IYGB

Special Extension Paper I

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core and the Advanced Level Further Pure Mathematics Syllabi of recent years.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $=T$, Number of non attempted questions $=N$, Percentage score $=P$.

$P=\frac{1}{2}T+N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

Solve the following logarithmic equation.

$$\ln\left(\frac{1}{12}-\frac{1}{3x^2}\right)-1=\ln\left(\frac{1}{12}+\frac{1}{4x}+\frac{1}{6x^2}\right),$$

giving the value of x in exact form. **(6)**

Question 2

Show that

$$\frac{\sqrt[3]{49}-2\sqrt[3]{7}-4}{\sqrt[3]{7}+1}$$

can be written in the form $a\sqrt[3]{7}+b$, where a and b are integers to be found. **(7)**

Question 3

Prove that the sum of the squares of two distinct positive integers, when doubled, it can be written as the sum of two distinct square numbers. **(4)**

Question 4

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1\times u_2\times u_3\times u_4\times\ldots\times u_{k-1}\times u_k .$$

Evaluate, showing a clear method

$$\prod_{n=2}^{\infty}\left[1+\frac{1}{n^2-1}\right].$$ **(6)**

Question 5

Differentiate with respect to x

$$\sin\left[\arctan\left[\frac{1}{\sqrt{1-x^2}}\right]\right].$$

Give a simplified answer in the form

$$\frac{A}{x^n},$$

where A and n are integers to be found. **(6)**

Question 6

Show that if n and m are natural numbers, then the equations

$$z^n = 1+\mathrm{i}$$

$$z^m = 2-\mathrm{i},$$

have no common solution for $z \in \mathbb{C}$. **(6)**

Question 7

The curve with equation $y = f(x)$ passes through the origin, and satisfies the relationship

$$\frac{d}{dx}\left[y\left(x^2+1\right)\right] = x^5 + 2x^3 + x + 3xy.$$

Determine a simplified expression for the equation of the curve. **(12)**

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Question 8

It is given that $0 < r < 1$, $0 < R < 1$ and $r < 2R$.

It is further given that

$$\sum_{n=0}^{\infty} R^n = \left(\sum_{n=0}^{\infty} r^n \right)^2 .$$

Show clearly that

$$\sum_{n=0}^{\infty} \left(\frac{r}{2R} \right)^n = \frac{2(2-r)}{3-2r} .$$

(8)

Question 9

By using appropriate substitutions, or otherwise, show that

$$\int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} \, dx = \frac{\pi \ln 2}{16} .$$

(14)

Question 10

A parabola has Cartesian equation

$$y^2 = 12x , \quad x \geq 0 .$$

The point P lies on the parabola and the point Q lies on the directrix of the parabola so that PQ is parallel to the x axis.

The area of the triangle PQF is $8\frac{2}{3}$ square units, where the point F represents the focus of the parabola.

Determine the coordinates of P, given further that the y coordinate of P is a positive integer.

(12)

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Question 11

Solve the following trigonometric equation.

$$2\sqrt{3}\sin\left(x+\frac{7\pi}{12}\right)=3\operatorname{cosec}\left(x+\frac{5\pi}{12}\right), \quad 0 \le x \le 2\pi .$$ **(10)**

Question 12

A shop opens on Saturdays at 09.00 and stays open for 9 hours has its sales modelled as follows.

The rate at which the sales are made, is **directly proportional** to the time left until the shop closes and **inversely proportional** to the sales already made until that time.

One hour after the shop opens it has made sales worth £500 and at that instant sales are made at the rate of £2000 per hour.

The owner knows that the shop is not profitable once the rate at which it makes sales drops under £200 per hour.

Use a detailed method to find the time the shop should close on Saturdays. **(12)**

Question 13

By considering

$$\frac{\sin\left[(2m+1)x\right]}{\sin x}-\frac{\sin\left[(2m-1)x\right]}{\sin x}, \ m \in \mathbb{N},$$

determine the exact value of

$$\int_0^{\frac{1}{2}\pi} \frac{\sin 7x}{\sin x}\, dx .$$ **(9)**

Question 14

If $0 < k < \sqrt{2} - 1$ prove that

$$\int_{k}^{\frac{1-k}{1+k}} \frac{\ln x}{x^2 - 1} \, dx = \int_{k}^{\frac{1-k}{1+k}} \frac{\operatorname{artanh} x}{x} \, dx .$$

You need not evaluate these integrals. **(10)**

Question 15

The triangle ABC is given.

The points D and E are such so that $\overrightarrow{AD} = \lambda \overrightarrow{BC}$ and $\overrightarrow{BE} = \mu \overrightarrow{AC}$, where λ and μ are positive scalar constants.

Given further that $\lambda\mu = 1$, show that D, C and E are collinear. **(8)**

Question 16

The complex number z is given by

$$z = \frac{2(a+b)(1+\mathrm{i})}{a+b\mathrm{i}}, \quad a+b \neq 0 ,$$

where a and b are real parameters.

Show, that for all allowable values of a and b, the point represented by z is tracing a circle, determining the coordinates of its centre and the size of its radius.

(12)

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Question 17

By considering the series expansions of $\ln\left(1-x^2\right)$ and $\ln\left(\dfrac{1+x}{1-x}\right)$, or otherwise, find the exact value of the following series.

$$\sum_{r=1}^{\infty}\left[\left(\frac{1}{2r}+\frac{1}{2r+1}\right)\left(\frac{1}{4}\right)^r\right].$$ **(12)**

Question 18

The curve C is defined in the greatest real domain by the equation

$$y=\frac{x}{(y-2)(y+1)(y-3)}.$$

a) Show that

$$\frac{dy}{dx}=\frac{1}{2(y-1)\left(ay^2+by+c\right)},$$ **(5)**

where a, b and c are integers to be found.

b) Determine the exact value of the gradient at the points on C, where $x=40$. **(5)**

c) Sketch the graph of C.

The sketch must include the coordinates of any points where C meets the coordinate axes, the coordinates of the points of infinite gradient. You must also find, with a full algebraic method, the line of symmetry of C. **(8)**

Question 19

A family of functions $f_n(x)$, where $n = 0, 1, 2, 3, 4, \ldots$, satisfies the equation

$$\sum_{n=0}^{\infty}\left[t^n f_n(x)\right] = \left(1-2xt+t^2\right)^{-\frac{1}{2}}.$$

By integrating both sides of the above equation with respect to t, from 0 to 1, show that

$$\sum_{n=0}^{\infty}\left[\frac{f_n(\cos\theta)}{n+1}\right] = \ln\left[1+\operatorname{cosec}\left(\tfrac{1}{2}\theta\right)\right].$$

You may assume in this question that integration and summation commute. **(14)**

Question 20

A curve C and a straight line L have respective equations

$$y = x^2 \quad \text{and} \quad y = x.$$

The finite region bounded by C and L is rotated around L by a full turn, forming a solid of revolution S.

Find, in exact form, the volume of S. **(14)**