} \end{cases}$$

$$4x+1 = \begin{cases} e^{\sqrt{y+2}} \\ e^{-\sqrt{y+2}} \end{cases}$$

$$4x = \begin{cases} e^{\sqrt{y+2}} - 1 & \leftarrow 4x > 0 \\ \cancel{e^{-\sqrt{y+2}} - 1} & \leftarrow 4x < 0 \end{cases}$$

$$x = \frac{1}{4}\left[e^{\sqrt{y+2}} - 1\right]$$

- THUS INTEGRATING W.R.T y, FROM $y=-2$ TO $y=0$

$$\Rightarrow \text{AREA} = \int_{-2}^{0} \frac{1}{4}e^{\sqrt{y+2}} - \frac{1}{4}\,dy$$

$$\Rightarrow \text{AREA} = \int_{-2}^{0} -\frac{1}{4}\,dy + \frac{1}{4}\int_{-2}^{0} e^{\sqrt{y+2}}\,dy$$

$$\Rightarrow \text{AREA} = \left[-\frac{1}{4}y\right]_{-2}^{0} + \frac{1}{4}\int_{0}^{\sqrt{2}} e^{u}(2u\,du)$$

$$\Rightarrow \text{AREA} = 0 - \frac{1}{2} + \frac{1}{4}\int_{0}^{\sqrt{2}} 2u e^{u}\,du$$

... SUBSTITUTION ...

$u = \sqrt{y+2}$

$u^2 = y+2$

$y = u^2 - 2$

$dy = 2u\,du$

$y=-2 \longmapsto u=0$

$y=0 \longmapsto u=\sqrt{2}$

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INTEGRATION BY PARTS

$$\Rightarrow \text{AREA} = -\frac{1}{2} + \frac{1}{4}\left\{\left[2ue^{u}\right]_{0}^{\sqrt{2}} - \int_{0}^{\sqrt{2}} 2e^{u}\,du\right\}$$

$2u$	2
e^{u}	e^{u}

$$\Rightarrow \text{AREA} = -\frac{1}{2} + \frac{1}{4}\left[2ue^{u} - 2e^{u}\right]_{0}^{\sqrt{2}}$$

$$\Rightarrow \text{AREA} = -\frac{1}{2} + \frac{1}{4}\left[\left(2\sqrt{2}e^{\sqrt{2}} - 2e^{\sqrt{2}}\right) - (0-2)\right]$$

$$\Rightarrow \text{AREA} = -\frac{1}{2} + \frac{1}{4}\left[2\sqrt{2}e^{\sqrt{2}} - 2e^{\sqrt{2}} + 2\right]$$

$$\Rightarrow \text{AREA} = \cancel{-\frac{1}{2}} + \frac{1}{2}\sqrt{2}e^{\sqrt{2}} - \frac{1}{2}e^{\sqrt{2}} + \cancel{\frac{1}{2}}$$

$$\Rightarrow \text{AREA} = \frac{1}{2}e^{\sqrt{2}}\left(\sqrt{2} - 1\right)$$