IYGB

Special Extension Paper K

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core and the Advanced Level Further Pure Mathematics Syllabi of recent years.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

i. Simplify the following expression.

$$9\log_{24} 2 + \log_{24} 27.$$

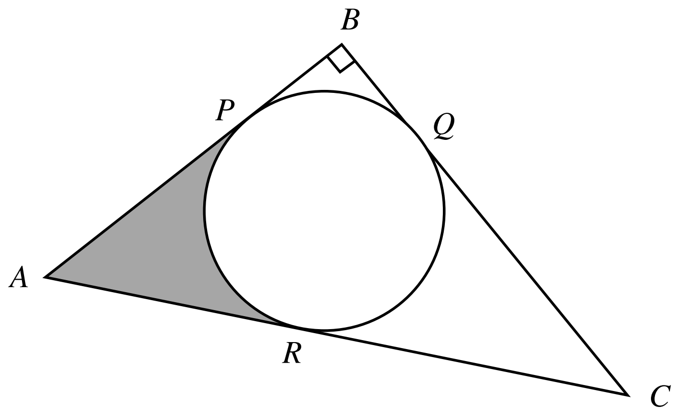
Show detailed workings in this simplification. **(4)**

ii. It is given that

$$5 \times 2^{t-1} = 2 \times t^{2t} \quad \Rightarrow \quad (10k)^t = k$$

Determine the value of k. **(4)**

Question 2



The triangle ABC has a right angle at B, with $|AB| = 21$ cm and $|BC| = 20$ cm.

A circle is drawn inside the triangle so that the three sides of the triangle are tangents to the circle.

The points P, Q and R are the respective points of tangency with AB, BC and AC.

Show that the area of the finite region bounded by AP, AR and the circular arc PR, shown shaded in the figure above, is

$$18\left[5 - \pi + \arctan\left(\frac{20}{21}\right)\right].$$ **(9)**

Question 3

The function f is defined as

$$f(x) \equiv \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\dots}}}}} \ , \quad x \in (0,\infty).$$

Determine the value of

$$\int_0^2 f(x)\ dx \ . \qquad \textbf{(6)}$$

Question 4

The quadratic equation

$$4x^2 + Px + Q = 0,$$

where P and Q are constants, has roots which differ by 2.

If another quadratic equation has repeated roots which are also the **squares of the roots** of the above given equation, find the value of P and the value of Q. **(9)**

Question 5

Find the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos x + \sin x}{\sqrt{\sin 2x}}\ dx . \qquad \textbf{(9)}$$

Question 6

It is given that

$$f(k) \equiv (k^3 - k)(2k^2 + 5k - 3),$$

where k is a positive integer.

Prove that $f(k)$ is divisible by 5.

*You may **not** use proof by induction in this question.* **(6)**

Question 7

A parabola has equation

$$y^2 = 4x, \quad 0 \le x \le 5$$

Show that the length this parabola is exactly $\ln\left(\sqrt{a} + \sqrt{b}\right) + \sqrt{ab}$ where a and b are positive integers. **(10)**

Question 8

An equation in x is summarized by the following determinant.

$$\begin{vmatrix} 1 & a-1 & (x-b)(x+b) \\ -1 & b+1 & (x-a)(x+a) \\ 1 & x-1 & (a-b)(a+b) \end{vmatrix} = 0$$

Give the solutions in terms of a and/or b where appropriate. **(7)**

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Question 9

A curve is given parametrically by the equations

$$x = 2t^2 - 3t + 1, \; x = t^2 + t + 1, \; t \in \mathbb{R}.$$

The tangents to the curve, at two distinct points P and Q, intersect each other at the point with coordinates $(2,9)$.

a) Determine the coordinates of P and Q. **(9)**

b) Show that the Cartesian equation of the curve is

$$25(y-1) = (2y - x - 1)(2y - x + 4).$$

You may not use a verification method in this part. **(4)**

Question 10

Solve the differential equation

$$\frac{d}{dx}\left(xy^2\right) = \frac{x^4 + x^2y^2 + y^4}{x^2}, \; y(\mathrm{e}) = \sqrt{2}\,\mathrm{e}.$$

Give the answer in the form $y^2 = f(x)$. **(12)**

Question 11

Use complex numbers to prove that $\cos\left(\frac{2}{7}\pi\right)$ is a solution of the cubic equation

$$x^3 + x^2 - 2x - 1 = 0.$$

*You may **not** use verification in this proof.* **(10)**

Question 12

If $a \in \mathbb{R}$, $b \in \mathbb{R}$, solve the following simultaneous equations.

$$3\left(a^2+b^2\right)^{\frac{3}{2}}-125a=0 \qquad \text{and} \qquad 4\left(a^2+b^2\right)+25b=0. \qquad \mathbf{(11)}$$

Question 13

Solve the following trigonometric equation.

$$\sin(2\theta+58)^\circ+2\sin^2(42^\circ)=1, \quad 0 \le \theta < 360. \qquad \mathbf{(9)}$$

Question 14

The complex number z has unit modulus and $\arg z=\theta$, $-\pi<\theta\le\pi$.

The complex conjugate of z is denoted by $\overline{z}$.

Using a detailed method, show that

$$\operatorname{Re}\left[\frac{z(1-\overline{z})}{\overline{z}(1+z)}\right]=-2\sin^2\left(\tfrac{1}{2}\theta\right). \qquad \mathbf{(8)}$$

Question 15

A curve has polar equation

$$r=1+\tan\theta, \quad 0\le\theta\le\frac{1}{2}\pi.$$

The point P lies on the curve where $\theta=\frac{1}{3}\pi$.

The point Q lies on the initial line so that the straight line L, which passes through P and Q meets the initial line at right angles.

Determine, in exact simplified form, the area of the finite region bounded by the curve and L.

(12)

Question 16

A curve C is defined in the largest real domain by the equation

$$x = \sqrt{\frac{y}{y+1}}.$$

Sketch a detailed graph of C, fully justifying its features. **(10)**

Question 17

$$I = \int_0^{\frac{1}{2}} \frac{\ln(1+2x)}{1+4x^2} \, dx.$$

a) Use an appropriate trigonometric substitution to show that

$$I = \int_0^{\frac{\pi}{4}} \frac{1}{2}\ln\sqrt{2} + \frac{1}{2}\ln\left[\frac{\cos\left(\theta - \frac{1}{4}\pi\right)}{\cos\theta}\right] d\theta.$$ **(10)**

b) Show further that

$$I = \frac{\pi \ln 2}{16} + \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \frac{1}{2}\ln\left[\frac{\cos\left(\varphi - \frac{1}{8}\pi\right)}{\cos\left(\varphi + \frac{1}{8}\pi\right)}\right] d\varphi.$$ **(4)**

c) Deduce that

$$I = \frac{\pi \ln 2}{16}.$$ **(3)**

Question 18

Show that

$$1+\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^9}{9!}+\frac{x^{12}}{12!}+\frac{x^{15}}{15!}+\ldots=\frac{1}{3}\left[e^x+2e^{\frac{1}{2}x}\cos\left(\frac{1}{2}\sqrt{3}x\right)\right].$$

You may find useful in this question the fact that if $z=e^{i\frac{2}{3}\pi}$ then $1+z+z^2=0$. **(15)**

Question 19

Three points in space A, B and K are such so that $\overrightarrow{KB} = 2\overrightarrow{AK}$.

Prove that if M is a fourth distinct arbitrary point in space, then

$$2\left|\overrightarrow{MA}\right|^2 + \left|\overrightarrow{MB}\right|^2 - 3\left|\overrightarrow{MK}\right|^2 = \text{constant}.$$ **(9)**

Question 20

Find the sum to infinity of the following series.

$$1+\frac{1}{3\times 4}+\frac{1}{5\times 4^2}+\frac{1}{7\times 4^3}+\frac{1}{9\times 4^4}+\ldots$$ **(10)**
