IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper M

Difficulty Rating: 4.02/0.7071

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 24 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

Solve the following inequality.

$$\frac{2}{x+1} < x \qquad (7)$$

Question 2

A circle C has equation

$$x^2 + y^2 - 6x - 10y + k = 0,$$

where k is a constant.

a) Determine the coordinates of the centre of C. **(3)**

The x axis is a tangent to C at the point P.

b) State the coordinates of P and find the value of k. **(2)**

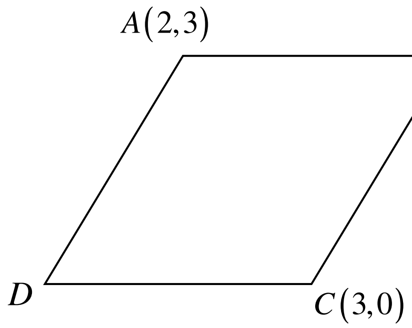
Question 3

$$f(x) = \frac{3x-1}{(1-2x)^2}, \quad |x| < \frac{1}{2}$$

Show that if x is small, then

$$f(x) \approx -1 - x + 4x^3. \qquad (6)$$

Question 4



The figure above shows a rhombus $ABCD$, where the vertices A and C have coordinates $(2,3)$ and $(3,0)$, respectively.

a) Show that an equation of the diagonal BD is

$$x - 3y + 2 = 0.$$ **(5)**

b) Given that an equation of the line through A and D is

$$3x - 4y + 6 = 0,$$

find the coordinates of D. **(3)**

c) State the coordinates of B. **(2)**

Question 5

Determine the value of k.

$$\frac{2^{399} - 2^{395}}{15} = 32^k.$$

You must show full workings. **(5)**

Question 6

A polynomial $p(x)$ is given by

$$p(x) = 4x^3 - 2x^2 + x + 5.$$

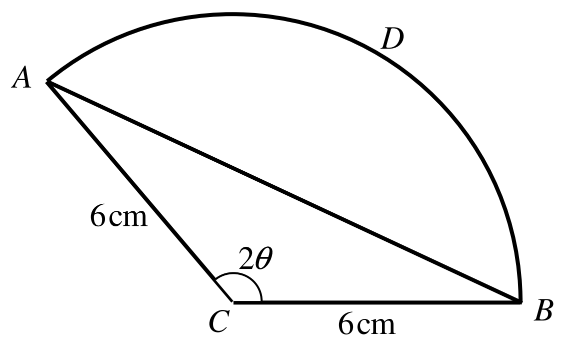
a) Find the remainder and the quotient when $p(x)$ is divided by $x^2 + 2x - 5$. **(3)**

A different polynomial $q(x)$ is defined as

$$q(x) = 4x^3 - 2x^2 + ax + b.$$

b) Find the value of the constants a and b so that when $q(x)$ is divided by $x^2 + 2x - 5$ there is no remainder. **(6)**

Question 7

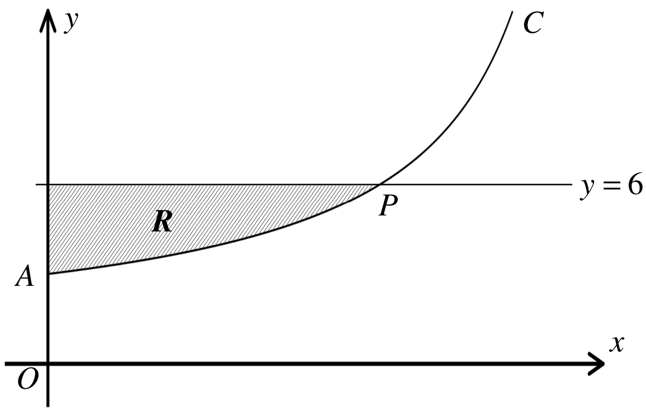


The figure above shows a sector $CADB$, of radius 6 cm and angle 2θ radians.

Given that the area of the triangle ABC and the area of segment ABD are in the ratio $4:1$, show that

$$8\theta - 5\sin 2\theta = 0.$$ **(6)**

Question 8



The figure above shows the curve C, with parametric equations

$$x = 6t\sin t, \quad y = 3\sec t \;, \; 0 \le t < \frac{\pi}{2}.$$

The curve meets the coordinate axes at the point A.

The line $y = 6$ meets C at the point P.

a) Show that the area **under** the arc of the curve between A and P, and the x axis is given by the integral

$$18\int_0^{\frac{\pi}{3}} t + \tan t \;\; dt. \qquad (5)$$

The shaded region R is bounded by C, the line $y = 6$ and the y axis.

b) Show that the area of R is approximately 10.3 square units. **(5)**

Question 9

Relative to a fixed origin O, the position vectors of three points A, B and C are

$$\overrightarrow{OA} = \mathbf{i} - 2\mathbf{k}, \quad \overrightarrow{AB} = 2\mathbf{i} + 10\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \overrightarrow{BC} = 6\mathbf{i} - 12\mathbf{j}.$$

a) Show that $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{AB}$. **(4)**

b) Show further that the area of the triangle ABC is $18\sqrt{6}$. **(2)**

c) Hence, or otherwise, determine the shortest distance of A from the straight line through B and C. **(2)**

Question 10

Use an appropriate substitution, followed by partial fractions, to show that

$$\int_{e^3}^{e^5} \frac{5}{2x\left[(\ln x)^2 + \ln x - 6\right]} \, dx = \ln\left(\frac{3}{2}\right).$$

You may assume that the integral converges. **(10)**

Question 11

$$f(x) = \ln(1 + \sin x), \ \sin x \neq \pm 1.$$

Show clearly that

$$f(x) - f(-x) = 2\ln(\sec x + \tan x). \quad \textbf{(6)}$$

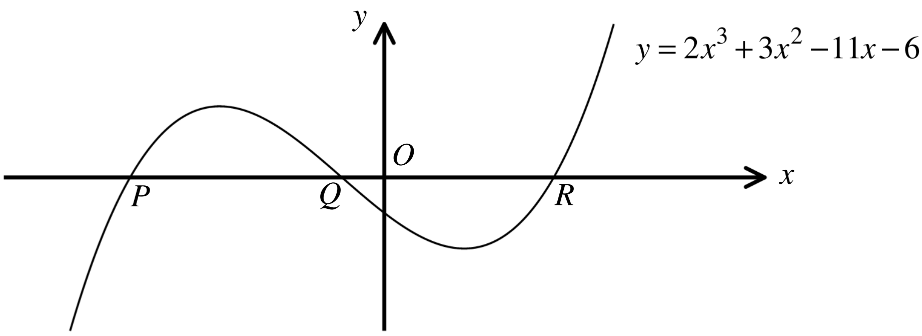
Question 12

Find, in exact simplified surd form, the roots of the following equation.

$$\sqrt{3}\left(x+\frac{6}{x}\right)=9,\ x\neq 0.$$

Detailed workings must be shown in this question. **(6)**

Question 13



The figure above shows the curve with equation

$$y=2x^3+3x^2-11x-6.$$

The curve crosses the x axis at the points P, Q and $R(2,0)$.

The tangent to the curve at R is the straight line L_1.

a) Find an equation of L_1. **(5)**

The normal to the curve at P is the straight line L_2.

The point S is the point of intersection between L_1 and L_2.

b) Show that $\measuredangle PSR=90°$. **(7)**

Question 14

The sum of the first n terms of a geometric series is denoted by S_n.

The common ratio of the series, r, is greater than 1.

a) If $S_4 = 5S_2$ find the value of r. **(6)**

b) Given further that $S_3 = 21$ determine the value of S_{10}. **(4)**

Question 15

A function f is defined by

$$f(x) = 2 + \frac{1}{x+1}, \quad x \in \mathbb{R}, \ x \geq 0.$$

a) Find an expression for $f^{-1}(x)$, as a simplified fraction. **(4)**

b) Find the domain and range of $f^{-1}(x)$. **(3)**

Question 16

Show that the following simultaneous equations

$$e^{2y} + 4 = x$$

$$\ln(x+1) = 2y - 1$$

are satisfied by the solution pair

$$x = \frac{4+e}{1-e}, \quad y = \frac{1}{2}\ln\left(\frac{5e}{1-e}\right)$$

and hence explain why the equations have no real solutions. **(7)**

Question 17

The curve C has equation

$$y = \frac{2x+3}{x-2}, \ x \in \mathbb{R}, \ x \neq 2.$$

a) Show clearly that

$$\frac{2x+3}{x-2} \equiv 2 + \frac{7}{x-2}. \qquad \textbf{(2)}$$

b) Find the coordinates of the points where C meets the coordinate axes. **(2)**

c) Sketch the graph of C showing clearly the equations of any asymptotes. **(4)**

d) Determine the coordinates of the points of intersection of C and the straight line with equation

$$y = 7x - 12. \qquad \textbf{(4)}$$

Question 18

Find the range of values that the constant k can take so that

$$2x^2 + (k+2)x + k = 0$$

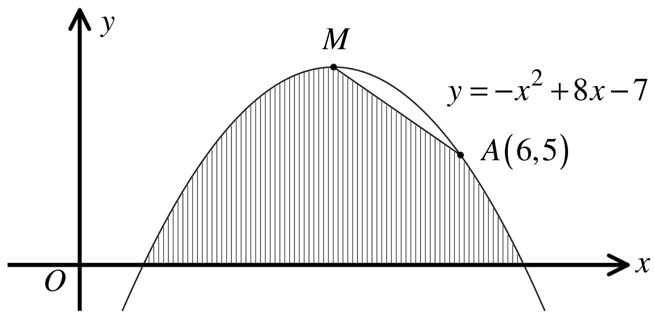
has two distinct real roots. **(6)**

Question 19

Find the set of values of x for which

$$\left| x^2 - 4 \right| > 3x. \qquad \textbf{(6)}$$

Question 20



The figure above shows the quadratic curve with equation

$$y=-x^2+8x-7.$$

The point M is the maximum point of the curve and A is another point on the curve whose coordinates are $(6,5)$.

Find the exact area of the shaded region, bounded by the curve, the x axis and the straight line segment from A to M. **(10)**

Question 21

A grass lawn has an area of 225 m^2 and has become host to a parasitic weed.

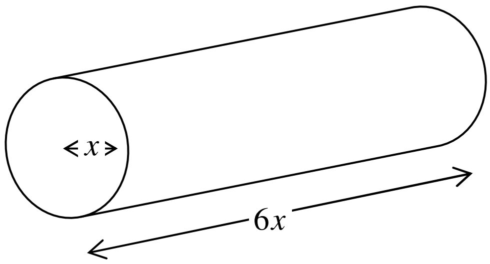
Let $A \text{ m}^2$ be the area covered by the parasitic weed, t days after it was first noticed.

The rate at which A is growing is proportional to the square root of the area of the lawn already covered by the weed.

Initially the parasitic weed has spread to an area of 1 m^2, and **at that instant** the parasitic weed is growing at the rate of 0.25 m^2 per day.

By forming and solving a suitable differential equation, calculate after how many days, the weed will have spread to the entire lawn. **(10)**

Question 22



A solid machine component, made of metal, is in the shape of a right circular cylinder, with radius x cm and length $6x$ cm.

The component is heated so that it is expanding at the constant rate of $\frac{6}{7}\pi \text{ cm}^3\text{s}^{-1}$.

Given that the initial volume of the component was $36\pi \text{ cm}^3$, find the rate at which the surface area of the component is increasing 14 s after the heating started.

You may assume that the shape of the component is mathematically similar to its original shape at all times. **(10)**

Question 23

It is given that

$$x = t^{\frac{1}{2}}, \ t > 0.$$

Given further that y is a function of x, show clearly that

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dt} + 4t\frac{d^2y}{dt^2}.$$ **(7)**

Question 24

$$f(x) = 3\sin x - \cos x + 3, \ x \in \mathbb{R}.$$

$$g(x) = \sin x + \cos x, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form

$$A \times g(x) + B \times g'(x) + 3,$$ **(5)**

where A and B are constants.

b) Express $g(x)$ in the form

$$R\cos(x - \varphi),$$

where R and φ are positive constants. **(3)**

c) Hence find a simplified expression for

$$\int \frac{f(x)}{g(x)} \, dx.$$ **(7)**