IYGB

Special Extension Paper L

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core and the Advanced Level Further Pure Mathematics Syllabi of recent years.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

$$f(x) \equiv \frac{8}{x^2} - x, \; x \neq 0.$$

Show that $f\left(-2^{\frac{4}{3}}\right) = 3\sqrt[3]{2}$. **(4)**

You must show detailed workings in this question.

Question 2

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k.$$

Solve the equation

$$\prod_{r=1}^{\infty}\left[\sqrt[2r]{2^x}\right] = 2^{-(x+2)}.$$

You may assume that the left hand side of the equation converges. **(6)**

Question 3

It is given that for $n \in \mathbb{N}$

$$U_n = \frac{2n}{2n+1}U_{n-1}, \quad U_1 = \frac{2}{3}.$$

Prove by induction that

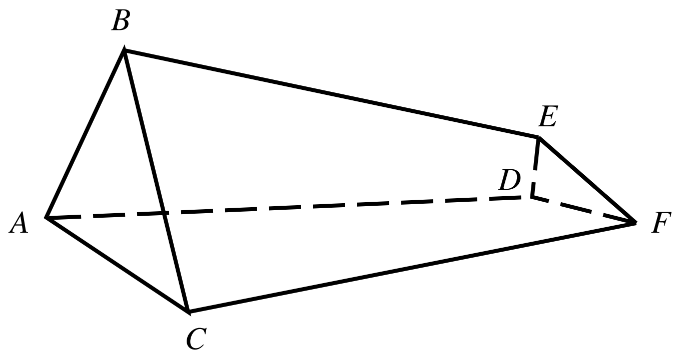
$$U_n \leq \left(\frac{2n}{2n+1}\right)^n.$$ **(7)**

Question 4

Find, in exact simplified form, the value of

$$\int_{\ln(\ln 2)}^{\ln[2\ln(e+1)]} e^{x+e^{x}} \, dx . \qquad (5)$$

Question 5



The figure above shows an irregular hollow shape, consisting of two non-congruent, non-parallel triangular faces ABC and DEF, and two non-congruent quadrilateral faces $ABED$ and $BCFE$.

The respective equations of the straight lines AD, DE and BC are

$$\mathbf{r}_1 = -5\mathbf{i} + 6\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j})$$
$$\mathbf{r}_2 = -\mathbf{i} + 12\mathbf{j} + \mathbf{k} + \mu(-2\mathbf{i} + 7\mathbf{j} - 7\mathbf{k})$$
$$\mathbf{r}_3 = -\mathbf{i} - 8\mathbf{j} + \mathbf{k} + \nu(-2\mathbf{i} + 7\mathbf{j} + 7\mathbf{k})$$

where λ, μ and ν are scalar parameters.

If the plane face $BCFE$ has equation $21x - 14y + 20z = 111$ and the point G has position vector $5\mathbf{i} + 7\mathbf{j}$, show that the acute angle between the plane face $BCFE$ and the straight line BG is

$$\frac{\pi}{2} - \arccos\left[\frac{13}{\sqrt{3111}}\right]. \qquad (10)$$

Question 6

a) Show that

$$\left(1+\mathrm{i}\tan\theta\right)^4+\left(1-\mathrm{i}\tan\theta\right)^4\equiv\frac{2\cos 4\theta}{\cos^4\theta}$$ **(3)**

b) By considering a suitable polynomial equation based on the result of part **(a)** show further

i. $\tan^2\left(\frac{1}{8}\pi\right)\tan^2\left(\frac{3}{8}\pi\right)=1$. **(4)**

ii. $\tan^2\left(\frac{1}{8}\pi\right)+\tan^2\left(\frac{3}{8}\pi\right)=6$. **(4)**

Question 7

A curve is defined implicitly as

$$y^3-x^2+x(3y+2)-3y=2.$$

The y axis is a tangent to the curve at the point A and the point B is another intercept of the curve with the y axis.

The tangent to the curve at the point B meets the curve again at the point C.

Determine the exact coordinates of C. **(12)**

Question 8

By considering the trigonometric identity for $\tan(A-B)$, with $A=\arctan(n+1)$ and $B=\arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty}\arctan\left(\frac{1}{n^2+n+1}\right).$$

You may assume the series converges. **(9)**

Question 9

$$w = \frac{2 - iz}{z}, \ z \in \mathbb{C}, \ z \neq 0 .$$

The complex function $w = f(z)$, maps the point $P(x, y)$ from the z complex plane onto the point $Q(u, v)$ on the w complex plane.

The curve C in the z complex plane is mapped in the w complex plane onto the curve with equation

$$\arg w = \tfrac{1}{3}\pi .$$

Determine a Cartesian equation of C, and hence find an exact simplified value for the area of the finite region bounded by C, and the y axis. **(12)**

Question 10

Find the exact value of

$$\int_0^1 \left[\sum_{n=1}^{\infty} \frac{(n+1)x^n}{(n+2)!} \right] dx . \qquad (7)$$

You may assume that integration and summation commutes..

Question 11

The straight line L and the circle C, have respective equations

$$L : \ y = \lambda(x - a) + a\sqrt{\lambda^2 + 1} \quad \text{and} \quad C : \ x^2 + y^2 = 2ax ,$$

where a is a positive constant and λ is a parameter.

Show that for all values of λ, L is a tangent to C. **(10)**

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Question 12

Solve the following trigonometric equation.

$$\cos\left(\arcsin\frac{1}{4}\right)\sin\left(\arccos x\right)=\frac{1}{4}(4-x)\ ,\quad x\in\mathbb{R}\ .\qquad (7)$$

Question 13

$$I=\int_0^{\frac{1}{2}\pi} x\cot x\ dx\ .$$

Use appropriate integration techniques to show that

$$I=\frac{1}{2}\pi\ln 2\ .\qquad (10)$$

Question 14

By sketching the graph of the integrand, or otherwise, determine the maximum value of the following function

$$F(a,b)\equiv\int_a^b 2\arcsin\sqrt{x+2}\ -\ \arcsin(2x+3)\ dx\ .\qquad (10)$$

Question 15

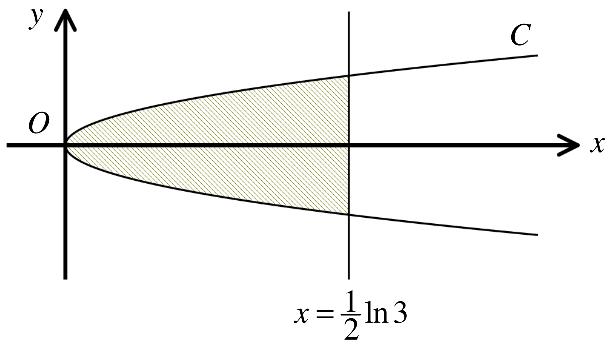
A curve C is defined, in the largest possible real domain, by the Cartesian equation

$$2y-1=(x-1)(y-1)^2\ .$$

By expressing the above equation in the form $y=f(x)$, sketch the graph of C.

Indicate the equations of any asymptotes, stationary points and any intersections with the coordinate axes. **(10)**

Question 16



The figure above shows the curve C whose parametric equations are

$$x = \operatorname{artanh}\left(\sin^2 t\right), \quad y = \sin t\,, \quad -\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi\,.$$

a) Use integration in Cartesian coordinates to find the exact area of the finite region bounded by the curve and the straight line with equation $x = \frac{1}{2}\ln 3$. **(12)**

b) Use integration in parametric to verify the validity of the result of part **(a)**. **(8)**

Question 17

The circle with equation

$$x^2 + y^2 = 4\,,$$

is rotated by 2π radians about the straight line with equation $x = 5$ axis to form a solid of revolution, known as a torus.

Use integration to show that the volume of the solid is

$$40\pi^2\,.$$

You may not use the formula for the volume of a torus or the theorem of Pappus. **(8)**

Question 18

The function with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right) = 2y\ln 3, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 2\ln 3.$$

Solve the above differential equation to show that $y = 3^{x^2+2x}$. **(12)**

Question 19

Solve the cubic equation

$$x^3 - 9x^2 + 3x - 3 = 0, \; x \in \mathbb{R}.$$

You may assume that this cubic equation only has one real root. **(12)**

Question 20

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

a) Show, with a detailed method, that $F(x) = f(\phi)x^{g(\phi)}$ is a solution of the differential equation,

$$F'(x) = F^{-1}(x),$$

where f and g are constant expressions of ϕ, to be found in simplified form. **(12)**

b) Verify the answer obtained in part **(a)** satisfies the differential equation, by differentiation and function inversion.

[You may assume that $F(x)$ is differentiable and invertible] **(6)**
