IYGB

Special Extension Paper J

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core and the Advanced Level Further Pure Mathematics Syllabi of recent years.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

Solve the following modulus inequality.

$$3|x+1|-|x-4| \le 11, \ x \in \mathbb{R}.$$ **(6)**

Question 2

Solve the quadratic equation

$$z^2 - 4z\text{i} + 4\text{i} = 7, \ z \in \mathbb{C}.$$ **(9)**

Question 3

It is given that

$$4p - \tfrac{1}{2}q = \log_6(3.6) \quad \text{and} \quad q - p + 1 = \log_6(75).$$

Solve these simultaneous equations, to show that

$$p = \log_6 k,$$

where k is a positive integer to be found. **(7)**

Question 4

The part of the curve with equation

$$y = \sin 2x, \ 0 \le x \le \frac{\pi}{2}$$

is rotated by $360°$ about the x axis.

Show that the area of the surface generated is

$$\pi\left[\tfrac{1}{2}\ln\left(2+\sqrt{5}\right)+\sqrt{5}\right].$$ **(11)**

Question 5

The complex number $z = z_1 + z_2$ where

$$z_1 = 3 + 4\text{i} \quad \text{and} \quad z_2 = 4\text{e}^{\text{i}\theta}, \ -\pi < \theta \le \pi$$

a) Sketch in an Argand diagram the locus of z. **(2)**

The complex number z_3 lies on the locus of z such that the argument of z_3 takes its maximum value.

b) State the value of $|z_3|$. **(1)**

c) Show clearly that

$$\arg z_3 = \pi - \arctan\frac{24}{7}.$$ **(5)**

d) Find z_3 in the form $x + \text{i}y$. **(4)**

Question 6

A curve is defined over the largest real domain by the equation

$$y = \frac{1}{x\text{e}^x\sqrt{x+1}}$$

Show that

$$\frac{dy}{dx} = \frac{f(x)\ \text{e}^{-x}}{2x^2(x+1)^{\frac{3}{2}}},$$

where $f(x)$ is a quadratic expression and hence find, in exact form, the coordinates of any stationary points of the curve. **(11)**

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Question 7

Prove by induction that

$$\cos x + \cos 3x + \cos 5x + ... + \cos\left[(2n-1)x\right] \equiv \frac{\sin(2nx)}{2\sin x}. \qquad (8)$$

Question 8

Use algebra to solve the following simultaneous equations

$$x + y + \sqrt{x+y} = 12 \qquad \text{and} \qquad x^2 + y^2 = 45,$$

given further that $x \in \mathbb{R}$ and $y \in \mathbb{R}$. **(8)**

Question 9

A curve C has equation

$$y = e^{\arctan x}, \; x \in \mathbb{R}.$$

a) Show, with detailed workings, that

$$\frac{d^3y}{dx^3} = \frac{\left(6x^2 - 6x - 1\right)e^{\arctan x}}{\left(1+x^2\right)^3}. \qquad (9)$$

b) Deduce that C has a point of inflection, stating its coordinates. **(4)**

Question 10

Use appropriate techniques to solve the following differential equation.

$$\frac{d^2y}{dx^2} = -\frac{144}{y^3}, \qquad y(0) = 6, \qquad \left.\frac{dy}{dx}\right|_{x=0} = 0. \qquad (11)$$

Question 11

The point $P(2p, p^2)$, where p is a parameter, lies on the parabola, with Cartesian equation

$$x^2 = 4y.$$

The point F is the focus of the parabola and O represents the origin.

The tangent to the parabola at P forms an angle θ with the positive x axis.

The straight line which passes through P and F forms an acute angle φ with the tangent to the parabola at P.

Show that $\theta + \varphi = \frac{1}{2}\pi$ and hence state the coordinates of P if $\theta = \varphi$. **(11)**

Question 12

The cubic equation

$$x^3 + px + q = 0, \quad p < 0$$

has 3 distinct real roots.

Show that $27q^2 + 4p^3 < 0$. **(7)**

Question 13

$$I = \int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \frac{\sqrt{3}\left(1 + \pi x^3\right)}{2 - \cos\left(|x| + \frac{1}{3}\pi\right)} \, dx.$$

Show that

$$I = 4\arctan\frac{1}{2}.$$ **(12)**

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Question 14

The curve C has equation

$$y = \frac{x^3 + 2}{x^2 - x + 1}.$$ **(12)**

Find the coordinates of the stationary point of C and determine its nature.

Question 15

The curve with equation $y = f(x)$ has the line $y = 1$ as an asymptote and satisfies the differential equation

$$x^3 \frac{dy}{dx} - x = xy + 1, \quad x \neq 0.$$

Solve the above differential equation, giving the solution in the form $y = f(x)$. **(10)**

Question 16

By considering a suitable binomial expansion, show that

$$\arcsin x = \sum_{r=0}^{\infty} \left[\binom{2r}{r} \frac{2}{2r+1} \left(\frac{x}{2} \right)^{2r+1} \right].$$ **(9)**

Question 17

It is given that

$$I = \int_0^{\frac{1}{3}} \frac{32x^2}{(x^2 - 1)(x+1)^3} \, dx.$$

Show that $I = \frac{7}{6} - 2\ln 2$. **(10)**

Question 18

Determine, in terms of k and n, a simplified expression for

$$\sum_{r=2}^{n}\left[\frac{r(1-k)-1}{r(r-1)k^{r}}\right]. \qquad (9)$$

Question 19

$$f(x)=\frac{\cos 2x}{\sqrt{1+\sin 2x}}, \ x\in\mathbb{R}, \ \sin 2x\neq -1.$$

a) Express $f(x)$ in the form

$$f(x)=\frac{g(x)\,g(-x)}{|g(x)|},$$

where $g(x)$ is a function to be found. (3)

b) Sketch the graph of $f(x)$ for $-2\pi \le x \le 2\pi$. (5)

c) Hence solve the trigonometric equation

$$\sqrt{2}f(x)=1, \quad -2\pi \le x \le 2\pi. \qquad (6)$$

Question 20

Find the modulus of $6\mathbf{a}-\mathbf{b}$, given that the equation $|x\mathbf{a}+\mathbf{b}| = 2\sqrt{3}$ has repeated roots in x, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. (10)
