IYGB

Special Extension Paper M

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core and the Advanced Level Further Pure Mathematics Syllabi of recent years.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

It is required to find the principal value of i^i, in exact simplified form, where i is the imaginary unit.

a) Show, with detailed workings, that

$$i^i = e^{-\frac{1}{2}\pi}. \qquad (2)$$

b) Use a different method to that used in part **(a)**, to verify the exact answer given in part **(a)**. (2)

Question 2

Solve the following simultaneous equations

$$a^{2x} \times b^{3y} = c^5$$
$$a^{3x} \times b^{2y} = c^{10}$$

Give the answers in exact form in terms of $\log a$, $\log b$ and $\log c$. (**8**)

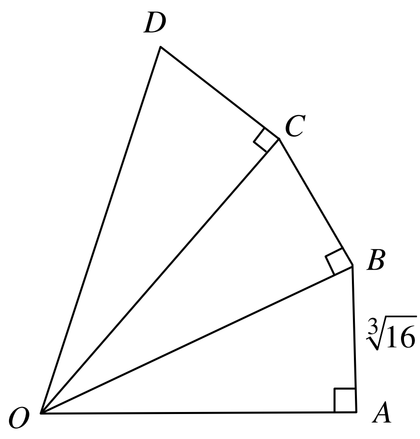
Question 3

The following information is given.

- The straight line l_1 is a tangent to a circle at the point T and the point C is another point on l_1.
- The straight line l_2 passes through C, intersecting the circle at two distinct points A and B.
- The straight line l_3 is the angle bisector of $\measuredangle TCA$.

Given further that l_3 intersects TA and TB at the points P and Q respectively, prove that the triangle TPQ is isosceles. (4)

Question 4



The figure above shows three right angle triangles, OAB, OBC and OCD.

It is given that $\measuredangle AOB = \measuredangle BOC = \measuredangle COD$ and $|OD| = 2|OA|$.

Given further that the length of AB is $\sqrt[3]{16}$, determine the length of DC. **(8)**

Question 5

The straight parallel lines l_1 and l_2 have respective equations

$$y = 2x + 11 \qquad \text{and} \qquad y = 3x + 3.$$

The straight line l_3, passing through the point $P(2,0)$, intersects l_1 and l_2 at the points P and Q respectively.

Given that $|PQ| = 8$ determine the possible equation of l_3. **(13)**

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Question 6

By using the substitution $u = 1 + \cos^4 x$, or otherwise, find the exact value of

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4\cot^3 x}{1 + 2\cot^2 x + 2\cot^4 x} \, dx .$$ **(8)**

Question 7

The finite sum C is given below.

$$C = \sum_{r=0}^{n} \left[\binom{n}{r} (-1)^n \cos^n \theta \cos n\theta \right].$$

Given that $n \in \mathbb{N}$ determine the 4 possible **simplified** expressions for C.

Give the answers in exact, fully simplified, form. **(10)**

Question 8

$$\frac{dy}{dx} = \tan\left(x^2 + 2y + \pi\right) - x, \quad y(0) = \frac{1}{4}\pi .$$

Solve the above differential equation to show that

$$y = -\frac{1}{2}\left[x^2 + \pi + \arcsin\left(e^{2x}\right)\right].$$ **(7)**

Question 9

Relative to a fixed origin O at $(0,0,0)$ the points A, B and C have coordinates $(0,4,6)$, $(3,5,4)$ and $(2,0,0)$, respectively.

- The straight line l_1 passes through A and B.
- The straight line l_2 passes through C and is parallel to l_1.
- The point D lies on l_1 so that $\measuredangle ACD = 90°$.
- The point E lies on l_2 so that $\measuredangle CDE = 90°$.
- The point F lies on l_2 so that $|EC| = 2|EF|$.

Determine the coordinates of the possible positions of F. **(11)**

Question 10

Find the sum of the first 16 terms of the following series.

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\frac{1^3+2^3+3^3+4^3}{1+3+5+7}+\ldots$$ **(8)**

Question 11

$$f(x)=\sqrt{\frac{1-x}{1+x}},\ x \in \mathbb{R},\ |x|<1.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = -\frac{1}{(1+x)\sqrt{1-x^2}}.$$ **(8)**

Question 12 (*****)

Find an exact simplified value for the following integral.

$$\int_0^1 \ln\left(\sqrt{x}+\sqrt{x+1}\right)\, dx \, . \qquad \textbf{(10)}$$

Question 13

A set of cartesian axes is superimposed over a set of polar axes, so that both set of axes have a common origin O, and the positive x axis coincides with the initial line.

A parabola P has Cartesian equation

$$y^2 = 8(2-x) \,, \quad x \le 2 \,.$$

A straight line L has polar equation

$$\tan\theta = \sqrt{3} \,, \; -\pi < \theta \Leftrightarrow \pi \,.$$

a) Use polar coordinates to determine, in exact simplified form, the area of the finite region bounded by P and L. **(10)**

b) Verify the answer of part **(a)** by using calculus in cartesian coordinates. **(10)**

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Question 14

A shear is defined by the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & 4 \end{pmatrix},$$

where a, b and c are scalar constants.

Under this transformation the point with coordinates $(1,2)$ is mapped onto the point with coordinates $(-8,11)$.

The shear defined by $\mathbf{M}$ has an invariant line L, which passes through the point with coordinates $(0,1)$.

Determine an equation of L. **(9)**

Question 15

The cubic equation

$$x^3 + px + q = 0,$$

has 2 distinct real roots.

a) Show that $27q^2 + 4p^3 < 0$. **(7)**

A parabola has Cartesian equation

$$y = x^2, \quad x \in \mathbb{R}.$$

Three distinct normals to this parabola pass through the point, which does not lie on the parabola, whose coordinates are (a,b).

b) Show further that

$$b > \tfrac{1}{2} + 3\left(\tfrac{1}{4}a\right)^{\frac{2}{3}}.$$ **(7)**

Question 16

A curve C is described implicitly by the equation

$$xy^2 = e^y .$$

a) Show, by a detailed method, that

$$\left(y^2-2y\right)\frac{d^2y}{dx^2}+\left(y^2-2\right)\left(\frac{dy}{dx}\right)^2-4y^3\frac{dy}{dx}e^{-y}=0 . \qquad \textbf{(5)}$$

b) Use an analytical method, with suitable boundary conditions, to obtain the equation of C by solving the above differential equation. **(16)**

Question 17

Sketch in separate sets of axes detailed graphs of the following curves, fully justifying their key features.

a) $y = \arccos(2x-1)$. **(2)**

b) $y^2 = \arccos|2x-1|$. **(4)**

c) $\cos y^2 = |2x-1|$ **(7)**

You may assume that each curve is defined in the largest real domain.

Question 18

Consider the following convergent infinite series.

$$\sum_{r=0}^{\infty} \frac{2^{r+4}}{(r+3)r!}$$

Show that the sum to infinity of the above series is exactly $4\left(e^2-1\right)$. **(13)**

Question 19

Determine the exact value of the following limit.

$$\lim_{h \to 0} \left[\frac{1}{h} \left[\int_{\frac{1}{6}\pi}^{\frac{1}{6}\pi + h} \frac{\sin x}{x} \, dx \right] \right]$$

You must justify the evaluation. (5)

Question 20

Prove that for all real numbers, a and b,

$$\sqrt{a^2 + b^2} \le \frac{\sqrt{4a^2 + b^2} + \sqrt{a^2 + 4b^2}}{3}.$$ (6)
