# DIFFERENTIAL EQUATIONS

**(by variation of parameters)**

**Question 1** (***)

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2xe^x .$$

$$\boxed{y = Ae^x + Bxe^x + \frac{1}{3}x^3 e^x}$$

FIRSTLY FIND THE COMPLEMENTARY FUNCTION

$$\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)^2 = 0$$
$$\lambda = 1 \qquad \therefore \quad y = Ae^x + Bxe^x$$

BY VARIATION OF PARAMETERS

$$1\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \underbrace{2xe^x}_{f(x)}$$

a(x)

$$\boxed{\begin{aligned} a(x) &= 1 \\ e_1 &= e^x \\ e_2 &= xe^x \end{aligned}}$$

OBTAIN THE WRONSKIAN, $W(x)$

$$W = \begin{vmatrix} e_1 & e_2 \\ e_1' & e_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}} = e^{2x}$$

HENCE WE CAN FIND THE PARTICULAR INTEGRAL, $y_P$

$$\Rightarrow y_P = -e_1 \int \frac{e_2 f(x)}{a(x)W(x)}\,dx + e_2 \int \frac{e_1 f(x)}{a(x)W(x)}\,dx$$

$$\Rightarrow y_P = -e^x \int \frac{(xe^x)(2xe^x)}{1 \times e^{2x}}\,dx + xe^x \int \frac{e^x(2xe^x)}{1 \times e^{2x}}\,dx$$

$$\Rightarrow y_P = -e^x \int \frac{2x^2e^{2x}}{e^{2x}}\,dx + xe^x \int \frac{2xe^{2x}}{e^{2x}}\,dx$$

$$\Rightarrow y_P = -e^x \int 2x^2\,dx + xe^{2x} \int 2x\,dx$$

$$\Rightarrow y_P = -e^x\left(\tfrac{2}{3}x^3\right) + xe^{2x}\left(x^2\right)$$

$$\Rightarrow y_P = \tfrac{1}{3}x^3e^x$$

THUS WE OBTAIN A GENERAL SOLUTION

$$y = Ae^x + Bxe^x + \tfrac{1}{3}x^3e^x$$

(NOTE THAT THE PARTICULAR INTEGRAL CAN ALSO BE FOUND BY INSPECTION)

**Question 2** (***)

Use the method of variation of parameters to find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 6xe^{2x}.$$

$$\boxed{y = Ae^{2x} + Bxe^{2x} + x^3e^{2x}}$$

START BY FINDING THE COMPLEMENTARY FUNCTION

$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$

$\Rightarrow (\lambda - 2)^2 = 0$

$\Rightarrow \lambda = 2$ (REPEATED)

COMPLEMENTARY FUNCTION: $y = Ae^{2x} + Bxe^{2x}$

BY THE METHOD OF VARIATION OF PARAMETERS

$$\underbrace{1}_{a(x)}\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = \underbrace{\left\{6xe^{2x}\right\}}_{f(x)}$$

- $a(x) = 1$
- $e_1 = e^{2x}$
- $e_2 = xe^{2x}$
- $f(x) = 6xe^{2x}$

PROCEED TO OBTAIN THE WRONSKIAN

$$W(x) = \begin{vmatrix} e_1 & e_2 \\ e_1' & e_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + \cancel{2xe^{4x}} - \cancel{2xe^{4x}} = e^{4x}$$

FINDING THE PARTICULAR INTEGRAL

$$\Rightarrow y_p = -e_1\int \frac{e_2 f(x)}{a(x)W(x)}\,dx + e_2\int \frac{e_1 f(x)}{a(x)W(x)}\,dx$$

$$\Rightarrow y_p = -e^{2x}\int \frac{(xe^{2x})(6xe^{2x})}{1 \times e^{4x}}\,dx + xe^{2x}\int \frac{e^{2x}(6xe^{2x})}{1 \times e^{4x}}\,dx$$

$$\Rightarrow y_p = -e^{2x}\int 6x^2\,dx + xe^{2x}\int 6x\,dx$$

$$\Rightarrow y_p = -2x^3e^{2x} + 3x^3e^{2x}$$

$$\Rightarrow y_p = x^3e^{2x}$$

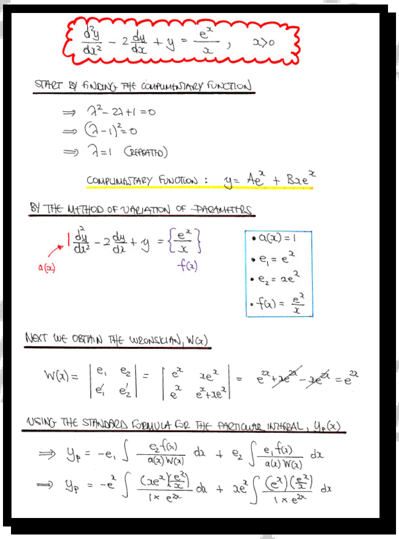
$$\therefore y = e^{2x}\left[x^3 + Bx + A\right]$$

**Question 3 (***)**

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}, \quad x > 0.$$

$\boxed{\phantom{xxxx}}$, $\boxed{y = Ae^x + xe^x(B + \ln x)}$



$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}, \quad x>0$$

START BY FINDING THE COMPLIMENTARY FUNCTION

$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$

$\Rightarrow (\lambda - 1)^2 = 0$

$\Rightarrow \lambda = 1$ (REPEATED)

COMPLIMENTARY FUNCTION : $y = Ae^x + Bxe^x$

BY THE METHOD OF VARIATION OF PARAMETERS

$$\underbrace{1}_{a(x)}\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \underbrace{\left\{\frac{e^x}{x}\right\}}_{f(x)}$$

- $a(x) = 1$
- $e_1 = e^x$
- $e_2 = xe^x$
- $f(x) = \frac{e^x}{x}$

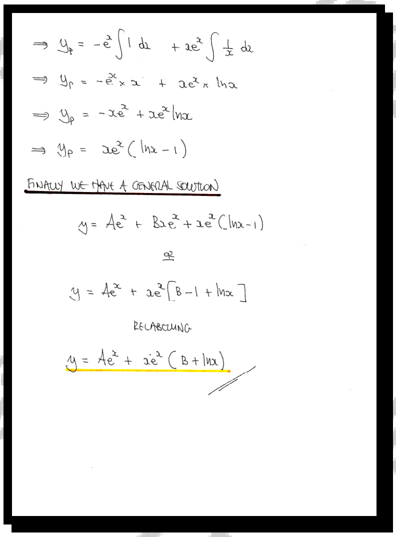
NEXT WE OBTAIN THE WRONSKIAN, $W(x)$

$$W(x) = \begin{vmatrix} e_1 & e_2 \\ e_1' & e_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}} = e^{2x}$$

USING THE STANDARD FORMULA FOR THE PARTICULAR INTEGRAL, $y_P(x)$

$\Rightarrow y_P = -e_1 \int \frac{e_2 f(x)}{a(x)W(x)}\,dx + e_2 \int \frac{e_1 f(x)}{a(x)W(x)}\,dx$

$\Rightarrow y_P = -e^x \int \frac{(xe^x)\left(\frac{e^x}{x}\right)}{1 \times e^{2x}}\,dx + xe^x \int \frac{(e^x)\left(\frac{e^x}{x}\right)}{1 \times e^{2x}}\,dx$



$\Rightarrow y_P = -e^x \int 1\,dx + xe^x \int \frac{1}{x}\,dx$

$\Rightarrow y_P = -e^x \times x + xe^x \times \ln x$

$\Rightarrow y_P = -xe^x + xe^x \ln x$

$\Rightarrow y_P = xe^x(\ln x - 1)$

FINALLY WE HAVE A GENERAL SOLUTION

$y = Ae^x + Bxe^x + xe^x(\ln x - 1)$

OR

$y = Ae^x + xe^x[B - 1 + \ln x]$

RELABELLING

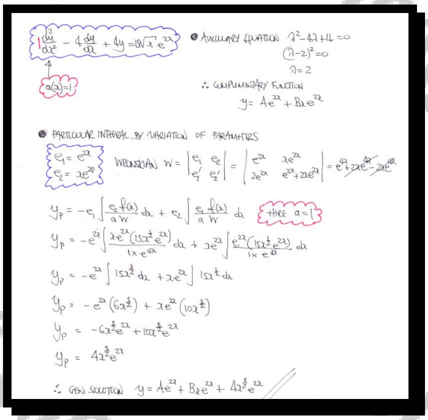
$y = Ae^x + xe^x(B + \ln x)$

**Question 4** **(***)**

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 15\sqrt{x}\,\mathrm{e}^{2x}.$$

$$\boxed{y = A\mathrm{e}^{2x} + Bx\mathrm{e}^{2x} + 4x^{\frac{5}{2}}\mathrm{e}^{2x}}$$

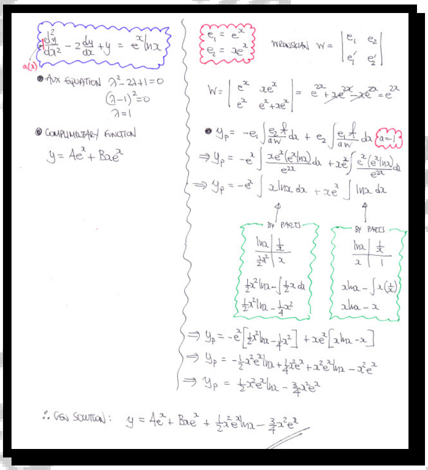


**Question 5** **(***+)**

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \mathrm{e}^x \ln x, \quad x > 0.$$

$$\boxed{y = A\mathrm{e}^x + Bx\mathrm{e}^x - \frac{3}{4}x^2\mathrm{e}^x + \frac{1}{2}x^2\mathrm{e}^x \ln x}$$

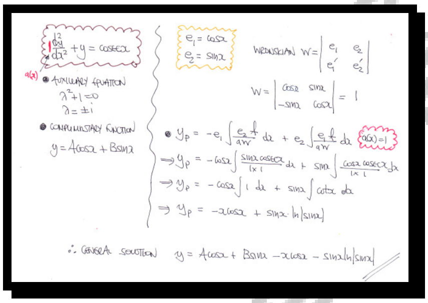


**Question 6** **(***+)**

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x .$$

$$\boxed{y = A\cos x + B\sin x - x\cos x - \sin x \ln|\sin x|}$$

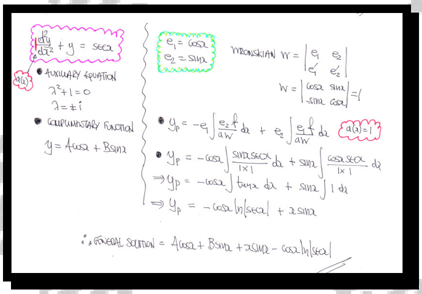


**Question 7** **(***+)**

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} + y = \sec x .$$

$$\boxed{y = A\cos x + B\sin x + x\sin x - \cos x \ln|\sec x|}$$



**Question 8** (***+)

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2+1}.$$

$$\boxed{y = e^x\left[A + Bx + x\arctan x - \frac{1}{2}\ln\left(x^2+1\right)\right]}$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2+1}$$

$a(x)=1$

FIRSTLY FIND THE COMPLIMENTARY FUNCTION

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda-1)^2 = 0$

$\lambda = 1$ REPEATED $\quad \therefore\ y = Ae^x + Bxe^x$

PARTICULAR INTEGRAL BY VARIATION OF PARAMETERS

$e_1 = e^x$

$e_2 = xe^x$

$$\text{WRONSKIAN} = \begin{vmatrix} e_1 & e_2 \\ e_1' & e_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$$

$$\text{WRONSKIAN} = e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}} = e^{2x}$$

$f(x) = \dfrac{e^x}{x^2+1}$

$a(x) = 1$

$$y_p = -e_1\int \frac{e_2 f}{aW}\,dx + e_2\int \frac{e_1 f}{aW}\,dx$$

$$y_p = -e^x\int \frac{xe^x\left(\frac{e^x}{x^2+1}\right)}{1\times e^{2x}}\,dx + xe^x\int \frac{e^x\left(\frac{e^x}{x^2+1}\right)}{1\times e^{2x}}\,dx$$

$$y_p = -e^x\int \frac{\frac{xe^{2x}}{x^2+1}}{e^{2x}}\,dx + xe^x\int \frac{\frac{e^{2x}}{x^2+1}}{e^{2x}}\,dx$$

$$y_p = -e^x\int \frac{x}{x^2+1}\,dx + xe^x\int \frac{1}{x^2+1}\,dx$$

$$y_p = -e^x \times \frac{1}{2}\ln(x^2+1) + xe^x\arctan x$$

$$y_p = xe^x\arctan x - \frac{1}{2}e^x\ln(x^2+1)$$

$$y = Ae^x + Bxe^x + xe^x\arctan x - \frac{1}{2}e^x\ln(x^2+1)$$

$$y = e^x\left[A + Bx + x\arctan x - \frac{1}{2}\ln(x^2+1)\right.$$

**Question 9** (***+)

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = \frac{(5x-2)e^{4x}}{x^3}.$$

You may assume that $\dfrac{d}{dx}\left(\dfrac{e^{5x}}{x^2}\right) = \dfrac{(5x-2)e^{5x}}{x^3}$.

$$\boxed{y = Ae^{4x} + Be^{-x} - \frac{e^{4x}}{x}}$$

$$\boxed{\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = \frac{(5x-2)e^{4x}}{x^3}}$$

AUXILIARY EQUATION FOR THE LHS

$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow (\lambda-4)(\lambda+1) = 0$

$\Rightarrow \lambda = \begin{cases} 4 \\ -1 \end{cases}$

C.F: $y = Ae^{4x} + Be^{-x}$

FIND THE PARTICULAR INTEGRAL BY VARIATION OF PARAMETERS

WRONSKIAN WITH $e_1 = e^{4x}$ & $e_2 = e^{-x}$

$$W = \begin{vmatrix} e_1 & e_2 \\ e_1' & e_2' \end{vmatrix} = \begin{vmatrix} e^{4x} & e^{-x} \\ 4e^{4x} & -e^{-x} \end{vmatrix} = -e^{3x} - 4e^{3x} = -5e^{3x}$$

THE PARTICULAR INTEGRAL IS GIVEN BY

$$y_p = -e_1\int \frac{e_2 f}{aW}\,dx + e_2\int \frac{e_1 f}{aW}\,dx$$

WHERE $f = f(x) = \dfrac{(5x-2)e^{4x}}{x^3}$

$a = a(x) = 1 \leftarrow$ COEFFICIENT OF $\dfrac{d^2y}{dx^2}$

$$\Rightarrow y_p = -e^{4x}\int \frac{\cancel{e^{-x}} \times \frac{(5x-2)\cancel{e^{4x}}}{x^3}}{1\times(-5\cancel{e^{3x}})}\,dx + e^{-x}\int \frac{e^{4x} \times \frac{(5x-2)e^{4x}}{x^3}}{1\times(-5e^{3x})}\,dx$$

$$\Rightarrow y_p = -e^{4x}\int \frac{5x-2}{-5x^3}\,dx + e^{-x}\int \frac{(5x-2)e^{5x}}{-5x^3}\,dx$$

$$\Rightarrow y_p = \frac{1}{5}e^{4x}\int \frac{5}{x^2} - \frac{2}{x^3}\,dx - \frac{1}{5}e^{-x}\int \frac{(5x-2)e^{5x}}{x^3}\,dx$$

$\dfrac{d}{dx}\left(\dfrac{e^{5x}}{x^2}\right) = \dfrac{(5x-2)e^{5x}}{x^3}$

$$\Rightarrow y_p = \frac{1}{5}e^{4x}\left[-\frac{5}{x} + \frac{1}{x^2}\right] - \frac{1}{5}e^{-x}\left[\frac{e^{5x}}{x^2}\right]$$

$$\Rightarrow y_p = \frac{1}{5}e^{4x}\left[\frac{1}{x^2} - \frac{5}{x}\right] - \frac{1}{5}e^{4x}\left[\frac{1}{x^2}\right]$$

$$\Rightarrow y_p = \frac{1}{5}e^{4x}\left[\cancel{\frac{1}{x^2}} - \frac{5}{x} - \cancel{\frac{1}{x^2}}\right]$$

$$\Rightarrow y_p = -\frac{e^{4x}}{x}$$

$$\therefore y = Ae^{4x} + Be^{-x} - \frac{e^{4x}}{x}$$

**Question 10** (****)

Find the general solution of the following differential equation.

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2\mathrm{e}^x,\quad x\neq 0.$$

$$\boxed{\phantom{xx}},\ \boxed{y=Ax+\frac{B}{x}+\mathrm{e}^x-\frac{1}{x}\mathrm{e}^x}$$

$$\boxed{x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x,\ x\neq 0}$$

$a(x)=x^2$

OBTAIN A COMPLEMENTARY FUNCTION BY TRIAL

- $y=x^n$
- $y'=nx^{n-1}$
- $y''=n(n-1)x^{n-2}$

SUB INTO THE O.D.E $x^2y''+xy'-y=0$

$\Rightarrow x^2\left(n(n-1)x^{n-2}\right)+x\left(nx^{n-1}\right)-x^n\equiv 0$

$\Rightarrow n(n-1)x^n+nx^n-x^n\equiv 0$

$\Rightarrow \left[n(n-1)+n-1\right]x^n\equiv 0$

$\Rightarrow n^2-n+n-1=0$

$\Rightarrow n^2=1$

$\Rightarrow n=\pm 1$

$\therefore\ y=Ax^1+Bx^{-1}=Ax+\frac{B}{x}$

PARTICULAR INTEGRAL BY VARIATION OF PARAMETERS

$e_1=x$

$e_2=\frac{1}{x}$

WRONSKIAN $=W(x)=\begin{vmatrix}e_1 & e_2\\ e_1' & e_2'\end{vmatrix}=\begin{vmatrix}x & \frac{1}{x}\\ 1 & -\frac{1}{x^2}\end{vmatrix}$

$=-\frac{1}{x}-\frac{1}{x}=-\frac{2}{x}$

NOW THE PARTICULAR INTEGRAL, $y_p$, MUST SATISFY

$\Rightarrow y_p=-e_1\int\frac{e_2f}{aW}\,dx+e_2\int\frac{e_1f}{aW}\,dx$

WHERE $f=f(x)=x^2e^x$

$W=W(x)=-\frac{2}{x}$

$a=a(x)=x^2$

$\Rightarrow y_p=-x\int\frac{\frac{1}{x}(x^2e^x)}{x^2\left(-\frac{2}{x}\right)}\,dx+\frac{1}{x}\int\frac{x(x^2e^x)}{x^2\left(-\frac{2}{x}\right)}\,dx$

$\Rightarrow y_p=x\int\frac{1}{2}e^x-\frac{1}{2x}\underbrace{\int x^2e^x\,dx}_{\text{BY PARTS TWICE}}$

| | |
|---|---|
| $x^2$ | $2x$ |
| $e^x$ | $e^x$ |

| | |
|---|---|
| $2x$ | $2$ |
| $e^x$ | $e^x$ |

$x^2e^x-\int 2xe^x\,dx$

$=x^2e^x-\left[2xe^x-\int 2e^x\,dx\right]$

$=x^2e^x-2xe^x+\int 2e^x\,dx$

$=x^2e^x-2xe^x+2e^x+C$

COLLECTING ALL THE RESULTS

$\Rightarrow y_p=x\left(\frac{1}{2}e^x\right)-\frac{1}{2x}\left[x^2e^x-2xe^x+2e^x\right]+C$

$\Rightarrow y_p=\cancel{\frac{1}{2}xe^x}-\cancel{\frac{1}{2}xe^x}+e^x-\frac{1}{x}e^x+C$

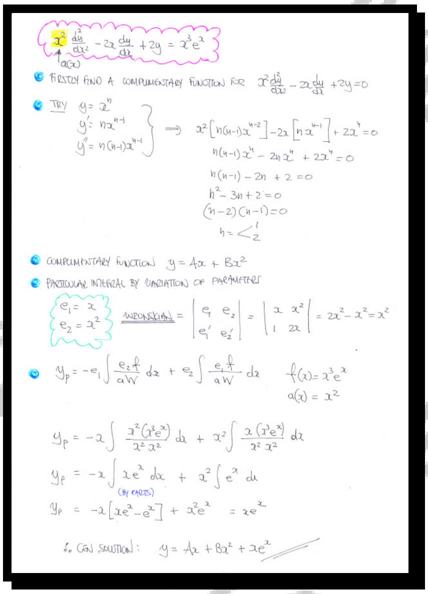
$\therefore\ y=Ax+\frac{B}{x}+e^x-\frac{1}{x}e^x$

**Question 11** (****)

Find the general solution of the following differential equation.

$$x^2 \frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3 \mathrm{e}^x, \quad x \neq 0.$$

$$\boxed{y = Ax + Bx^2 + x\mathrm{e}^x}$$



$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3 e^x$$

$a(x)$

- FIRSTLY FIND A COMPLEMENTARY FUNCTION FOR $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$
- TRY $\left.\begin{array}{l} y = x^n \\ y' = nx^{n-1} \\ y'' = n(n-1)x^{n-1} \end{array}\right\} \Longrightarrow x^2\left[n(n-1)x^{n-2}\right] - 2x\left[nx^{n-1}\right] + 2x^n = 0$

$$n(n-1)x^n - 2nx^n + 2x^n = 0$$
$$n(n-1) - 2n + 2 = 0$$
$$n^2 - 3n + 2 = 0$$
$$(n-2)(n-1) = 0$$
$$n = <^1_2$$

- COMPLEMENTARY FUNCTION $y = Ax + Bx^2$
- PARTICULAR INTEGRAL BY VARIATION OF PARAMETERS

$$\begin{array}{l} e_1 = x \\ e_2 = x^2 \end{array} \quad \underline{\underline{\text{WRONSKIAN}}} = \begin{vmatrix} e_1 & e_2 \\ e_1' & e_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

- $y_p = -e_1\int \frac{e_2 f}{aW}\,dx + e_2\int \frac{e_1 f}{aW}\,dx \qquad \begin{array}{l} f(x) = x^3e^x \\ a(x) = x^2 \end{array}$

$$y_p = -x\int \frac{x^2(x^3e^x)}{x^2x^2}\,dx + x^2\int \frac{x(x^3e^x)}{x^2x^2}\,dx$$

$$y_p = -x\int xe^x\,dx + x^2\int e^x\,dx$$

(BY PARTS)

$$y_p = -x\left[xe^x - e^x\right] + x^2e^x = xe^x$$

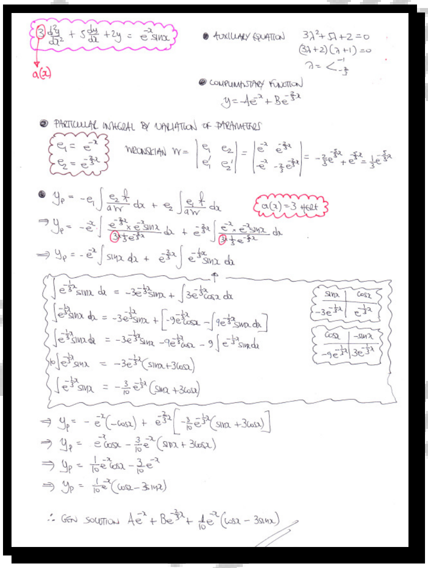
$\therefore$ GEN SOLUTION: $y = Ax + Bx^2 + xe^x$

**Question 12 (****)**

Find the general solution of the following differential equation.

$$3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = e^{-x}\sin x .$$

$$\boxed{y = A\,e^{-x} + B\,e^{x} + B\,e^{-\frac{2}{3}x} + \frac{1}{10}e^{-x}\left(\cos x - 3\sin x\right)}$$



$3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = e^{-x}\sin x$

$a(x)$

• AUXILIARY EQUATION $3\lambda^2 + 5\lambda + 2 = 0$

$(3\lambda+2)(\lambda+1) = 0$

$\lambda = \begin{cases} -1 \\ -\frac{2}{3} \end{cases}$

• COMPLEMENTARY FUNCTION

$y = Ae^{-x} + Be^{-\frac{2}{3}x}$

• PARTICULAR INTEGRAL BY VARIATION OF PARAMETERS

$e_1 = e^{-x}$

$e_2 = e^{-\frac{2}{3}x}$

WRONSKIAN $W = \begin{vmatrix} e_1 & e_2 \\ e_1' & e_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-\frac{2}{3}x} \\ -e^{-x} & -\frac{2}{3}e^{-\frac{2}{3}x} \end{vmatrix} = -\frac{2}{3}e^{-\frac{5}{3}x} + e^{-\frac{5}{3}x} = \frac{1}{3}e^{-\frac{5}{3}x}$

• $y_p = -e_1\int \frac{e_2 f}{aW}\,dx + e_2\int \frac{e_1 f}{aW}\,dx$ $\quad$ $a(x) = 3$ HERE

$\Rightarrow y_p = -e^{-x}\int \frac{e^{-\frac{2}{3}x} \times e^{-x}\sin x}{3 \times \frac{1}{3}e^{-\frac{5}{3}x}}\,dx + e^{-\frac{2}{3}x}\int \frac{e^{-x} \times e^{-x}\sin x}{3 \times \frac{1}{3}e^{-\frac{5}{3}x}}\,dx$

$\Rightarrow y_p = -e^{-x}\int \sin x\,dx + e^{-\frac{2}{3}x}\int e^{-\frac{1}{3}x}\sin x\,dx$

$\int e^{-\frac{1}{3}x}\sin x\,dx = -3e^{-\frac{1}{3}x}\sin x + \int 3e^{-\frac{1}{3}x}\cos x\,dx$

$\int e^{-\frac{1}{3}x}\sin x\,dx = -3e^{-\frac{1}{3}x}\sin x + \left[-9e^{-\frac{1}{3}x}\cos x - \int 9e^{-\frac{1}{3}x}\sin x\,dx\right]$

$\int e^{-\frac{1}{3}x}\sin x\,dx = -3e^{-\frac{1}{3}x}\sin x - 9e^{-\frac{1}{3}x}\cos x - 9\int e^{-\frac{1}{3}x}\sin x\,dx$

$10\int e^{-\frac{1}{3}x}\sin x = -3e^{-\frac{1}{3}x}(\sin x + 3\cos x)$

$\int e^{-\frac{1}{3}x}\sin x = -\frac{3}{10}e^{-\frac{1}{3}x}(\sin x + 3\cos x)$

| $\sin x$ | $\cos x$ |
|---|---|
| $-3e^{-\frac{1}{3}x}$ | $e^{-\frac{1}{3}x}$ |

| $\cos x$ | $-\sin x$ |
|---|---|
| $-9e^{-\frac{1}{3}x}$ | $3e^{-\frac{1}{3}x}$ |

$\Rightarrow y_p = -e^{-x}(-\cos x) + e^{-\frac{2}{3}x}\left[-\frac{3}{10}e^{-\frac{1}{3}x}(\sin x + 3\cos x)\right]$

$\Rightarrow y_p = e^{-x}\cos x - \frac{3}{10}e^{-x}(\sin x + 3\cos x)$

$\Rightarrow y_p = \frac{1}{10}e^{-x}\cos x - \frac{3}{10}e^{-x}$

$\Rightarrow y_p = \frac{1}{10}e^{-x}(\cos x - 3\sin x)$

$\therefore$ GEN SOLUTION $Ae^{-x} + Be^{-\frac{2}{3}x} + \frac{1}{10}e^{-x}(\cos x - 3\sin x)$

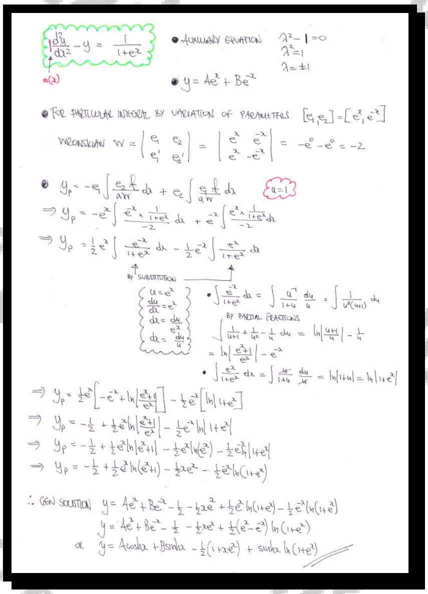
**Question 13** (****)

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - y = \frac{1}{1+e^x}.$$

$$\boxed{y = Ae^x + Be^{-x} - \frac{1}{2}\left(1 + xe^x\right) + \frac{1}{2}\left(e^x - e^{-x}\right)\ln\left(1+e^x\right)},$$

$$\boxed{y = A\cosh x + B\sinh x - \frac{1}{2}\left(1 + xe^x\right) + \sinh x \ln\left(1+e^x\right)}$$



$\frac{d^2y}{dx^2} - y = \frac{1}{1+e^x}$

$a(x)$

• AUXILIARY EQUATION $\lambda^2 - 1 = 0$, $\lambda^2 = 1$, $\lambda = \pm 1$

• $y = Ae^x + Be^{-x}$

• FOR PARTICULAR INTEGRAL BY VARIATION OF PARAMETERS $[e_1, e_2] = [e^x, e^{-x}]$

WRONSKIAN $W = \begin{vmatrix} e_1 & e_2 \\ e_1' & e_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^0 - e^0 = -2$

• $y_p = -e_1\int \frac{e_2 f}{aW}\,dx + e_2\int \frac{e_1 f}{aW}\,dx$ $\quad a=1$

$\Rightarrow y_p = -e^x\int \frac{e^{-x} \times \frac{1}{1+e^x}}{-2}\,dx + e^{-x}\int \frac{e^x \times \frac{1}{1+e^x}}{-2}\,dx$

$\Rightarrow y_p = \frac{1}{2}e^x\int \frac{e^{-x}}{1+e^x}\,dx - \frac{1}{2}e^{-x}\int \frac{e^x}{1+e^x}\,dx$

BY SUBSTITUTION

$u = e^x$, $\frac{du}{dx} = e^x$, $dx = \frac{du}{e^x}$, $dx = \frac{du}{u}$

• $\int \frac{e^{-x}}{1+e^x}\,dx = \int \frac{u^{-1}}{1+u}\frac{du}{u} = \int \frac{1}{u^2(u+1)}\,du$

BY PARTIAL FRACTIONS

$\int \frac{1}{u+1} + \frac{1}{u^2} - \frac{1}{u}\,du = \ln\left|\frac{u+1}{u}\right| - \frac{1}{u}$

$= \ln\left|\frac{e^x+1}{e^x}\right| - e^{-x}$

• $\int \frac{e^x}{1+e^x}\,dx = \int \frac{u}{1+u}\frac{du}{u} = \ln|1+u| = \ln|1+e^x|$

$\Rightarrow y_p = \frac{1}{2}e^x\left[-e^{-x} + \ln\left|\frac{e^x+1}{e^x}\right|\right] - \frac{1}{2}e^{-x}\left[\ln|1+e^x|\right]$

$\Rightarrow y_p = -\frac{1}{2} + \frac{1}{2}e^x\ln\left|\frac{e^x+1}{e^x}\right| - \frac{1}{2}e^{-x}\ln|1+e^x|$

$\Rightarrow y_p = -\frac{1}{2} + \frac{1}{2}e^x\ln|e^x+1| - \frac{1}{2}e^x\ln(e^x) - \frac{1}{2}e^{-x}\ln|1+e^x|$

$\Rightarrow y_p = -\frac{1}{2} + \frac{1}{2}e^x\ln(e^x+1) - \frac{1}{2}xe^x - \frac{1}{2}e^{-x}\ln(1+e^x)$

$\therefore$ GEN SOLUTION $y = Ae^x + Be^{-x} - \frac{1}{2} - \frac{1}{2}xe^x + \frac{1}{2}e^x\ln(1+e^x) - \frac{1}{2}e^{-x}\ln(1+e^x)$

$y = Ae^x + Be^{-x} - \frac{1}{2} - \frac{1}{2}xe^x + \frac{1}{2}(e^x - e^{-x})\ln(1+e^x)$

or $y = A\cosh x + B\sinh x - \frac{1}{2}(1+xe^x) + \sinh x\ln(1+e^x)$

**Question 14** (****)

Use variation of parameters to determine the specific solution of the following differential equation

$$x^2\frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 16y = 16\ln x\,,$$

given further that $y = \frac{1}{2}$, $\frac{dy}{dx} = 2$ at $x = 1$.

$$\boxed{y = \frac{1}{2} + \left(1 + x^4\right)\ln x}$$

$x^2\frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 16y = 16\ln x$ SUBJECT TO $x=1$, $y=\frac{1}{2}$, $y'=2$

- ASSUME SOLUTION OF THE FORM

$y = x^{\lambda} \Rightarrow y' = \lambda x^{\lambda-1}$ & $y'' = \lambda(\lambda-1)x^{\lambda-2}$

SUB INTO THE O.D.E WITH R.H.S ZERO

$\lambda(\lambda-1)x^{\lambda} - 7\lambda x^{\lambda} + 16x^{\lambda} = 0$

$(\lambda^2 - \lambda - 7\lambda + 16)x^{\lambda} = 0$

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda-4)^2 = 0$

$\lambda = 4$ REPEATED $\therefore$ COMPLEMENTARY FUNCTION

$y = Ax^4 + Bx^4\ln x$

- PARTICULAR INTEGRAL BY VARIATION OF PARAMETERS

$x^2\frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 16y = 16\ln x$

$a(x) = x^2$

$e_1 = x^4$

$e_2 = x^4\ln x$

WRONSKIAN $= \begin{vmatrix} e_1 & e_2 \\ e_1' & e_2' \end{vmatrix}$

$= \begin{vmatrix} x^4 & x^4\ln x \\ 4x^3 & 4x^3\ln x + x^3 \end{vmatrix}$

$= 4x^7\ln x + x^7 - 4x^7\ln x$

$= x^7$

- THUS THE PARTICULAR INTEGRAL IS GIVEN BY

$y_p = -e_1\int \frac{e_2 f}{aW}\,dx + e_2\int\frac{e_1 f}{aW}\,dx$

$y_p = -x^4\int\frac{x^4\ln x \times 16\ln x}{x^2 \times x^7}\,dx + x^4\ln x\int\frac{x^4 \times 16\ln x}{x^2\times x^7}\,dx$

$y_p = -x^4\int\frac{16(\ln x)^2}{x^5}\,dx + x^4\ln x\int\frac{16\ln x}{x^5}\,dx$

- EACH BY PARTS

- $\int 16x^{-5}(\ln x)^2\,dx$

| $(\ln x)^2$ | $(2\ln x)\frac{1}{x}$ |
|---|---|
| $-4x^{-4}$ | $16x^{-5}$ |

$= -\frac{4}{x^4}(\ln x)^2 + \int 8x^{-5}(\ln x)\,dx$

BY PARTS AGAIN

| $\ln x$ | $\frac{1}{x}$ |
|---|---|
| $-2x^{-4}$ | $8x^{-5}$ |

$= -\frac{4}{x^4}(\ln x)^2 - \frac{2}{x^4}\ln x + \int 2x^{-5}\,dx$

$= -\frac{4}{x^4}(\ln x)^2 - \frac{2}{x^4}\ln x - \frac{1}{2}x^{-4}$

- $\int 16x^{-5}(\ln x)\,dx$

| $\ln x$ | $\frac{1}{x}$ |
|---|---|
| $-4x^{-4}$ | $16x^{-5}$ |

$= -\frac{4}{x^4}\ln x + \int 4x^{-5}\,dx$

$= -\frac{4}{x^4}\ln x - x^{-4}$

$\therefore y_p = -x^4\left[-\frac{4}{x^4}(\ln x)^2 - \frac{2}{x^4}\ln x - \frac{1}{2}x^{-4}\right] + x^4\ln x\left[-\frac{4}{x^4}\ln x - x^{-4}\right]$

$y_p = \cancel{4(\ln x)^2} + 2\ln x + \frac{1}{2} - \cancel{4(\ln x)^2} - \ln x$

SO $y_p = \ln x + \frac{1}{2}$

- GEN SOLUTION

$y = Ax^4 + Bx^4\ln x + \ln x + \frac{1}{2}$

- APPLY CONDITION $x=1$ $y=\frac{1}{2}$

$\frac{1}{2} = A + \frac{1}{2} \Rightarrow \boxed{A=0}$

$\therefore y = Bx^4\ln x + \ln x + \frac{1}{2}$

$\frac{dy}{dx} = 4Bx^3\ln x + Bx^3 + \frac{1}{x}$

- APPLY CONDITION $x=1$ $\frac{dy}{dx} = 2$

$2 = B + 1$

$\boxed{B=1}$

$\therefore y = x^4\ln x + \ln x + \frac{1}{2}$

$y = \frac{1}{2} + (x^4+1)\ln x$