JACOBIANS

CURVILINEAR COORDINATES

Question 1

a) Determine, by a Jacobian matrix, an expression for the area element in plane polar coordinates, (r,θ).

b) Verify the answer of part **(a)** by performing the same operation in reverse.

$$\boxed{dA = r\,dr\,d\theta}$$

a) $\boxed{dx\,dy = \frac{\partial(x,y)}{\partial(r,\theta)}\,dr\,d\theta}$ $\quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta$$

$$dx\,dy = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta$$

$$dx\,dy = \left[r\cos^2\theta - (r\sin^2\theta)\right] dr\,d\theta$$

$$dx\,dy = r\left(\cos^2\theta + \sin^2\theta\right) dr\,d\theta$$

$$dx\,dy = r\,dr\,d\theta$$

b) $\boxed{dr\,d\theta = \frac{\partial(r,\theta)}{\partial(x,y)}\,dx\,dy}$ $\quad \begin{cases} r^2 = x^2+y^2 \\ r = (x^2+y^2)^{\frac{1}{2}} \\ \tan\theta = \frac{y}{x} \\ \theta = \arctan\frac{y}{x} \end{cases}$

$$\Rightarrow dr\,d\theta = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} dx\,dy$$

$$\Rightarrow dr\,d\theta = \begin{vmatrix} \frac{x}{(x^2+y^2)^{\frac{1}{2}}} & \frac{y}{(x^2+y^2)^{\frac{1}{2}}} \\ \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(\frac{-y}{x^2}\right) & \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x} \end{vmatrix} dx\,dy$$

$$\Rightarrow dr\,d\theta = \begin{vmatrix} \frac{x}{(x^2+y^2)^{\frac{1}{2}}} & \frac{y}{(x^2+y^2)^{\frac{1}{2}}} \\ -\frac{\frac{y}{x^2}}{1+\frac{y^2}{x^2}} & \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}} \end{vmatrix} dx\,dy$$

$$\Rightarrow dr\,d\theta = \begin{vmatrix} \frac{x}{(x^2+y^2)^{\frac{1}{2}}} & \frac{y}{(x^2+y^2)^{\frac{1}{2}}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} dx\,dy$$

$$\Rightarrow dr\,d\theta = \left[\frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} - \left(\frac{-y^2}{(x^2+y^2)^{\frac{3}{2}}}\right)\right] dx\,dy$$

$$\Rightarrow dr\,d\theta = \frac{x^2+y^2}{(x^2+y^2)^{3/2}}\,dx\,dy$$

$$\Rightarrow dr\,d\theta = \frac{1}{(x^2+y^2)^{\frac{1}{2}}}\,dx\,dy$$

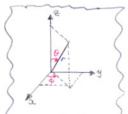
$$\Rightarrow dr\,d\theta = \frac{1}{r}\,dx\,dy$$

$$\Rightarrow r\,dr\,d\theta = dx\,dy$$

Question 2

Determine, by a Jacobian matrix, an expression for the volume element in spherical polar coordinates, (r,θ,φ).

$$\boxed{dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi}$$



$$dV = dx\,dy\,dz \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| dr\,d\theta\,d\phi$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$
$(x^2+y^2+z^2=r^2)$

THUS

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

... EXPAND BY THE BOTTOM ROW ...

$$= \cos\theta \begin{vmatrix} r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} + r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix}$$

... FACTORIZE TERMS OUT OF EACH DETERMINANT ...

$$= \cos\theta\,(r^2\cos\theta\sin\theta) \begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix} + r\sin\theta\,(r\sin^2\theta) \begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix}$$

$$= \left[r^2\cos^2\theta\sin\theta + r^2\sin^3\theta\right] \begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix}$$

$$= r^2\sin\theta\left[\cancel{\cos^2\theta + \sin^2\theta}\right]\left[\cancel{\cos^2\phi + \sin^2\phi}\right]$$

$$= r^2\sin\theta$$

$$\therefore \quad dV = dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Question 3

Two sets of variables are related by the equations

$$x = r\cosh\theta \quad \text{and} \quad y = r\sinh\theta ,$$

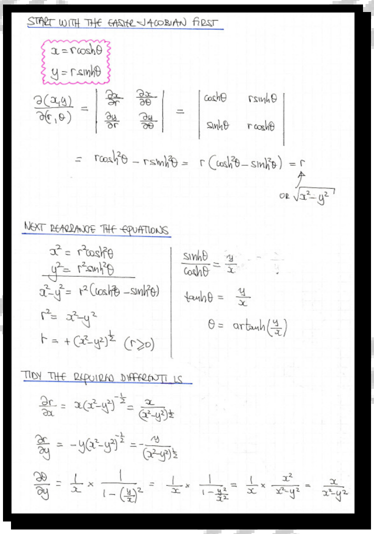
where $r \geq 0$.

Evaluate independently Jacobians

$$I = \frac{\partial(x,y)}{\partial(r,\theta)} \quad \text{and} \quad J = \frac{\partial(r,\theta)}{\partial(x,y)} ,$$

and hence show that $I = \dfrac{1}{J}$.

$$\boxed{I = \sqrt{x^2+y^2} = r}, \quad \boxed{J = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}}$$



START WITH THE EASIER JACOBIAN FIRST

$x = r\cosh\theta$

$y = r\sinh\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cosh\theta & r\sinh\theta \\ \sinh\theta & r\cosh\theta \end{vmatrix}$$

$$= r\cosh^2\theta - r\sinh^2\theta = r(\cosh^2\theta - \sinh^2\theta) = r$$

OR $\sqrt{x^2-y^2}$

NEXT REARRANGE THE EQUATIONS

$x^2 = r^2\cosh^2\theta$

$y^2 = r^2\sinh^2\theta$

$x^2-y^2 = r^2(\cosh^2\theta - \sinh^2\theta)$

$r^2 = x^2-y^2$

$r = +(x^2-y^2)^{\frac{1}{2}} \quad (r \geq 0)$

$\frac{\sinh\theta}{\cosh\theta} = \frac{y}{x}$

$\tanh\theta = \frac{y}{x}$

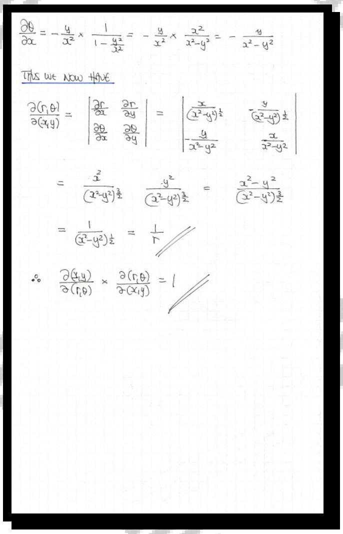
$\theta = \text{artanh}\left(\frac{y}{x}\right)$

TIDY THE REQUIRED DIFFERENTIALS

$$\frac{\partial r}{\partial x} = x(x^2-y^2)^{-\frac{1}{2}} = \frac{x}{(x^2-y^2)^{\frac{1}{2}}}$$

$$\frac{\partial r}{\partial y} = -y(x^2-y^2)^{-\frac{1}{2}} = -\frac{y}{(x^2-y^2)^{\frac{1}{2}}}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{x} \times \frac{1}{1-\left(\frac{y}{x}\right)^2} = \frac{1}{x} \times \frac{1}{1-\frac{y^2}{x^2}} = \frac{1}{x} \times \frac{x^2}{x^2-y^2} = \frac{x}{x^2-y^2}$$



$$\frac{\partial\theta}{\partial x} = -\frac{y}{x^2} \times \frac{1}{1-\frac{y^2}{x^2}} = -\frac{y}{x^2} \times \frac{x^2}{x^2-y^2} = -\frac{y}{x^2-y^2}$$

THUS WE NOW HAVE

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{(x^2-y^2)^{\frac{1}{2}}} & -\frac{y}{(x^2-y^2)^{\frac{1}{2}}} \\ -\frac{y}{x^2-y^2} & \frac{x}{x^2-y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2-y^2)^{\frac{3}{2}}} \quad \frac{y^2}{(x^2-y^2)^{\frac{3}{2}}} = \frac{x^2-y^2}{(x^2-y^2)^{\frac{3}{2}}}$$

$$= \frac{1}{(x^2-y^2)^{\frac{1}{2}}} = \frac{1}{r}$$

$$\therefore \quad \frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

Question 4

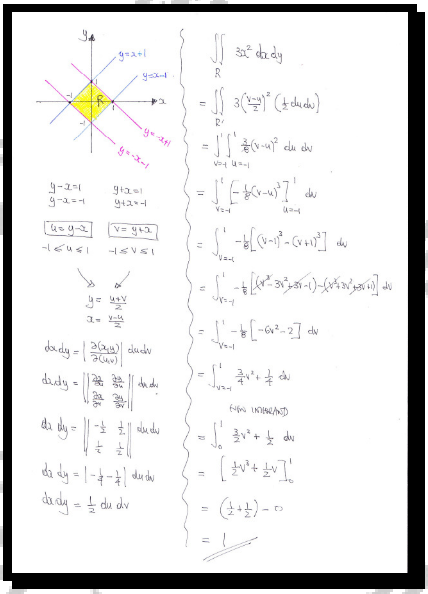
The finite region R is bounded by the straight lines with equations

$$y = x - 1, \quad y = x + 1, \quad y = -x - 1 \quad \text{and} \quad y = -x + 1.$$

Find an exact value for

$$\iint_R 3x^2 \, dx\, dy .$$

$\boxed{1}$



Question 5

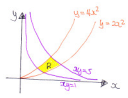
The finite region R is bounded by the curves with equations

$$y = 2x^2, \quad y = 4x^2, \quad xy = 1 \quad \text{and} \quad xy = 5.$$

Find an exact value for

$$\iint_R xy \, dx\, dy.$$

$\boxed{4\ln 2}$



• CURVES ONLY INTERSECT IN THE FIRST QUADRANT

• $\int_R xy \, dx\,dy$ ... TRANSFORMATIONS

$= \int_{R'} u \left(\frac{1}{3v}\,du\,dv\right)$

$= \int_{v=2}^{4} \int_{u=1}^{5} \frac{u}{3v}\,du\,dv$

$= \int_{v=2}^{4} \left[\frac{u^2}{6v}\right]_{u=1}^{5} dv$

$= \int_{v=2}^{4} \frac{1}{6v}[25-1]\,dv$

$= \int_{v=2}^{4} \frac{4}{v}\,dv$

$= \left[4\ln v\right]_2^4$

$= 4[\ln 4 - \ln 2]$

$= 4\ln 2$

LET $u = xy$ AND $v = \frac{y}{x^2}$

$1 \leqslant u \leqslant 5$ $\quad$ $2 \leqslant v \leqslant 4$

$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$

$du\,dv = \left\| \begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix} \right\| dx\,dy$

$du\,dv = \left\| \begin{matrix} y & x \\ -\frac{2y}{x^3} & \frac{1}{x^2} \end{matrix} \right\| dy\,dx$

$du\,dv = \left| \frac{y}{x^2} + \frac{2y}{x^2} \right| dy\,dx$

$du\,dv = \frac{3y}{x^2}\,dy\,dx$

$dy\,dx = \frac{x^2}{3y}\,du\,dv$

$dy\,dx = \frac{1}{3v}\,du\,dv$

Question 6

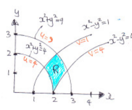
The finite region R in the first quadrant is defined by the inequalities

$$4 \le x^2 + y^2 \le 9 \quad \text{and} \quad 1 \le x^2 - y^2 \le 4.$$

Evaluate the following integral

$$\iint_R xy \, dx\, dy .$$

$\boxed{\frac{15}{8}}$



• LET $u = x^2+y^2$
$v = x^2-y^2$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left\|\begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{matrix}\right\| = \left\|\begin{matrix} 2x & 2y \\ 2x & -2y\end{matrix}\right\|$$

$$= \left| -4xy - 4xy \right| = 8xy$$

$$du\,dv = \frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy$$

$$du\,dv = 8xy\,dx\,dy$$

$$\boxed{dx\,dy = \frac{du\,dv}{8xy}}$$

ELIMINATING FROM THE EQUATIONS ABOVE
$u+v = 2x^2$ & $u-v = 2y^2$
$\boxed{x^2 = \frac{u+v}{2}}$ $\boxed{y^2 = \frac{u-v}{2}}$
(NOT ACTUALLY NEEDED HERE)

THUS

$$\iint_R xy\,dx\,dy = \iint_R xy\,\frac{du\,dv}{8xy} = \iint_R \frac{1}{8}\,du\,dv$$

$$= \int_{v=1}^{4}\int_{u=4}^{u=9} \frac{1}{8}\,du\,dv = \frac{1}{8}\times \boxed{\text{AREA}} = \frac{15}{8}$$

ALTERNATIVE JACOBIAN IS NOT SENSIBLE

e.g. $u = x^2+y^2$, $v = x^2-y^2$ ELIMINATE $x^2 = \frac{u+v}{2}$, $y^2 = \frac{u-v}{2}$ $\Rightarrow$ $x = \frac{1}{\sqrt{2}}(u+v)^{\frac{1}{2}}$ $x>0$, $y = \frac{1}{\sqrt{2}}(u-v)^{\frac{1}{2}}$ $y>0$

$$\therefore\ xy = \frac{1}{\sqrt{2}}(u+v)^{\frac{1}{2}} \times \frac{1}{\sqrt{2}}(u-v)^{\frac{1}{2}} = \frac{1}{2}(u^2-v^2)^{\frac{1}{2}}$$

NOW

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \left|\begin{matrix} \frac{1}{2\sqrt{2}}(u+v)^{-\frac{1}{2}} & \frac{1}{2\sqrt{2}}(u+v)^{-\frac{1}{2}} \\ \frac{1}{2\sqrt{2}}(u-v)^{-\frac{1}{2}} & -\frac{1}{2\sqrt{2}}(u-v)^{-\frac{1}{2}}\end{matrix}\right| du\,dv$$

$$= \left| -\frac{1}{8}(u^2-v^2)^{-\frac{1}{2}} - \frac{1}{8}(u^2-v^2)^{-\frac{1}{2}} \right| du\,dv = \left[\frac{1}{4}(u^2-v^2)^{-\frac{1}{2}}\right] du\,dv$$

THUS

$$\iint_R xy\,dx\,dy = \int_{v=1}^{4}\int_{u=4}^{9} \frac{1}{2}(u^2-v^2)^{\frac{1}{2}} \times \frac{1}{4}(u^2-v^2)^{-\frac{1}{2}}\,du\,dv$$

$$= \int_{v=1}^{4}\int_{u=4}^{9} \frac{1}{8}\,du\,dv$$

$$= \int_{v=1}^{4} \left[\frac{1}{8}u\right]_4^9 dv$$

$$= \int_{v=1}^{4} \left(\frac{9}{8} - \frac{4}{8}\right) dv$$

$$= \int_1^4 \frac{5}{8}\,dv$$

$$= \left[\frac{5}{8}v\right]_1^4$$

$$= \frac{20}{8} - \frac{5}{8}$$

$$= \frac{15}{8}$$

AS BEFORE

Question 7

The finite region R is bounded by the straight lines with equations

$$x+y=1, \quad x+y=2, \quad y=x \quad \text{and} \quad y=0.$$

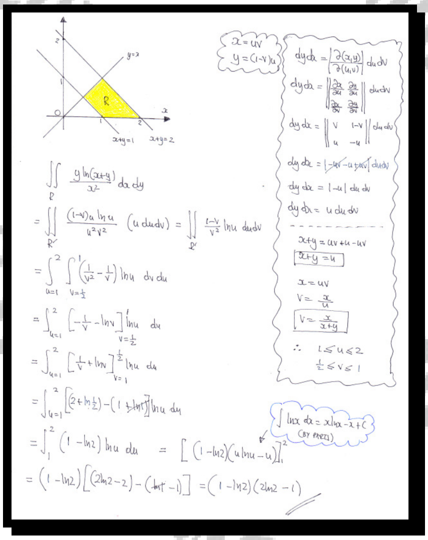
Use the transformation equations

$$x=uv \quad \text{and} \quad y=u(1-v),$$

to find an exact value for

$$\iint_R \frac{y\ln(x+y)}{x^2}\,dx\,dy.$$

$$\boxed{(1-\ln 2)(-1+2\ln 2)}$$



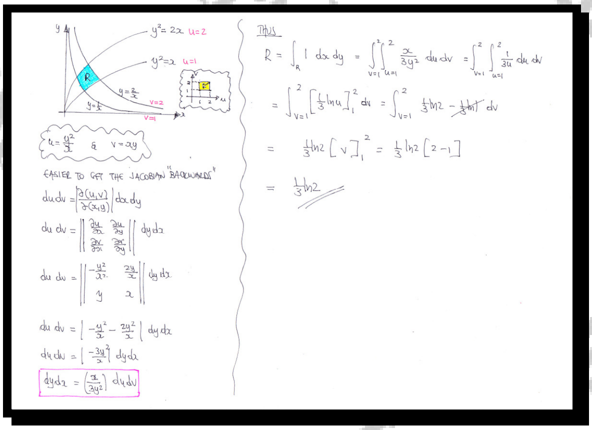
Question 8

The finite region R satisfies the inequalities

$$x \le y^2 \le 2x \quad \text{and} \quad \frac{1}{x} \le y \le \frac{2}{x}.$$

Find the area of R, giving the answer as an exact simplified logarithm.

$\boxed{\frac{1}{3}\ln 2}$



Question 9

An ellipse has Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are positive constants.

Use the transformation equations

$$x = r\cos\theta \quad \text{and} \quad y = f(r)\sin\theta,$$

where f is a function to be found, to determine the area enclosed by the ellipse.

$\boxed{\pi ab}$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

- TRANSFORM THE ELLIPSE INTO A CIRCLE, AS FOLLOWS

$$\Rightarrow x^2 + \frac{y^2a^2}{b^2} = a^2$$

$$\Rightarrow (x)^2 + \left(\frac{a}{b}y\right)^2 = a^2$$

$$\Rightarrow X^2 + Y^2 = a^2$$

WHERE $X = x$, $Y = \frac{a}{b}y$

- SWITCH INTO POLARS

$$\left.\begin{array}{l} X = x = r\cos\theta \\ Y = \frac{a}{b}y = r\sin\theta \end{array}\right\} \Rightarrow \begin{array}{l} X = r\cos\theta \\ Y = \frac{b}{a}r\sin\theta \end{array} \quad \text{if } f(r) = \frac{b}{a}r$$

- WORK OUT THE JACOBIAN OF THE TRANSFORMATION

$$J = \begin{vmatrix} \frac{\partial X}{\partial r} & \frac{\partial X}{\partial \theta} \\ \frac{\partial Y}{\partial r} & \frac{\partial Y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \frac{b}{a}\sin\theta & \frac{b}{a}r\cos\theta \end{vmatrix} = \frac{b}{a}r\cos^2\theta + \frac{b}{a}r\sin^2\theta$$

$$= \frac{b}{a}r(\cos^2\theta + \sin^2\theta) = \frac{b}{a}r$$

- THUS THE ELLIPSE IS TRANSFORMED INTO A CIRCLE OF RADIUS a

$$\Rightarrow \text{AREA} = \int_R 1 \, dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{r=a} 1 \left(\frac{b}{a}r \, dr \, d\theta\right)$$

$$\Rightarrow \text{AREA} = \int_{\theta=0}^{2\pi} \left[\frac{b}{2a}r^2\right]_{r=0}^{r=a} d\theta = \int_{\theta=0}^{2\pi} \frac{b}{2a}(a^2 - 0) \, d\theta$$

$$\Rightarrow \text{AREA} = \frac{1}{2}ab \int_{\theta=0}^{2\pi} 1 \, d\theta$$

$$\Rightarrow \text{AREA} = \frac{1}{2}ab \times 2\pi = \pi ab$$

Question 10

The finite region R is bounded by the straight lines with equations

$$y = x \quad \text{and} \quad y = 4x\,,$$

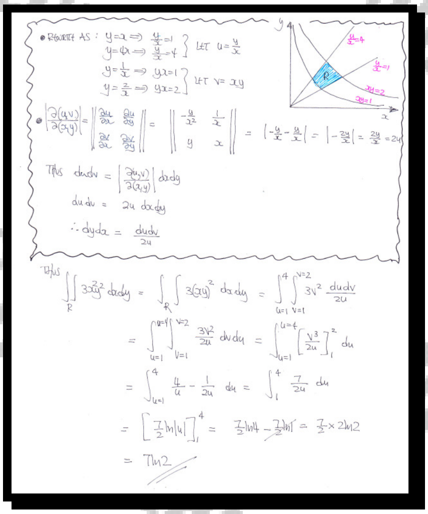
and the hyperbolae with equations

$$y = \frac{1}{x} \quad \text{and} \quad y = \frac{2}{x}\,,\ x \neq 0\,.$$

Show clearly that

$$\iint_R 3x^2y^2 \ dx\,dy = 7\ln 2\,.$$

proof



• REWRITE AS: $y = x \Rightarrow \frac{y}{x} = 1$, $y = 4x \Rightarrow \frac{y}{x} = 4$ } LET $u = \frac{y}{x}$

$y = \frac{1}{x} \Rightarrow yx = 1$, $y = \frac{2}{x} \Rightarrow yx = 2$ } LET $v = xy$

• $\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left\|\begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix}\right\| = \left\|\begin{matrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{matrix}\right\| = \left|-\frac{y}{x} - \frac{y}{x}\right| = \left|-\frac{2y}{x}\right| = \frac{2y}{x} = 2u$

Thus $du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$

$du\,dv = 2u\ dx\,dy$

$\therefore dy\,dx = \frac{du\,dv}{2u}$

Thus $\iint_R 3x^2y^2\,dx\,dy = \iint_R 3(xy)^2\,dx\,dy = \int_{u=1}^{4}\int_{v=1}^{v=2} 3v^2\,\frac{du\,dv}{2u}$

$= \int_{u=1}^{u=4}\int_{v=1}^{v=2} \frac{3v^2}{2u}\,dv\,du = \int_{u=1}^{u=4}\left[\frac{v^3}{2u}\right]_1^2 du$

$= \int_{u=1}^{4} \frac{4}{u} - \frac{1}{2u}\ du = \int_1^4 \frac{7}{2u}\ du$

$= \left[\frac{7}{2}\ln|u|\right]_1^4 = \frac{7}{2}\ln 4 - \cancel{\frac{7}{2}\ln 1} = \frac{7}{2}\times 2\ln 2$

$= 7\ln 2$

Question 11

The **unbounded** region R is defined by the curves with equations

$$y = x^2, \quad y = 2x^2 \quad \text{and} \quad y = \frac{1}{4x^2}.$$

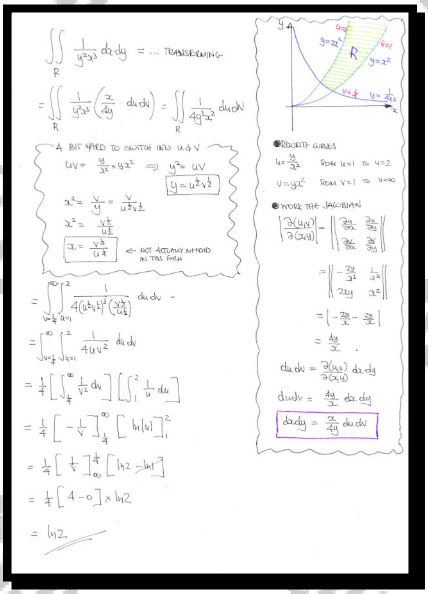
Use the transformation equations

$$u = \frac{y}{x^2} \quad \text{and} \quad v = yx^2,$$

to find an exact value for

$$\iint_R \frac{1}{y^2 x^3} \, dx \, dy.$$

$\boxed{\ln 2}$



$$\iint_R \frac{1}{y^2x^3}\,dx\,dy = \dots \text{TRANSFORMING}$$

$$= \iint_R \frac{1}{y^2x^3}\left(\frac{x}{4y}\,du\,dv\right) = \iint_R \frac{1}{4y^3x^2}\,du\,dv$$

A BIT HARD TO SWITCH INTO u & v

$$uv = \frac{y}{x^2} \times yx^2 \implies y^2 = uv \qquad \boxed{y = u^{\frac{1}{2}}v^{\frac{1}{2}}}$$

$$x^2 = \frac{v}{y} = \frac{v}{u^{\frac{1}{2}}v^{\frac{1}{2}}}$$

$$x^2 = \frac{v^{\frac{1}{2}}}{u^{\frac{1}{2}}}$$

$$\boxed{x = \frac{v^{\frac{1}{4}}}{u^{\frac{1}{4}}}} \leftarrow \text{NOT ACTUALLY NEEDED IN THIS FORM}$$

$$= \int_{v=\frac{1}{4}}^{\infty}\int_{u=1}^{2} \frac{1}{4\left(u^{\frac{1}{2}}v^{\frac{1}{2}}\right)^3\left(\frac{v^{\frac{1}{2}}}{u^{\frac{1}{2}}}\right)}\,du\,dv$$

$$= \int_{v=\frac{1}{4}}^{\infty}\int_{u=1}^{2} \frac{1}{4uv^2}\,du\,dv$$

$$= \frac{1}{4}\left[\int_{\frac{1}{4}}^{\infty} \frac{1}{v^2}\,dv\right]\left[\int_1^2 \frac{1}{u}\,du\right]$$

$$= \frac{1}{4}\left[-\frac{1}{v}\right]_{\frac{1}{4}}^{\infty}\left[\ln|u|\right]_1^2$$

$$= \frac{1}{4}\left[\frac{1}{v}\right]_{\infty}^{\frac{1}{4}}\left[\ln 2 - \ln 1\right]$$

$$= \frac{1}{4}\left[4 - 0\right] \times \ln 2$$

$$= \ln 2$$

● REWRITE CURVES

$u = \frac{y}{x^2}$ FROM $u=1$ TO $u=2$

$v = yx^2$ FROM $v=1$ TO $v=\infty$

● WORK THE JACOBIAN

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left\|\begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix}\right\|$$

$$= \left\|\begin{matrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ 2xy & x^2 \end{matrix}\right\|$$

$$= \left| -\frac{2y}{x} - \frac{2y}{x} \right|$$

$$= \frac{4y}{x}.$$

$$du\,dv = \frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy$$

$$du\,dv = \frac{4y}{x}\,dx\,dy$$

$$\boxed{dx\,dy = \frac{x}{4y}\,du\,dv}$$

Question 12

The finite region R, in the first quadrant, satisfies the inequalities

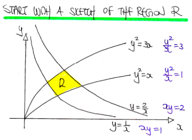
$$x \le y^2 \le 3x \quad \text{and} \quad \frac{1}{x} \le y \le \frac{2}{x}.$$

Find the exact value of

$$\int_R y^6 \; dx\,dy.$$

$\boxed{}$, $\boxed{\frac{28}{9}}$

START WITH A SKETCH OF THE REGION R



- LET $u=\frac{y^2}{x}$, $v=xy$
- AND $1 \le u \le 3$, $1 \le v \le 2$

OBTAIN THE JACOBIAN "BACKWARDS" AS IT IS EASIER

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \left\| \begin{matrix} \partial u/\partial x & \partial v/\partial x \\ \partial u/\partial y & \partial v/\partial y \end{matrix} \right\| = \left\| \begin{matrix} -\frac{y^2}{x^2} & y \\ \frac{2y}{x} & x \end{matrix} \right\| = \left| -\frac{y^2}{x} - \frac{2y^2}{x} \right|$$

$$= \left| -\frac{3y^2}{x} \right| = \frac{3y^2}{x} = 3u$$

$$\Rightarrow du\,dv = \frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy$$

$$\Rightarrow du\,dv = 3u\,dx\,dy$$

$$\Rightarrow \boxed{dx\,dy = \frac{du\,dv}{3u}}$$

ALSO SINCE THE INTEGRANT IS A FUNCTION OF y

$$\left.\begin{aligned} u &= \frac{y^2}{x} \\ v &= xy \end{aligned}\right\} \Rightarrow uv = \frac{y^2}{x}(xy)$$

$$uv = y^3$$

$$\boxed{y^6 = u^2v^2}$$

TRANSFORMING THE INTEGRAL NOW

$$\iint_R y^6\,dx\,dy = \int_{v=1}^{2}\int_{u=1}^{3} (u^2v^2)\,\frac{du\,dv}{3u}$$

$$= \frac{1}{3}\int_{v=1}^{2}\int_{u=1}^{3} uv^2\,du\,dv$$

$$= \frac{1}{3}\int_{v=1}^{2} \left[\frac{1}{2}u^2v^2\right]_{u=1}^{3} dv$$

$$= \frac{1}{3}\int_{v=1}^{2} \frac{9}{2}v^2 - \frac{1}{2}v^2\,dv$$

$$= \frac{1}{3}\int_1^2 4v^2\,dv$$

$$= \frac{1}{3}\left[\frac{4}{3}v^3\right]_1^2$$

$$= \frac{4}{9}(8-1)$$

$$= \frac{28}{9}$$

Question 13

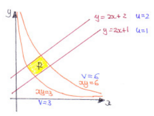
The finite region R, in the first quadrant, is bounded by the curves with equations

$$y = 2x + 1, \quad y = 2x + 2, \quad y = \frac{3}{x} \quad \text{and} \quad y = \frac{6}{x}.$$

Show clearly that

$$\iint_R (y + 2x)^3 \ dx\,dy = 115.$$

proof



LET $\boxed{\begin{matrix} u = y - 2x \\ v = xy \end{matrix}}$ $\quad 1 \le u \le 2$, $\quad 3 \le v \le 6$

$dx\,dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$

EASIER

$du\,dv = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx\,dy$

$du\,dv = \left\| \begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix} \right\| dx\,dy$

$du\,dv = \left\| \begin{matrix} -2 & 1 \\ y & x \end{matrix} \right\| dx\,dy$

$du\,dv = |-2x - y|\, dx\,dy$

$\boxed{dx\,dy = \frac{du\,dv}{2x+y}}$

$$\iint_R (y+2x)^3 \, dx\,dy$$
$$= \iint_{R'} (y+2x)^3 \, \frac{du\,dv}{y+2x}$$
$$= \iint_{R'} (y+2x)^2 \, du\,dv$$
$$= \iint_{R'} y^2 + 4xy + 4x^2 \, du\,dv$$
$$= \iint_{R'} (y^2 - 4xy + 4x^2) + 8xy \, du\,dv$$
$$= \iint_{R'} (y-2x)^2 + 8xy \, du\,dv$$
$$= \iint_{R} u^2 + 8v \, du\,dv$$
$$= \int_{v=3}^{v=6} \int_{u=1}^{u=2} u^2 + 8v \, du\,dv$$
$$= \int_{v=3}^{6} \left[\tfrac{1}{3}u^3 + 8vu \right]_{u=1}^{u=2} dv$$
$$= \int_{v=3}^{6} \left(\tfrac{8}{3} + 16v \right) - \left(\tfrac{1}{3} + 8v \right) dv$$
$$= \int_{3}^{6} \tfrac{7}{3} + 8v \, dv$$
$$= \left[\tfrac{7}{3}v + 4v^2 \right]_3^6$$
$$= (14 + 144) - (7 + 36)$$
$$= 115$$

Question 14

The finite region R is defined by the inequalities

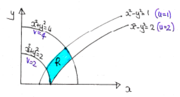
$$2 \le x^2 + y^2 \le 4 \quad \text{and} \quad 1 \le x^2 - y^2 \le 2 .$$

Given further that $x > 0$ and $x > 0$, evaluate the following integral

$$\iint_R x^3 y^3 \, dx\,dy .$$

$\boxed{}$, $\boxed{\frac{7}{16}}$

STARTING WITH A DIAGRAM IN THE FIRST QUADRANT



- LET $u = x^2 - y^2$
  $v = x^2 + y^2$
- AND $1 \le u \le 2$
  $2 \le v \le 4$

OBTAIN THE "JACOBIAN"

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \left\| \begin{matrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{matrix} \right\| = \left\| \begin{matrix} 2x & -2y \\ 2x & 2y \end{matrix} \right\| = |4xy + 4xy| = 8xy$$

$$\Rightarrow du\,dv = \frac{\partial(u,v)}{\partial(x,y)} dx\,dy$$

$$\Rightarrow du\,dv = 8xy\,dx\,dy$$

$$\Rightarrow dx\,dy = \frac{du\,dv}{8xy}$$ ← LEAVE IT IN THIS FORM AS IT WILL CANCEL

ALSO GET x & y (OR RATHER x^2 & y^2) IN TERMS OF u & v

$u = x^2 - y^2$
$v = x^2 + y^2$

$\therefore \; u + v = 2x^2$ & $v - u = 2y^2$

$\boxed{x^2 = \frac{1}{2}(u+v)}$ $\boxed{y^2 = \frac{1}{2}(v-u)}$

TRANSFORMING THE INTEGRAL NOW

$$\iint_R x^3y^3\,dx\,dy = \text{.... CHANGE COORDS ...} \int_{v=2}^{4}\int_{u=1}^{2} x^3y^3 \frac{du\,dv}{8xy}$$

$$= \int_{v=2}^{4}\int_{u=1}^{2} \frac{1}{8}x^2y^2\,du\,dv = \int_{v=2}^{4}\int_{u=1}^{2} \frac{1}{8}\times\frac{1}{2}(v+u)\times\frac{1}{2}(v-u)\,du\,dv$$

$$= \frac{1}{32}\int_{v=2}^{4}\int_{u=1}^{2}(v^2-u^2)\,du\,dv = \frac{1}{32}\int_{v=2}^{4}\left[v^2u - \frac{1}{3}u^3\right]_{u=1}^{u=2} dv$$

$$= \frac{1}{32}\int_{v=2}^{4}\left(2v^2-\frac{8}{3}\right)-\left(v^2-\frac{1}{3}\right) dv = \frac{1}{32}\int_2^4 v^2 - \frac{7}{3}\,dv$$

$$= \frac{1}{32}\left[\frac{1}{3}v^3 - \frac{7}{3}v\right]_2^4 = \frac{1}{32}\left[\left(\frac{64}{3}-\frac{28}{3}\right)-\left(\frac{8}{3}-\frac{14}{3}\right)\right]$$

$$= \frac{1}{32}\left[\frac{64-28-8+14}{3}\right] = \frac{1}{32}\times\frac{42}{3} = \frac{1}{32}\times 14$$

$$= \frac{7}{16}$$

Question 15

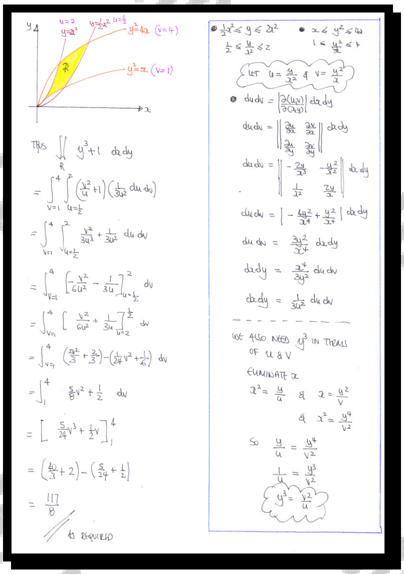
The finite region R is bounded by the parabolas with equations

$$y = \tfrac{1}{2}x^2, \quad y = 2x^2, \quad y^2 = x \quad \text{and} \quad y^2 = 4x .$$

Show clearly that

$$\iint_R y^3 + 1 \; dx\,dy = \frac{117}{8} .$$

$\boxed{\text{proof}}$



Question 16

The finite region R is bounded by the straight lines with equations

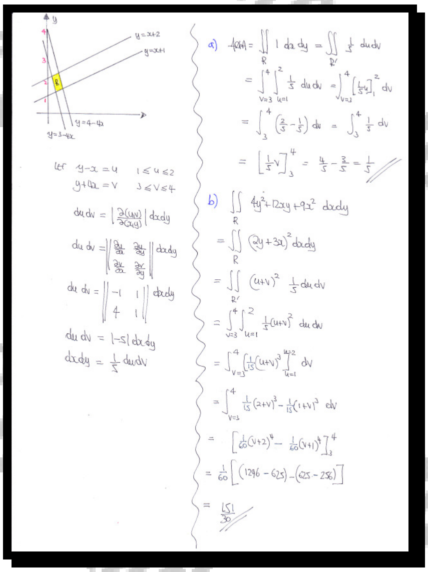
$$y = x + 1, \quad y = x + 2, \quad y = 3 - 4x \quad \text{and} \quad y = 4 - 4x.$$

a) Find the exact area of R.

b) Show clearly that

$$\iint_R 4y^2 + 12xy + 9x^2 \; dx\,dy = \frac{151}{30}.$$

$\boxed{\text{area} = \frac{1}{5}}$



let $y - x = u \qquad 1 \leqslant u \leqslant 2$

$y + 4x = v \qquad 3 \leqslant v \leqslant 4$

$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$

$du\,dv = \left\| \begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix} \right\| dx\,dy$

$du\,dv = \left\| \begin{matrix} -1 & 1 \\ 4 & 1 \end{matrix} \right\| dx\,dy$

$du\,dv = |-5|\,dx\,dy$

$dx\,dy = \frac{1}{5}\,du\,dv$

a) $\cancel{\text{AREA}} = \iint_R 1\,dx\,dy = \iint_{R'} \frac{1}{5}\,du\,dv$

$= \int_{v=3}^{4}\int_{u=1}^{2} \frac{1}{5}\,du\,dv = \int_{v=3}^{4}\left[\frac{1}{5}u\right]_1^2 dv$

$= \int_3^4 \left(\frac{2}{5} - \frac{1}{5}\right) dv = \int_3^4 \frac{1}{5}\,dv$

$= \left[\frac{1}{5}v\right]_3^4 = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$

b) $\iint_R 4y^2 + 12xy + 9x^2 \; dx\,dy$

$= \iint_R (2y + 3x)^2\,dx\,dy$

$= \iint_{R'} (u+v)^2 \, \frac{1}{5}\,du\,dv$

$= \int_{v=3}^{4}\int_{u=1}^{2} \frac{1}{5}(u+v)^2\,du\,dv$

$= \int_{v=3}^{4} \left[\frac{1}{15}(u+v)^3\right]_{u=1}^{u=2} dv$

$= \int_{v=3}^{4} \frac{1}{15}(2+v)^3 - \frac{1}{15}(1+v)^3\,dv$

$= \left[\frac{1}{60}(v+2)^4 - \frac{1}{60}(v+1)^4\right]_3^4$

$= \frac{1}{60}\left[(1296 - 625) - (625 - 256)\right]$

$= \frac{151}{30}$

Question 17

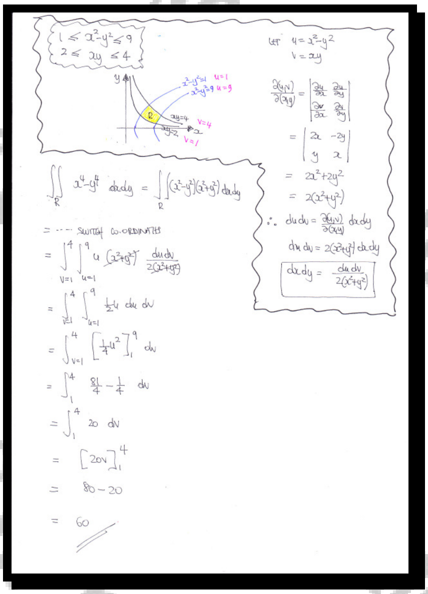
The finite region R is defined by the inequalities

$$1 \le x^2 - y^2 \le 9 \quad \text{and} \quad 2 \le xy \le 4 .$$

Given further that $x > 0$ and $x > 0$, evaluate the following integral

$$\iint_R \left(x^4 - y^4 \right) \, dx \, dy .$$

$\boxed{60}$



Question 18

The finite region R is bounded by the straight lines with equations

$$y = x - 1 \quad \text{and} \quad y = x - 3,$$

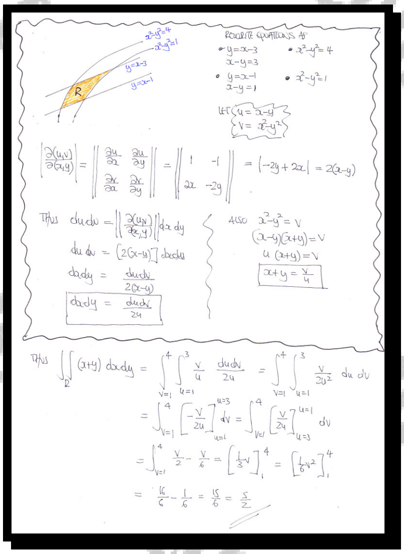
and the hyperbolae with equations

$$x^2 - y^2 = 1 \quad \text{and} \quad x^2 - y^2 = 4.$$

Evaluate the integral

$$\iint_R (x + y)\, dx\, dy.$$

$$\boxed{\frac{5}{2}}$$



Question 19

The finite region R is bounded by the curves with equations

$$6xy = \pi \quad \text{and} \quad 2xy = \pi ,$$

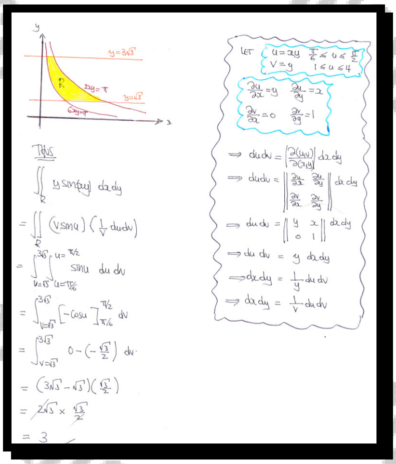
and the straight lines with equations

$$y = \sqrt{3} \quad \text{and} \quad y = 3\sqrt{3} .$$

evaluate the following integral

$$\iint_R y \sin(xy)\, dx\, dy .$$

$\boxed{3}$



Question 20

The finite region R satisfies the inequalities

$$1 \le x+y \le 2 \quad \text{and} \quad 0 \le y \le x .$$

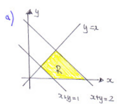
a) Use plane polar coordinates (r,θ) to find the value of

$$\iint_R \frac{y(x+y)^2}{x^3}\,dx\,dy .$$

b) Verify the answer obtained in part **(a)** by transforming the integral to different coordinate system.

$\boxed{\frac{3}{4}}$

a)



IN POLARS

- $x+y=1$

$r\cos\theta + r\sin\theta = 1$

$r(\cos\theta + \sin\theta) = 1$

$r = \frac{1}{\cos\theta + \sin\theta}$

& SIMILARLY $x+y=2$

$r = \frac{2}{\cos\theta + \sin\theta}$

- x AXIS IS $\theta = 0$
- $y = x$ IS $\theta = \frac{\pi}{4}$

$\therefore dx\,dy = r\,dr\,d\theta$

$\frac{1}{\cos\theta+\sin\theta} \le r \le \frac{2}{\cos\theta+\sin\theta}$

$0 \le \theta \le \frac{\pi}{4}$

THUS

$$\iint_R \frac{y(x+y)^2}{x^3}\,dx\,dy = \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=\frac{1}{\cos\theta+\sin\theta}}^{r=\frac{2}{\cos\theta+\sin\theta}} \frac{(r\sin\theta)(r\cos\theta+r\sin\theta)^2}{(r\cos\theta)^3}\,r\,dr\,d\theta$$

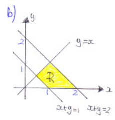
$$= \int_{\theta=0}^{\theta=\frac{\pi}{4}} \int_{r=\frac{1}{\cos\theta+\sin\theta}}^{r=\frac{2}{\cos\theta+\sin\theta}} r\,\frac{\sin\theta(\cos\theta+\sin\theta)^2}{\cos^3\theta}\,dr\,d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{4}} \left[\frac{1}{2}r^2 \times \frac{\sin\theta(\cos\theta+\sin\theta)^2}{\cos^3\theta}\right]_{r=\frac{1}{\cos\theta+\sin\theta}}^{r=\frac{2}{\cos\theta+\sin\theta}} d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} \frac{1}{2} \times \frac{\sin\theta(\cos\theta+\sin\theta)^2}{\cos^3\theta}\left[\frac{4}{(\cos\theta+\sin\theta)^2} - \frac{1}{(\cos\theta+\sin\theta)^2}\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}\,\frac{\sin\theta(\cos\theta+\sin\theta)^2}{\cos^3\theta}\left[\frac{3}{(\cos\theta+\sin\theta)^2}\right] d\theta = \int_0^{\frac{\pi}{4}} \frac{3}{2}\,\frac{\sin\theta}{\cos^3\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{3}{2}\tan\theta\sec^2\theta\,d\theta = \left[\frac{3}{4}\tan^2\theta\right]_0^{\frac{\pi}{4}} = \frac{3}{4}$$

b)



$x+y=u$

$v = \frac{y}{x}$

- FIND THE JACOBIAN BACKWARDS (EASIER)

$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$

$du\,dv = \left\| \begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix} \right\| dx\,dy$

$du\,dv = \left\| \begin{matrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{matrix} \right\| dx\,dy$

$du\,dv = \left|\frac{1}{x} + \frac{y}{x^2}\right| dx\,dy$

$du\,dv = \left|\frac{x+y}{x^2}\right| dx\,dy$

$dx\,dy = \left|\frac{x^2}{x+y}\right| du\,dv$

- LIMITS $1 \le u \le 2$

$0 \le v \le 1$

THUS

$$\iint_R \frac{y(x+y)^2}{x^3}\,dy\,dx$$

$$= \iint_{R'} \frac{y(x+y)^2}{x^3} \times \frac{x^2}{x+y}\,du\,dv$$

$$= \iint_{R'} \frac{y(x+y)}{x}\,du\,dv$$

$$= \int_{v=0}^{1}\int_{u=1}^{2} uv\,du\,dv$$

$$= \int_{v=0}^{1} \left[\frac{1}{2}u^2 v\right]_{u=1}^{u=2} dv$$

$$= \int_0^1 2v - \frac{1}{2}v\,dv$$

$$= \int_0^1 \frac{3}{2}v\,dv$$

$$= \left[\frac{3}{4}v^2\right]_0^1$$

$$= \frac{3}{4}$$

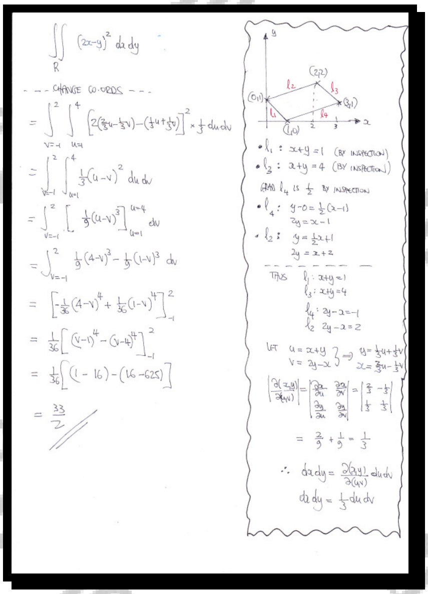
Question 21

The finite region R in the x-y plane is defined as the region enclosed by the straight line segments joining the points with coordinates at $(1,0)$, $(1,0)$, $(1,0)$ and $(1,0)$, in that order.

Evaluate the following integral

$$\iint_R (2x-y)^2 \, dx\,dy .$$

$\boxed{\dfrac{33}{2}}$



Question 22

The finite region R in the x-y plane, is defined as the interior of a parallelogram with vertices at $(4,0)$, $(0,1)$, $(-2,7)$ and $(2,6)$.

Evaluate the integral

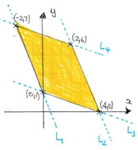
$$\int_R x^2 \ dxdy \ .$$

$$\boxed{}, \ \boxed{\frac{176}{3}}$$

- START WITH A SKETCH OF THE REGION

GRADIENT OF L_1: $\frac{7-1}{-2-0} = \frac{6}{-2} = -3$

GRADIENT OF L_3: $\frac{0-1}{4-0} = -\frac{1}{4}$



- NOW WE HAVE THE EQUATIONS OF ALL 4 LINES WHICH DEFINE THE REGION

L_1: $y = -3x+1 \Rightarrow 3x+y = 1$

L_2: $y = -3x+12 \Rightarrow 3x+y = 12$

L_3: $y = -\frac{1}{4}x+1 \Rightarrow x+4y = 4$

L_4: $y = -\frac{1}{4}x+\frac{13}{2} \Rightarrow x+4y = 26$

- DEFINE A NEW CO-ORDINATE SYSTEM, PARALLEL TO THE SIDES L_1 TO L_4

$$\boxed{\begin{array}{ll} u = 3x+y & 1 \leqslant u \leqslant 12 \\ v = x+4y & 4 \leqslant v \leqslant 26 \end{array}}$$

$\Rightarrow du\,dv = \frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy$

$\Rightarrow du\,dv = \left\| \begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix} \right\| dx\,dy$

$\Rightarrow du\,dv = \left\| \begin{matrix} 3 & 1 \\ 1 & 4 \end{matrix} \right\| dx\,dy$

$\Rightarrow du\,dv = 11\,dx\,dy$

$\Rightarrow \boxed{dx\,dy = \frac{1}{11}\,du\,dv}$

- FINALLY GETTING A SIMPLIFIED EXPRESSION FOR THE INTEGRAND IN TERMS OF u & v

$$\left.\begin{array}{l} u = 3x+y \\ v = x+4y \end{array}\right\} \Rightarrow \begin{array}{l} -4u = -12x-4y \\ v = x+4y \end{array}$$

$\Rightarrow v-4u = -11x$

$\Rightarrow 11x = 4u-v$

$\Rightarrow 121x^2 = (4u-v)^2$

$\Rightarrow \boxed{x^2 = \frac{1}{121}(4u-v)^2}$

- RETURNING TO THE ACTUAL INTEGRAL

$$\int_R x^2\,dx\,dy = \int_{v=4}^{26}\int_{u=1}^{12} \frac{1}{121}(4u-v)^2 \left(\frac{1}{11}\,du\,dv\right)$$

$$= \frac{1}{11^3}\int_{v=4}^{26}\int_{u=1}^{12} (4u-v)^2\,du\,dv = \frac{1}{11^3}\int_{v=4}^{26}\left[\frac{1}{12}(4u-v)^3\right]_{u=1}^{12} dv$$

$$= \frac{1}{11^3\times 12}\int_4^{26} (48-v)^3-(4-v)^3\,dv = \frac{1}{11^3\times 12}\left[-\frac{1}{4}(48-v)^4+\frac{1}{4}(4-v)^4\right]_4^{26}$$

$$= \frac{1}{11^3\times 12\times 4}\left[(4-v)^4-(48-v)^4\right]_4^{26} = \frac{1}{11^3\times 12\times 4}\left[\cancel{\left[(-22)^4-(22)^4\right]}-\left[0-(44)^4\right]\right]$$

$$= \frac{1}{11^3\times 12\times 4}(44)^4 = \frac{4^4\times 11^4}{4\times 4\times 3\times 11^3} = \frac{4^2\times 11}{3} = \frac{16\times 11}{3}$$

$$= \frac{176}{3}$$

Question 23

Given that R is the finite region in the x-y plane, defined as

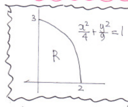
$$\frac{x^2}{4}+\frac{y^2}{9}\le 1,\ x\ge 0\,,\ y\ge 0\,,$$

evaluate the integral

$$\int_R yx^3\ dxdy\,.$$

$\boxed{6}$

REGION IS THE ELLIPSE & ITS INTERIOR: $\frac{x^2}{4}+\frac{y^2}{9}\le 1$ (FIRST QUADRANT)



USE A "STRETCH" TRANSFORMATION TO SWITCH INTO STANDARD PLANE POLARS

$x = 2X \qquad \frac{x}{2}=X \qquad dx = 2dX$

$y = 3Y \qquad \frac{y}{3}=Y \qquad dy = 3dY$

$$\iint_R x^3y\,dx\,dy = \iint_{R'} (2X)^3(3Y)\,(2dX)(3dY) = \iint_{R'} 144X^3Y\,dX\,dY$$

WHERE R' IS $X^2+Y^2=1$

SWITCH INTO PLANE POLARS

$$\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} 144(r^3\cos^3\theta)(r\sin\theta)\,r\,dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} 144r^5\cos^3\theta\sin\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\Big[24r^6\cos^3\theta\sin\theta\Big]_0^1\,d\theta = \int_{\theta=0}^{\frac{\pi}{2}} 24\cos^3\theta\sin\theta = \Big[-6\cos^4\theta\Big]_0^{\frac{\pi}{2}}$$

$$= \Big[6\cos^4\theta\Big]_{\frac{\pi}{2}}^{0} = 6-0 = 6$$

ALTERNATIVE BY JACOBIAN TRANSFORMATION (ESSENTIALLY THE SAME)

$x = 2r\cos\theta \qquad 0\le r\le 1$

$y = 3r\sin\theta \qquad 0\le\theta\le\frac{\pi}{2}$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{matrix}\right\| = \left\|\begin{matrix}2\cos\theta & -2r\sin\theta\\ 3\sin\theta & 3r\cos\theta\end{matrix}\right\| = 6r\cos^2\theta+6r\sin^2\theta = 6r$$

$\therefore \boxed{dx\,dy = 6r\,dr\,d\theta}$

THUS

$$\iint_R x^3y\,dx\,dy = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{r=1}(2r\cos\theta)^3(3r\sin\theta)\,(6r\,dr\,d\theta)$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} 144r^5\cos^3\theta\sin\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\Big[24r^6\cos^3\theta\sin\theta\Big]_0^1\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} 24\cos^3\theta\sin\theta\,d\theta$$

$$= \Big[-6\cos^4\theta\Big]_0^{\frac{\pi}{2}}$$

$$= \Big[6\cos^4\theta\Big]_{\frac{\pi}{2}}^{0}$$

$$= 6-0$$

$$= 6$$

Question 24

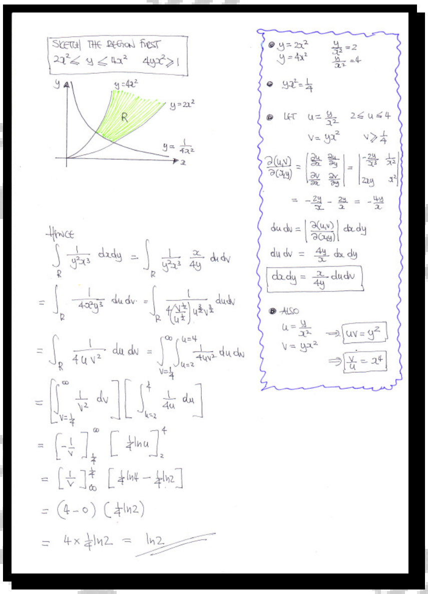
Given that R is the region of the x-y plane, defined as

$$2x^2 \le y \le 4x^2 \quad \text{and} \quad 4yx^2 \ge 1,$$

evaluate the integral

$$\int_R \frac{1}{y^2 x^3} \, dxdy \, .$$

$\boxed{\ln 2}$



SKETCH THE REGION FIRST
$2x^2 \le y \le 4x^2 \quad 4yx^2 \ge 1$

HENCE

$\int_R \frac{1}{y^2x^3}\,dxdy = \int_R \frac{1}{y^2x^3}\frac{x}{4y}\,du\,dv$

$= \int_R \frac{1}{4x^2y^3}\,du\,dv = \int_R \frac{1}{4\left(\frac{v^{\frac{1}{2}}}{u^{\frac{1}{2}}}\right)u^{\frac{3}{2}}v^{\frac{3}{2}}}\,du\,dv$

$= \int_R \frac{1}{4uv^2}\,du\,dv = \int_{v=\frac{1}{4}}^{\infty}\int_{u=2}^{u=4} \frac{1}{4uv^2}\,du\,dv$

$= \left[\int_{v=\frac{1}{4}}^{\infty} \frac{1}{v^2}\,dv\right]\left[\int_{u=2}^{4} \frac{1}{4u}\,du\right]$

$= \left[-\frac{1}{v}\right]_{\frac{1}{4}}^{\infty}\left[\frac{1}{4}\ln u\right]_2^4$

$= \left[\frac{1}{v}\right]_{\infty}^{\frac{1}{4}}\left[\frac{1}{4}\ln 4 - \frac{1}{4}\ln 2\right]$

$= (4-0)\left(\frac{1}{4}\ln 2\right)$

$= 4 \times \frac{1}{4}\ln 2 = \ln 2$

- $y = 2x^2 \quad \frac{y}{x^2} = 2$
  $y = 4x^2 \quad \frac{y}{x^2} = 4$
- $yx^2 = \frac{1}{4}$
- LET $u = \frac{y}{x^2} \quad 2 \le u \le 4$
  $v = yx^2 \quad v \ge \frac{1}{4}$

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ 2xy & x^2 \end{vmatrix}$

$= -\frac{2y}{x} - \frac{2y}{x} = -\frac{4y}{x}$

$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$

$du\,dv = \frac{4y}{x}\,dx\,dy$

$\boxed{dx\,dy = \frac{x}{4y}\,du\,dv}$

- ALSO
  $u = \frac{y}{x^2}$, $v = yx^2 \implies \boxed{uv = y^2}$
  $\implies \boxed{\frac{v}{u} = x^4}$

Question 25

By suitably changing coordinates, find the volume of the solid defined as

$$0 \le \sqrt{x} + \sqrt{y} + \sqrt{z} \le \sqrt{3}.$$

$\boxed{\frac{3}{10}}$

THIS LOOKS LIKE A SPHERE OR TRANSFORMED TO ALMOST A SPHERE (NOTE x, y, z HAVE TO BE NON NEGATIVE)

USE THE TRANSFORMATIONS

$$\sqrt{x} = u^2 \implies x = u^4$$
$$\sqrt{y} = v^2 \implies y = v^4$$
$$\sqrt{z} = w^2 \implies z = w^4$$

AND $\sqrt{3} = a^2$

WORK THE JACOBIAN

$$dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

$$dx\,dy\,dz = \left\|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{matrix}\right\| du\,dv\,dw$$

$$dx\,dy\,dz \left\|\begin{matrix} 4u^3 & 0 & 0 \\ 0 & 4v^3 & 0 \\ 0 & 0 & 4w^3 \end{matrix}\right\| du\,dv\,dw$$

AND BY PROPERTIES OF DETERMINANTS

$$dx\,dy\,dz = (64u^3v^3w^3)\,du\,dv\,dw$$

NOW FROM $0 \le \sqrt{x} + \sqrt{y} + \sqrt{z} \le \sqrt{3}$ $\quad (a^2 = \sqrt{3})$

.... TO $0 \le u^2 + v^2 + w^2 \le a^2$

.... TO $0 \le X^2 + Y^2 + Z^2 \le a^2$

.... TO SPHERICAL POLARS

$$\left.\begin{aligned} X = u &= r\sin\theta\cos\phi \\ Y = v &= r\sin\theta\sin\phi \\ Z = w &= r\cos\theta \end{aligned}\right\} \quad \begin{aligned} 0 &\le r \le a \\ 0 &\le \theta \le \tfrac{\pi}{2} \\ 0 &\le \phi \le \tfrac{\pi}{2} \end{aligned}$$

$$dX\,dY\,dZ = r^2\sin\theta\,dr\,dr\,d\phi$$

THUS

$$V = \int_V 1\,dx\,dy\,dz = \int_V 64u^3v^3w^3\,du\,dv\,dw$$

$$V = \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} 64(r\sin\theta\cos\phi)^3(r\sin\theta\sin\phi)^3(r\cos\theta)^3(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

$$V = \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} 64r^{11}\sin^7\theta\cos^3\theta\cos^3\phi\sin^3\phi\,dr\,d\theta\,d\phi$$

SPLIT BACK INTO THREE INTEGRALS

$$V = \left[\int_{\phi=0}^{\frac{\pi}{2}} 2\cos^3\phi\sin^3\phi\,d\phi\right]\left[\int_{\theta=0}^{\frac{\pi}{2}} 2\sin^7\theta\cos^3\theta\,d\theta\right]\left[\int_{r=0}^{a} 16r^{11}\,dr\right]$$

$$V = \left[\int_0^{\frac{\pi}{2}} 2(\cos\phi)^{2\times2-1}(\sin\phi)^{2\times2-1}\,d\phi\right]\left[\int_0^{\frac{\pi}{2}} 2(\sin\theta)^{2\times4-1}(\cos\theta)^{2\times2-1}\,d\theta\right]\left[\int_0^{a} 16r^{11}\,dr\right]$$

$$V = B(2,2) \times B(4,2) \times \left[\tfrac{4}{3}r^{12}\right]_0^a$$

$$V = \frac{\cancel{\Gamma(2)}\Gamma(2)}{\cancel{\Gamma(4)}} \times \frac{\cancel{\Gamma(4)}\cancel{\Gamma(2)}}{\Gamma(6)} \times \frac{4}{3}\cdot a^{12}$$

$$V = \frac{\Gamma(2)}{\Gamma(6)} \times \frac{4}{3}(a^2)^6$$

$$V = \frac{1!}{5!} \times \frac{4}{3} \times (\sqrt{3})^6$$

$$V = \frac{1}{\cancel{120}_{30}} \times \frac{\cancel{4}^{1}}{\cancel{3}} \times \cancel{27}^{9}$$

$$V = \frac{9}{30}$$

$$V = \frac{3}{10}$$

Question 26

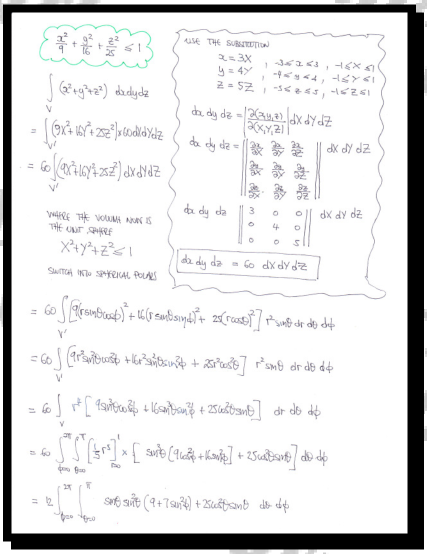
The finite region R is defined as the region enclosed by the ellipsoid with Cartesian equation

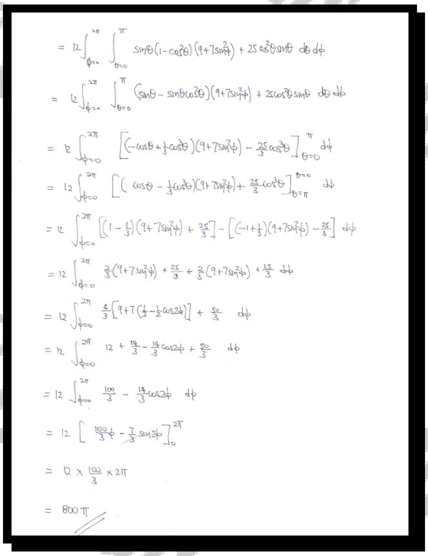
$$\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{25}=1.$$

By first transforming the Cartesian coordinates into a new Cartesian coordinate system, use spherical polar coordinates, (r,θ,φ), find the value of

$$\iiint_R \left(x^2+y^2+z^2\right)\ dx\,dy\,dz.$$

$\boxed{800\pi}$





Question 27

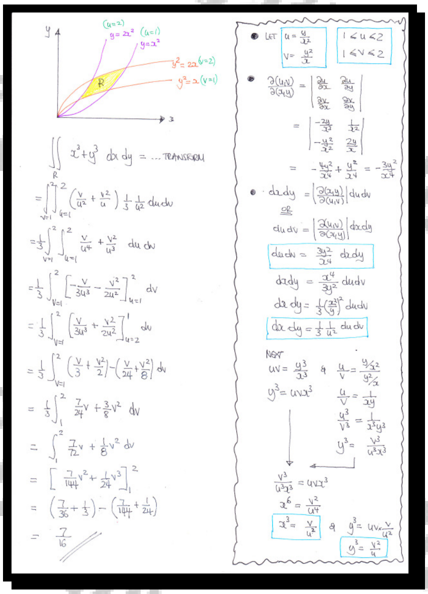
The finite region R in the x-y plane, is defined

$$x^2 \le y \le 2x^2 \quad \text{and} \quad x \le y^2 \le 2x \,.$$

Evaluate the integral

$$\int_R x^3 + y^3 \; dxdy \,.$$

$$\boxed{\frac{7}{16}}$$



Question 28

The finite region R is bounded by the coordinate axes and the straight line with Cartesian equation

$$x + y = 1$$

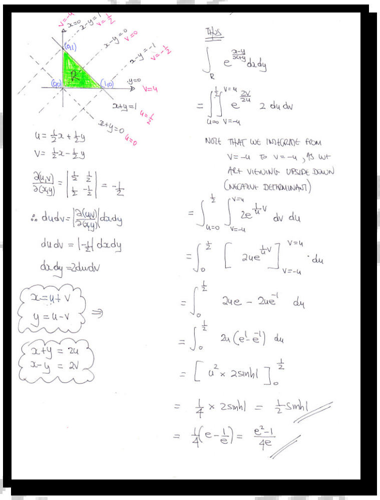
Use the coordinate transformation equations

$$u = \frac{1}{2}x + \frac{1}{2}y \quad \text{and} \quad v = \frac{1}{2}x - \frac{1}{2}y$$

to find an exact value for

$$\int_R \mathrm{e}^{\frac{x-y}{x+y}} \, dxdy \, .$$

$$\boxed{\frac{\mathrm{e}^2 - 1}{4\mathrm{e}} = \frac{1}{2}\sinh 1}$$



$u = \frac{1}{2}x + \frac{1}{2}y$

$v = \frac{1}{2}x - \frac{1}{2}y$

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

$\therefore \; du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$

$du\,dv = \left|-\frac{1}{2}\right| dx\,dy$

$dx\,dy = 2\,du\,dv$

$x = u + v$, $y = u - v$ $\Rightarrow$ $x + y = 2u$, $x - y = 2v$

THUS

$\int_R e^{\frac{x-y}{x+y}} dx\,dy$

$= \int_{u=0}^{\frac{1}{2}} \int_{v=-u}^{v=u} e^{\frac{2v}{2u}} \, 2\,du\,dv$

NOTE THAT WE INTEGRATE FROM $v=-u$ TO $v=-u$, AS WE ARE VIEWING UPSIDE DOWN (NEGATIVE DETERMINANT)

$= \int_{u=0}^{\frac{1}{2}} \int_{v=-u}^{v=u} 2e^{\frac{1}{u}v} \, dv\,du$

$= \int_0^{\frac{1}{2}} \left[2ue^{\frac{1}{u}v} \right]_{v=-u}^{v=u} du$

$= \int_0^{\frac{1}{2}} 2ue - 2ue^{-1} \, du$

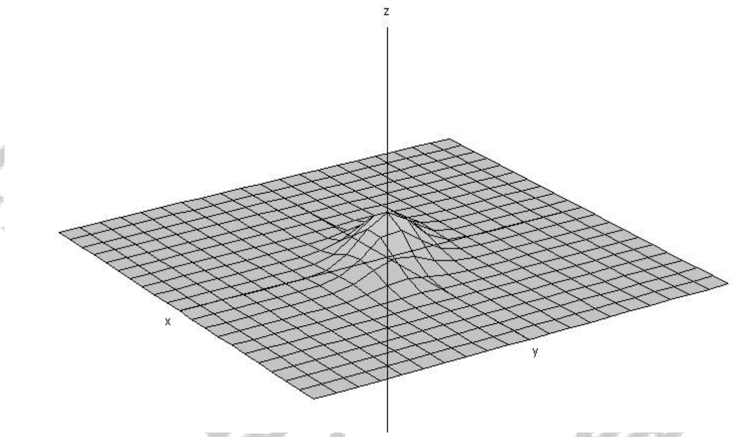
$= \int_0^{\frac{1}{2}} 2u(e^1 - e^{-1}) \, du$

$= \left[u^2 \times 2\sinh 1 \right]_0^{\frac{1}{2}}$

$= \frac{1}{4} \times 2\sinh 1 = \frac{1}{2}\sinh 1$

$= \frac{1}{4}\left(e - \frac{1}{e}\right) = \frac{e^2-1}{4e}$

Question 29



The figure above shows the graph of a "hill", modelled by the function $z = f(x, y)$, defined in the entire x-y plane by

$$z = e^{-\left(\frac{5}{4}x^2 - xy + 2y^2\right)}.$$

Use the transformation equations

$$x = u + 2v \quad \text{and} \quad y = u - v$$

to show that the volume of the "hill" is $\dfrac{2\pi}{3}$.

You may assume without proof that $\displaystyle\int_{-\infty}^{\infty} e^{-s^2}\,ds = \sqrt{\pi}$.

$\boxed{}$, $\boxed{\text{proof}}$

START BY MANIPULATING THE EXPONENT

$$\frac{5}{4}x^2 - xy + 2y^2 = \frac{5}{4}(u+2v)^2 - (u+2v)(u-v) + 2(u-v)^2$$
$$= \frac{5}{4}(u^2 + 4uv + 4v^2) - (u^2 + uv - 2v^2) + 2(u^2 - 2uv + v^2)$$
$$= \frac{5}{4}u^2 + \cancel{5uv} + 5v^2 - u^2 - \cancel{uv} + 2v^2 + 2u^2 - \cancel{4uv} + 2v^2$$
$$= \frac{9}{4}u^2 + 9v^2$$

NEXT CALCULATE THE "SCALING FACTOR"

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \left\| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right\| du\,dv$$
$$= \left\| \begin{matrix} 1 & 2 \\ 1 & -1 \end{matrix} \right\| du\,dv = |-3|\,du\,dv$$
$$\therefore \boxed{dx\,dy = 3\,du\,dv}$$

HENCE WE HAVE THE FOLLOWING VOLUME INTEGRAL

$$V = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(\frac{5}{4}x^2 - xy + 2y^2\right)}\,dx\,dy$$

CHANGE THE VARIABLES INTO THE u–v PLANE, NOTING THE LIMITS ARE UNCHANGED

$$\dots = \int_{v=-\infty}^{\infty}\int_{u=-\infty}^{\infty} e^{-\left(\frac{9}{4}u^2 + 9v^2\right)}\,(3\,du\,dv)$$

SPLIT THE INTEGRAL, AS THERE IS NO DEPENDENCE, BETWEEN u & v

$$\dots = \left[\int_{-\infty}^{\infty} 3e^{-\frac{9}{4}u^2}\,du\right]\left[\int_{-\infty}^{\infty} e^{-9v^2}\,dv\right]$$

BY SUBSTITUTION

$t = \frac{3}{2}u$
$dt = \frac{3}{2}\,du$
$du = \frac{2}{3}\,dt$
LIMITS UNCHANGED

$s = 3v$
$ds = 3\,dv$
$dv = \frac{1}{3}\,ds$
LIMITS UNCHANGED

... TRANSFORMING THE TWO INTEGRALS

$$= \left[\int_{-\infty}^{\infty} 3e^{-t^2}\left(\tfrac{2}{3}\,dt\right)\right]\left[\int_{-\infty}^{\infty} e^{-s^2}\left(\tfrac{1}{3}\,ds\right)\right]$$
$$= \frac{2}{3}\left[\int_{-\infty}^{\infty} e^{-t^2}\,dt\right]\left[\int_{-\infty}^{\infty} e^{-s^2}\,ds\right]$$
$$= \frac{2}{3}\sqrt{\pi}\sqrt{\pi}$$
$$= \frac{2}{3}\pi$$

Question 30

The finite region R is bounded by the coordinate axes and the straight line with Cartesian equation

$$x + y = 1$$

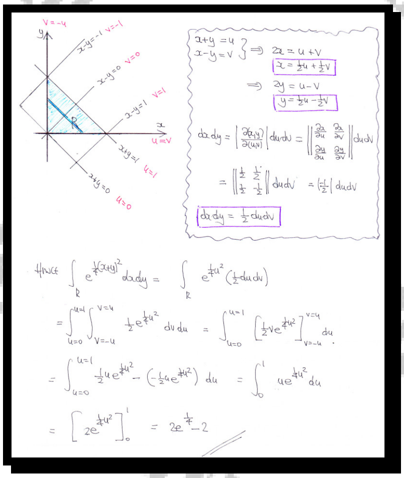
Use the transformation equations

$$u = x + y \qquad \text{and} \qquad v = x - y$$

to find an exact value for

$$\int_R e^{\frac{1}{4}(x+y)^2} \, dxdy \, .$$

$$\boxed{2\left(e^{\frac{1}{4}} - 1\right)}$$



$$\left.\begin{array}{l} x+y=u \\ x-y=v \end{array}\right\} \Rightarrow 2x = u+v \qquad \boxed{x = \tfrac{1}{2}u + \tfrac{1}{2}v}$$

$$\Rightarrow 2y = u - v \qquad \boxed{y = \tfrac{1}{2}u - \tfrac{1}{2}v}$$

$$dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \left\|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right\| dudv$$

$$= \left\|\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix}\right\| dudv = \left|-\tfrac{1}{2}\right| dudv$$

$$\boxed{dxdy = \tfrac{1}{2} dudv}$$

Hence $\displaystyle\int_R e^{\frac{1}{4}(x+y)^2} dxdy = \int_R e^{\frac{1}{4}u^2}\left(\tfrac{1}{2}dudv\right)$

$$= \int_{u=0}^{u=1}\int_{v=-u}^{v=u} \tfrac{1}{2}e^{\frac{1}{4}u^2} \, dv\,du = \int_{u=0}^{u=1} \left[\tfrac{1}{2}ve^{\frac{1}{4}u^2}\right]_{v=-u}^{v=u} du$$

$$= \int_{u=0}^{u=1} \tfrac{1}{2}ue^{\frac{1}{4}u^2} - \left(-\tfrac{1}{2}ue^{\frac{1}{4}u^2}\right) du = \int_0^1 ue^{\frac{1}{4}u^2} du$$

$$= \left[2e^{\frac{1}{4}u^2}\right]_0^1 = 2e^{\frac{1}{4}} - 2$$

Question 31

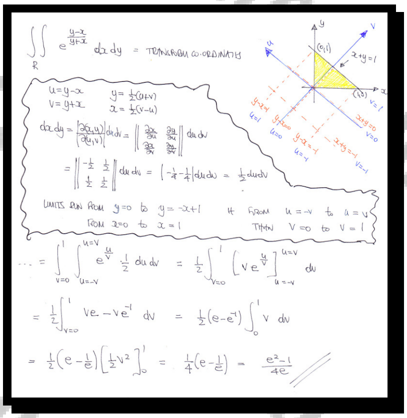
The finite region R is bounded by the coordinate axes and the straight line with Cartesian equation

$$x + y = 1$$

Use a suitable coordinate transformation to find an exact value for

$$\int_R e^{\frac{y-x}{y+x}} \, dx\,dy \,.$$

$$\boxed{\frac{e^2-1}{4e} = \frac{1}{2}\sinh 1}$$



Question 32

The finite region R satisfies the inequalities

$$1 \le x + y \le 2 \quad \text{and} \quad 0 \le y \le x .$$

Show clearly that

$$\iint_R \frac{y \ln(x+y)}{x^2} \, dx \, dy = (1 - \ln 2)(-1 + 2\ln 2) .$$

$\boxed{}$, $\boxed{\text{proof}}$

$$\left.\begin{array}{l} v = \frac{y}{x} \\ u = x+y \end{array}\right\} \Rightarrow x = u - y$$

$\Rightarrow v = \frac{y}{u-y}$

$\Rightarrow uv - vy = y$

$\Rightarrow uv = y + vy$

$\Rightarrow uv = y(v+1)$

$\Rightarrow \boxed{y = \frac{uv}{v+1}}$

RETURNING TO THE INTEGRAL WE OBTAIN

$$\ldots = \iint_{R'} \cancel{\left(\frac{uv}{v+1}\right)} \frac{\ln u}{\cancel{u}} \, du \, dv = \int_{u=1}^{2} \int_{v=0}^{1} \frac{v}{v+1} \ln u \, dv \, du$$

$$= \left[\int_1^2 \ln u \, du\right] \left[\int_0^1 \frac{v}{v+1} \, dv\right]$$

BY PARTS / STANDARD RESULT

$$= \left[u \ln u - u\right]_1^2 \int_0^1 \frac{v+1-1}{v+1} \, dv$$

$$= \left[(2\ln 2 - 2) - (\cancel{\ln 1} - 1)\right] \int_0^1 1 - \frac{1}{v+1} \, dv$$

$$= (2\ln 2 - 1)\left[v - \ln|v+1|\right]_0^1$$

$$= (2\ln 2 - 1)\left((1 - \ln 2) - (0 - \cancel{\ln 1})\right)$$

$$= (2\ln 2 - 1)(1 - \ln 2)$$

AS REQUIRED

Question 33

The finite region R is defined by the inequalities

$$y \le x, \quad y \le 1 - x \quad \text{and} \quad y \ge 0$$

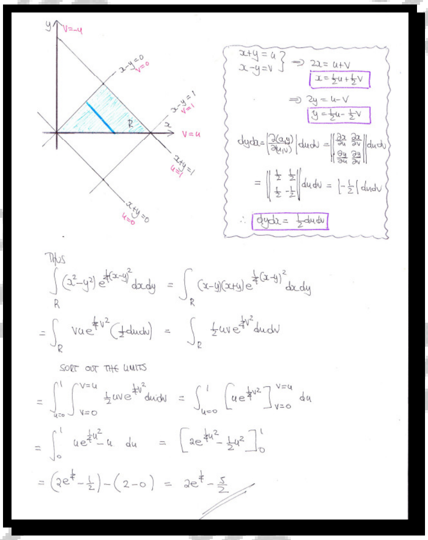
Use the transformation equations

$$u = x + y \qquad \text{and} \qquad v = x - y$$

to find an exact value for

$$\int_R \left(x^2 - y^2\right) \mathrm{e}^{\frac{1}{4}(x-y)^2} \, dx\,dy .$$

$$\boxed{\frac{1}{2}\left(4\mathrm{e}^{\frac{1}{4}} - 5\right)}$$



Question 34

The finite region R is bounded by the straight lines with equations

$$y = x, \quad x = 1 \quad \text{and} \quad y = 0.$$

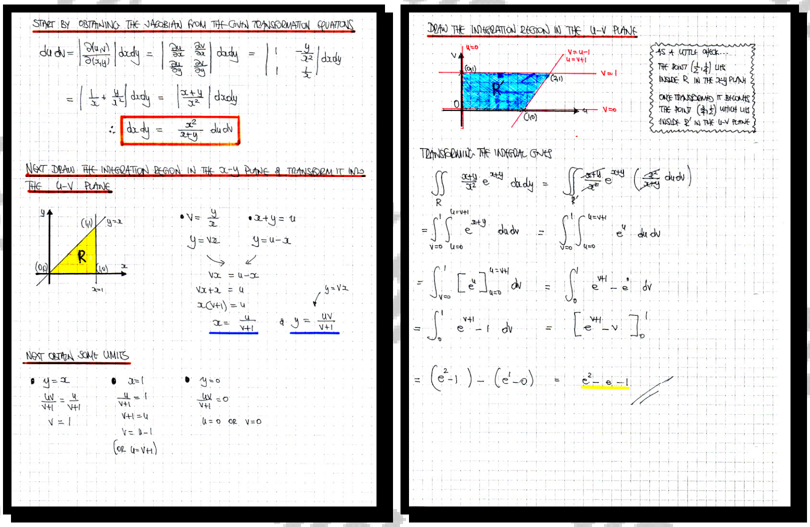
Use the transformation equations

$$u = x + y \quad \text{and} \quad v = \frac{y}{x},$$

to find an exact value for

$$\iint_R \left(\frac{x+y}{x^2}\right) e^{x+y} \, dx \, dy.$$

$\boxed{}$, $\boxed{e^2 - e - 1}$



START BY OBTAINING THE JACOBIAN FROM THE GIVEN TRANSFORMATION EQUATIONS

$$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy = \left|\begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{matrix}\right| dx\,dy = \left|\begin{matrix} 1 & -\frac{y}{x^2} \\ 1 & \frac{1}{x} \end{matrix}\right| dx\,dy$$

$$= \left|\frac{1}{x} + \frac{y}{x^2}\right| dx\,dy = \left|\frac{x+y}{x^2}\right| dx\,dy$$

$$\therefore \boxed{dx\,dy = \frac{x^2}{x+y}\,du\,dv}$$

NEXT DRAW THE INTEGRATION REGION IN THE x-y PLANE & TRANSFORM IT INTO THE u-v PLANE

- $v = \frac{y}{x}$, $y = vx$
- $x + y = u$, $y = u - x$

$vx = u - x$

$vx + x = u$

$x(v+1) = u$

$x = \frac{u}{v+1}$ & $y = \frac{uv}{v+1}$ ($y = vx$)

NEXT OBTAIN SOME LIMITS

- $y = x$: $\frac{uv}{v+1} = \frac{u}{v+1}$, $v = 1$
- $x = 1$: $\frac{u}{v+1} = 1$, $v+1 = u$, $v = u - 1$ (OR $u = v+1$)
- $y = 0$: $\frac{uv}{v+1} = 0$, $u = 0$ OR $v = 0$

DRAW THE INTEGRATION REGION IN THE u-v PLANE

AS A LITTLE CHECK... THE POINT $\left(\frac{1}{2},\frac{1}{4}\right)$ LIES INSIDE R IN THE x-y PLANE. ONCE TRANSFORMED IT BECOMES THE POINT $\left(\frac{3}{4},\frac{1}{2}\right)$ WHICH LIES INSIDE R' IN THE u-v PLANE

TRANSFORMING THE INTEGRAL GIVES

$$\iint_R \frac{x+y}{x^2} e^{x+y}\,dx\,dy = \iint_{R'} \cancel{\frac{x+y}{x^2}} e^{x+y} \left(\cancel{\frac{x^2}{x+y}}\,du\,dv\right)$$

$$= \int_{v=0}^{1}\int_{u=0}^{u=v+1} e^{x+y}\,du\,dv = \int_{v=0}^{1}\int_{u=0}^{u=v+1} e^{u}\,du\,dv$$

$$= \int_{v=0}^{1} \left[e^{u}\right]_{u=0}^{u=v+1} dv = \int_0^1 e^{v+1} - e^0\,dv$$

$$= \int_0^1 e^{v+1} - 1\,dv = \left[e^{v+1} - v\right]_0^1$$

$$= (e^2 - 1) - (e^1 - 0) = e^2 - e - 1$$

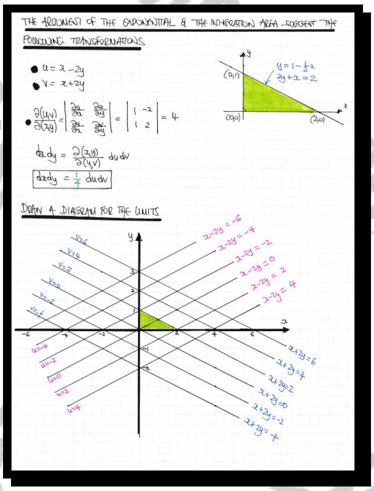
Question 35

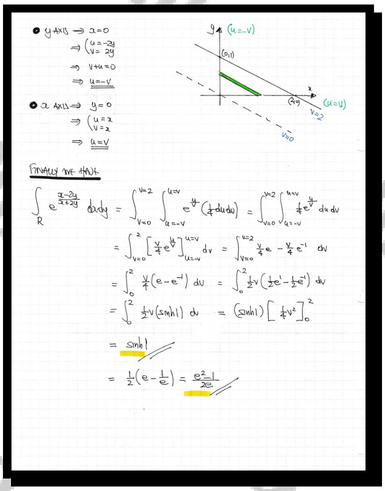
The finite region R in the x-y plane is enclosed by the rectilinear triangle with vertices at $(0,0)$, $(0,1)$ and $(2,0)$.

Use a suitable coordinate transformation to find an exact value for

$$\int_R e^{\frac{x-2y}{x+2y}} \, dxdy .$$

$$\boxed{\frac{e^2-1}{2e} = \sinh 1}$$





Question 36

$$I = \int_0^\infty \int_0^\infty \mathrm{e}^{-(x+y)^2}\,dx\,dy\,.$$

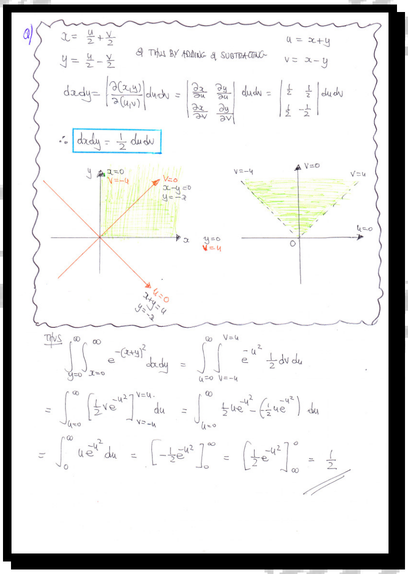
a) Use the coordinate transformation equations

$$x = \frac{1}{2}u + \frac{1}{2}v \quad \text{and} \quad y = \frac{1}{2}u - \frac{1}{2}v\,,$$

to find the value of I.

b) Evaluate I in plane polar coordinates, (r, θ), and hence verify the answer of part **(a)**.

$$\boxed{\frac{1}{2}}$$



a) $x = \frac{u}{2} + \frac{v}{2}$, $y = \frac{u}{2} - \frac{v}{2}$ & THUS BY ADDING & SUBTRACTING $u = x+y$, $v = x-y$

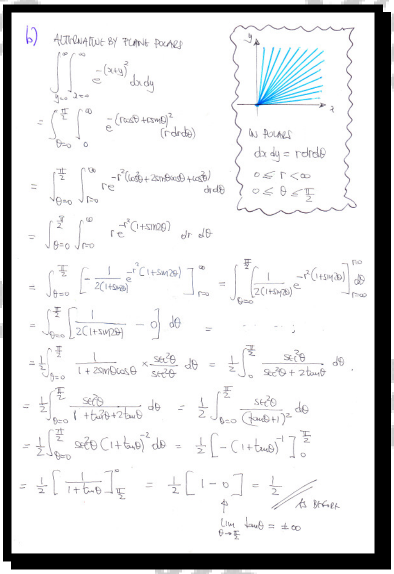
$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{matrix}\right| du\,dv = \left|\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\end{matrix}\right| du\,dv$

$\therefore \boxed{dx\,dy = \frac{1}{2}\,du\,dv}$

THUS $\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-(x+y)^2}\,dx\,dy = \int_{u=0}^{\infty}\int_{v=-u}^{v=u} e^{-u^2}\,\frac{1}{2}\,dv\,du$

$= \int_{u=0}^{\infty}\left[\frac{1}{2}ve^{-u^2}\right]_{v=-u}^{v=u} du = \int_{u=0}^{\infty} \frac{1}{2}ue^{-u^2} - \left(-\frac{1}{2}ue^{-u^2}\right) du$

$= \int_0^\infty ue^{-u^2}\,du = \left[-\frac{1}{2}e^{-u^2}\right]_0^\infty = \left[\frac{1}{2}e^{-u^2}\right]_\infty^0 = \frac{1}{2}$



b) ALTERNATIVE BY PLANE POLARS

$\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-(x+y)^2}\,dx\,dy$

$= \int_{\theta=0}^{\frac{\pi}{2}}\int_0^\infty e^{-(r\cos\theta + r\sin\theta)^2}\,(r\,dr\,d\theta)$

IN POLARS
$dx\,dy = r\,dr\,d\theta$
$0 \leqslant r < \infty$
$0 \leqslant \theta \leqslant \frac{\pi}{2}$

$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} re^{-r^2(\cos^2\theta + 2\sin\theta\cos\theta + \cos^2\theta)}\,dr\,d\theta$

$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} re^{-r^2(1+\sin 2\theta)}\,dr\,d\theta$

$= \int_{\theta=0}^{\frac{\pi}{2}}\left[-\frac{1}{2(1+\sin 2\theta)}e^{-r^2(1+\sin 2\theta)}\right]_{r=0}^{\infty} = \int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{1}{2(1+\sin 2\theta)}e^{-r^2(1+\sin 2\theta)}\right]_{r=\infty}^{r=0} d\theta$

$= \int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{1}{2(1+\sin 2\theta)} - 0\right] d\theta =$

$= \frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} \frac{1}{1 + 2\sin\theta\cos\theta} \times \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{\sec^2\theta + 2\tan\theta}\,d\theta$

$= \frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} \frac{\sec^2\theta}{1 + \tan^2\theta + 2\tan\theta}\,d\theta = \frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} \frac{\sec^2\theta}{(\tan\theta + 1)^2}\,d\theta$

$= \frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} \sec^2\theta\,(1+\tan\theta)^{-2}\,d\theta = \frac{1}{2}\left[-(1+\tan\theta)^{-1}\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\frac{1}{1+\tan\theta}\right]_{\frac{\pi}{2}}^{0} = \frac{1}{2}\left[1 - 0\right] = \frac{1}{2}$ AS BEFORE

$\lim_{\theta \to \frac{\pi}{2}} \tan\theta = \pm\infty$

Question 37

The finite region R is bounded by the curve with Cartesian equation

$$x^4 + y^4 = 1, \ x \geq 0, \ y \geq 0.$$

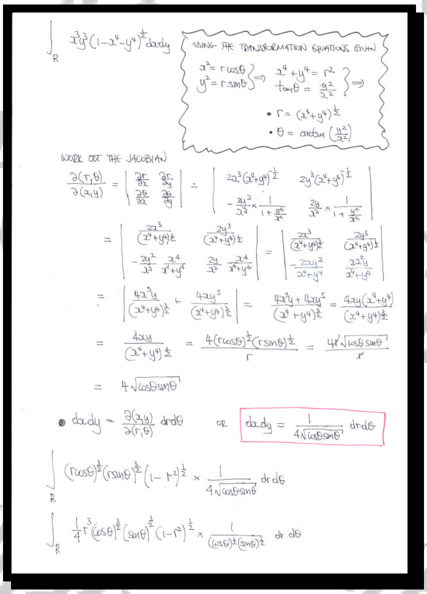
Use the transformation equations

$$x^2 = r\cos\theta \quad \text{and} \quad x^2 = r\cos\theta,$$

to find the value of

$$\iint_R x^3 y^3 \sqrt{1 - x^4 - y^4} \ dxdy.$$

$$\boxed{\frac{1}{60}}$$



$$\int_R x^3y^3(1-x^4-y^4)^{\frac{1}{2}}\,dxdy$$

USING THE TRANSFORMATION EQUATIONS GIVEN

$$\left.\begin{array}{l}x^2 = r\cos\theta\\ y^2 = r\sin\theta\end{array}\right\} \Longrightarrow \left.\begin{array}{l}x^4+y^4 = r^2\\ \tan\theta = \frac{y^2}{x^2}\end{array}\right\} \Longrightarrow$$

- $r = (x^4+y^4)^{\frac{1}{2}}$
- $\theta = \arctan\left(\frac{y^2}{x^2}\right)$

WORK OUT THE JACOBIAN

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y}\end{vmatrix} = \begin{vmatrix}2x^3(x^4+y^4)^{-\frac{1}{2}} & 2y^3(x^4+y^4)^{-\frac{1}{2}}\\ -\frac{2y^2}{x^3}\times\frac{1}{1+\frac{y^4}{x^4}} & \frac{2y}{x^2}\times\frac{1}{1+\frac{y^4}{x^4}}\end{vmatrix}$$

$$= \begin{vmatrix}\frac{2x^3}{(x^4+y^4)^{\frac{1}{2}}} & \frac{2y^3}{(x^4+y^4)^{\frac{1}{2}}}\\ -\frac{2y^2}{x^3}\,\frac{x^4}{x^4+y^4} & \frac{2y}{x^2}\,\frac{x^4}{x^4+y^4}\end{vmatrix} = \begin{vmatrix}\frac{2x^3}{(x^4+y^4)^{\frac{1}{2}}} & \frac{2y^3}{(x^4+y^4)^{\frac{1}{2}}}\\ -\frac{2xy^2}{x^4+y^4} & \frac{2x^2y}{x^4+y^4}\end{vmatrix}$$

$$= \left|\frac{4x^5y}{(x^4+y^4)^{\frac{3}{2}}} + \frac{4xy^5}{(x^4+y^4)^{\frac{3}{2}}}\right| = \frac{4x^5y+4xy^5}{(x^4+y^4)^{\frac{3}{2}}} = \frac{4xy(x^4+y^4)}{(x^4+y^4)^{\frac{3}{2}}}$$

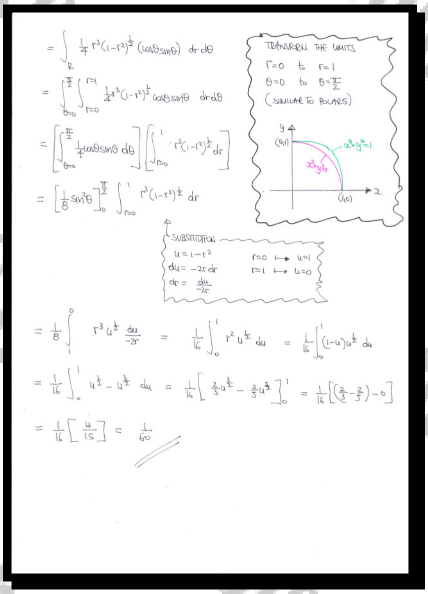
$$= \frac{4xy}{(x^4+y^4)^{\frac{1}{2}}} = \frac{4(r\cos\theta)^{\frac{1}{2}}(r\sin\theta)^{\frac{1}{2}}}{r} = \frac{4\cancel{r}\sqrt{\cos\theta\sin\theta}}{\cancel{r}}$$

$$= 4\sqrt{\cos\theta\sin\theta}$$

- $dxdy = \frac{\partial(x,y)}{\partial(r,\theta)}\,drd\theta$ OR $\boxed{dxdy = \frac{1}{4\sqrt{\cos\theta\sin\theta}}\,drd\theta}$

$$\int_R (r\cos\theta)^{\frac{3}{2}}(r\sin\theta)^{\frac{3}{2}}(1-r^2)^{\frac{1}{2}}\times\frac{1}{4\sqrt{\cos\theta\sin\theta}}\,drd\theta$$

$$\int_R \frac{1}{4}r^3(\cos\theta)^{\frac{3}{2}}(\sin\theta)^{\frac{3}{2}}(1-r^2)^{\frac{1}{2}}\times\frac{1}{(\cos\theta)^{\frac{1}{2}}(\sin\theta)^{\frac{1}{2}}}\,dr\,d\theta$$



$$= \int_R \frac{1}{4}r^3(1-r^2)^{\frac{1}{2}}(\cos\theta\sin\theta)\,dr\,d\theta$$

TRANSFORM THE LIMITS

$r=0$ to $r=1$

$\theta=0$ to $\theta=\frac{\pi}{2}$

(SIMILAR TO POLARS)

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{r=1}\frac{1}{4}r^3(1-r^2)^{\frac{1}{2}}\cos\theta\sin\theta\,drd\theta$$

$$= \left[\int_{\theta=0}^{\frac{\pi}{2}}\frac{1}{4}\cos\theta\sin\theta\,d\theta\right]\left[\int_{r=0}^{1}r^3(1-r^2)^{\frac{1}{2}}\,dr\right]$$

$$= \left[\frac{1}{8}\sin^2\theta\right]_0^{\frac{\pi}{2}}\int_{r=0}^{1}r^3(1-r^2)^{\frac{1}{2}}\,dr$$

SUBSTITUTION

$u = 1-r^2$ $\quad r=0 \mapsto u=1$

$du = -2r\,dr$ $\quad r=1 \mapsto u=0$

$dr = \frac{du}{-2r}$

$$= \frac{1}{8}\int_1^0 r^3u^{\frac{1}{2}}\,\frac{du}{-2r} = \frac{1}{16}\int_0^1 r^2u^{\frac{1}{2}}\,du = \frac{1}{16}\int_0^1(1-u)u^{\frac{1}{2}}\,du$$

$$= \frac{1}{16}\int_0^1 u^{\frac{1}{2}}-u^{\frac{3}{2}}\,du = \frac{1}{16}\left[\frac{2}{3}u^{\frac{3}{2}}-\frac{2}{5}u^{\frac{5}{2}}\right]_0^1 = \frac{1}{16}\left[\left(\frac{2}{3}-\frac{2}{5}\right)-0\right]$$

$$= \frac{1}{16}\left[\frac{4}{15}\right] = \frac{1}{60}$$

Question 38

The finite region V is enclosed by the surface with Cartesian equation

$$x^4 + y^4 + z^4 = 64.$$

By first transforming the Cartesian coordinates into a new Cartesian coordinate system, use spherical polar coordinates, (r,θ,φ), to show that the volume of V is

$$\frac{8}{3\pi}\left[\Gamma\left(\tfrac{1}{4}\right)\right]^4.$$

proof

$$x^4+y^4+z^4=64$$

TRANSFORM INTO A "SPHERE" BY $u=x^2$, $v=y^2$, $Z=w^2$ $\Longrightarrow$ $x=u^{\frac{1}{2}}$, $y=v^{\frac{1}{2}}$, $w=z^{\frac{1}{2}}$

$$dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw \Longrightarrow dx\,dy\,dz = \left\|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{matrix}\right\| du\,dv\,dw$$

$$\Rightarrow dx\,dy\,dz = \left\|\begin{matrix} \frac{1}{2}u^{-\frac{1}{2}} & 0 & 0 \\ 0 & \frac{1}{2}v^{-\frac{1}{2}} & 0 \\ 0 & 0 & \frac{1}{2}w^{-\frac{1}{2}} \end{matrix}\right\| du\,dv\,dw$$

$$\therefore dx\,dy\,dz = \tfrac{1}{8}(uvw)^{-\frac{1}{2}}\,du\,dv\,dw$$

$$dx\,dy\,dz = \frac{du\,dv\,dw}{8(uvw)^{\frac{1}{2}}}$$

SO THE VOLUME IN CARTESIAN

$$V = \int_V 1\,dx\,dy\,dz = \int_V \frac{1}{8(uvw)^{\frac{1}{2}}}\,du\,dv\,dw = \dots$$ SWITCH INTO SPHERICAL POLARS

$$\dots = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{8} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{8\sqrt{(r\sin\theta\cos\phi)(r\sin\theta\sin\phi)(r\cos\theta)}}$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{8} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{8\sqrt{r^3\sin^2\theta\cos\phi\sin\phi\cos\theta}}$$

$$= \frac{1}{8}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{8} r^{\frac{1}{2}}(\cos\phi\sin\phi\cos\theta)^{-\frac{1}{2}}\,dr\,d\theta\,d\phi$$

$X = u = r\sin\theta\cos\phi$
$Y = v = r\sin\theta\sin\phi$
$Z = w = r\cos\theta$
$w^2+v^2+u^2=r^2$
$du\,dv\,dw = r^2\sin\theta\,dr\,d\theta\,d\phi$
AND THE SPHERE IN THE PROBLEM HAS EQUATION
$u^2+v^2+w^2=64$
$\therefore\ 0 \le r \le 8$
$0 \le \theta \le \pi$
$0 \le \phi \le 2\pi$

AS THE LIMITS ARE NOT FUNCTIONS WE CAN SPLIT THE INTEGRAL INTO 3

$$\dots = \frac{1}{8}\left[\int_{\phi=0}^{2\pi}(\cos\phi\sin\phi)^{-\frac{1}{2}}\,d\phi\right]\left[\int_{\theta=0}^{\pi}(\cos\theta)^{-\frac{1}{2}}\,d\theta\right]\left[\int_{r=0}^{8} r^{\frac{1}{2}}\,dr\right]$$

THE SOLID IS SYMMETRICAL (EVEN IN x,y,z) SO WE CAN WORK OUT THE VOLUME OF THE FIRST OCTANT AND MULTIPLY BY 8. THIS WILL ALLOW BETA FUNCTIONS IN THE EVALUATION

$$= \frac{1}{8}\left[4\int_{\phi=0}^{\frac{\pi}{2}}(\cos\phi)^{2\times\frac{1}{4}-1}(\sin\phi)^{2\times\frac{1}{4}-1}\,d\phi\right]\left[2\int_{\theta=0}^{\frac{\pi}{2}}(\cos\theta)^{2\times\frac{1}{4}-1}\,d\theta\right]\left[\int_{r=0}^{8} r^{\frac{1}{2}}\,dr\right]$$

$$= \left[\frac{1}{2}\int_0^{\frac{\pi}{2}} 2(\cos\phi)^{2\times\frac{1}{4}-1}(\sin\phi)^{2\times\frac{1}{4}-1}\,d\phi\right]\left[\frac{1}{2}\int_0^{\frac{\pi}{2}} 2(\cos\theta)^{2\times\frac{1}{4}-1}(\sin\theta)^{2\times\frac{1}{2}-1}\,d\theta\right]\times\left[\tfrac{2}{3}r^{\frac{3}{2}}\right]_0^8$$

$$= \frac{1}{2}B\left(\tfrac{1}{4},\tfrac{1}{4}\right)\times\frac{1}{2}B\left(\tfrac{1}{4},\tfrac{1}{2}\right)\times\frac{2}{3}\times 8^{\frac{3}{2}}$$

$$= \frac{1}{6}\times 8^{\frac{3}{2}}\times\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{4})}{\cancel{\Gamma(\frac{1}{2})}}\times\frac{\Gamma(\frac{1}{4})\cancel{\Gamma(\frac{1}{2})}}{\Gamma(\frac{3}{4})}$$

$$= \frac{1}{6}\times 8\sqrt{8}\,\frac{(\Gamma(\frac{1}{4}))^3}{\Gamma(\frac{3}{4})}$$

MULTIPLY TOP & BOTTOM BY $\Gamma(\frac{1}{4})$

$$= \frac{1}{6}\times 16\sqrt{2}\times\frac{[\Gamma(\frac{1}{4})]^4}{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}$$

$$= \frac{8}{3}\cancel{\sqrt{2}}\times\frac{(\Gamma(\frac{1}{4}))^4}{\pi\cancel{\sqrt{2}}}$$

$$= \frac{8}{3\pi}\left[\Gamma(\tfrac{1}{4})\right]^4$$

AS REQUIRED

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}$$
$$\Gamma(\tfrac{1}{4})\Gamma(\tfrac{3}{4}) = \frac{\pi}{\sin\frac{\pi}{4}} = \pi\sqrt{2}$$