IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper K

Difficulty Rating: 3.85/0.6512

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 23 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

I.Y.G.B. www.madasmaths.com

Question 1

The points $A(7,4,3)$, B and $C(1,2,-1)$ form the parallelogram $OABC$, where the above coordinates are measured relative to a fixed origin O.

a) Find the coordinates of B. **(2)**

The side OC is extended in the $\overrightarrow{OC}$ direction to a point D.

The point M is the midpoint of AC.

b) Given further that $\overrightarrow{MD} = \mathbf{i} + 7\mathbf{j} - 6\mathbf{k}$, determine $\left|\overrightarrow{OC}\right| : \left|\overrightarrow{CD}\right|$. **(5)**

Question 2

$$f(x) = x^4 + 3x - 1, \; x \in \mathbb{R}.$$

a) By sketching two suitable graphs in the same set of axes, determine the number of real roots of the equation $f(x) = 0$. **(3)**

b) Show that the equation $f(x) = 0$ has a root α between 0 and 1. **(2)**

The recurrence relation

$$x_{n+1} = \frac{1 - x_n^4}{3}$$

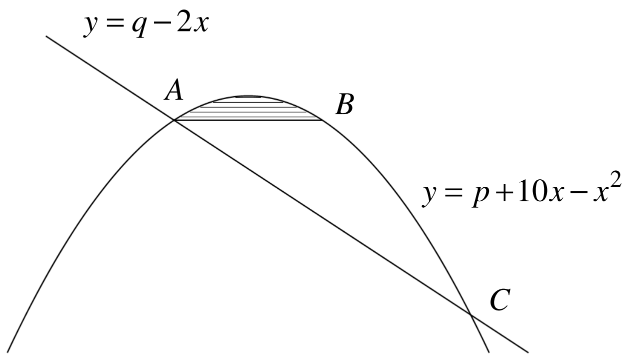
starting with $x_0 = 0.3$ is to be used to find α.

c) Find to 4 decimal places, the value of x_1, x_2, x_3 and x_4. **(1)**

d) Explain whether the convergence to the root α, can be represented by a cobweb or a staircase diagram. **(1)**

e) By considering the sign of $f(x)$ in a suitable interval, show that $\alpha = 0.32941$, correct to 5 decimal places. **(2)**

Question 3



The figure above shows a curve and a straight line with respective equations

$$y = p + 10x - x^2 \quad \text{and} \quad y = q - 2x,$$

where p and q are constants.

The curve meets the straight line at the points A and C, and the point B lies on the curve so that AB is parallel to the x axis.

a) Given the coordinates of C are $(10,0)$ find …

i. … the value of p and the value of q.

ii. … the coordinates of A and B. **(6)**

a) Determine the value of the area of the finite region bounded by the curve and the straight line segment AB. **(6)**

Question 4

It is given that

$$N = k^2 - 1 \quad \text{and} \quad k = 2^p - 1, \quad p \in \mathbb{N}.$$

Use direct proof to show that 2^{p+1} is a factor of N. **(4)**

Question 5

A circle C has equation

$$x^2 + y^2 - 14x - 14y + 49 = 0.$$

a) Find the radius of the circle and the coordinates of its centre. **(3)**

b) Sketch the circle, indicating clearly all the relevant details. **(2)**

The point P has coordinates $(15,8)$.

A tangent drawn from P touches the circle at the point Q.

c) Determine the distance PQ. **(4)**

Question 6

In the series expansion of

$$(1+ax)^n, \ |ax|<1,$$

the coefficient of x is -10 and the coefficient of x^2 is 75.

a) Show that $n=-2$ and find the value of a. **(6)**

b) Find the coefficient of x^3. **(2)**

c) State the range of values of x for which the above expansion is valid. **(1)**

Question 7

$$\frac{d}{dx}(\sin x) = \cos x.$$

Prove by first principles the validity of the above result by using the small angle approximations for $\sin x$ and $\cos x$.

(6)

Question 8

The point P lies on the curve with equation

$$y = x\sqrt{\ln x}, \quad x > 1.$$

Determine the two possible sets of coordinates of P given further that the gradient of the curve at P is $\frac{3}{2}$.

(10)

Question 9

Five numbers are consecutive terms of an arithmetic progression.

The arithmetic mean of these numbers is 7, while the arithmetic mean of the **squares** of these numbers is 67.

Determine these five numbers.

(8)

Question 10

Solve the following indicial equation

$$\frac{2^n}{2^{\sqrt{n}} \times 2^6} = 1.$$

You must show full workings.

(7)

Question 11

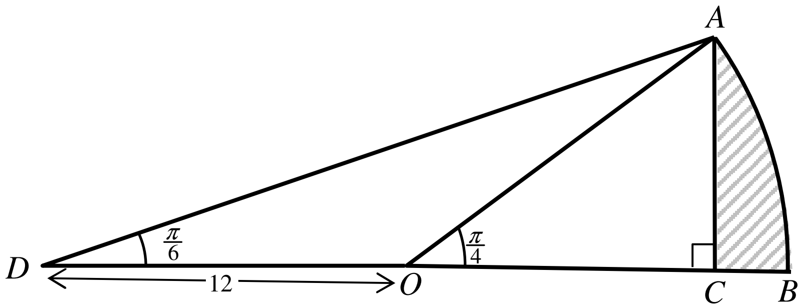
Solve, without the use of any calculating aid, the quadratic equation

$$5x^2 - 9x - 1 = 0,$$

giving the answers correct to one decimal place.

Detailed workings must be shown in this question. **(4)**

Question 12



The figure above shows a triangle OAC with $\measuredangle ACO = \frac{1}{2}\pi$ and $\measuredangle AOC = \frac{1}{4}\pi$.

Another triangle AOD is drawn next to the triangle OAC, so that DOC is a straight line, $|DO| = 12$ units and $\measuredangle ADO = \frac{1}{6}\pi$.

Finally a circular sector OAB is drawn, centred at O, with radius OA, so that $DOCB$ is a straight line.

a) Find the area of the sector OAB. **(5)**

b) Hence show that the area of the shaded region ACB is approximately 67 square units. **(5)**

Question 13

A curve C is given parametrically by

$$x = \frac{1}{t}, \quad y = t^2, \quad t \in \mathbb{R}, \quad t \neq 0.$$

The point P lies on C at the point where $t = 1$.

a) Show that an equation of the tangent to C at P is

$$y + 2x = 3. \qquad \textbf{(4)}$$

The tangent to C at P meets the curve again at the point Q.

b) Determine the coordinates of Q. **(6)**

Question 14

A curve C is given by the implicit equation

$$x^2 + 2xy - 3y^2 = 4x + 4y - 20.$$

a) Show clearly that

$$\frac{dy}{dx} = \frac{x + y - 2}{3y - x + 2}. \qquad \textbf{(5)}$$

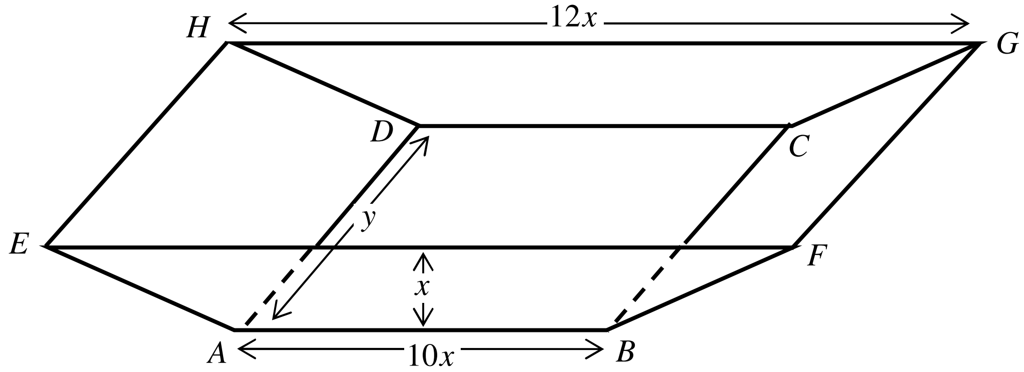
b) Find the coordinates of the stationary points of C. **(6)**

c) Show further that

$$(x - 3y - 2)\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} + 1 = 0. \qquad \textbf{(3)}$$

d) Hence determine the nature of these turning points. **(3)**

Question 15



The figure above shows the design of a baking tray with a horizontal rectangular base $ABCD$, measuring $10x$ cm by y cm.

The faces $ABFE$ and $DCGH$ are isosceles trapeziums, parallel to each other.

The lengths of the edges EF and HG are $12x$ cm.

The faces $ADHE$ and $BCGF$ are identical rectangles.

The height of the tray is x cm.

The capacity of the tray is 1980 cm^3.

a) Show that the surface area, $A \text{ cm}^2$, of the tray is given by

$$A = 22x^2 + \frac{360}{x}\left(5+\sqrt{2}\right).$$ **(6)**

b) Determine the value of x for which A is stationary, showing that this value of x minimizes the value for A. **(6)**

c) Calculate the minimum surface area of the tray. **(1)**

Question 16

$$f(x) = a - |x - 2a|, \ x \in \mathbb{R},$$

where a is a positive constant.

a) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph meets the coordinate axes, and the coordinates of the cusp of the curve. **(4)**

b) Find the value of $\int_0^{3a} f(x) \, dx$. **(3)**

Question 17

The function f satisfies

$$f(x) \equiv x^2 - 4x + 1, \ x \in \mathbb{R}, \ x > 4.$$

a) Find an expression for $f^{-1}(x)$. **(4)**

b) Determine the domain and range of $f^{-1}(x)$. **(1)**

c) Solve the equation

$$f(x) = f^{-1}(x).$$ **(5)**

Question 18

A large water tank is in the shape of a cuboid with a rectangular base measuring 10 m by 5 m, and a height of 5 m.

Let h m be the height of the water in the tank and t the time in hours.

At a certain instant, water begins to pour into the tank at the constant rate of 50 m^3 per hour and at the same time water begins to drain from a tap at the bottom of the tank at the rate of $10h \text{ m}^3$ per hour.

a) Show clearly that

$$5\frac{dh}{dt} = 5 - h.$$ **(5)**

b) Show further that it takes $5\ln 3$ hours for the height of the water to rise from 2 m to 4 m. **(8)**

Question 19

Solve the following trigonometric equation

$$2\arctan\left(\frac{3}{x}\right) = \arctan\left(\frac{2x}{9}\right).$$ **(6)**

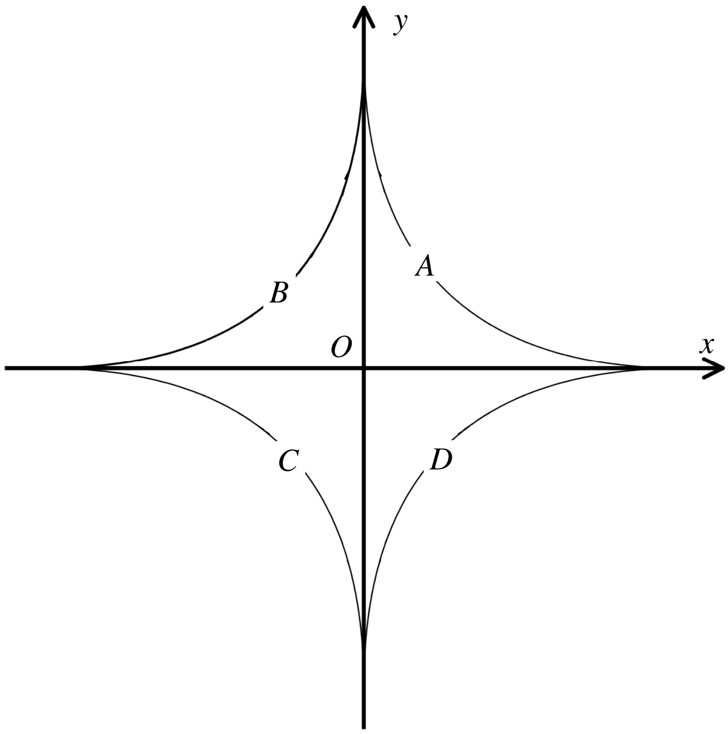
Question 20

Solve the following exponential equation.

$$4^{x+1} \times 3^{1-2x} = 24.$$

Give the answer correct to 3 decimal places. **(7)**

Question 21



The figure above shows a star shaped curve consisting of four distinct sections, each in a separate quadrant, labelled as A, B, C and D.

The equation of A is

$$\sqrt{x}+\sqrt{y}=1, \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Determine the equations for each of the remaining sections B, C and D. **(3)**

Question 22

By using trigonometric identities, show that

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}\, dx = \frac{1}{8}(16-3\pi). \qquad \mathbf{(12)}$$

Question 23

The point $A(a,b)$, where a and b are constants, lie on the straight line with equation

$$y = 2x + 1.$$

The point $B(a,c)$, where c is a constant, lie on the straight line with equation

$$2y + 3x = 6.$$

The point $P\left[a, \frac{1}{2}(b+c)\right]$ lies on the straight line L.

Determine an equation of L. **(7)**