BESSEL EQUATION and BESSEL FUNCTIONS

Summary of Bessel Functions

Bessel's Equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right) y = 0 .$$

If n is an integer, the two independent solutions of Bessel's Equation are

- $J_n(x)$, Bessel function of the first kind,

  $$J_n(x) = \sum_{p=0}^{\infty} \left[\frac{(-1)^p}{(n+p)!\,p!} \left(\frac{x}{2}\right)^{2p+n} \right]$$

  Generating function for $J_n(x)$

  $$\mathrm{e}^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} \left[t^n J_n(x) \right]$$

- $Y_n(x)$, Bessel function of the second kind

  $$Y_n(x) = \frac{2}{\pi} \ln\left(\frac{1}{2}x\right) J_n(x) - \frac{1}{\pi}\left(\frac{1}{2}x\right)^{-n} \sum_{p=0}^{n-1} \left[\frac{(n-1-p)!}{p!} \left(\frac{1}{2}x\right)^{2p} \right]$$

  $$+\frac{1}{\pi}\left(\frac{1}{2}x\right)^{n} \sum_{p=0}^{n-1} \left[\frac{(-1)^p}{(p+n)!\,p!} \left(\frac{1}{2}x\right)^{2p} \left[2\gamma - \sum_{m=1}^{p}\left(\frac{1}{m}\right) - \sum_{m=1}^{p+n}\left(\frac{1}{m}\right) \right] \right]$$

Other relations for $J_n(x)$, $n \in \mathbb{Z}$.

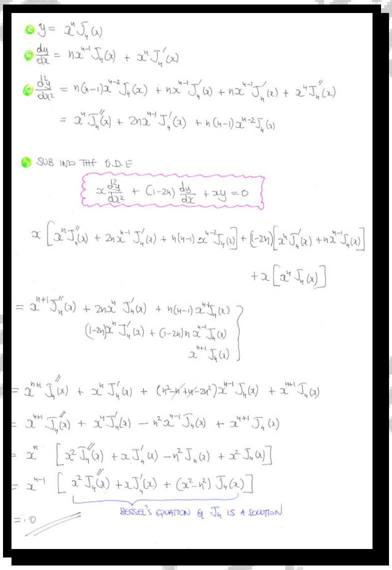
- $J_{-n}(x) = (-1)^n J_n(x)$.
- $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$
- $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$
- $J_n(x) = \sum_{p=0}^{\infty}\left[\frac{(-1)^p}{(n+p)!\,p!}\left(\frac{x}{2}\right)^{2p+n}\right]$
- $J_0(x) = \sum_{p=0}^{\infty}\left[\frac{(-1)^p}{(p!)^2}\left(\frac{x}{2}\right)^{2p}\right] = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \ldots$
- $J_1(x) = \sum_{p=0}^{\infty}\left[\frac{(-1)^p}{(p+1)!\,p!}\left(\frac{x}{2}\right)^{2p+1}\right] = \frac{x}{2^1 0!1!} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + \ldots$

Question 1

$$x\frac{d^2y}{dx^2}+(1-2n)\frac{dy}{dx}+xy=0\,, \quad x\neq 0\,.$$

Show that $x^n J_n(x)$ is a solution of the above differential equation.

proof



$y = x^n J_n(x)$

$\frac{dy}{dx} = nx^{n-1}J_n(x) + x^n J_n'(x)$

$\frac{d^2y}{dx^2} = n(n-1)x^{n-2}J_n(x) + nx^{n-1}J_n'(x) + nx^{n-1}J_n'(x) + x^n J_n''(x)$

$= x^n J_n''(x) + 2nx^{n-1}J_n'(x) + n(n-1)x^{n-2}J_n(x)$

SUB INTO THE O.D.E

$$x\frac{d^2y}{dx^2} + (1-2n)\frac{dy}{dx} + xy = 0$$

$x\left[x^n J_n''(x) + 2nx^{n-1}J_n'(x) + n(n-1)x^{n-2}J_n(x)\right] + (1-2n)\left[x^n J_n'(x) + nx^{n-1}J_n(x)\right]$

$+ x\left[x^n J_n(x)\right]$

$= x^{n+1}J_n''(x) + 2nx^n J_n'(x) + n(n-1)x^{n-1}J_n(x)$
$(1-2n)x^n J_n'(x) + (1-2n)nx^{n-1}J_n(x)$
$x^{n+1}J_n(x)$

$= x^{n+1}J_n''(x) + x^n J_n'(x) + (n^2 - \cancel{n} + \cancel{n} - 2n^2)x^{n-1}J_n(x) + x^{n+1}J_n(x)$

$= x^{n+1}J_n''(x) + x^n J_n'(x) - n^2 x^{n-1}J_n(x) + x^{n+1}J_n(x)$

$= x^n \left[x^2 J_n''(x) + xJ_n'(x) - n^2 J_n(x) + x^2 J_n(x)\right]$

$= x^{n-1}\left[\underbrace{x^2 J_n''(x) + xJ_n'(x) + (x^2-n^2)J_n(x)}_{\text{BESSEL'S EQUATION \& } J_n \text{ IS A SOLUTION}}\right]$

$= 0$

Question 2

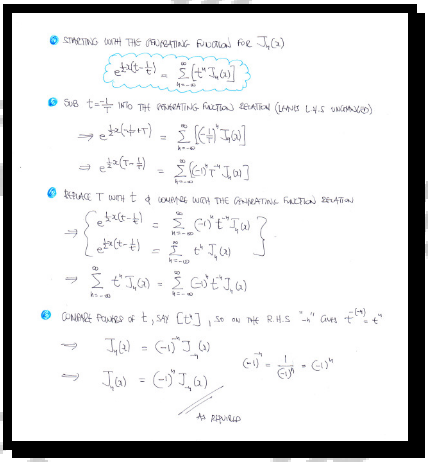
Starting from the generating function of the Bessel function of the first kind

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right],\ n \in \mathbb{Z},$$

show that

$$J_n(x) = (-1)^n J_{-n}(x).$$

$\boxed{\text{proof}}$



- STARTING WITH THE GENERATING FUNCTION FOR $J_n(x)$

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right]$$

- SUB $t = -\frac{1}{T}$ INTO THE GENERATING FUNCTION RELATION (LEAVES L.H.S UNCHANGED)

$$\Rightarrow e^{\frac{1}{2}x\left(-\frac{1}{T}+T\right)} = \sum_{n=-\infty}^{\infty}\left[\left(-\frac{1}{T}\right)^n J_n(x)\right]$$

$$\Rightarrow e^{\frac{1}{2}x\left(T-\frac{1}{T}\right)} = \sum_{n=-\infty}^{\infty}\left[(-1)^n T^{-n} J_n(x)\right]$$

- REPLACE T WITH t & COMPARE WITH THE GENERATING FUNCTION RELATION

$$\Rightarrow \left\{\begin{array}{l} e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}(-1)^n t^{-n} J_n(x) \\ e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x) \end{array}\right\}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} t^n J_n(x) = \sum_{n=-\infty}^{\infty}(-1)^n t^{-n} J_n(x)$$

- COMPARE POWERS OF t, SAY $[t^n]$, SO ON THE R.H.S "$-n$" GIVES $t^{-(-n)} = t^n$

$$\Rightarrow J_n(x) = (-1)^{-n} J_{-n}(x)$$

$$\Rightarrow J_n(x) = (-1)^n J_{-n}(x)$$

$$(-1)^{-n} = \frac{1}{(-1)^n} = (-1)^n$$

AS REQUIRED

Question 3

Starting from the series definition of the Bessel function of the first kind

$$J_n(x) = \sum_{r=0}^{\infty} \left[\frac{(-1)^r}{(n+r)!\,r!} \left(\frac{x}{2} \right)^{2r+n} \right], \; n \in \mathbb{Z},$$

show that

$$J_{-n}(x) = (-1)^n J_n(x).$$

$\boxed{\text{proof}}$

$$J_n(x) \equiv \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(r+n)!} \left(\frac{x}{2}\right)^{2r+n} \quad \& \quad J_{-n}(x) \equiv \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(r-n)!} \left(\frac{x}{2}\right)^{2r-n}$$

LET $q = r-n$ IN $J_{-n}(x)$

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(r-n)!} \left(\frac{x}{2}\right)^{2r-n} = \sum_{q=0}^{\infty} \frac{(-1)^{q+n}}{(q+n)!\,q!} \left(\frac{x}{2}\right)^{2(q+n)-n}$$

$$= (-1)^n \sum_{q=0}^{\infty} \frac{(-1)^q}{q!(q+n)!} \left(\frac{x}{2}\right)^{2q+n} = (-1)^n J_n(x)$$

Question 4

Starting from the generating function of the Bessel function of the first kind

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right], \quad n \in \mathbb{Z},$$

show that

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right].$$

proof

- STARTING WITH THE GENERATING FUNCTION FOR $J_n(x)$

$$\boxed{e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right]}$$

- DIFFERENTIATE BOTH SIDES W.R.T. x

$$\Rightarrow \frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \frac{1}{2}t\,e^{\frac{1}{2}x(t-\frac{1}{t})} - \frac{1}{2t}e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \frac{1}{2}t\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] - \frac{1}{2t}\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[t^{n+1} J_n(x)\right] - \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[t^{n-1} J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

- EQUATE POWER OF t, SAY $[t^n]$

$$\Rightarrow \frac{1}{2}J_{n-1}(x) - \frac{1}{2}J_{n+1}(x) = J_n'(x)$$

$$\Rightarrow J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

// AS REQUIRED

Question 5

Starting from the generating function of the Bessel function of the first kind

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right], \; n \in \mathbb{Z},$$

show that

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x).$$

$\boxed{\text{proof}}$

- STARTING WITH THE GENERATING FUNCTION FOR $J_n(x)$

$$e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right]$$

- DIFFERENTIATE BOTH SIDES W.R.T t

$$\Rightarrow \frac{1}{2}x\left(1+\frac{1}{t^2}\right)e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[nt^{n-1}J_n(x)\right]$$

$$\Rightarrow \frac{1}{2}x\,e^{\frac{1}{2}x(t-\frac{1}{t})} + \frac{1}{2t^2}x\,e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[nt^{n-1}J_n(x)\right]$$

$$\Rightarrow \frac{1}{2}x\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] + \frac{1}{2t^2}x\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = n\sum_{n=-\infty}^{\infty}\left[t^{n-1}J_n(x)\right]$$

$$\Rightarrow \frac{1}{2}x\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] + \frac{1}{2}x\sum_{n=\infty}^{\infty} t^{n-2}J_n(x) = n\sum_{n=-\infty}^{\infty}\left[t^{n-1}J_n(x)\right]$$

- EQUATE POWERS OF t, SAY $[t^n]$

$$\frac{1}{2}x\,J_n(x) + \frac{1}{2}x\,J_{n+2}(x) = (n+1)J_{n+1}(x)$$

- "SHIFT" POWERS DOWN BY 1

$$\frac{1}{2}x\,J_{n-1}(x) + \frac{1}{2}x\,J_{n+1}(x) = nJ_n(x)$$

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

// AS REQUIRED

Question 6

The generating function of the Bessel function of the first kind is

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right], \quad n \in \mathbb{Z}.$$

a) By differentiating the generating function relation with respect to x, show that

$$\tfrac{1}{2}J_{n-1}(x) - \tfrac{1}{2}J_{n+1}(x) = J_n'(x).$$

b) By differentiating the generating function relation with respect to t, show that

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right].$$

c) Hence find a simplified expression for

$$\frac{d}{dx}\left[\left(x^n + x^{-n}\right)J_n(x)\right].$$

$$\boxed{\quad}, \quad \boxed{\frac{d}{dx}\left[\left(x^n + x^{-n}\right)J_n(x)\right] = x^n J_{n-1}(x) - x^{-n} J_{n+1}(x)}$$

a) START BY DIFFERENTIATING THE GENERATING FUNCTION FOR $J_n(x)$, WITH RESPECT TO x

$$\Rightarrow e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right]$$

$$\Rightarrow \tfrac{1}{2}\left(t-\tfrac{1}{t}\right)e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \tfrac{1}{2}te^{\frac{1}{2}x(t-\frac{1}{t})} - \tfrac{1}{2t}e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \tfrac{1}{2}t\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] - \tfrac{1}{2t}\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\left[t^{n+1} J_n(x)\right] - \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\left[t^{n-1} J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

EQUATING POWERS OF t, SAY $[t^n]$, GIVES

$$\Rightarrow \tfrac{1}{2}t^n J_{n-1}(x) - \tfrac{1}{2}t^n J_{n+1}(x) = t^n J_n'(x)$$

$$\Rightarrow \tfrac{1}{2}J_{n-1}(x) - \tfrac{1}{2}J_{n+1}(x) = J_n'(x)$$

AS REQUIRED

b) NEXT DIFFERENTIATE WITH RESPECT TO t

$$\Rightarrow e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right]$$

$$\Rightarrow \tfrac{1}{2}x\left(1+\tfrac{1}{t^2}\right)e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[nt^{n-1} J_n(x)\right]$$

$$\Rightarrow \tfrac{1}{2}xe^{\frac{1}{2}x(t-\frac{1}{t})} + \tfrac{x}{2t^2}e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty}\left[nt^{n-1} J_n(x)\right]$$

$$\Rightarrow \tfrac{1}{2}x\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] + \tfrac{x}{2t^2}\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[nt^{n-1} J_n(x)\right]$$

$$\Rightarrow \tfrac{1}{2}x\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] + \tfrac{1}{2}x\sum_{n=-\infty}^{\infty}\left[t^{n-2} J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[nt^{n-1} J_n(x)\right]$$

EQUATING POWERS OF t, SAY $[t^{n-1}]$ GIVES

$$\Rightarrow \tfrac{1}{2}x\left[t^{n-1}J_{n-1}(x)\right] + \tfrac{1}{2}x\left[t^{n-1}J_{n+1}(x)\right] = nt^{n-1}J_n(x)$$

$$\Rightarrow \tfrac{1}{2}xJ_{n-1}(x) + \tfrac{1}{2}xJ_{n+1}(x) = nJ_n(x)$$

$$\Rightarrow J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

AS REQUIRED

c) DIFFERENTIATING, USING THE PRODUCT RULE, WITH RESPECT TO x

$$\frac{d}{dx}\left[\left(x^n+x^{-n}\right)J_n(x)\right] = \left(nx^{n-1}-nx^{-n-1}\right)J_n(x) + \left(x^n+x^{-n}\right)J_n'(x)$$

USING PART (b) AND PART (a) IN THE RESPECTIVE TERMS GIVES

$$= \left(nx^{n-1}-nx^{-n-1}\right)\frac{x}{2n}\left[J_{n-1}(x)+J_{n+1}(x)\right] + \left(x^n+x^{-n}\right)\left[\tfrac{1}{2}J_{n-1}(x)-\tfrac{1}{2}J_{n+1}(x)\right]$$

$$= \tfrac{1}{2}\left(x^n-x^{-n}\right)\left[J_{n-1}(x)+J_{n+1}(x)\right] + \tfrac{1}{2}\left(x^n+x^{-n}\right)\left[J_{n-1}(x)-J_{n+1}(x)\right]$$

$$= \tfrac{1}{2}\left[\begin{array}{l} x^nJ_{n-1}(x) + \cancel{x^nJ_{n+1}(x)} - \cancel{x^{-n}J_{n-1}(x)} - x^{-n}J_{n+1}(x) \\ x^nJ_{n-1}(x) - \cancel{x^nJ_{n+1}(x)} + \cancel{x^{-n}J_{n-1}(x)} - x^{-n}J_{n+1}(x) \end{array}\right]$$

$$= x^nJ_{n-1}(x) - x^{-n}J_{n+1}(x)$$

Question 7

Starting from the generating function of the Bessel function of the first kind

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right], \ n \in \mathbb{Z},$$

determine the series expansion of $J_n(x)$, and hence show that

- $J_0(x) = 1 - \dfrac{x^2}{2^2(1!)^2} + \dfrac{x^4}{2^4(2!)^2} - \dfrac{x^6}{2^6(3!)^2} + \ldots$
- $J_1(x) = \dfrac{x}{2^1 0!1!} - \dfrac{x^3}{2^3 1!2!} + \dfrac{x^5}{2^5 2!3!} - \dfrac{x^7}{2^7 3!4!} + \ldots$

$$\boxed{J_n(x) = \sum_{p=0}^{\infty}\left[\frac{(-1)^p}{(n+p)!\,p!}\left(\frac{x}{2}\right)^{2p+n}\right]}$$

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = e^{\frac{1}{2}xt} \cdot e^{-\frac{1}{2}xt^{-1}} = \left[\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}xt\right)^p}{p!}\right]\left[\sum_{q=0}^{\infty}\frac{\left(-\frac{1}{2}xt^{-1}\right)^q}{q!}\right]$$

$$= \left[\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}x\right)^p t^p}{p!}\right]\left[\sum_{q=0}^{\infty}\frac{\left(-\frac{1}{2}x\right)^q t^{-q}}{q!}\right] = \left[\sum_{p=0}^{\infty}\frac{\left(\frac{1}{2}x\right)^p t^p}{p!}\right]\left[\sum_{q=0}^{\infty}\frac{(-1)^q\left(\frac{1}{2}x\right)^q t^{-q}}{q!}\right]$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left[\frac{(-1)^q\left(\frac{1}{2}x\right)^{p+q}}{p!\,q!}t^{p-q}\right]$$

LET $n = p - q \implies -\infty < n < \infty$ SINCE $0 \leqslant p, q < \infty$
$(p = n + q)$

$$\ldots = \sum_{n=-\infty}^{\infty}\sum_{q=0}^{\infty}\left[\frac{(-1)^q\left(\frac{x}{2}\right)^{2q+n}t^n}{(n+q)!\,q!}\right]$$

$$= \sum_{n=-\infty}^{\infty}\left[t^n \underbrace{\sum_{q=0}^{\infty}\frac{(-1)^q\left(\frac{x}{2}\right)^{2q+n}}{(n+q)!\,q!}}_{J_n(x)}\right]$$

THUS

$$J_n(x) = \sum_{q=0}^{\infty}\frac{(-1)^q\left(\frac{1}{2}x\right)^{2q+n}}{(n+q)!\,q!}$$

$$J_0(x) = \sum_{q=0}^{\infty}\frac{(-1)^q\left(\frac{1}{2}x\right)^{2q}}{(q!)^2} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \ldots$$

$$J_1(x) = \sum_{q=0}^{\infty}\frac{(-1)^q\left(\frac{1}{2}x\right)^{2q+1}}{q!\,(q+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + \ldots$$

Question 8

The generating function of the Bessel function of the first kind is

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right], \; n \in \mathbb{Z}.$$

Use the generating function relation, to show that for $n \geq 0$

a) $J_{-n}(x) = (-1)^n J_n(x)$

b) $J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$.

c) $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$.

Use parts **(b)** and **(c)** to find simplified expressions for

d) $\frac{d}{dx}\left[x^n J_n(x)\right]$.

e) $\frac{d}{dx}\left[x^{-n} J_n(x)\right]$

f) Use parts **(d)** and **(e)** to show that the positive zeros of $J_n(x)$ interlace with those of $J_{n+1}(x)$.

$$\boxed{\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)}, \; \boxed{\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)}$$

[solution overleaf]

a) STARTING WITH THE GENERATING FUNCTION

$$\boxed{e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)}$$

LET $t = -\frac{1}{T}$

$$\Rightarrow e^{\frac{1}{2}x\left(-\frac{1}{T}+T\right)} = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{T}\right)^n J_n(x)$$

LHS IS UNCHANGED

$$\Rightarrow e^{\frac{1}{2}x\left(T-\frac{1}{T}\right)} = \sum_{n=-\infty}^{\infty} (-1)^n T^{-n} J_n(x)$$

AS T IS A DUMMY VARIABLE, REPLACE IT BACK WITH t

$$\Rightarrow e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} (-1)^n t^{-n} J_n(x)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} t^n J_n(x) = \sum_{n=-\infty}^{\infty} (-1)^n t^{-n} J_n(x)$$

COMPARE POWERS OF t, SAY $[t^n]$, NOTING ON RHS "$n=-n$"

$$\Rightarrow J_n(x) = (-1)^{-n} J_{-n}(x)$$

$$\Rightarrow J_n(x) = \frac{1}{(-1)^n} J_{-n}(x)$$

$$\Rightarrow J_{-n}(x) = (-1)^n J_n(x)$$

b) STARTING WITH THE GENERATING FUNCTION

$$\boxed{e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)}$$

DIFFERENTIATE WITH RESPECT TO x

$$\Rightarrow \frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n'(x)$$

$$\Rightarrow \frac{1}{2}t e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} - \frac{1}{2t}e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n'(x)$$

$$\Rightarrow \frac{1}{2}t\sum_{n=-\infty}^{\infty} t^n J_n(x) - \frac{1}{2t}\sum_{n=-\infty}^{\infty} t^n J_n(x) = \sum_{n=-\infty}^{\infty} t^n J_n'(x)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} t^{n+1} J_n(x) - \sum_{n=-\infty}^{\infty} t^{n-1} J_n(x) = 2\sum_{n=-\infty}^{\infty} t^n J_n'(x)$$

EQUATE POWERS OF t, SAY $[t^n]$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

c) STARTING FROM THE GENERATING FUNCTION FOR $J_n(x)$

$$\boxed{e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)}$$

DIFFERENTIATE BOTH SIDES WITH RESPECT TO t

$$\Rightarrow \frac{1}{2}x\left(1+\frac{1}{t^2}\right)e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} n t^{n-1} J_n(x)$$

$$\Rightarrow x e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} + \frac{x}{t^2}e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = 2\sum_{n=-\infty}^{\infty} n t^{n-1} J_n(x)$$

$$\Rightarrow x\sum_{n=-\infty}^{\infty} t^n J_n(x) + \frac{x}{t^2}\sum_{n=-\infty}^{\infty} t^n J_n(x) = 2\sum_{n=-\infty}^{\infty} n t^{n-1} J_n(x)$$

$$\Rightarrow x\sum_{n=-\infty}^{\infty} t^n J_n(x) + x\sum_{n=-\infty}^{\infty} t^{n-2} J_n(x) = 2\sum_{n=-\infty}^{\infty} n t^{n-1} J_n(x)$$

EQUATE POWERS OF t, HERE $[t^{n-1}]$

$$\Rightarrow x J_{n-1}(x) + x J_{n+1}(x) = 2n J_n(x)$$

$$\Rightarrow J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

d) $\frac{d}{dx}\left[x^n J_n(x)\right]$ .... PRODUCT RULE $= nx^{n-1}J_n(x) + x^n J_n'(x)$

USING (b) & (c) TO SIMPLIFY

$$= nx^{n-1}\left[\frac{x}{2n}\left(J_{n-1}(x) + J_{n+1}(x)\right)\right] + x^n \times \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= \frac{1}{2}x^n\left[J_{n-1}(x) + J_{n+1}(x)\right] + \frac{1}{2}x^n\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= \frac{1}{2}x^n\left[J_{n-1}(x) + \cancel{J_{n+1}(x)} + J_{n-1}(x) - \cancel{J_{n+1}(x)}\right]$$

$$= x^n J_{n-1}(x)$$

e) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \dots$ PRODUCT RULE $\dots -nx^{-n-1}J_n(x) + x^{-n} J_n'(x)$

USING (b) & (c) TO SIMPLIFY

$$= -nx^{-n-1} \times \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] + x^{-n} \times \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= -\frac{1}{2}x^{-n}\left[J_{n-1}(x) + J_{n+1}(x)\right] + \frac{1}{2}x^{-n}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$= \frac{1}{2}x^{-n}\left[-\cancel{J_{n-1}(x)} - J_{n+1}(x) + \cancel{J_{n-1}(x)} - J_{n+1}(x)\right]$$

$$= -x^{-n} J_{n+1}(x)$$

f) WE FOUND

$$\left.\begin{aligned} \frac{d}{dx}\left(x^n J_n(x)\right) &= x^n J_{n-1}(x) \\ \frac{d}{dx}\left(x^{-n} J_n(x)\right) &= -x^{-n} J_{n+1}(x) \end{aligned}\right\} \Rightarrow \begin{aligned} \int x^n J_{n-1}(x)\,dx &= x^n J_n(x) + C \quad (\mathrm{I}) \\ -\int x^{-n} J_{n+1}(x)\,dx &= x^{-n} J_n(x) + D \quad (\mathrm{II}) \end{aligned}$$

LET x_1 & x_2 BE TWO ADJACENT ROOTS OF THE EQUATION $J_n(x) = 0$, WITH $x_2 > x_1$

THEN BY (I)

$$\int_{x_1}^{x_2} x^n J_{n-1}(x)\,dx = \left[x^n J_n(x)\right]_{x_1}^{x_2} = 0 \quad , \text{ SINCE } J_n(x_1) = J_2(x_2) = 0$$

NEXT SUPPOSE THAT $J_{n-1}(x)$ HAS A ZERO SAY X_1, $x_1 < X_1 < x_2$ & THERE IS ANOTHER ZERO X_2 ALSO $x_1 < X_2 < x_1$

BY (II) $\quad x^{-n} J_n(x) + D = -\int x^{-n} J_{n+1}(x)\,dx$

$$x^{-n-1} J_{n-1}(x) + D = -\int x^{-n-1} J_n(x)\,dx$$

$$\left[x^{-(n+1)} J_{n-1}(x)\right]_{X_1}^{X_2} = -\int_{X_1}^{X_2} x^{-(n+1)} J_n(x)\,dx$$

$$0 = -\int_{X_1}^{X_2} x^{-(n+1)} J_n(x)\,dx \quad , \text{ SINCE } J_{n-1}(X_1) = J_{n-1}(X_2) = 0$$

BUT THIS NOW IMPLIES THAT $J_n(x)$ HAS A ZERO BETWEEN X_1 & X_2

I.E CALL THIS ZERO $x \Rightarrow x_1 < X_1 < x < X_2 < x_2$

THIS IS A CONTRADICTION TO THE ASSERTION, THAT x_1 & x_2 ARE ADJACENT, AND HENCE THERE CAN ONLY BE ONE ZERO OF $J_{n-1}(x)$ BETWEEN SUCCESSIVE POSITIVE ZEROS FOR $J_n(x)$

Question 9

The Bessel function of the first kind is defined by the series

$$J_n(x) = \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(n+r)!\,r!}\left(\frac{x}{2}\right)^{2r+n}\right], \quad n \in \mathbb{Z}.$$

Use the above definition to show

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right], \qquad n \in \mathbb{Z}.$$

proof

STARTING FROM THE GENERATING FUNCTION

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = e^{\frac{1}{2}xt} \times e^{-\frac{1}{2}\frac{x}{t}} = \left[\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}xt\right)^k}{k!}\right]\left[\sum_{m=0}^{\infty}\frac{\left(-\frac{x}{2t}\right)^m}{m!}\right]$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}xt\right)^k\left(-\frac{x}{2t}\right)^m}{k!\,m!}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{1}{2}\right)^{k+m}x^{k+m}t^k t^{-m}}{k!\,m!}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[(-1)^m\left(\frac{1}{2}\right)^{k+m}\frac{x^{k+m}t^{k-m}}{k!\,m!}\right]$$

NOW WE SPLIT INTO TWO CASES – POWER OF t IS POSITIVE, SAY $n \geqslant 0$

$k - m = n \geqslant 0$

$k = m + n$ & $k + m = 2m + n$

$$\ldots = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[(-1)^m\left(\frac{1}{2}\right)^{2m+n}\frac{x^{2m+n}t^n}{(n+m)!\,m!}\right]$$

$$= \sum_{n=0}^{\infty}\left[t^n\left[\sum_{m=0}^{\infty}\frac{(-1)^m}{(n+m)!\,m!}\left(\frac{x}{2}\right)^{2m+n}\right]\right]$$

$$= \sum_{n=0}^{\infty} t^n J_n(x)$$

CASE B – THE POWER OF t IS A NEGATIVE INTEGER, SAY $-n < 0$

LET $k - m = -n \implies k = m - n$

$\implies k + m = 2m - n$

$$\ldots = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[(-1)^m\left(\frac{1}{2}\right)^{k+m}\frac{x^{k+m}t^{k-m}}{k!\,m!}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[(-1)^m\left(\frac{1}{2}\right)^{2m-n}\frac{x^{2m-n}t^{-n}}{(m-n)!\,m!}\right]$$

$$= \sum_{n=0}^{\infty}\left[t^{-n}\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{(m-n)!\,m!}\left(\frac{x}{2}\right)^{2m-n}\right]\right]$$

$$= \sum_{n=0}^{\infty}\left[t^{-n}J_{-n}(x)\right]$$

$$\therefore\ e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

Question 10

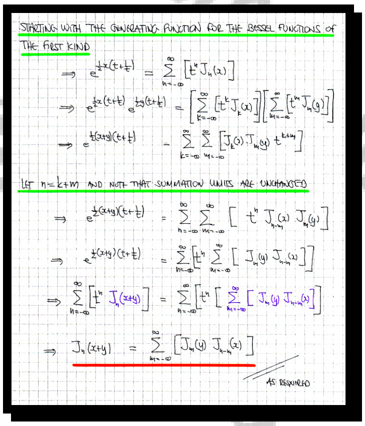
The generating function of the Bessel function of the first kind is

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right], \; n \in \mathbb{Z}.$$

Use the generating function relation, to show that

$$J_n(x+y) = \sum_{m=-\infty}^{\infty}\left[J_m(y)\,J_{n-m}(x)\right].$$

MM, proof



STARTING WITH THE GENERATING FUNCTION FOR THE BESSEL FUNCTIONS OF THE FIRST KIND

$$\Rightarrow e^{\frac{1}{2}x\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right]$$

$$\Rightarrow e^{\frac{1}{2}x\left(t+\frac{1}{t}\right)} e^{\frac{1}{2}y\left(t+\frac{1}{t}\right)} = \left[\sum_{k=-\infty}^{\infty}\left[t^k J_k(x)\right]\right]\left[\sum_{m=-\infty}^{\infty}\left[t^m J_m(y)\right]\right]$$

$$\Rightarrow e^{\frac{1}{2}(x+y)\left(t+\frac{1}{t}\right)} = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[J_k(x)\,J_m(y)\,t^{k+m}\right]$$

LET $n=k+m$ AND NOTE THAT SUMMATION LIMITS ARE UNCHANGED

$$\Rightarrow e^{\frac{1}{2}(x+y)\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[t^n J_{n-m}(x)\,J_m(y)\right]$$

$$\Rightarrow e^{\frac{1}{2}(x+y)\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n \sum_{m=-\infty}^{\infty}\left[J_m(y)\,J_{n-m}(x)\right]\right]$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}\left[t^n J_n(x+y)\right] = \sum_{n=-\infty}^{\infty}\left[t^n\left[\sum_{m=-\infty}^{\infty}\left[J_m(y)\,J_{n-m}(x)\right]\right]\right]$$

$$\Rightarrow J_n(x+y) = \sum_{m=-\infty}^{\infty}\left[J_m(y)\,J_{n-m}(x)\right]$$

AS REQUIRED

Question 11

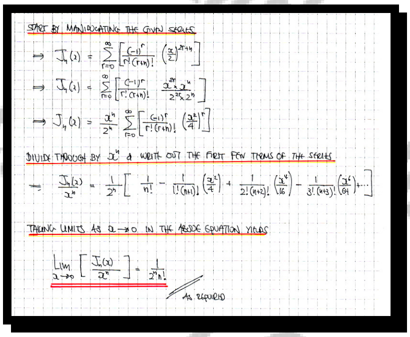
The Bessel function of the first kind is defined by the series

$$J_n(x) = \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(n+r)!\,r!}\left(\frac{x}{2}\right)^{2r+n}\right], \quad n \in \mathbb{Z}.$$

Use the above definition to show

$$\lim_{x \to 0}\left[\frac{J_n(x)}{x^n}\right] = \frac{1}{2^n\ n!}, \quad n \in \mathbb{Z}.$$

[ ], [proof]



START BY MANIPULATING THE GIVEN SERIES

$$\Rightarrow J_n(x) = \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!\,(r+n)!}\left(\frac{x}{2}\right)^{2r+n}\right]$$

$$\Rightarrow J_n(x) = \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!\,(r+n)!}\ \frac{x^{2r}\times x^n}{2^{2r}\times 2^n}\right]$$

$$\Rightarrow J_n(x) = \frac{x^n}{2^n}\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!\,(r+n)!}\left(\frac{x^2}{4}\right)^r\right]$$

DIVIDE THROUGH BY x^n & WRITE OUT THE FIRST FEW TERMS OF THE SERIES

$$\Rightarrow \frac{J_n(x)}{x^n} = \frac{1}{2^n}\left[\frac{1}{n!} - \frac{1}{1!\,(n+1)!}\left(\frac{x^2}{4}\right) + \frac{1}{2!\,(n+2)!}\left(\frac{x^4}{16}\right) - \frac{1}{3!\,(n+3)!}\left(\frac{x^6}{64}\right) + \dots\right]$$

TAKING LIMITS AS $x \to 0$ IN THE ABOVE EQUATION YIELDS

$$\lim_{x \to 0}\left[\frac{J_n(x)}{x^n}\right] = \frac{1}{2^n n!}$$

AS REQUIRED

Question 12

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\left(x^2+n^2\right)y=0.$$

The above differential equation is known as modified Bessel's Equation.

Use the Frobenius method to show that the general solution of this differential equation, for $n=\frac{1}{2}$, is

$$y=x^{-\frac{1}{2}}\left[A\cosh x+B\sinh x\right].$$

$\boxed{\text{proof}}$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+n^2)y=0$$
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-x^2y-\frac{1}{4}y=0$$

- ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_r x^{r+p}$, $a_0\neq 0$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+p)x^{r+p-1}$$
$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+p)(r+p-1)x^{r+p-2}$$

- SUBSTITUTE INTO THE O.D.E

$$\sum_{r=0}^{\infty}a_r(r+p)(r+p-1)x^{r+p}+\sum_{r=0}^{\infty}a_r(r+p)x^{r+p}-\sum_{r=0}^{\infty}a_r x^{r+p+2}-\frac{1}{4}\sum_{r=0}^{\infty}a_r x^{r+p}=0$$

- WHEN $r=0$ THE LOWEST POWER OF x IS x^p, AND THE HIGHEST x^{p+2}. PULL x^p AND x^{p+1} OUT OF THE SUMMATIONS

$$\underbrace{\left[a_0p(p-1)+a_0p-\frac{1}{4}a_0\right]x^p}_{\text{INDICIAL EQUATION}}+\left[a_1(p+1)p+a_1(p+1)-\frac{1}{4}a_1\right]x^{p+1}$$
$$+\sum_{r=2}^{\infty}a_r(r+p)(r+p-1)x^{r+p}+\sum_{r=2}^{\infty}a_r(r+p)x^{r+p}-\sum_{r=0}^{\infty}a_r x^{r+p+2}-\frac{1}{4}\sum_{r=2}^{\infty}a_r x^{r+p}=0$$

- INDICIAL EQUATION, $a_0\neq 0$

$$p(p-1)+p-\frac{1}{4}=0$$
$$p^2-\frac{1}{4}=0$$
$p=\pm\frac{1}{2}$ TWO DISTINCT SOLUTIONS DIFFERING BY AN INTEGER

- CHECK THE NEXT POWER UP FOR UNDETERMINED COEFFICIENTS

$$\left[(p+1)p+(p+1)-\frac{1}{4}\right]a_1=0$$
$$\left[p^2+p+p+1-\frac{1}{4}\right]a_1=0$$
$$\left[p^2+2p+\frac{3}{4}\right]a_1=0$$

IF $p=-\frac{1}{2}$

$$a_1\left[p^2+2p+\frac{3}{4}\right]=0$$
$$a_1\left[\frac{1}{4}-1+\frac{3}{4}\right]=0$$
$$a_1\times 0=0$$
$\therefore a_1$ IS UNDETERMINED

THUS THE ENTIRE SOLUTION WILL BE OBTAINED FROM $p=-\frac{1}{2}$ ($p=\frac{1}{2}$ MERELY PRODUCES PART OF THE SOLUTION)

- ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

$$\sum_{r=0}^{\infty}a_{r+2}(r+p+2)(r+p+1)x^{r+p+2}+\sum_{r=0}^{\infty}a_{r+2}(r+p+2)x^{r+p+2}-\sum_{r=0}^{\infty}a_r x^{r+p+2}-\frac{1}{4}\sum_{r=0}^{\infty}a_{r+2}x^{r+p+2}=0$$

$$a_{r+2}\left[(r+p+2)(r+p+1)+(r+p+2)-\frac{1}{4}\right]=a_r$$
$$a_{r+2}\left[4(r+p+2)(r+p+1)+4(r+p+2)-1\right]=4a_r$$
$$a_{r+2}=\frac{4a_r}{4(r+p+2)(r+p+1)+4(r+p+2)-1}$$

TIDY $\quad 4(w+2)(w+1)+4(w+2)-1 \qquad (w=r+p)$
$$=4w^2+12w+8+4w+8-1$$
$$=4w^2+16w+15$$
$$=(2w+3)(2w+5)$$
$$=\left[2(r+p)+3\right]\left[2(r+p)+5\right]$$
$$=\left[2r+2p+3\right]\left[2r+2p+5\right]$$
BUT $p=-\frac{1}{2}$
$$=\left[2r-1+3\right]\left[2r-1+5\right]$$
$$=(2r+2)(2r+4)$$
$$=4(r+1)(r+2)$$

$$\therefore a_{r+2}=\frac{4a_r}{4(r+1)(r+2)}$$

$$a_{r+2}=\frac{a_r}{(r+1)(r+2)}$$

$r=0$: $a_2=\frac{a_0}{1\times 2}$

$r=1$: $a_3=\frac{a_1}{2\times 3}$

$r=2$: $a_4=\frac{a_2}{3\times 4}=\frac{a_0}{1\times 2\times 3\times 4}$

$r=3$: $a_5=\frac{a_3}{4\times 5}=\frac{a_1}{1\times 2\times 3\times 4\times 5}$

$r=4$: $a_6=\frac{a_4}{5\times 6}=\frac{a_0}{1\times 2\times 3\times 4\times 5\times 6}$

$r=5$: $a_7=\frac{a_5}{6\times 7}=\frac{a_1}{1\times 2\times 3\times 4\times 5\times 6\times 7}$

$r=6$: $a_8=\frac{a_6}{7\times 8}=\frac{a_0}{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8}$ ETC

THUS

$$y=x^{-\frac{1}{2}}\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7+a_8x^8+\dots\right]$$
$$y=x^{-\frac{1}{2}}\left[a_0+a_1x+\frac{a_0}{2!}x^2+\frac{a_1}{3!}x^3+\frac{a_0}{4!}x^4+\frac{a_1}{5!}x^5+\frac{a_0}{6!}x^6+\dots\right]$$
$$y=a_0x^{-\frac{1}{2}}\left[1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots\right]+a_1x^{-\frac{1}{2}}\left[x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\dots\right]$$
$$y=\frac{A}{\sqrt{x}}\sum_{r=0}^{\infty}\frac{x^{2r}}{(2r)!}+\frac{B}{\sqrt{x}}\sum_{r=0}^{\infty}\frac{x^{2r+1}}{(2r+1)!}$$
$$y=\frac{A}{\sqrt{x}}\cosh x+\frac{B}{\sqrt{x}}\sinh x$$

Question 13

Find the two independent solutions of Bessel's equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\nu^2\right)y=0\,,\ \nu\notin\mathbb{Z}\,.$$

Give the answer as exact simplified summations.

$$\boxed{y=Ax^{\nu}\sum_{r=0}^{\infty}\left[\frac{(-1)^r\,\nu!}{r!(\nu+r)!}\left(\tfrac{1}{2}x\right)^{2r}\right]\ \text{or}\ J_{\nu}(x)=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!(\nu+r)!}\left(\tfrac{1}{2}x\right)^{2r+\nu}\right]}$$

$$\boxed{y=Bx^{-\nu}\sum_{r=0}^{\infty}\left[\frac{(-1)^r(-\nu)!}{r!(r-\nu)!}\left(\tfrac{1}{2}x\right)^{2r}\right]\ \text{or}\ J_{-\nu}(x)=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!(r-\nu)!}\left(\tfrac{1}{2}x\right)^{2r-\nu}\right]}$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y=0 \qquad \nu\notin\mathbb{Z}$$

- TRY A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_rx^{r+k}$, $a_0\neq0$, $k\in\mathbb{R}$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}$$
$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+k-1)(r+k)x^{r+k-2}$$

- SUB INTO THE O.D.E.

$$\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k}+\sum_{r=0}^{\infty}a_r(r+k)x^{r+k}+\sum_{r=0}^{\infty}a_rx^{r+k+2}-\sum_{r=0}^{\infty}\nu^2a_rx^{r+k}=0$$

$$\left[a_0k(k-1)x^k+a_0kx^k-\nu^2a_0x^k\right]+\left[a_1(k+1)kx^{k+1}+a_1(k+1)x^{k+1}-\nu^2a_1x^{k+1}\right]$$
$$+\sum_{r=2}^{\infty}a_r(r+k)(r+k-1)x^{r+k}+\sum_{r=2}^{\infty}a_r(r+k)x^{r+k}+\sum_{r=0}^{\infty}a_rx^{r+k+2}-\sum_{r=2}^{\infty}\nu^2a_rx^{r+k}=0$$

- INDICIAL EQUATION

$$a_0x^k\left[k(k-1)+k-\nu^2\right]=0 \quad a_0\neq0$$
$$k^2-\nu^2=0$$
$$k=\pm\nu \quad \text{TWO DISTINCT ROOTS NOT DIFFERING BY INTEGER}$$

- ADJUST THE SUMMATIONS SO THEY ALL START FROM ZERO

$$\sum_{r=0}^{\infty}a_{r+2}(r+k+2)(r+k+1)x^{r+k+2}+\sum_{r=0}^{\infty}a_{r+2}(r+k+2)x^{r+k+2}+\sum_{r=0}^{\infty}a_rx^{r+k+2}-\sum_{r=0}^{\infty}a_{r+2}\nu^2x^{r+k+2}=0$$

$$a_{r+2}\left[(r+k+2)(r+k+1)+(r+k+2)-\nu^2\right]=-a_r$$

$$a_{r+2}=-\frac{a_r}{(r+k+2)(r+k+1)+(r+k+2)-\nu^2}$$

$$a_{r+2}=-\frac{a_r}{(r+k+1)(r+k+2)-\nu^2}$$

$$a_{r+2}=-\frac{a_r}{(r+k+2)^2-\nu^2}$$

- NOW LET $k=\nu$

$$a_{r+2}=-\frac{a_r}{(r+\nu+2)^2-\nu^2}=-\frac{a_r}{(r+\nu+2-\nu)(r+\nu+2+\nu)}$$

$$a_{r+2}=-\frac{a_r}{(r+2)(r+2\nu+2)}$$

ALSO $\left[a_1k(k+1)+a_1(k+1)-\nu^2a_1\right]=0$

$$a_1\left[k^2+k+k+1-\nu^2\right]=0$$

$$\boxed{a_1=0}\implies a_3,a_5,a_7,a_9\ldots=0$$

THUS

$r=0 \quad a_2=\dfrac{-a_0}{2(2\nu+2)}$

$r=2 \quad a_4=\dfrac{-a_2}{4(2\nu+4)} \quad \dfrac{a_0}{4(2\nu+4)\times2(2\nu+2)}$

$r=4 \quad a_6=\dfrac{-a_4}{6(2\nu+6)}=\dfrac{-a_0}{6(2\nu+6)4(2\nu+4)\,2(2\nu+2)}$

$r=6 \quad a_8=\dfrac{-a_6}{8(2\nu+8)}=\dfrac{a_0}{8(2\nu+8)\,6(2\nu+6)\,4(2\nu+4)\,2(2\nu+2)}$ ETC

LOOKING FOR A PATTERN FROM $a_8=\dfrac{a_0}{8\cdot6\cdot4\cdot2\cdot(2\nu+8)(2\nu+6)(2\nu+4)(2\nu+2)}$

$$a_8=\frac{a_0}{2^4(4\cdot3\cdot2\cdot1)\,2^4(\nu+4)(\nu+3)(\nu+2)(\nu+1)}$$

$$a_8=\frac{a_0}{2^8\times4!\,(\nu+1)(\nu+2)(\nu+3)(\nu+4)}$$

$$\therefore\ a_r=\frac{a_0(-1)^r}{2^{2r}\times r!\times(\nu+1)(\nu+2)(\nu+3)\ldots(\nu+r)}$$

$$a_r=\frac{a_0(-1)^r\,\Gamma(\nu+1)}{2^{2r}\times r!\times\Gamma(\nu+1)\times(\nu+1)(\nu+2)(\nu+3)\ldots(\nu+r)}$$

$$a_r=\frac{a_0(-1)^r\,\Gamma(\nu+1)}{2^{2r}\times r!\,\Gamma(\nu+r+1)}$$

NOW THE FIRST SOLUTION $y_1=x^k\left[a_0+\cancel{a_1x}+a_2x^2+\cancel{a_3x^3}+a_4x^4+\ldots\right]$

$$y_1=x^{\nu}\left[a_0\sum_{r=0}^{\infty}\frac{(-1)^r\,\Gamma(\nu+1)}{2^{2r}\times r!\,\Gamma(\nu+r+1)}(x)^{2r}\right]$$

$$y_1=x^{\nu}a_0\sum_{r=0}^{\infty}\frac{(-1)^r\times\nu!}{r!\times(\nu+r)!}\left(\frac{x}{2}\right)^{2r}$$

$$y_1=Ax^{\nu}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{\nu!}{(\nu+r)!}\left(\frac{x}{2}\right)^{2r}$$

- SIMILARLY FOR THE SECOND SOLUTION $k=-\nu$

$$a_r=\frac{a_0(-1)^r}{2^{2r}\times r!\times(-\nu+1)(-\nu+2)(-\nu+3)\ldots(-\nu+r)}$$

$$a_r=\frac{a_0(-1)^r}{2^{2r}\times r!\,(1-\nu)(2-\nu)(3-\nu)\ldots(r-\nu)}$$

$$a_r=\frac{\Gamma(1-\nu)\,a_0(-1)^r}{2^{2r}\times r!\,\Gamma(1-\nu)(1-\nu)(2-\nu)(3-\nu)\ldots(r-\nu)}$$

$$a_r=\frac{\Gamma(1-\nu)\,a_0(-1)^r}{2^{2r}\,r!\,\Gamma(r-\nu+1)}$$

$$a_r=\frac{(-\nu)!\,(-1)^r a_0}{2^{2r}\,r!\,(r-\nu)!}$$

THE SECOND SOLUTION $y_2=x^k\left(a_0+\cancel{a_1x}+a_2x^2+\cancel{a_3x^3}+a_4x^4+\ldots\right)$

$$y_2=x^{-\nu}\left[a_0\sum_{r=0}^{\infty}\frac{(-1)^r(-\nu)!}{r!\,2^{2r}(r-\nu)!}x^{2r}\right]$$

$$y_2=Bx^{-\nu}\sum_{r=0}^{\infty}\frac{(-1)^r(-\nu)!}{r!\,(r-\nu)!}\left(\frac{x}{2}\right)^{2r}$$

NOTE IF $\nu\notin\mathbb{R}$

- IF $A=\dfrac{1}{2^{\nu}}\dfrac{1}{\nu!}$

$$y_1=\frac{x^{\nu}}{2^{\nu}}\frac{1}{\cancel{\nu!}}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{\cancel{\nu!}}{(\nu+r)!}\left(\frac{x}{2}\right)^{2r}$$

$$J_{\nu}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(\nu+r)!}\left(\frac{x}{2}\right)^{2r+\nu}$$

- IF $B=\dfrac{1}{2^{-\nu}}\dfrac{1}{(-\nu)!}$

$$y_2=\frac{x^{-\nu}}{2^{-\nu}}\frac{1}{\cancel{(-\nu)!}}\sum_{r=0}^{\infty}\frac{(-1)^r\cancel{(-\nu)!}}{r!\,(r-\nu)!}\left(\frac{x}{2}\right)^{2r}$$

$$J_{-\nu}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(r-\nu)!}\left(\frac{x}{2}\right)^{2r-\nu}$$

Question 14

Find the two independent solutions of Bessel's equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-n^2\right)y=0\,,\ n\in\mathbb{Z}\,.$$

Give the answer as exact simplified summations.

$$\boxed{y=Ax^n\sum_{r=0}^{\infty}\left[\frac{(-1)^r\,n!}{r!(n+r)!}\left(\tfrac{1}{2}x\right)^{2r}\right]\ \text{ or }\ J_n(x)=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!(n+r)!}\left(\tfrac{1}{2}x\right)^{2r+n}\right]}$$

$$\boxed{y=B(\ln x)\,x^n\sum_{r=0}^{\infty}\left[\frac{(-1)^r\,n!}{r!(n+r)!}\left(\tfrac{1}{2}x\right)^{2r}\right]+Bx^n\sum_{r=1}^{\infty}\left[\frac{(-1)^r\,n!}{r!(n+r)!}\left(\tfrac{1}{2}x\right)^{2r}\left[\frac{1}{2}\sum_{m=1}^{r}\frac{1}{m}+\frac{1}{2}\sum_{m=1}^{r}\frac{1}{m+n}\right]\right]}$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y(x^2-n^2)=0\,,\quad n\in\mathbb{Z},\ n\neq 0$$

- ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_r x^{r+k}$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}$$
$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}$$

- SUBSTITUTE INTO THE O.D.E.

$$\Rightarrow \sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k}+\sum_{r=0}^{\infty}a_r(r+k)x^{r+k}+\sum_{r=0}^{\infty}a_rx^{r+k+2}-\sum_{r=0}^{\infty}n^2a_rx^{r+k}=0$$

$$\Rightarrow \left[a_0k(k-1)x^k+a_0kx^k-n^2a_0x^k\right]+\left[a_1(k+1)kx^{k+1}+a_1(k+1)x^{k+1}-n^2a_1x^{k+1}\right]$$
$$+\sum_{r=2}^{\infty}a_r(r+k)(r+k-1)x^{r+k}+\sum_{r=2}^{\infty}a_r(r+k)x^{r+k}+\sum_{r=0}^{\infty}a_rx^{r+k+2}-\sum_{r=2}^{\infty}n^2a_rx^{r+k}=0$$

- INDICIAL EQUATION

$$a_0x^k\left[k(k-1)+k-n^2\right]=0,\quad a_0\neq 0$$
$$k^2-n^2=0$$
$k=\pm n$ TWO DISTINCT SOLUTIONS, BOTH INTEGERS

- ADJUST THE SUMMATIONS SO THEY ALL START FROM ZERO

$$\sum_{r=0}^{\infty}a_{r+2}(r+k+2)(r+k+1)x^{r+k+2}+\sum_{r=0}^{\infty}a_{r+2}(r+k+2)x^{r+k+2}+\sum_{r=0}^{\infty}a_rx^{r+k+2}-n^2\sum_{r=0}^{\infty}a_{r+2}x^{r+k+2}=0$$

$$a_{r+2}\left[(r+k+2)(r+k+1)+(r+k+2)-n^2\right]=-a_r$$
$$a_{r+2}=-\frac{a_r}{(r+k+2)(r+k+1)+(r+k+2)-n^2}$$
$$a_{r+2}=-\frac{a_r}{(r+k+2)(r+k+2)-n^2}$$
$$a_{r+2}=-\frac{a_r}{(r+k+2)^2-n^2}$$

IF $k=n\in\mathbb{Z}$

$$a_{r+2}=-\frac{a_0}{(r+n+2)^2-n^2}=-\frac{a_0}{(r+n+2-n)(r+n+2+n)}=-\frac{a_0}{(r+2)(r+2n+2)}$$

THUS

$$a_{r+2}=-\frac{a_0}{(r+2)(r+2n+2)}$$

IF $k=n$
$$a_1x^{n+1}\left[(n+1)n+(n+1)-n^2\right]=0$$
$$a_1x^{n+1}\left[\cancel{n^2}+n+n+1-\cancel{n^2}\right]=0$$
$$a_1x^{n+1}(2n+1)=0$$
$a_1=0 \qquad 2n+1\neq 0$

$\therefore a_1,a_3,a_5,a_7\ldots=0$

THUS OBTAINING

$r=0 \quad a_2=-\frac{a_0}{2(2n+2)}$

$r=2 \quad a_4=-\frac{a_2}{4(2n+4)}=\frac{a_0}{4\cdot 2(2n+4)(2n+2)}$

$r=4 \quad a_6=-\frac{a_4}{6(2n+6)}=\frac{-a_0}{6\cdot 4\cdot 2(2n+6)(2n+4)(2n+2)}$

$r=6 \quad a_8=-\frac{a_6}{8(2n+8)}=\frac{a_0}{8\cdot 6\cdot 4\cdot 2(2n+8)(2n+6)(2n+4)(2n+2)}$ ETC

LOOKING FOR A PATTERN IN $a_8=\frac{a_0}{2^4(4\times3\times2\times1)\times 2^4(n+4)(n+3)(n+2)(n+1)}$

$$a_8=\frac{a_0\,n!}{2^8\times 4!\,(n+4)(n+3)(n+2)(n+1)\,n!}$$
$$a_8=\frac{n!\,a_0}{2^8\times 4!\,(n+4)!}$$
$$a_r=\frac{n!(-1)^r a_0}{2^{2r}r!(n+r)!}$$

$$y=x^k\left(a_0+\cancel{a_1x}+a_2x^2+\cancel{a_3x^3}+a_4x^4+\cancel{a_5x^5}+a_6x^6+\ldots\right)$$
$$y=x^n\times a_0\sum_{r=0}^{\infty}\frac{n!(-1)^r x^{2r}}{2^{2r}r!(r+n)!}=A\sum_{r=0}^{\infty}\frac{n!(-1)^r x^{2r+n}}{2^{2r}r!(r+n)!}$$

$$\therefore y=A\sum_{r=0}^{\infty}\frac{n!(-1)^r x^{2r+n}}{2^{2r}r!(r+n)!}$$

IF $A=\frac{1}{n!\,2^n}$ $\quad J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(r+n)!}\left(\frac{x}{2}\right)^{2r+n}$

TO GET A SECOND INDEPENDENT SOLUTION, RETURN TO $a_2,a_4,a_6,\ldots$ IN TERMS OF k

$$a_{r+2}=-\frac{a_r}{(r+k+2)^2-n^2}=-\frac{a_r}{(r+k+2-n)(r+k+2+n)}$$

- IF $r=0 \quad a_2=-\frac{a_0}{(k+2-n)(k+2+n)}$
- $r=2 \quad a_4=-\frac{a_2}{(k+4-n)(k+4+n)}=\frac{a_0}{(k+4-n)(k+2-n)(k+4+n)(k+2+n)}$
- $r=4 \quad a_6=-\frac{a_4}{(k+6-n)(k+6+n)}=-\frac{a_0}{(k+6-n)(k+4-n)(k+2-n)(k+6+n)(k+4+n)(k+2+n)}$

ETC

$$y=x^n\left[a_0+\cancel{a_1x}+a_2x^2+\cancel{a_3x^3}+a_4x^4+\cancel{a_5x^5}+a_6x^6+\ldots\right]$$

$\frac{d}{dk}(a_0)=0$ EVALUATED AT $k=n$ IS STILL ZERO $\leftarrow$ CONSTANT TERM

$$\frac{d}{dk}(a_2)=-a_0\frac{d}{dk}\left(\frac{1}{(k+2-n)(k+2+n)}\right)=-a_0\frac{d}{dk}(t)\qquad t=\frac{1}{(k+2-n)(k+2+n)}$$
$$\ln t=-\ln(k+2-n)-\ln(k+2+n)$$
$$\frac{1}{t}\frac{dt}{dk}=-\frac{1}{k+2-n}-\frac{1}{k+2+n}$$
$$\frac{dt}{dk}=-t\left[\frac{1}{k+2-n}+\frac{1}{k+2+n}\right]$$
$$\therefore -a_0\frac{dt}{dk}\bigg|_{k=n}=-\frac{1}{2(2n+2)}\left[\frac{1}{2}+\frac{1}{2n+2}\right]$$

$$=\frac{a_0}{2(2n+2)}\left[\frac{1}{2}+\frac{1}{2n+2}\right]\leftarrow \text{coeff of } x^2$$

$$\frac{d}{dk}(a_4)=a_0\frac{dt}{dk}\quad \text{WHERE } t=\frac{1}{(k+4-n)(k+2-n)(k+4+n)(k+2+n)}$$
$$\ln t=-\ln(k+4-n)-\ln(k+2-n)-\ln(k+4+n)-\ln(k+2+n)$$
$$\frac{1}{t}\frac{dt}{dk}=-\frac{1}{k+4-n}-\frac{1}{k+2-n}-\frac{1}{k+4+n}-\frac{1}{k+2+n}$$
$$\frac{dt}{dk}=-t\left[\frac{1}{k+4-n}+\frac{1}{k+2-n}+\frac{1}{k+4+n}+\frac{1}{k+2+n}\right]$$
$$\therefore a_0\frac{dt}{dk}\bigg|_{k=n}=-\frac{a_0}{4\cdot 2\cdot(2n+4)(2n+2)}\left[\frac{1}{4}+\frac{1}{2}+\frac{1}{2n+4}+\frac{1}{2n+2}\right]$$

COEFFICIENT OF x^4 ETC.

LOOKING FOR A PATTERN SAY FROM ABOVE

$$a_4=-\frac{a_0}{2^2(2\times1)\times 2^2(n+2)(n+1)}\left[\frac{1}{2}\left(1+\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)\right]$$

$$y_2=B(\ln x)J_n+Bx^n\sum_{r=1}^{\infty}\left[\frac{(-1)^n n!}{2^{2r}(n+r)!}x^{2r}\left[\frac{1}{2}\sum_{m=1}^{r}\frac{1}{m}+\frac{1}{2}\sum_{m=1}^{r}\frac{1}{m+n}\right]\right]$$

NOTE THE "NORMALIZED" TABULATED Y_n IS

$$Y_n(x)=\frac{2}{\pi}\ln\frac{x}{2}J_n(x)-\frac{1}{\pi}\left(\frac{x}{2}\right)^{-n}\sum_{r=0}^{n-1}\frac{(n-1-r)!}{r!}\left(\frac{x}{2}\right)^{2r}$$
$$+\frac{1}{\pi}\left(\frac{x}{2}\right)^{n}\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r!(r+n)!}\left(\frac{x}{2}\right)^{2r}\left[2\gamma-\sum_{m=1}^{r}\frac{1}{r}-\sum_{m=1}^{r+n}\frac{1}{m}\right]\right]$$

Question 15

Find the two independent solutions of Bessel's equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-n^2\right)y=0,\ n=0.$$

Give the answer as exact simplified summations.

$$\boxed{y=A\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(r!)^2}\left(\frac{1}{2}x\right)^{2r}\right]\ \text{ or }\ J_0(x)=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(r!)^2}\left(\frac{1}{2}x\right)^{2r}\right]}$$

$$\boxed{y=B(\ln x)\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(r!)^2}\left(\frac{1}{2}x\right)^{2r}\right]+B\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(r!)^2}\left(\frac{1}{2}x\right)^{2r}\sum_{m=1}^{r}\frac{1}{m}\right]}$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y(x^2-n^2)=0,\quad n=0$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+yx^2=0$$

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+yx=0$$

- ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_r x^{r+k}$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-2}$$

- SUBSTITUTE INTO THE O.D.E

$$\Rightarrow \sum_{r=0}^{\infty}a_r(r+k)(r+k-1)x^{r+k-1}+\sum_{r=0}^{\infty}a_r(r+k)x^{r+k-1}+\sum_{r=0}^{\infty}a_r x^{r+k+1}=0$$

$$\Rightarrow \left[a_0k(k-1)x^{k-1}+a_0kx^{k-1}\right]+\left[a_1(k+1)kx^{k}+a_1(k+1)x^{k}\right]$$
$$+\sum_{r=2}^{\infty}a_r(r+k)(r+k-1)x^{r+k-1}+\sum_{r=2}^{\infty}a_r(r+k)x^{r+k-1}+\sum_{r=0}^{\infty}a_r x^{r+k+1}=0$$

- INDICIAL EQUATION

$a_0x^{k-1}\left[k(k-1)+k\right]=0,\ a_0\neq 0$ IF $k=0$, $a_1=0$

$k^2=0$

$k=0$ REPEATED

- ADJUST THE SUMMATIONS SO THEY ALL START FROM ZERO

$$\sum_{r=0}^{\infty}a_{r+2}(r+k+2)(r+k+1)x^{r+k+1}+\sum_{r=0}^{\infty}a_{r+2}(r+k+2)x^{r+k+1}+\sum_{r=0}^{\infty}a_r x^{r+k+1}=0$$

$$a_{r+2}\left[(r+k+2)(r+k+1)+(r+k+2)\right]+a_r=0$$

$$a_{r+2}\left[(r+k+2)(r+k+1+1)\right]=-a_r$$

$$a_{r+2}(r+k+2)^2=-a_r$$

$$a_{r+2}=-\frac{a_r}{(r+k+2)^2}\quad \text{OR}\quad a_{r+2}=-\frac{a_r}{(r+2)^2}$$

IF $r=0$ $a_2=-\frac{a_0}{2^2}$

$r=1$ $a_3=-\frac{a_1}{3^2}=0$ (SINCE $a_1=0$)

$r=2$ $a_4=-\frac{a_2}{4^2}=\frac{a_0}{4^2\times 2^2}$

$r=3$ $a_5=-\frac{a_3}{5^2}=0$

$r=4$ $a_6=-\frac{a_4}{6^2}=-\frac{a_0}{6^2\times4^2\times2^2}$ ETC

THUS THE SOLUTION IS

$$y=x^k\left(a_0+\cancel{a_1x}+a_2x^2+\cancel{a_3x^3}+a_4x^4+\cancel{a_5x^5}+a_6x^6+\dots\right)$$

$$y=x^0\left(a_0-\frac{a_0x^2}{2^2}+\frac{a_0x^4}{4^2\times2^2}-\frac{a_0x^6}{6^2\times4^2\times2^2}+\frac{a_0x^8}{8^2\times6^2\times4^2\times2^2}-\dots\right)$$

$$y=a_0\left[1-\frac{x^2}{2^2}+\frac{x^4}{4^2\times2^2}-\frac{x^6}{6^2\times4^2\times2^2}+\frac{x^8}{8^2\times6^2\times4^2\times2^2}-\dots\right]$$

$$y=a_0\left[1-\frac{x^2}{2^2\times1^2}+\frac{x^4}{2^4(2\times1)^2}-\frac{x^6}{2^6(3\times2\times1)}+\frac{x^8}{2^8(4\times3\times2\times1)}-\dots\right]$$

$$y=a_0\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{2^{2r}(r!)^2}$$

$$y=A\sum_{r=0}^{\infty}\frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r}$$

TAKE $A=1$ SO $J_0(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r}$

TO GET A SECOND INDEPENDENT SOLUTION RETURN TO

$$a_{r+2}=-\frac{a_r}{(r+k+2)^2}$$

OBTAIN THE FIRST FEW COEFFICIENTS IN GENERAL FORM.

$r=0$ $a_2=-\frac{a_0}{(k+2)^2}$

$r=1$ $a_3=-\frac{a}{(k+3)^2}=0$

$r=2$ $a_4=-\frac{a_2}{(k+4)^2}=\frac{a_0}{(k+4)^2(k+2)^2}$

$r=3$ $a_5=-\frac{a_3}{(k+5)^2}=0$

$r=4$ $a_6=-\frac{a_4}{(k+6)^2}=-\frac{a_0}{(k+6)^2(k+4)^2(k+2)^2}$ ETC

$$\therefore\ y=x^k\left[a_0+\cancel{a_1x}+a_2x^2+\cancel{a_3x^3}+a_4x^4+\cancel{a_5x^5}+a_6x^6+\dots\right]$$

- $\frac{d}{dk}(a_0)=0$ EVALUATED AT $k=0$ GIVES 0 ← CONSTANT TERM

- $\frac{d}{dk}(a_2)=-a_0\frac{d}{dk}\left(\frac{1}{(k+2)^2}\right)=-a_0\frac{-2}{(k+2)^3}=\frac{2a_0}{(k+2)^3}$ EVALUATED AT $k=0$ GIVES $\frac{2a_0}{2^3}=\frac{a_0}{2^2}$ ← COEFFICIENT OF x^2

- $\frac{d}{dk}(a_4)=a_0\frac{d}{dk}\left[\frac{1}{(k+4)^2(k+2)^2}\right]=a_0\frac{d}{dk}(t)$ WHERE $t=\frac{1}{(k+4)^2(k+2)^2}$

$\ln t=-2\ln(k+4)-2\ln(k+2)$

$$\frac{dt}{dk}\frac{1}{t}=-\frac{2}{k+4}-\frac{2}{k+2}$$

$$\frac{dt}{dk}=-2t\left[\frac{1}{k+4}+\frac{1}{k+2}\right]$$

$$\frac{dt}{dk}=\frac{-2}{(k+4)^2(k+2)}\left[\frac{1}{k+4}+\frac{1}{k+2}\right]$$

EVALUATED AT $k=0$ GIVES $a_0\left(\frac{-2}{4^2\times2^2}\right)\left(\frac{1}{4}+\frac{1}{2}\right)$ ← COEFFICIENT OF x^4

- $\frac{d}{dk}(a_6)=-a_0\frac{d}{dk}\left[\frac{1}{(k+6)^2(k+4)^2(k+2)^2}\right]=a_0\frac{d}{dk}(t)$

WHERE $t=\frac{1}{(k+6)^2(k+4)^2(k+2)^2}$

$\ln t=-2\ln(k+6)-2\ln(k+4)-2\ln(k+2)$

$$\frac{dt}{dk}\frac{1}{t}=\frac{-2}{k+6}-\frac{2}{k+4}-\frac{2}{k+2}$$

$$\frac{dt}{dk}=-2t\left[\frac{1}{k+6}+\frac{1}{k+4}+\frac{1}{k+2}\right]$$

$$\frac{dt}{dk}=-2\left(\frac{1}{(k+6)^2(k+4)^2(k+2)^2}\right)\left(\frac{1}{k+6}+\frac{1}{k+4}+\frac{1}{k+2}\right)$$

WHEN $k=0$

$$\frac{da_6}{dk}=-\left[-2\left(\frac{1}{6^2\cdot4^2\cdot2^2}\right)\left(\frac{1}{6}+\frac{1}{4}+\frac{1}{2}\right)\right]\ \text{ETC.}$$ ← COEFFICIENT OF x^6

SO THE SECOND SOLUTION IS GIVEN BY

$$y=A(\ln x)J_0+A\left[\frac{2x^2}{2}-\frac{2x^4}{4^2\times2^2}\left(\frac{1}{4}+\frac{1}{2}\right)+\frac{2x^6}{6^2\times4^2\times2^2}\left(\frac{1}{6}+\frac{1}{4}+\frac{1}{2}\right)+\dots\right]$$

$$y=A(\ln x)J_0+A\left[\frac{2x^2}{2^2\times1^2}\left(\frac{1}{2}\right)-\frac{2x^4}{2^4(2\times1)^2}\left(\frac{1}{4}+\frac{1}{2}\right)+\frac{2x^6}{2^6(3\times2\times1)^2}\left(\frac{1}{6}+\frac{1}{4}+\frac{1}{2}\right)+\dots\right]$$

$$y=A(\ln x)J_0+A\sum_{r=1}^{\infty}\frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r}\sum_{m=1}^{r}\frac{1}{m}$$

THIS SOLUTION AFTER SOME NORMALISATION IS KNOWN AS Y_0

$$Y_0(x)=\frac{2}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r}\left[\ln\frac{x}{2}+\gamma-\sum_{n=1}^{r}\frac{1}{n}\right]$$

Question 16

The generating function of the Bessel function of the first kind is

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right], \ n \in \mathbb{Z}.$$

a) Use the generating function, to show that for $n \geq 0$

i. $J_{-n}(x) = (-1)^n J_n(x)$

ii. $J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$.

iii. $J_{n-1}(x) + J_{n+1}(x) = \dfrac{2n}{x} J_n(x)$.

b) Use part **(a)** deduce that

i. $\dfrac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.

ii. $\dfrac{d}{dx}\left[x^{1-n} J_{n-1}(x)\right] = -x^{1-n} J_n(x)$

c) Use part **(b)** to show further that

$$x^2 J_n''(x) + x J_n'(x) + \left(x^2 - n^2\right) J_n(x) = 0.$$

proof

[solution overleaf]

a) $\boxed{e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right]}$

I) STARTING FROM THE GENERATING FUNCTION REPLACE t FOR $-\frac{1}{t}$

$$\Rightarrow e^{\frac{1}{2}x\left(-\frac{1}{t}+t\right)} = \sum_{n=-\infty}^{\infty}\left[\left(-\frac{1}{t}\right)^n J_n(x)\right]$$

$$\Rightarrow e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[(-1)^n t^{-n} J_n(x)\right]$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[(-1)^n t^{-n} J_n(x)\right]$$

- COMPARING POWERS OF t, SAY THE COEFFICIENTS OF t^n

$$\Rightarrow J_n(x) = (-1)^{-n} J_{-n}(x)$$

$$\Rightarrow (-1)^n J_n(x) = (-1)^n(-1)^{-n} J_{-n}(x)$$

$$\Rightarrow (-1)^n J_n(x) = (-1)^0 J_{-n}(x)$$

$$\Rightarrow J_{-n}(x) = (-1)^n J_n(x)$$

// AS REQUIRED

II) DIFFERENTIATE THE GENERATING FUNCTION W.R.T. x

$$\frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \frac{1}{2}\left(t-\frac{1}{t}\right)\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}\left[t^{n+1} J_n(x) - t^{n-1} J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[2t^n J_n'(x)\right]$$

LOOKING AT POWERS OF t, SAY THE COEFFICIENT OF t^n

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

// AS REQUIRED

III) DIFFERENTIATE THE GENERATING FUNCTION W.R.T t

$$\frac{1}{2}x\left[1+\frac{1}{t^2}\right]e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[nt^{n-1}J_n(x)\right]$$

$$\left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = \frac{2}{x}\sum_{n=-\infty}^{\infty}\left[nt^{n-1}J_n(x)\right]$$

$$\sum_{n=-\infty}^{\infty}\left[t^n J_n(x) + t^{n-2}J_n(x)\right] = \frac{2}{x}\sum_{n=-\infty}^{\infty}\left[nt^{n-1}J_n(x)\right]$$

LOOKING AT POWERS OF t, SAY HERE t^{n-1}

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

// AS REQUIRED

b) I) $\frac{d}{dx}\left[x^n J_n(x)\right] = nx^{n-1}J_n(x) + x^n J_n'(x)$

$$\Rightarrow \frac{2}{x}\frac{d}{dx}\left[x^n J_n(x)\right] = \frac{2}{x}\left[nx^{n-1}J_n(x) + x^n J_n'(x)\right]$$

$$\Rightarrow \frac{2}{x}\frac{d}{dx}\left[x^n J_n(x)\right] = x^{n-1}\left[\frac{2n}{x}J_n(x)\right] + 2x^{n-1}J_n'(x)$$

$\downarrow a_{III}$ $\qquad$ $\downarrow a_{II}$

$$\Rightarrow \frac{2}{x}\frac{d}{dx}\left[x^n J_n(x)\right] = x^{n-1}\left[J_{n-1}(x) + J_{n+1}(x)\right] + x^{n-1}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$\Rightarrow \frac{2}{x}\frac{d}{dx}\left[x^n J_n(x)\right] = x^{n-1}\left[2J_{n-1}(x)\right]$$

$$\Rightarrow \frac{d}{dx}\left(x^n J_n(x)\right) = \frac{x}{2}x^{n-1}\left(2J_{n-1}(x)\right)$$

$$\Rightarrow \frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

// AS REQUIRED

II) $\frac{d}{dx}\left[x^{1-n}J_{n-1}(x)\right] = (1-n)x^{-n}J_{n-1}(x) + x^{1-n}J_{n-1}'(x)$

$$a_{II} \rightarrow 2J_{n-1}'(x) = J_{n-2}(x) - J_n(x)$$

$$a_{III} \rightarrow \frac{2(n-1)}{x}J_{n-1}(x) = J_{n-2}(x) + J_n(x)$$

$$\Rightarrow -\frac{2}{x}\frac{d}{dx}\left[x^{1-n}J_{n-1}(x)\right] = -\frac{2(1-n)}{x}x^{-n}J_{n-1}(x) - \frac{2}{x}x^{1-n}J_{n-1}'(x)$$

$$\Rightarrow -\frac{2}{x}\frac{d}{dx}\left[x^{1-n}J_{n-1}(x)\right] = \frac{2(n-1)}{x}x^{-n}J_{n-1} - 2x^{-n}J_{n-1}'(x)$$

$$\Rightarrow -\frac{2}{x}\frac{d}{dx}\left[x^{1-n}J_{n-1}(x)\right] = x^{-n}\left[J_{n-2}(x) + J_n(x)\right] - x^{-n}\left[J_{n-2}(x) - J_n(x)\right]$$

$$\Rightarrow -\frac{2}{x}\frac{d}{dx}\left[x^{1-n}J_{n-1}(x)\right] = x^{-n}\left[2J_n(x)\right]$$

$$\Rightarrow \frac{d}{dx}\left[x^{1-n}J_{n-1}(x)\right] = -\frac{x}{2}x^{-n}\left[2J_n(x)\right]$$

$$\Rightarrow \frac{d}{dx}\left[x^{1-n}J_{n-1}(x)\right] = -x^{1-n}J_n(x)$$

// AS REQUIRED

c) COMBINING THE LAST TWO RESULTS AS FOLLOWS

(b_I): $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

$$J_{n-1}(x) = \boxed{\frac{1}{x^n}\frac{d}{dx}\left[x^n J_n(x)\right]}$$

$b_{(II)}$: $\frac{d}{dx}\left[x^{1-n}J_{n-1}(x)\right] = -x^{1-n}J_n(x)$

$$\Rightarrow \frac{d}{dx}\left[x^{1-n}\frac{1}{x^n}\frac{d}{dx}\left[x^n J_n(x)\right]\right] = -x^{1-n}J_n(x)$$

$$\Rightarrow \frac{d}{dx}\left[x^{1-2n}\frac{d}{dx}\left[x^n J_n(x)\right]\right] = -x^{1-n}J_n(x)$$

DROPPING THE ARGUMENTS FOR "SPACE-SAKE"

$$\Rightarrow (1-2n)x^{-2n}\frac{d}{dx}\left[x^n J_n\right] + x^{1-2n}\frac{d^2}{dx^2}\left[x^n J_n\right] = -x^{1-n}J_n$$

MULTIPLY BY x^n

$$\Rightarrow (1-2n)x^{-n}\frac{d}{dx}\left[x^n J_n\right] + x^{1-n}\frac{d^2}{dx^2}\left[x^n J_n\right] = -xJ_n$$

$$\Rightarrow (1-2n)x^{-n}\left[nx^{n-1}J_n + x^n J_n'\right] + x^{1-n}\frac{d}{dx}\left[nx^{n-1}J_n + x^n J_n'\right] = -xJ_n$$

$$\Rightarrow (1-2n)nx^{-1}J_n + (1-2n)J_n'$$
$$+$$
$$x^{1-n}\left[n(n-1)x^{n-2}J_n + nx^{n-1}J_n' + nx^{n-1}J_n' + x^n J_n''\right] = -xJ_n$$

$$\Rightarrow (n-2n^2)x^{-1}J_n + (1-2n)J_n'$$
$$+$$
$$x^{1-n}\left[(n^2-n)x^{n-2}J_n + 2nx^{n-1}J_n' + x^n J_n''\right] = -xJ_n$$

$$\Rightarrow (n-2n^2)x^{-1}J_n + (1-2n)J_n'$$
$$(n^2-n)x^{-1}J_n + 2nJ_n' + xJ_n'' = -xJ_n.$$

$$\Rightarrow -n^2x^{-1}J_n + J_n' + xJ_n'' = -xJ_n$$

MULTIPLY THE EQUATION BY x & REARRANGE

$$\Rightarrow -n^2J_n + xJ_n' + x^2J_n'' = -x^2J_n$$

$$\Rightarrow x^2J_n'' + xJ_n' + (x^2-n^2)J_n = 0$$

// AS REQUIRED

Question 17

$$t\frac{d^2y}{dt^2}+\frac{dy}{dt}+ty\ ,\ t>0\,.$$

The Bessel function of order zero, $J_0(t)$, is a solution of the above differential equation.

It is further given that $\lim_{t\to 0}\left[J_0(t)\right]=1$.

By taking the Laplace transform of the above differential equation, show that

$$\mathcal{L}\left[J_0(t)\right]=\frac{1}{\sqrt{s^2+1}}\,.$$

proof

TAKE THE LAPLACE TRANSFORM OF THE O.D.E

$$\boxed{t\frac{d^2y}{dt^2}+\frac{dy}{dt}+ty=0}$$

WHOSE SOLUTION $y(t)=J_0(t)$ SUCH THAT $J_0(0)=1$

$$\Rightarrow -\frac{d}{ds}\left[s^2\bar{y}-sy_0-\dot{y}_0\right]+\left[s\bar{y}-y_0\right]-\frac{d}{ds}(\bar{y})=0$$

IT IS IRRELEVANT WHAT $\dot{y}_0$ IS, AS IT VANISHES ON DIFFERENTIATION. ALSO $y_0=1$

$$\Rightarrow -\frac{d}{ds}\left[s^2\bar{y}-s-\dot{y}_0\right]+s\bar{y}-1-\frac{d\bar{y}}{ds}=0$$

$$\Rightarrow -\left[2s\bar{y}+s^2\frac{d\bar{y}}{ds}-1+0\right]+s\bar{y}-1-\frac{d\bar{y}}{ds}=0$$

$$\Rightarrow -2s\bar{y}-s^2\frac{d\bar{y}}{ds}\cancel{+1}+s\bar{y}\cancel{-1}-\frac{d\bar{y}}{ds}=0$$

$$\Rightarrow -s\bar{y}=(s^2+1)\frac{d\bar{y}}{ds}$$

$$\Rightarrow \boxed{\frac{d\bar{y}}{ds}=-\frac{s\bar{y}}{s^2+1}}$$

SOLVE THE ODE BY SEPARATING VARIABLES

$$\Rightarrow \frac{1}{\bar{y}}\,d\bar{y}=-\frac{s}{s^2+1}\,ds$$

$$\Rightarrow \ln\bar{y}=-\frac{1}{2}\ln(s^2+1)+C$$

$$\Rightarrow \ln\bar{y}=\ln\left(\frac{A}{\sqrt{s^2+1}}\right)$$

$$\Rightarrow \boxed{\bar{y}=\frac{A}{\sqrt{s^2+1}}}$$

NOW WE USE THESE RESULTS TO EVALUATE THE CONSTANT A

$$\boxed{\begin{aligned}\lim_{t\to 0}f(t)&=\lim_{s\to\infty}\left(s\bar{f}(s)\right)\\ \lim_{t\to\infty}f(t)&=\lim_{s\to 0}\left(s\bar{f}(s)\right)\end{aligned}}$$

HERE WE OBTAIN

$$\lim_{s\to\infty}\left[s\bar{y}\right]=\lim_{t\to 0}\left[y(t)\right]=\lim_{t\to 0}\left[J_0(t)\right]=1$$

THUS

$$\lim_{s\to\infty}\left[\frac{As}{\sqrt{s^2+1}}\right]=1 \qquad \therefore\ \boxed{A=1}$$

RETURNING TO THE PROBLEM

$$\bar{y}=\frac{1}{\sqrt{s^2+1}}$$

$$\therefore\ \mathcal{L}\left[J_0(t)\right]=\frac{1}{\sqrt{s^2+1}}$$

Question 18

It can be shown that for $n \in \mathbb{N}$

$$\int_{-1}^{1} \left(1-t^2\right)^{n-\frac{1}{2}} \mathrm{e}^{\mathrm{i}xt} \ dt = \sum_{m=0}^{\infty} \left[\frac{(-1)^m \ \Gamma\left(m+\frac{1}{2}\right) \Gamma\left(n+\frac{1}{2}\right)}{(2m)! \ \Gamma(m+n+1)} \right].$$

Use Legendre's duplication formula for the Gamma Function to show

$$J_n(x) = \frac{x^n}{2^{n-1}\sqrt{\pi}\ \Gamma\left(n+\frac{1}{2}\right)} \int_0^1 \left(1-t^2\right)^{n-\frac{1}{2}} \cos(xt) \ dt .$$

MM , proof

START BY MANIPULATING LEGENDRE'S DUPLICATION FORMULA

$$\Gamma\left(m+\tfrac{1}{2}\right) = \frac{\Gamma(2m)\sqrt{\pi}}{2^{2m-1}\ \Gamma(m)}$$

$$\Gamma\left(m+\tfrac{1}{2}\right) = \frac{(2m-1)!\sqrt{\pi}}{2^{2m-1}(m-1)!} = \frac{2m \times (m-1)!\sqrt{\pi}}{2^{2m-1}\times 2 \times m\,(m-1)!} = \frac{(2m)!\sqrt{\pi}}{2^{2m}\times m!}$$

NOW USING THE GIVEN RESULT

$$\Rightarrow \int_{-1}^{1}(1-t^2)^{n-\frac{1}{2}} e^{ixt}\,dt = \sum_{m=0}^{\infty}\left[\frac{(-1)^m x^{2m}\,\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{(2m)!\ \Gamma(m+n+1)}\right]$$

$$\Rightarrow \int_{-1}^{1}(1-t^2)^{n-\frac{1}{2}}(\cos xt + i\sin xt)\,dt = \sum_{m=0}^{\infty}\left[\frac{(-1)^m x^{2m}\,\Gamma(n+\frac{1}{2})}{\cancel{(2m)!}\ \underbrace{(m+n)!}_{\Gamma(m+n+1)}} \times \underbrace{\frac{\cancel{(2m)!}\sqrt{\pi}}{2^{2m}\times m!}}_{\Gamma(m+\frac{1}{2})}\right]$$

EVEN EVEN ODD

TIDYING UP BOTH SIDES

$$\Rightarrow 2\int_0^1 (1-t^2)^{n-\frac{1}{2}}\cos xt\,dt = \sum_{m=0}^{\infty}\left[\frac{(-1)^m\,\Gamma(n+\frac{1}{2})\sqrt{\pi}}{(m+n)!\ m!}\left(\frac{x}{2}\right)^{2m}\right]$$

$$\Rightarrow 2\int_0^1 (1-t^2)^{n-\frac{1}{2}}\cos xt\,dt = \Gamma(n+\tfrac{1}{2})\sqrt{\pi}\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{(m+n)!\,m!}\left(\frac{x}{2}\right)^{2m}\right]$$

NOW THE SUMMATION IS ALMOST A BESSEL — MANIPULATE FURTHER

$$\Rightarrow 2\int_0^1 (1-t^2)^{n-\frac{1}{2}}\cos xt\,dt = \Gamma(n+\tfrac{1}{2})\sqrt{\pi}\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{(m+n)!\,m!}\times\left(\frac{x}{2}\right)^{2m}\times\left(\frac{x}{2}\right)^{n}\times\left(\frac{x}{2}\right)^{-n}\right]$$

$$\Rightarrow 2\int_0^1 (1-t^2)^{n-\frac{1}{2}}\cos xt\,dt = \left(\frac{x}{2}\right)^{-n}\Gamma(n+\tfrac{1}{2})\sqrt{\pi}\ \underbrace{\sum_{m=0}^{\infty}\left[\frac{(-1)^m}{(m+n)!\,m!}\left(\frac{x}{2}\right)^{2m+n}\right]}_{J_n(x)}$$

$$\Rightarrow 2\int_0^1 (1-t^2)^{n-\frac{1}{2}}\cos xt\,dt = \Gamma(n+\tfrac{1}{2})\sqrt{\pi}\left(\frac{x}{2}\right)^{-n} J_n(x)$$

$$\Rightarrow J_n(x) = \frac{2}{\Gamma(n+\frac{1}{2})\sqrt{\pi}\left(\frac{x}{2}\right)^{-n}}\int_0^1 (1-t^2)^{n-\frac{1}{2}}\cos xt\,dt$$

$$\Rightarrow J_n(x) = \frac{2}{\Gamma(n+\frac{1}{2})\sqrt{\pi}\left(\frac{2}{x}\right)^{n}}\int_0^1 (1-t^2)^{n-\frac{1}{2}}\cos xt\,dt$$

$$\Rightarrow J_n(x) = \frac{2x^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}\ 2^n}\int_0^1 (1-t^2)^{n-\frac{1}{2}}\cos xt\,dt$$

$$\Rightarrow J_n(x) = \frac{x^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}\ 2^{n-1}}\int_0^1 (1-t^2)^{n-\frac{1}{2}}\cos xt\,dt$$

AS REQUIRED

Question 19

Legendre's duplication formula for the Gamma Function states

$$\Gamma\left(n+\frac{1}{2}\right) \equiv \frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\Gamma(n)}, \quad n \in \mathbb{N}.$$

a) Prove the validity of the above formula.

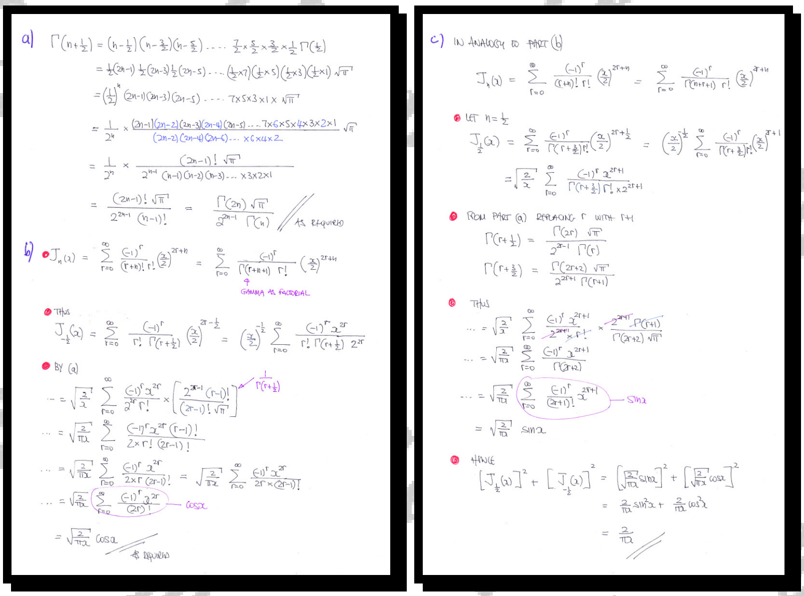
b) Hence show that

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

c) Determine an exact simplified expression for

$$\left[J_{-\frac{1}{2}}(x)\right]^2 + \left[J_{\frac{1}{2}}(x)\right]^2.$$

$$\boxed{\frac{2}{\pi x}}$$



Question 20

a) By using techniques involving the Beta function and the Gamma function, show that

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{2k+1}\ d\theta = \frac{(k!)^2\, 2^{2k}}{(2k+1)!}.$$

The series definition of the Bessel function of the first kind

$$J_n(x) = \sum_{r=0}^{\infty}\left[\frac{(-1)^r}{(n+r)!\,r!}\left(\frac{x}{2}\right)^{2r+n}\right],\ n\in\mathbb{Z}.$$

b) Use the above definition and the result of part **(a)**, to show that

$$\int_0^{\frac{\pi}{2}} J_1(x\cos\theta)\ d\theta = \frac{1-\cos\theta}{x},$$

proof

a)

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{2k+1}\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 2(\cos\theta)^{\overset{2(k+1)-1}{2k+1}}(\sin\theta)^{2\times\frac{1}{2}-1}\,d\theta$$

$$= \frac{1}{2}B\left(k+1,\tfrac{1}{2}\right) = \frac{1}{2}\,\frac{\Gamma(k+1)\Gamma(\frac{1}{2})}{\Gamma(k+\frac{3}{2})}$$

$$= \frac{1}{2}\times\frac{k!\,\cancel{\Gamma(\frac{1}{2})}}{(k+\frac{1}{2})(k-\frac{1}{2})(k-\frac{3}{2})\dots\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\cancel{\Gamma(\frac{1}{2})}}$$

$$= \frac{k!}{2}\times\frac{1}{\frac{1}{2}(2k+1)\frac{1}{2}(2k-1)\frac{1}{2}(2k-3)\dots\frac{1}{2}\times5\times\frac{1}{2}\times3\times\frac{1}{2}\times1}$$

$$= \frac{k!}{2}\times\frac{1}{(\frac{1}{2})^{k+1}\times(2k+1)(2k-1)(2k-3)\dots5\times3\times1}$$

$$\begin{matrix} 1 & 3 & 5 & 7 & \dots & 2k+1 \\ 1 & 2 & 3 & 4 & & k+1 \end{matrix}$$

$$= \frac{k!\,2^{k+1}}{2}\times\frac{2k(2k-2)\dots6\times4\times2}{(2k+1)2k(2k-1)(2k-2)(2k-3)\dots6\times5\times4\times3\times2\times1}$$

$$= \frac{k!\,2^k}{(2k+1)!}\times 2^k\left[k(k-1)(k-2)\dots3\times2\times1\right]$$

$$= \frac{k!\,2^{2k}\,k!}{(2k+1)!} = \frac{(k!)^2\,2^{2k}}{(2k+1)!}$$ AS REQUIRED

b)

- NOW STARTING FROM THE SERIES DEFINITION OF $J_n(x)$

$$\boxed{J_n(x) = \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{k!\,(n+k)!}\left(\frac{x}{2}\right)^{n+2k}\right]}$$

- LET $n=1$

$$\Longrightarrow J_1(x) = \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{k!\,(k+1)!}\left(\frac{x}{2}\right)^{2k+1}\right]$$

- LET $x \mapsto x\cos\theta$

$$\Longrightarrow J_1(x\cos\theta) = \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{k!\,(k+1)!}\left(\frac{x\cos\theta}{2}\right)^{2k+1}\right]$$

$$\Longrightarrow J_1(x\cos\theta) = \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{k!\,(k+1)!}\left(\frac{x}{2}\right)^{2k+1}(\cos\theta)^{2k+1}\right]$$

- INTEGRATE WITH RESPECT TO θ, FROM 0 TO $\pi/2$

$$\Longrightarrow \int_0^{\pi/2} J_1(x\cos\theta)\,d\theta = \int_0^{\frac{\pi}{2}}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(k+1)!}\left(\frac{x}{2}\right)^{2k+1}(\cos\theta)^{2k+1}\right]d\theta$$

$$\Longrightarrow \int_0^{\frac{\pi}{2}} J_1(x\cos\theta)\,d\theta = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(k+1)!}\left(\frac{x}{2}\right)^{2k+1}\int_0^{\frac{\pi}{2}}(\cos\theta)^{2k+1}\,d\theta$$

- USING PART (a)

$$\Longrightarrow \int_0^{\frac{\pi}{2}} J_1(x\cos\theta)\,d\theta = \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{k!\,(k+1)!}\left(\frac{x}{2}\right)^{2k+1}\times\frac{2^{2k}\times(k!)^2}{(2k+1)!}\right]$$

$$\Longrightarrow \int_0^{\frac{\pi}{2}} J_1(x\cos\theta)\,d\theta = \sum_{k=0}^{\infty}\left[\frac{(-1)^k x^{2k+1}\times2^{2k}\times\cancel{(k!)^2}}{2^{2k+1}\cancel{k!}\times(k+1)\cancel{k!}\times(2k+1)!}\right]$$

$$\Longrightarrow \int_0^{\frac{\pi}{2}} J_1(x\cos\theta)\,d\theta = \sum_{k=0}^{\infty}\left[\frac{(-1)^k x^{2k+1}}{2(k+1)(2k+1)!}\right]$$

$$\Longrightarrow \int_0^{\frac{\pi}{2}} J_1(x\cos\theta)\,d\theta = \sum_{k=0}^{\infty}\left[\frac{(-1)^k x^{2k+1}}{(2k+2)!}\right]$$

$$\Longrightarrow \int_0^{\frac{\pi}{2}} J_1(x\cos\theta)\,d\theta = \frac{1}{x}\sum_{k=0}^{\infty}\left[\frac{(-1)^k x^{2k+2}}{(2k+2)!}\right]$$

$$\Longrightarrow \int_0^{\frac{\pi}{2}} J_1(x\cos\theta)\,d\theta = \frac{1}{x}\left[\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\dots\right]$$

$$\Longrightarrow \int_0^{\frac{\pi}{2}} J_1(x\cos\theta)\,d\theta = \frac{1}{x}\left[1-\left(\underbrace{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ }_{\cos\theta}\right)\right]$$

$$\Longrightarrow \int_0^{\frac{\pi}{2}} J_1(x\cos\theta)\,d\theta = \frac{1-\cos\theta}{x}$$ AS REQUIRED

Question 21

The Bessel function $J_n(\alpha x)$ satisfies the differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(\alpha^2x^2-n^2\right)y=0,\ n\in\mathbb{Z},$$

where α is a non zero constant.

If $J_n(\alpha_1 x)$ and $J_n(\alpha_2 x)$ satisfy $J_n(\alpha_1)=J_n(\alpha_2)=0$, with $\alpha_1\neq\alpha_2$, show that

$$\int_0^1 x\,J_n(\alpha_1 x)J_n(\alpha_2 x)\ dx=0.$$

$\boxed{\text{proof}}$

$$\boxed{x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\alpha^2x^2-n^2)y=0}$$

SOLUTIONS ARE

$y_1=J_n(\alpha_1 x)$

$y_2=J_n(\alpha_2 x)$

$\alpha_1\neq\alpha_2$

$J_n(\alpha_1)=J_n(\alpha_2)=0$

y_1 & y_2 MUST SATISFY THE O.D.E

$$x^2\frac{d^2y_1}{dx^2}+x\frac{dy_1}{dx}+(\alpha_1^2x^2-n^2)y_1=0 \quad \times y_2$$

$$x^2\frac{d^2y_2}{dx^2}+x\frac{dy_2}{dx}+(\alpha_2^2x^2-n^2)y_2=0 \quad \times y_1$$

THEN SUBTRACT

$$x^2\left[y_2\frac{d^2y_1}{dx^2}-y_1\frac{d^2y_2}{dx^2}\right]+x\left[y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right]+y_1y_2\left[\alpha_1^2x^2-\alpha_2^2x^2\right]=0$$

DIVIDE BY x

$$\underbrace{x\left[y_2\frac{d^2y_1}{dx^2}-y_1\frac{d^2y_2}{dx^2}\right]+\left[y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right]}_{\text{THIS IS AN EXACT DERIVATIVE}}+\underbrace{y_1y_2x}_{\text{REQUIRED INTEGRAND}}(\alpha_1^2-\alpha_2^2)=0$$

$$\frac{d}{dx}\left[x\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)\right]=(\alpha_2^2-\alpha_1^2)\,y_1y_2x$$

$$\int_0^1\frac{d}{dx}\left[x\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)\right]dx=\int_0^1(\alpha_2^2-\alpha_1^2)\,y_1y_2x\ dx$$

$$\left[x\left(y_2\frac{dy_1}{dx}-y_1\frac{dy_2}{dx}\right)\right]_0^1=(\alpha_2^2-\alpha_1^2)\int_0^1 y_1y_2x\ dx$$

NOW $y_1=J_n(\alpha_1x)\Rightarrow\frac{dy_1}{dx}=\alpha_1J_n'(\alpha_1x)$

$y_2=J_n(\alpha_2x)\Rightarrow\frac{dy_2}{dx}=\alpha_2J'(\alpha_2x)$

SO

$$\Rightarrow(\alpha_2^2-\alpha_1^2)\int_0^1 xy_1y_2\ dx=\left[x\left[\alpha_1J_n'(\alpha_1x)J_n(\alpha_2x)-\alpha_2J'(\alpha_2x)J_n(\alpha_1x)\right]\right]_0^1$$

$$\Rightarrow(\alpha_2^2-\alpha_1^2)\int_0^1 xy_1y_2\ dx=\left[\alpha_1J'(\alpha_1)\underbrace{J(\alpha_2)}_{\text{ZERO}}-\alpha_2J'(\alpha_2)\underbrace{J(\alpha_1)}_{\text{ZERO}}\right]-[0]$$

$$\Rightarrow(\alpha_2^2-\alpha_1^2)\int_0^1 xy_1y_2\ dx=0$$

& AS $\alpha_1=\alpha_2$

$$\Rightarrow\int_0^1 xy_1y_2\ dx=0$$

Question 22

The series definition of the Bessel function of the first kind

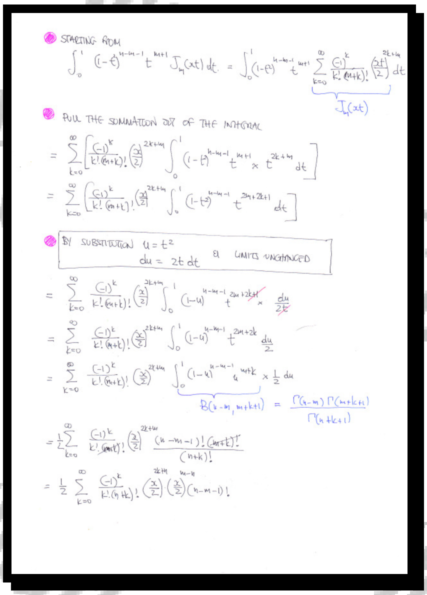
$$J_n(x) = \sum_{r=0}^{\infty} \left[\frac{(-1)^r}{(n+r)!\,r!} \left(\frac{x}{2} \right)^{2r+n} \right], \quad n \in \mathbb{Z}.$$

Use the above definition to show that

$$J_n(x) = \frac{2\,I}{(n-m-1)!} \left(\frac{x}{2} \right)^{n-m},$$

where $I = \int_0^1 (1-t)^{n-m-1}\, t^{m+1}\, J_m(xt)\, dt\,, \quad n > m > -1.$

$\boxed{\text{proof}}$



STARTING FROM

$$\int_0^1 (1-t^2)^{n-m-1} t^{m+1} J_m(xt)\,dt = \int_0^1 (1-t^2)^{n-m-1} t^{m+1} \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(m+k)!} \left(\frac{xt}{2}\right)^{2k+m}}_{J_m(xt)} dt$$

PULL THE SUMMATION OUT OF THE INTEGRAL

$$= \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{k!\,(m+k)!} \left(\frac{x}{2}\right)^{2k+m} \int_0^1 (1-t^2)^{n-m-1} t^{m+1} \times t^{2k+m}\,dt \right]$$

$$= \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{k!\,(m+k)!} \left(\frac{x}{2}\right)^{2k+m} \int_0^1 (1-t^2)^{n-m-1} t^{2m+2k+1}\,dt \right]$$

$\boxed{\begin{array}{l} \text{BY SUBSTITUTION } u = t^2 \\ \qquad du = 2t\,dt \quad \text{\& LIMITS UNCHANGED} \end{array}}$

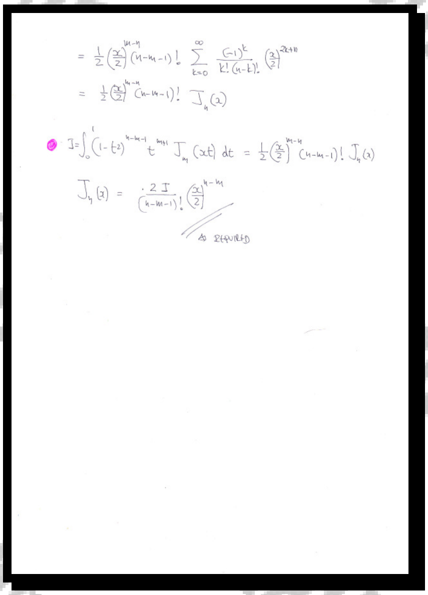
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(m+k)!} \left(\frac{x}{2}\right)^{2k+m} \int_0^1 (1-u)^{n-m-1} t^{2m+2k+\cancel{1}} \times \frac{du}{2\cancel{t}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(m+k)!} \left(\frac{x}{2}\right)^{2k+m} \int_0^1 (1-u)^{n-m-1} t^{2m+2k} \frac{du}{2}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(m+k)!} \left(\frac{x}{2}\right)^{2k+m} \underbrace{\int_0^1 (1-u)^{n-m-1} u^{m+k}}_{B(n-m,\,m+k+1) \,=\, \frac{\Gamma(n-m)\,\Gamma(m+k+1)}{\Gamma(n+k+1)}} \times \frac{1}{2}\,du$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\cancel{(m+k)!}} \left(\frac{x}{2}\right)^{2k+m} \frac{(n-m-1)!\,\cancel{(m+k)!}}{(n+k)!}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{2k+n} \left(\frac{x}{2}\right)^{m-n} (n-m-1)!$$



$$= \frac{1}{2} \left(\frac{x}{2}\right)^{m-n} (n-m-1)! \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n-k)!} \left(\frac{x}{2}\right)^{2k+n}$$

$$= \frac{1}{2} \left(\frac{x}{2}\right)^{m-n} (n-m-1)!\; J_n(x)$$

$$I = \int_0^1 (1-t^2)^{n-m-1} t^{m+1} J_m(xt)\,dt = \frac{1}{2}\left(\frac{x}{2}\right)^{m-n} (n-m-1)!\, J_n(x)$$

$$J_n(x) = \frac{2I}{(n-m-1)!} \left(\frac{x}{2}\right)^{n-m}$$

AS REQUIRED

Question 23

$$I = \int_{-1}^{1} \left(1-t^2\right)^{n-\frac{1}{2}} \mathrm{e}^{\mathrm{i}\,xt}\ dt$$

a) By using the series definition of the exponential function and converting the integrand into a Beta function, show that

$$I = \sum_{m=0}^{\infty} \left[\frac{(-1)^m x^{2m}}{(2m)!} \, \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(n+m+1)} \right].$$

Legendre's duplication formula for the Gamma Function states

$$\Gamma\left(m+\frac{1}{2}\right) \equiv \frac{\Gamma(2m)\sqrt{\pi}}{2^{2m-1}\Gamma(m)}, \quad m \in \mathbb{N}.$$

b) Use the above formula and the result of part **(a)** to show further

$$J_n(x) = \frac{x^n}{2^{n-1}\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)} \int_0^1 \left(1-t^2\right)^{n-\frac{1}{2}} \cos(xt)\ dt.$$

proof

a) $I = \int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}} e^{ixt}\,dt = \int_{-1}^{1} \left[(1-t^2)^{n-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(ixt)^k}{k!}\right] dt$

$\Rightarrow I = \sum_{k=0}^{\infty} \left[\frac{(ix)^k}{k!} \int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}} t^k \, dt\right]$

NOW IF k IS ODD INTEGRAND IS ODD
IF k IS EVEN INTEGRAND IS EVEN.
SO LET $k = 2m$

$\Rightarrow I = \sum_{m=0}^{\infty} \left[\frac{(ix)^{2m}}{(2m)!} \times 2\int_0^1 (1-t^2)^{n-\frac{1}{2}} t^{2m}\,dt\right]$

NOW LET $u = t^2$
$du = 2t\,dt$ & LIMITS UNCHANGED
$dt = \frac{du}{2t}$

$\Rightarrow I = \sum_{m=0}^{\infty} \left[\frac{(-1)^m x^{2m}}{2m!} \cancel{2} \int_0^1 (1-u)^{n-\frac{1}{2}} u^m \frac{du}{\cancel{2}t}\right]$

$\Rightarrow I = \sum_{m=0}^{\infty} \left[\frac{(-1)^m x^{2m}}{(2m)!} \int_0^1 (1-u)^{n-\frac{1}{2}} u^m \frac{du}{u^{\frac{1}{2}}}\right]$

$\Rightarrow I = \sum_{m=0}^{\infty} \left[\frac{(-1)^m x^{2m}}{(2m)!} \int_0^1 (1-u)^{n-\frac{1}{2}} u^{m-\frac{1}{2}}\,du\right]$

$\Rightarrow I = \sum_{m=0}^{\infty} \left[\frac{(-1)^m x^{2m}}{(2m)!} B\left(n+\tfrac{1}{2}, m+\tfrac{1}{2}\right)\right]$

$B(\lambda,\mu) = \int_0^1 (1-u)^{\lambda-1} u^{\mu-1}\,du$

$B(\lambda,\mu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}$

$\Rightarrow I = \sum_{m=0}^{\infty} \left[\frac{(-1)^m x^{2m}}{(2m)!} \frac{\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(n+m+1)}\right]$

b) LEGENDRE'S DUPLICATION FORMULA.
$\Gamma(m+\tfrac{1}{2}) = \frac{\Gamma(2m)\sqrt{\pi}}{2^{2m-1}\Gamma(m)}$

$I = \sum_{m=0}^{\infty} \left[\frac{(-1)^m x^{2m}}{(2m)!} \frac{\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(n+m+1)}\right]$

$I = \Gamma(n+\tfrac{1}{2}) \sum_{m=0}^{\infty} \left[\frac{(-1)^m x^{2m}}{(2m)!\,\Gamma(m+n+1)} \times \frac{\Gamma(2m)\sqrt{\pi}}{2^{2m-1}\Gamma(m)}\right]$

$I = \sqrt{\pi}\,\Gamma(n+\tfrac{1}{2}) \sum_{m=0}^{\infty} \left[\frac{(-1)^m x^{2m}}{(2m)!\,(m+n)!} \frac{(2m-1)!}{2^{2m-1}(m-1)!}\right]$

$I = \sqrt{\pi}\,\Gamma(n+\tfrac{1}{2}) \sum_{m=0}^{\infty} \left[\frac{(-1)^m x^{2m}}{(2m)!\,(m+n)!} \times \frac{2m\,(2m-1)!}{2^{2m-1} \times 2 \times m\,(m-1)!}\right]$

$I = \sqrt{\pi}\,\Gamma(n+\tfrac{1}{2}) \sum_{m=0}^{\infty} \left[\frac{(-1)^m x^{2m}}{\cancel{(2m)!}\,(m+n)!} \times \frac{\cancel{(2m)!}}{2^{2m}\,m!}\right]$

$I = \sqrt{\pi}\,\Gamma(n+\tfrac{1}{2}) \sum_{m=0}^{\infty} \left[\frac{(-1)^m}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m}\right]$

$I = \sqrt{\pi}\,\Gamma(n+\tfrac{1}{2}) \sum_{m=0}^{\infty} \left[\frac{(-1)^m}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m+n-n}\right]$

$I = \sqrt{\pi}\,\Gamma(n+\tfrac{1}{2}) \left(\frac{x}{2}\right)^{-n} \underbrace{\sum_{m=}^{\infty} \left[\frac{(-1)^m}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m+n}\right]}_{J_n(x)}$

$\int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}} e^{ixt}\,dt = \sqrt{\pi}\,\Gamma(n+\tfrac{1}{2}) \left(\frac{x}{2}\right)^{-n} J_n(x)$

$\int_{-1}^{1} \overset{\text{EVEN}}{(1-t^2)^{n-\frac{1}{2}}} \left[\overset{\text{EVEN}}{\cos xt} + \overset{\text{ODD}}{\cancel{i\sin xt}}\right] dt = \sqrt{\pi}\,\Gamma(n+\tfrac{1}{2}) \left(\frac{x}{2}\right)^{-n} J_n(x)$

$J_n(x) = \frac{1}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2}) \left(\frac{x}{2}\right)^{-n}} \times 2\int_0^1 (1-t)^{n-\frac{1}{2}} \cos xt\,dt = \frac{x^n \int_0^1 (1-t^2)^{n-\frac{1}{2}} \cos xt\,dt}{\sqrt{\pi}\,2^{n-1}\,\Gamma(n+\frac{1}{2})}$

Question 24

The Bessel function of the first kind $J_n(x)$, satisfies

$$J_n(x)=\sum_{p=0}^{\infty}\left[\frac{(-1)^p}{(n+p)!\,p!}\left(\frac{x}{2}\right)^{2p+n}\right].$$

Show that

$$J_n(x)=\frac{I}{\sqrt{\pi}\;\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^n,$$

where $I=\displaystyle\int_0^{\pi}\cos(x\sin\theta)\cos^{2n}\theta\ d\theta$.

proof

$$J_n(x)=\frac{I}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n \quad \text{WHERE } I=\int_0^{\pi}\cos(x\sin\theta)\cos^{2n}\theta\,d\theta$$

- STARTING FROM

$$\Rightarrow I=\int_0^{\pi}\cos(x\sin\theta)\cos^{2n}\theta\,d\theta$$

AS $\sin\theta$ IS EVEN ABOUT $\theta=\frac{\pi}{2}$ & $\cos^{2n}\theta$ IS EVEN ABOUT $\theta=\frac{\pi}{2}$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}2\cos(x\sin\theta)\cos^{2n}\theta\,d\theta$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}2\left[1-\frac{x^2\sin^2\theta}{2!}+\frac{x^4\sin^4\theta}{4!}-\frac{x^6\sin^6\theta}{6!}+\dots\right]\cos^{2n}\theta\,d\theta$$

- NOW

$$\int_0^{\frac{\pi}{2}}2\cos^{2p-1}x\sin^{2q-1}x\,dx\equiv B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}2\cos^{2n}\theta\,d\theta-\frac{x^2}{2!}\int_0^{\frac{\pi}{2}}2\cos^{2n}\theta\sin^2\theta\,d\theta+\frac{x^4}{4!}\int_0^{\frac{\pi}{2}}2\cos^{2n}\theta\sin^4\theta\,d\theta-\frac{x^6}{6!}\int_0^{\frac{\pi}{2}}2\cos^{2n}\theta\sin^6\theta\,d\theta\dots$$

$$\Rightarrow I=B\left(n+\tfrac{1}{2},\tfrac{1}{2}\right)-\frac{x^2}{2!}B\left(n+\tfrac{1}{2},\tfrac{3}{2}\right)+\frac{x^4}{4!}B\left(n+\tfrac{1}{2},\tfrac{5}{2}\right)-\frac{x^6}{6!}B\left(n+\tfrac{1}{2},\tfrac{7}{2}\right)+\dots$$

$$\Rightarrow I=\frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(n+1)}-\frac{x^2}{2!}\frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(n+2)}+\frac{x^4}{4!}\frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{5}{2})}{\Gamma(n+3)}-\frac{x^6}{6!}\frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{7}{2})}{\Gamma(n+4)}+\dots$$

$$\Rightarrow I=\Gamma\left(n+\tfrac{1}{2}\right)\left[\frac{\Gamma(\frac{1}{2})}{\Gamma(n+1)}-\frac{x^2}{2!}\frac{\Gamma(\frac{3}{2})}{\Gamma(n+2)}+\frac{x^4}{4!}\frac{\Gamma(\frac{5}{2})}{\Gamma(n+3)}-\frac{x^6}{6!}\frac{\Gamma(\frac{7}{2})}{\Gamma(n+4)}+\dots\right]$$

$$\Rightarrow I=\Gamma\left(n+\tfrac{1}{2}\right)\left[\frac{\sqrt{\pi}}{n!}-\frac{x^2}{2!}\frac{\frac{1}{2}\sqrt{\pi}}{(n+1)!}+\frac{x^4}{4!}\frac{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{(n+2)!}-\frac{x^6}{6!}\frac{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{(n+3)!}+\dots\right]$$

$$\Gamma(p)=(p-1)\Gamma(p-1)$$
$$\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}$$

$$\Rightarrow I=\frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}{n!}\left[1-\frac{\frac{1}{2}x^2}{2!(n+1)}+\frac{\frac{3}{2}\times\frac{1}{2}x^4}{4!(n+2)(n+1)}-\frac{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}x^6}{6!(n+3)(n+2)(n+1)}+\dots\right]$$

$$\Rightarrow I=\frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}{n!}\left[1-\frac{x^2}{2^1 2!(n+1)}+\frac{3\times1\,x^4}{2^2\times4!(n+1)(n+2)}-\frac{5\times3\times1\times x^6}{2^3 6!(n+1)(n+2)(n+3)}+\dots\right]$$

$$\Rightarrow I=\frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}{n!}\left[1-\frac{x^2}{4(n+1)}+\frac{x^4}{4\times8\times(n+1)(n+2)}-\frac{x^6}{4\times8\times12\,(n+1)(n+2)(n+3)}+\dots\right]$$

$$\Rightarrow I=\frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}{n!}\left[1-\left(\frac{x^2}{4}\right)^1\frac{1}{1!(n+1)}+\left(\frac{x^2}{4}\right)^2\frac{1}{2!(n+1)(n+2)}-\left(\frac{x^2}{4}\right)^3\frac{1}{3!(n+1)(n+2)(n+3)}+\dots\right]$$

$$\Rightarrow I=\sqrt{\pi}\,\Gamma\left(n+\tfrac{1}{2}\right)\left[\frac{1}{n!}-\frac{1}{(n+1)!\,1!}\left(\frac{x^2}{2^2}\right)^1+\frac{1}{(n+2)!\,2!}\left(\frac{x^2}{2^2}\right)^2-\frac{1}{(n+3)!\,3!}\left(\frac{x^2}{2^2}\right)^3+\dots\right]$$

$$\Rightarrow I=\sqrt{\pi}\,\Gamma\left(n+\tfrac{1}{2}\right)\left[\frac{1}{n!}-\frac{1}{(n+1)!\,1!}\left(\frac{x}{2}\right)^2+\frac{1}{(n+2)!\,2!}\left(\frac{x}{2}\right)^4-\frac{1}{(n+3)!\,3!}\left(\frac{x}{2}\right)^6+\dots\right]$$

- MULTIPLY BY $\left(\frac{x}{2}\right)^n$

$$\Rightarrow\left(\frac{x}{2}\right)^n I=\sqrt{\pi}\,\Gamma\left(n+\tfrac{1}{2}\right)\underbrace{\left[\frac{1}{n!}\left(\frac{x}{2}\right)^n-\frac{1}{(n+1)!\,1!}\left(\frac{x}{2}\right)^{2+n}+\frac{1}{(n+2)!\,2!}\left(\frac{x}{2}\right)^{4+n}-\dots\right]}_{J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(n+k)!\,k!}\left(\frac{x}{2}\right)^{2k+n}}$$

$$\Rightarrow\left(\frac{x}{2}\right)^n I=\sqrt{\pi}\,\Gamma\left(n+\tfrac{1}{2}\right)J_n(x)$$

$$\Rightarrow J_n(x)=\frac{I}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n$$

$$\Rightarrow J_n(x)=\frac{I}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n\int_0^{\pi}\cos(x\sin\theta)\cos^{2n}\theta\,d\theta$$

AS REQUIRED

Question 25

The generating function of the Bessel function of the first kind is

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right], \ n \in \mathbb{Z}.$$

a) Use the generating function, to show that for $n \geq 0$

i. $J_{-n}(x) = (-1)^n J_n(x)$

ii. $J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$.

iii. $J_{n-1}(x) + J_{n+1}(x) = \dfrac{2n}{x} J_n(x)$.

b) Given that $y = J_n(\lambda x)$ satisfies the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(\lambda^2 x^2 - n^2\right) y = 0, \quad n = 0, 1, 2, 3, \ldots$$

verify that

$$\frac{d}{dx}\left[x^2\left(\frac{dy}{dx}\right)^2\right] + \left(\lambda^2 x^2 - n^2\right)\frac{d}{dx}\left(y^2\right) = 0,$$

and hence show that if λ_i is a non zero root of $J_n(\lambda) = 0$

$$2\int_0^1 x\left[J_n(\lambda_i x)\right]^2 \, dx = \left[J_{n-1}(\lambda_i)\right]^2 = \left[J_{n+1}(\lambda_i)\right]^2.$$

$\boxed{\text{proof}}$

[solution overleaf]

a) I) STARTING FROM THE GENERATING FUNCTION

$$\boxed{e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right]}$$

- REPLACE t WITH $-\frac{1}{t}$

$$\Rightarrow e^{\frac{1}{2}x\left(-\frac{1}{t}+t\right)} = \sum_{n=-\infty}^{\infty}\left[\left(-\frac{1}{t}\right)^n J_n(x)\right]$$

$$\Rightarrow e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[(-1)^n t^{-n} J_n(x)\right]$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[(-1)^n t^{-n} J_n(x)\right]$$

- COMPARING COEFFICIENTS OF t, SAY $[t^n]$

$$\Rightarrow J_n(x) = (-1)^{-n} J_{-n}(x)$$

$$\Rightarrow (-1)^n J_n(x) = (-1)^n (-1)^{-n} J_{-n}(x)$$

$$\Rightarrow J_{-n}(x) = (-1)^n J_n(x)$$

II) - DIFFERENTIATE THE GENERATING FUNCTION WITH RESPECT TO x

$$\Rightarrow \frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \frac{1}{2}\left(t-\frac{1}{t}\right)\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[t^n J_n'(x)\right]$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}\left[t^{n+1} J_n(x)\right] - \sum_{n=-\infty}^{\infty}\left[t^{n-1} J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[2t^n J_n'(x)\right]$$

- COMPARING COEFFICIENTS OF t, SAY $[t^n]$

$$\Rightarrow J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

III) - DIFFERENTIATE THE GENERATING FUNCTION W.R.T t

$$\Rightarrow \frac{1}{2}x\left(1+\frac{1}{t^2}\right)e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty}\left[nt^{n-1} J_n(x)\right]$$

$$\Rightarrow \frac{1}{2}x\left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = \sum_{n=-\infty}^{\infty}\left[nt^{n-1} J_n(x)\right]$$

$$\Rightarrow \left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] = \frac{2}{x}\sum_{n=-\infty}^{\infty}\left[nt^{n-1} J_n(x)\right]$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}\left[t^n J_n(x)\right] + \sum_{n=-\infty}^{\infty}\left[t^{n-2} J_n(x)\right] = \frac{2}{x}\sum_{n=-\infty}^{\infty}\left[nt^{n-1} J_n(x)\right]$$

- EQUATE POWERS OF t, SAY t^n

$$\Rightarrow J_n(x) + J_{n+2} = \frac{2}{x}(n+1)J_{n+1}(x)$$

$n \mapsto n-1$

$$\Rightarrow J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

b)

$$\frac{d}{dx}\left[x^2\left(\frac{dy}{dx}\right)^2\right] + (\lambda^2x^2-n^2)\frac{d}{dx}\left[y^2\right] = 0$$

$$\Rightarrow 2x\left(\frac{dy}{dx}\right)^2 + 2x^2\frac{dy}{dx}\frac{d^2y}{dx^2} + (\lambda^2x^2-n^2)\left(2y\frac{dy}{dx}\right) = 0$$

DIVIDE THROUGH BY $2\frac{dy}{dx}$

$$\Rightarrow x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} + (\lambda^2x^2-n^2)y = 0$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2-n^2)y = 0$$

WHICH INDEED IS SATISFIED BY $J_n(\lambda x)$

NOW INTEGRATE THE O.D.E WITH RESPECT TO x, FROM $x=0$ TO $x=1$

$$\Rightarrow \int_0^1 x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2-n^2)y \, dx = \int_0^1 0 \, dx$$

$$\Rightarrow \int_0^1 \frac{d}{dx}\left[x^2\left(\frac{dy}{dx}\right)^2\right] + (\lambda^2x^2-n^2)\frac{d}{dx}\left[y^2\right] dx = 0$$

$$\Rightarrow \int_0^1 \frac{d}{dx}\left[x^2\left(\frac{dy}{dx}\right)^2\right] - n^2\frac{d}{dx}\left[y^2\right] + \lambda^2x^2\frac{d}{dx}\left[y^2\right] dx = 0$$

$$\Rightarrow \left[x^2\left(\frac{dy}{dx}\right)^2\right]_0^1 - n^2\left[y^2\right]_0^1 + \lambda^2\underbrace{\int_0^1 x^2\frac{d}{dx}(y^2)\, dx}_{\text{BY PARTS}} = 0$$

x^2	$2x$
y^2	$\frac{d}{dx}(y^2)$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2\Bigg|_{x=1} - n^2\left[y^2\right]_0^1 + \lambda^2\left\{\left[x^2y^2\right]_0^1 - 2\int_0^1 xy^2\, dx\right\} = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2\Bigg|_{x=1} - n^2\left[y^2\right]_0^1 + \lambda^2y^2\Big|_{x=1} - 2\lambda^2\int_0^1 xy^2\, dx = 0$$

$$\Rightarrow 2\lambda^2\int_0^1 xy^2\, dx = \lambda^2y^2\Big|_{x=1} + \left(\frac{dy}{dx}\right)^2\Bigg|_{x=1} - n^2\left[y^2\right]_{x=0}^1$$

BUT $y = J_n(\lambda x)$

$\frac{dy}{dx} = \lambda J_n'(\lambda x)$

$$\Rightarrow 2\lambda^2\int_0^1 xy^2\, dx = \lambda^2\left[J_n(\lambda x)\right]^2\Big|_{x=1} + \lambda^2\left[J_n'(\lambda x)\right]^2\Big|_{x=1} - n^2\left[J_n^2(\lambda x)\right]_0^1$$

$$\Rightarrow 2\lambda_i^2\int_0^1 xy^2\, dx = \lambda_i^2\left[J_n(\lambda_i x)\right]^2\Big|_{x=1} + \lambda_i^2\left[J_n'(\lambda_i x)\right]^2\Big|_{x=1} - n^2\left[J_n^2(\lambda_i x)\right]_0^1$$

$$\Rightarrow 2\lambda_i^2\int_0^1 x\left[J_n^2(\lambda_i x)\right] dx = \lambda_i\cancel{\left[J_n^2(\lambda_i)\right]} + \lambda_i^2\left[J_n'(\lambda_i)\right]^2 - \cancel{n^2J_n^2(\lambda_i)} + \cancel{n^2J_n^2(0)}$$

$J_n(\lambda_i)=0$ $\quad$ $J_n(\lambda_i)=0$

$$\Rightarrow 2\int_0^1 xJ_n^2(\lambda_i x)\, dx = \left[J_n'(\lambda_i)\right]^2$$

FINALLY USING THE RESULTS OF a_{II} & a_{III}

$$\left.\begin{aligned} J_{n-1}(x) - J_{n+1}(x) &= 2J_n'(x) \\ J_{n-1}(x) + J_{n+1}(x) &= \frac{2n}{x}J_n(x) \end{aligned}\right\} \Rightarrow \text{ADDING \& SUBTRACTING}$$

$$\left.\begin{aligned} J_{n-1}(x) &= J_n'(x) + \frac{n}{x}J_n(x) \\ J_{n+1}(x) &= \frac{n}{x}J_n(x) - J_n'(x) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} J_n'(x) &= J_{n-1}(x) - \frac{n}{x}J_n(x) \\ J_n'(x) &= \frac{n}{x}J_n(x) - J_{n+1}(x) \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} J_n'(\lambda_i) &= J_{n-1}(\lambda_i) - \frac{n}{\lambda_i}\cancel{J_n(\lambda_i)} \\ J_n'(\lambda_i) &= \frac{n}{\lambda_i}\cancel{J_n(\lambda_i)} - J_{n+1}(\lambda_i) \end{aligned}\right\} \Rightarrow \begin{aligned} J_n'(\lambda_i) &= J_{n-1}(\lambda_i) \\ J_n'(\lambda_i) &= -J_{n+1}(\lambda_i) \end{aligned}$$

RETURNING TO THE MAIN PART

$$\Rightarrow 2\int_0^1 x\left[J_n(\lambda_i x)\right]^2 dx = \left[J_{n-1}(\lambda_i)\right]^2 = \left[J_{n+1}(\lambda_i)\right]^2$$

AS REQUIRED