IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper S

Difficulty Rating: 3.9250/ 0.6747

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

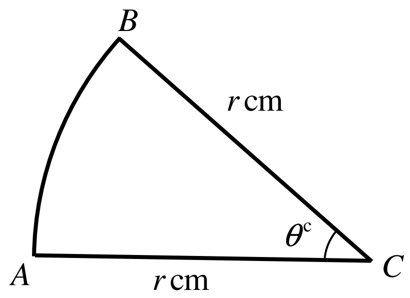
This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. (**2**).
There are 23 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1



The figure above shows a circular sector ABC of radius r cm subtending an angle θ radians at C.

The length of the arc AB is $\frac{2}{9}$ of the perimeter of the sector.

Show that $\theta = \frac{4}{7}$ radians. **(4)**

Question 2

The k^{th} term of a geometric progression is given by

$$u_k = 15625 \times 1.25^{-k}.$$

a) Find the first three terms of the progression. **(1)**

b) Find the sum to infinity of the progression. **(2)**

c) Evaluate the sum

$$\sum_{k=1}^{10} u_k ,$$

giving the answer to the nearest integer. **(2)**

Question 3

The straight lines l_1 and l_2 have equations

$$l_1:\ 2x+y=10,$$

$$l_2:\ 3x-4y=10.$$

a) Sketch l_1 and l_2 in a single set of axes.

The sketch must include the coordinates of all the points where each of these straight lines meet the coordinate axes. **(5)**

The two lines intersect at the point P.

b) Use algebra to determine the exact coordinates of P. **(3)**

Question 4

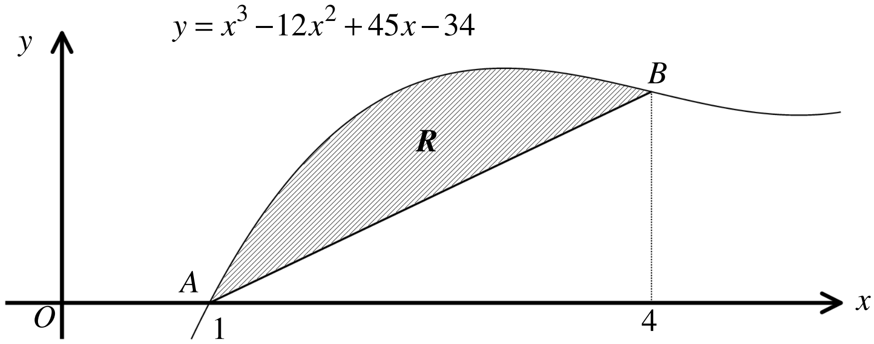
The function f is defined

$$f: x \mapsto \frac{2x-3}{x-2},\ x \in \mathbb{R},\ x \neq 2.$$

a) Find an expression for $f^{-1}(x)$ in its simplest form. **(4)**

b) Hence, or otherwise, find in its simplest form $ff(k+2)$. **(2)**

Question 5



The figure above shows the curve with equation

$$y = x^3 - 12x^2 + 45x - 34.$$

The points A and B lie on the curve, where $x = 1$ and $x = 4$, respectively.

The finite region R is bounded by the curve and the straight line segment AB.

Show that the area of R, shown shaded in the figure, is exactly $\frac{81}{4}$. **(8)**

Question 6

$$f(x) \equiv 2x^3 - 9x^2 - 11x + 30.$$

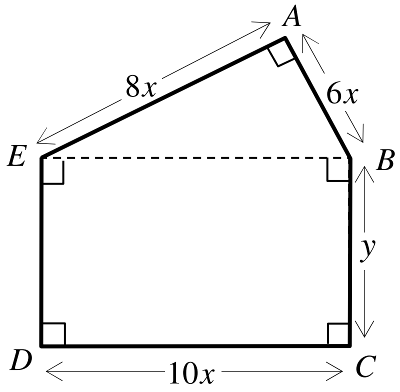
a) Show, by using the factor theorem, that $(x-5)$ is a factor of $f(x)$ and hence factorize $f(x)$ into product of three linear factors. **(5)**

b) Sketch the graph of $f(x)$.

The sketch must include the coordinates of all the points where the graph meets the coordinate axes. **(3)**

c) Find the x coordinates of the points where the line with equation $y = 7x + 30$ meets the graph of $f(x)$. **(4)**

Question 7



The figure above shows a pentagon $ABCDE$ whose measurements, in cm, are given in terms of x and y.

a) If the perimeter of the pentagon is 120 cm, show clearly that its area, A cm^2, is given by

$$A = 600x - 96x^2.$$ **(4)**

b) Use a method based on differentiation to calculate the maximum value for A, fully justifying the fact that it is indeed the maximum value. **(7)**

Question 8

The half life of a radioactive isotope is the time it takes for a given mass to reduce to half the size of that given mass.

A radioactive substance reduces from 12 g to exactly 10.24 g in 30 days.

Assuming that the isotope decays exponentially, determine the half life of the isotope, correct to the nearest day. **(8)**

Question 9

The points $P(-2,5)$ and $Q(6,-1)$ lie on a circle so that the chord PQ is a diameter of this circle.

a) Find an equation for this circle. **(4)**

The straight line with equation $y=6$ intersects the circle at the points A and B.

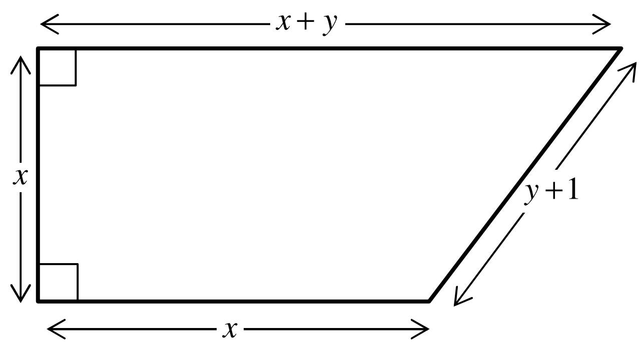
b) Determine the shortest distance of AB from the centre of the circle and hence, or otherwise, find the distance AB. **(4)**

Question 10

Simplify $(\tan x + \cot x)\sin 2x$ and hence prove that

$$\tan\left(\tfrac{1}{8}\pi\right) + \tan\left(\tfrac{5}{12}\pi\right) + \cot\left(\tfrac{1}{8}\pi\right) + \cot\left(\tfrac{5}{12}\pi\right) = 4+2\sqrt{2}.$$ **(7)**

Question 11



The figure above shows a right angled trapezium whose measurements are given in terms of x and y.

The trapezium has and a perimeter of 27 and an area of 30.

Determine the value x and the value of y, and hence show that the above trapezium does **not** exist. **(12)**

Question 12

Solve the inequality

$$x^2 + 2y^2 < 3xy,$$

and hence indicate the solution in a suitable sketch. **(8)**

Question 13

The variables x, y and t satisfy

$$\frac{dx}{dt} = kx \qquad \text{and} \qquad 2\left(x^2 + y^2\right) = 5xy,$$

where k is a non zero constant.

Find, in terms of k, the possible values of $\frac{dy}{dt}$ when $x = 2$. **(10)**

Question 14

If x is sufficiently small find the series expansion of

$$\frac{10x^2 - x - 6}{(2+3x)\left(1-2x^2\right)},$$

up and including the term in x^3. **(12)**

Question 15

A curve has parametric equations

$$x = \frac{3}{t^2}, \quad y = 5t^2 \ , \ t > 0.$$

If the tangent to the curve at the point P passes through the point with coordinates $\left(\frac{9}{2}, \frac{5}{2}\right)$, determine the possible coordinates of P. **(10)**

Question 16

A curve has equation

$$y = f(2x+3).$$

a) Describe the two geometric transformations which map the graph of $y = f(2x+3)$ onto the graph of $y = f(x)$. **(3)**

b) Describe a **different** set of two geometric transformations which map the graph of $y = f(2x+3)$ onto the graph of $y = f(x)$. **(3)**

The description must be formal, clearly indicating the order in which the two transformations take place.

Question 17

$$f(x) = 2^{4x}$$

Show that the solution of the equation $f(x-1) = \frac{5}{8}$ is given by

$$x = \frac{1}{4\log_{10} 2}.$$ **(7)**

Question 18

It is required to find the real solutions of the equation

$$x^2 = 2^x .$$

a) State the 2 integer solutions of the equation. **(1)**

b) Sketch in the same set of axes the graph of $y = x^2$ and the graph of $y = 2^x$. **(2)**

c) Use the Newton Raphson method, with a suitable function and an appropriate starting value, to find the third real root of this equation correct to 4 decimal places.

You may use as many steps as necessary in part **(c)**, to obtain the required accuracy. **(7)**

Question 19

$$f(x) = \sec^2 x, \ x \in \mathbb{R}, \ x \neq \frac{\pi}{2}(2n+1), \ n \in \mathbb{N} .$$

Show that if

$$f(x) = \frac{1}{2} f\left(x + \frac{\pi}{4}\right),$$

then either $\sin x = 0$ or $\tan x = 2$. **(9)**

Question 20

A large cylindrical water tank has a height of 16 m and a horizontal cross section of constant area 20 m^2.

Water is pouring into the tank at a constant rate of 10 m^3 per hour and leaking out of a tap at the base of the tank at a rate $\sqrt{x} \text{ m}^3$ per hour, where x is the height of the water in the tank, in m, at time t hours.

a) Show that

$$20\frac{dx}{dt} = 10 - \sqrt{x}. \qquad (5)$$

The water in the cylinder had an initial height of 9 m.

b) Solve the differential equation of part **(a)** to find, correct to the nearest hour, the time it takes to fill up the tank. **(12)**

Question 21

An arithmetic progression has first term 11.

The sum of its **first** 20 terms is 1360, and the sum of its **last** 20 terms is 4720.

Determine the number of terms in the progression. **(8)**

Question 22

A triangle OAB is given.

The point M is the midpoint of OA.

The point N lies on OB so that $|ON| : |NB| = 1 : 5$

If the point P is the intersection of the straight lines AN and BM, use vector algebra to find the ratio of $|AP| : |EP|$. **(8)**

Question 23

The function f is defined as

$$f(A,B) \equiv A^4 + 4B^4, \quad A \in \mathbb{R}, \quad B \in \mathbb{R}.$$

a) By completing the square, or otherwise, factorize f into 2 quadratic factors **(4)**

b) Hence factorize $x^4 + 64$. **(2)**