IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper P

Difficulty Rating: 3.9925/0.7472

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 22 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

Determine the value of the positive constant k given further that

$$\int_{k}^{8} \frac{4}{2x-1}\, dx = 1.90038 .$$

Give the value of k to an appropriate degree of accuracy. **(5)**

Question 2

A circle has equation

$$x^2 + y^2 + ax + by = 0,$$

where a and b are constants.

The straight lines with equations

$$y = x - 4 \qquad \text{and} \qquad x + y = 2$$

are both diameters of this circle.

Determine the length of the radius of the circle. **(6)**

Question 3

A triangle PQR has $|PQ| = x + 2$ cm, $|QR| = (2-x)^2$ cm and $\measuredangle PQR = 30°$.

a) Show that the area of the triangle, A cm^2, is given by

$$A = \tfrac{1}{4}\left(x^3 - 2x^2 - 4x + 8\right).$$ **(4)**

b) Determine the value of x for which A is stationary and hence find, with justification, the greatest value of A. **(8)**

Question 4

$$f(x) = (1-2x)^{-\frac{1}{2}}.$$

a) Expand $f(x)$ up and including the term in x^2. **(3)**

b) State the values of x for which the expansion is valid. **(1)**

c) By substituting $x = \frac{1}{8}$ in the expansion of part **(a)** show that

$$\sqrt{3} \approx \frac{256}{147}.$$ **(3)**

Question 5

A cubic function is defined in terms of the constants a, b and c as

$$f(x) = x^3 + ax^2 + bx + c,\ x \in \mathbb{R}.$$

a) Given that $(x-1)$ is a factor of $f(x)$ show that

$$a+b+c = -1.$$ **(1)**

It is further given that when $f(x)$ is divided by $(x-2)$ the remainder is -4 and when $f(x)$ is divided by $(x-3)$ the remainder is -12.

b) Find the values of a, b and c. **(5)**

c) Hence express $f(x)$ as the product of three linear factors. **(3)**

d) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes. **(3)**

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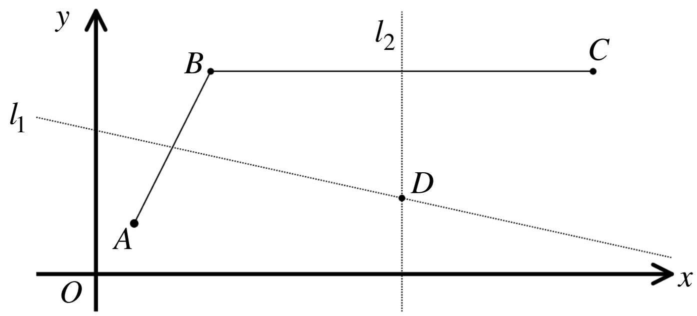
Question 6



The points $A(1,2)$, $B(3,8)$ and $C(13,8)$ are shown in the figure above.

The straight lines l_1 and l_2 are the perpendicular bisectors of the straight line segments AB and BC, respectively.

a) Find an equation for l_1. **(5)**

The point D is the intersection of l_1 and l_2.

b) Show by a direct algebraic method that D is equidistant from A, B and C.
You may not use any circle theorems in this part of the question. **(8)**

Question 7

Prove that if we subtract 1 from a positive odd square number, the answer is always divisible by 8. **(5)**

Question 8

Given that a is positive constant greater than 1, solve the following logarithmic equation

$$\log_a x = \log_{a^2}(x+20). \qquad (7)$$

Question 9

Find the value of the constant p, so that

$$\sum_{n=1}^{20} (25 + np) = 80 .$$ **(6)**

Question 10

Determine, in exact simplified form where appropriate, the solutions for each of the following equations.

a) $e^{2x} + 2 = 3e^{x}$. **(4)**

b) $e^{2y-2} + 2 = 3e^{y-1}$. **(3)**

c) $e^{t} = 3^{\frac{3}{\ln 3}}$. **(3)**

Question 11

The piecewise continuous function f is **even** with domain $x \in \mathbb{R}$, $-6 \le x \le 6$.

It is defined by

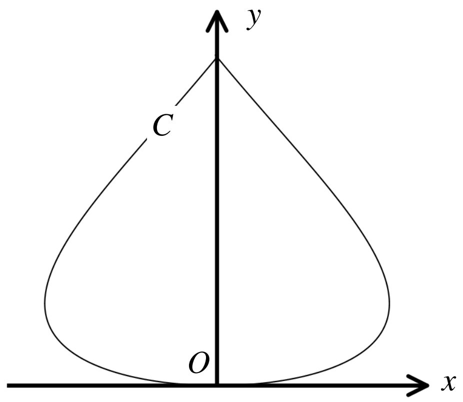
$$f(x) \equiv \begin{cases} x & 0 \le x \le 2 \\ 3 - \frac{1}{2}x & 2 \le x \le 6 \end{cases}$$

a) Sketch the graph of f for $-6 \le x \le 6$. **(4)**

b) Hence, solve the equation

$$x = 4 + 5f(x) .$$ **(7)**

Question 12



The figure above shows the curve C with parametric equations

$$x = \sin t, \quad y = t^2, \quad 0 \le t \le 2\pi.$$

It is given that C is symmetrical about the y axis.

Show that the area enclosed by C can be found by the integral

$$\int_0^{\pi} 4t \sin t \; dt,$$

and hence find an exact value for this area. **(10)**

Question 13

$$f(x) = k + 12x - 4x^2,$$

where k is a constant.

It is further given that $f(x) > 5$ for **some** values of x.

Show by suitable discriminant calculations, or otherwise, that

$$k > -4.$$ **(6)**

Question 14

$$f(x) = ax^2 + bx + c,$$

where a, b and c are non zero constants.

Given that $f(-1) = f(5) = 30$ and that the minimum value of $f(x)$ is -6, solve the equation $f(x) = 3$. **(7)**

Question 15

$$f(x) = a - \frac{1}{b-x}, \quad x \in \mathbb{R}, \quad x \neq b,$$

where a and b are positive constants such that $ab > 1$.

Sketch the graph of $f(x)$.

The sketch must include, in terms of a and b, …

- … the coordinates of the points where $f(x)$ meets the coordinate axes.
- … the equations of any asymptotes of the curve. **(6)**

Question 16

The triangle ABC is right angled at the vertex B.

The point D lies on AC so that $|BD| = |BC|$.

Given that the area of the triangle BDC is 3 times as large as the area of the triangle ABD, show that

$$4\sin\theta = 3\tan\theta,$$

where θ denotes the angle BCA. **(7)**

Question 17

It is given that

$$\tan 3x \equiv \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}, \quad x \neq \frac{n\pi}{3}, \quad n = 0,1,2,3,...$$

a) Use the above identity to express $\cot 3x$ in terms of $\cot x$. **(3)**

b) Show clearly that

$$\frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} \equiv \cot x, \quad \cos x \neq \tfrac{1}{2}. \quad \textbf{(3)}$$

a) Hence, or otherwise, given that $\cos 3x \neq \tfrac{1}{2}$ solve the trigonometric equation

$$\cos 6x + \sin 6x - \cos 3x - \sin 3x + 1 = 0,$$

for $0 < x < \pi$, giving the answers in terms of π. **(8)**

Question 18

$$f(x) \equiv \frac{2}{x+\sqrt{2x-1}}, \ x \geq \frac{1}{2}.$$

a) Use the substitution $u=\sqrt{2x-1}$ transforms to show

$$\int_{1}^{5} f(x) \ dx \equiv \int_{u_1}^{u_2} \frac{4u}{(u+1)^2} \ du, \qquad \textbf{(4)}$$

where u_1 and u_2 are constants to be found.

b) By using another suitable substitution, or otherwise, show that

$$\int_{1}^{5} f(x) \ dx = -1+\ln 16. \qquad \textbf{(7)}$$

Question 19

A snowball is melting and its shape remains spherical at all times.

The volume of the snowball, V cm^3, is decreasing at constant rate.

Let t be the time in hours since the snowball's radius was 18 cm.

Ten hours later its radius has reduced to 9 cm.

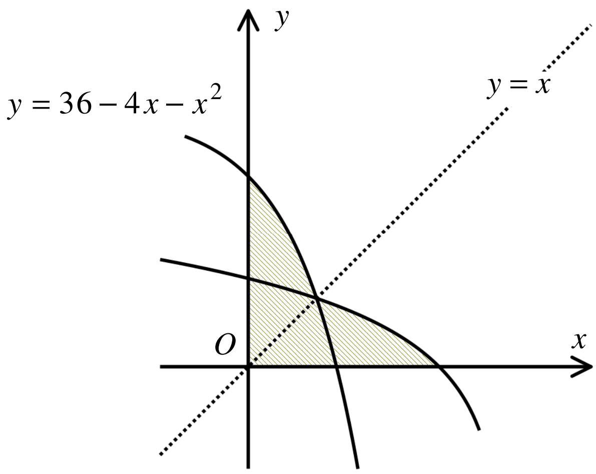
Show that the volume V of the melting snowball satisfies

$$V = 97.2\pi(80-7t),$$

and hence find the value of t when the radius of the snowball has reduced to 4.5 cm.

$\left[\text{volume of a sphere of radius } r \text{ is given by } \frac{4}{3}\pi r^3\right]$ **(10)**

Question 20



The figure above shows the graph of the curve with equation $y = 36 - 4x - x^2$ and its reflection about the straight line with equation $y = x$.

a) Write down the equation of the curve which is the reflection of the curve with equation $y = 36 - 4x - x^2$ about the straight line with equation $y = x$. **(1)**

b) Determine an exact value for the area of the finite region bounded by the curve with equation $y = 36 - 4x - x^2$, its reflection about the straight line with equation $y = x$, and the positive x and y axes.

This region is shown shaded in the figure above. **(8)**

Question 21

Solve the following equation

$$\sum_{r=2}^{\infty}\left(2^{x-r}\right) = \sqrt{1+3\times 2^{x-2}}\,.$$

You may assume that the left hand side of the equation converges. **(9)**

Question 22

The point P has x coordinate 2 and lies on the curve with equation

$$xy = e^x, \qquad xy > 0.$$

a) Determine an equation of the tangent to the curve at P.

The tangent to the curve found in part **(a)** meets the curve again at the point Q. **(6)**

b) Show that the x coordinate of Q is -0.6, correct to one significant figure. **(4)**

c) Use the Newton Raphson method **twice** to find a better approximation for the x coordinate of Q, giving the answer correct to 4 significant figures. **(7)**