DIFFERENTIAL EQUATIONS

2nd order or higher

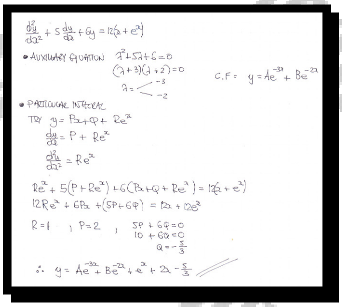
2^{ND} ORDER WITH CONSTANT COEFFICIENTS

Question 1 **(**)**

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=12\left(x+\mathrm{e}^x\right).$$

$$y = A\mathrm{e}^{-3x} + B\mathrm{e}^{-2x} + \mathrm{e}^{x} + 2x - \frac{5}{3}$$



$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=12(x+e^x)$

• AUXILIARY EQUATION $\lambda^2+5\lambda+6=0$

$(\lambda+3)(\lambda+2)=0$

$\lambda = -3, -2$

C.F: $y=Ae^{-3x}+Be^{-2x}$

• PARTICULAR INTEGRAL

TRY $y=Px+Q+Re^x$

$\frac{dy}{dx}=P+Re^x$

$\frac{d^2y}{dx^2}=Re^x$

$Re^x+5(P+Re^x)+6(Px+Q+Re^x)=12(x+e^x)$

$12Re^x+6Px+(5P+6Q)=12x+12e^x$

$R=1$, $P=2$, $5P+6Q=0$, $10+6Q=0$, $Q=-\frac{5}{3}$

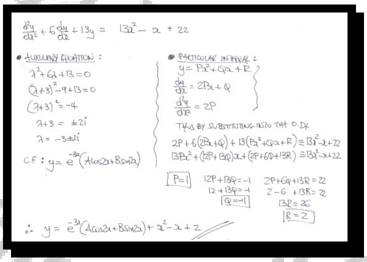
$\therefore y=Ae^{-3x}+Be^{-2x}+e^x+2x-\frac{5}{3}$

Question 2 **(**)**

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=13x^2-x+22.$$

$$y=\mathrm{e}^{-3x}\left(A\cos 2x+B\sin 2x\right)+x^2-x+2$$



$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=13x^2-x+22$

• AUXILIARY EQUATION:

$\lambda^2+6\lambda+13=0$

$(\lambda+3)^2-9+13=0$

$(\lambda+3)^2=-4$

$\lambda+3=\pm 2i$

$\lambda=-3\pm 2i$

C.F: $y=e^{-3x}(A\cos 2x+B\sin 2x)$

• PARTICULAR INTEGRAL:

$y=Px^2+Qx+R$

$\frac{dy}{dx}=2Px+Q$

$\frac{d^2y}{dx^2}=2P$

THUS BY SUBSTITUTING INTO THE O.D.E

$2P+6(2Px+Q)+13(Px^2+Qx+R)\equiv 13x^2-x+22$

$13Px^2+(12P+13Q)x+(2P+6Q+13R)\equiv 13x^2-x+22$

$P=1$; $12P+13Q=-1$, $12+13Q=-1$, $Q=-1$; $2P+6Q+13R=22$, $2-6+13R=22$, $13R=26$, $R=2$

$\therefore y=e^{-3x}(A\cos 2x+B\sin 2x)+x^2-x+2$

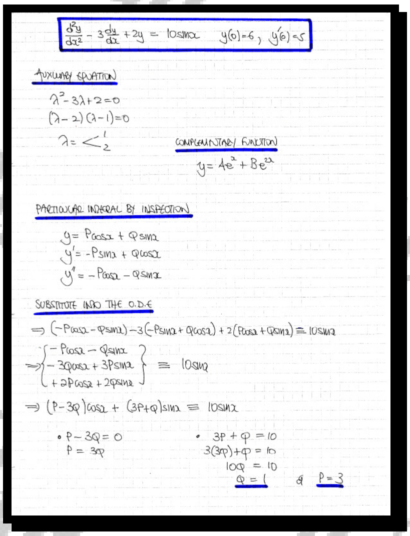
Question 3 (**)

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10\sin x,$$

subject to the boundary conditions $y = 6$ and $\frac{dy}{dx} = 5$ at $x = 0$.

$\boxed{}$, $\boxed{y = 2e^{x} + e^{2x} + 3\cos x + \sin x}$



$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10\sin x \quad y(0)=6,\ y'(0)=5$

AUXILIARY EQUATION

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda = 1, 2$

COMPLEMENTARY FUNCTION

$y = Ae^{x} + Be^{2x}$

PARTICULAR INTEGRAL BY INSPECTION

$y = P\cos x + Q\sin x$

$y' = -P\sin x + Q\cos x$

$y'' = -P\cos x - Q\sin x$

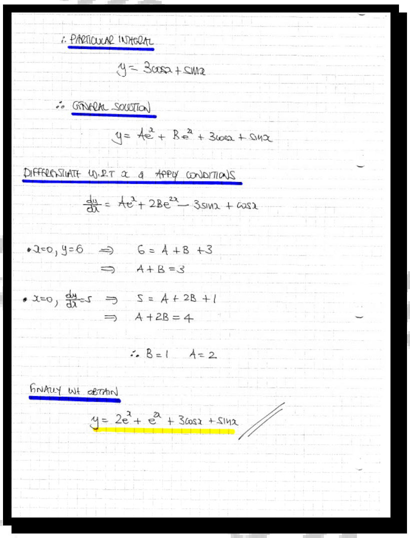
SUBSTITUTE INTO THE O.D.E

$\Rightarrow (-P\cos x - Q\sin x) - 3(-P\sin x + Q\cos x) + 2(P\cos x + Q\sin x) \equiv 10\sin x$

$\Rightarrow \left\{ \begin{array}{l} -P\cos x - Q\sin x \\ -3Q\cos x + 3P\sin x \\ +2P\cos x + 2Q\sin x \end{array} \right\} \equiv 10\sin x$

$\Rightarrow (P - 3Q)\cos x + (3P + Q)\sin x \equiv 10\sin x$

- $P - 3Q = 0$, $P = 3Q$
- $3P + Q = 10$, $3(3Q) + Q = 10$, $10Q = 10$, $Q = 1$ & $P = 3$



∴ PARTICULAR INTEGRAL

$y = 3\cos x + \sin x$

∴ GENERAL SOLUTION

$y = Ae^{x} + Be^{2x} + 3\cos x + \sin x$

DIFFERENTIATE W.R.T x & APPLY CONDITIONS

$\frac{dy}{dx} = Ae^{x} + 2Be^{2x} - 3\sin x + \cos x$

- $x=0,\ y=6 \Rightarrow 6 = A + B + 3 \Rightarrow A + B = 3$
- $x=0,\ \frac{dy}{dx}=5 \Rightarrow 5 = A + 2B + 1 \Rightarrow A + 2B = 4$

∴ $B = 1 \quad A = 2$

FINALLY WE OBTAIN

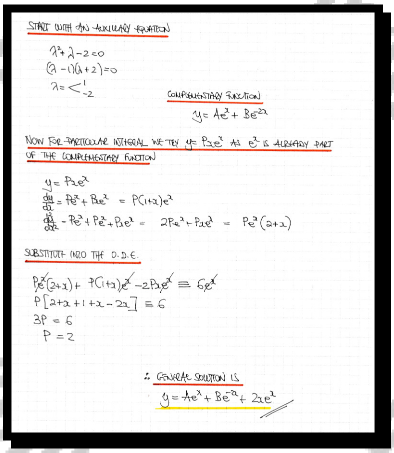
$y = 2e^{x} + e^{2x} + 3\cos x + \sin x$

Question 4 (**)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=6e^{x}.$$

$\boxed{}$, $\boxed{y=(A+2x)e^{x}+Be^{-2x}}$



START WITH AN AUXILIARY EQUATION

$\lambda^2+\lambda-2=0$

$(\lambda-1)(\lambda+2)=0$

$\lambda = \begin{cases}1\\-2\end{cases}$

COMPLEMENTARY FUNCTION

$y=Ae^{x}+Be^{-2x}$

NOW FOR PARTICULAR INTEGRAL WE TRY $y=Pxe^{x}$ AS e^{x} IS ALREADY PART OF THE COMPLEMENTARY FUNCTION

$y=Pxe^{x}$

$\frac{dy}{dx}=Pe^{x}+Pxe^{x}=P(1+x)e^{x}$

$\frac{d^2y}{dx^2}=Pe^{x}+Pe^{x}+Pxe^{x}=2Pe^{x}+Pxe^{x}=Pe^{x}(2+x)$

SUBSTITUTE INTO THE O.D.E.

$Pe^{x}(2+x)+P(1+x)e^{x}-2Pxe^{x}\equiv 6e^{x}$

$P[2+x+1+x-2x]\equiv 6$

$3P=6$

$P=2$

$\therefore$ GENERAL SOLUTION IS

$y=Ae^{x}+Be^{-2x}+2xe^{x}$

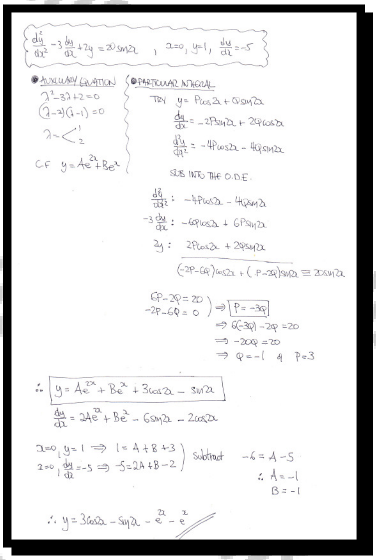
Question 5 **(**)**

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin 2x,$$

subject to the boundary conditions $y = 1$ and $\frac{dy}{dx} = -5$ at $x = 0$.

$$\boxed{y = 3\cos 2x - \sin 2x - e^{2x} - e^{x}}$$



$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin 2x$, $x=0$, $y=1$, $\frac{dy}{dx} = -5$

• AUXILIARY EQUATION

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda-2)(\lambda-1) = 0$

$\lambda = 1, 2$

C.F $y = Ae^{2x} + Be^{x}$

• PARTICULAR INTEGRAL

TRY $y = P\cos 2x + Q\sin 2x$

$\frac{dy}{dx} = -2P\sin 2x + 2Q\cos 2x$

$\frac{d^2y}{dx^2} = -4P\cos 2x - 4Q\sin 2x$

SUB INTO THE O.D.E.

$\frac{d^2y}{dx^2}$: $-4P\cos 2x - 4Q\sin 2x$

$-3\frac{dy}{dx}$: $-6Q\cos 2x + 6P\sin 2x$

$2y$: $2P\cos 2x + 2Q\sin 2x$

$(-2P-6Q)\cos 2x + (P-2Q)\sin 2x \equiv 20\sin 2x$

$6P - 2Q = 20$, $-2P - 6Q = 0$ $\Rightarrow$ $\boxed{P = -3Q}$

$\Rightarrow 6(-3Q) - 2Q = 20$

$\Rightarrow -20Q = 20$

$\Rightarrow Q = -1$ & $P = 3$

$\therefore \boxed{y = Ae^{2x} + Be^{x} + 3\cos 2x - \sin 2x}$

$\frac{dy}{dx} = 2Ae^{2x} + Be^{x} - 6\sin 2x - 2\cos 2x$

$x=0, y=1 \Rightarrow 1 = A + B + 3$

$x=0, \frac{dy}{dx} = -5 \Rightarrow -5 = 2A + B - 2$

subtract $-6 = A - 5$ $\therefore A = -1$, $B = -1$

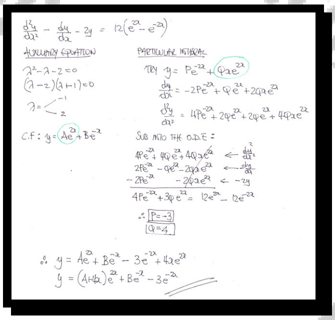
$\therefore y = 3\cos 2x - \sin 2x - e^{2x} - e^{x}$

Question 6 **(**)**

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 12\left(e^{2x} - e^{-2x}\right).$$

$$\boxed{y = (A + 4x)e^{2x} + Be^{-x} - 3e^{-2x}}$$



$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 12(e^{2x} - e^{-2x})$

AUXILIARY EQUATION

$\lambda^2 - \lambda - 2 = 0$

$(\lambda-2)(\lambda+1) = 0$

$\lambda = -1, 2$

C.F: $y = Ae^{2x} + Be^{-x}$

PARTICULAR INTEGRAL

TRY $y = Pe^{-2x} + Qxe^{2x}$

$\frac{dy}{dx} = -2Pe^{-2x} + Qe^{2x} + 2Qxe^{2x}$

$\frac{d^2y}{dx^2} = 4Pe^{-2x} + 2Qe^{2x} + 2Qe^{2x} + 4Qxe^{2x}$

SUB INTO THE O.D.E:

$4Pe^{-2x} + 4Qe^{2x} + 4Qxe^{2x}$ $\leftarrow \frac{d^2y}{dx^2}$

$2Pe^{-2x} - Qe^{2x} - 2Qxe^{2x}$ $\leftarrow -\frac{dy}{dx}$

$-2Pe^{-2x} \quad - 2Qxe^{2x}$ $\leftarrow -2y$

$4Pe^{-2x} + 3Qe^{2x} = 12e^{2x} - 12e^{-2x}$

$\therefore \boxed{P = -3}$ $\boxed{Q = 4}$

$\therefore y = Ae^{2x} + Be^{-x} - 3e^{-2x} + 4xe^{2x}$

$y = (A + 4x)e^{2x} + Be^{-x} - 3e^{-2x}$

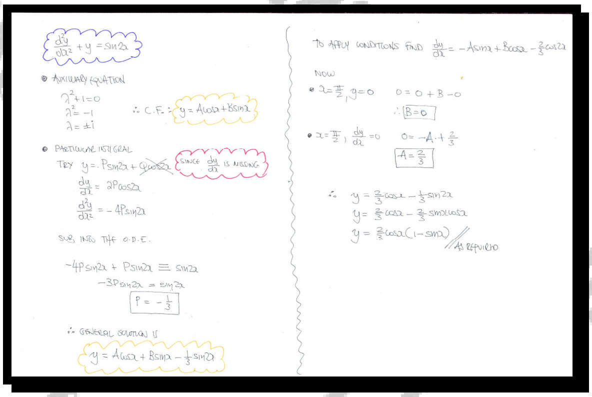
Question 7 (**)

$$\frac{d^2y}{dx^2} + y = \sin 2x, \quad \text{with } y = 0, \ \frac{dy}{dx} = 0 \text{ at } x = \frac{\pi}{2}.$$

Show that a solution of the above differential equation is

$$y = \frac{2}{3}\cos x(1 - \sin x).$$

proof


$\frac{d^2y}{dx^2} + y = \sin 2x$

• AUXILIARY EQUATION

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\lambda = \pm i$

$\therefore$ C.F.: $y = A\cos x + B\sin x$

• PARTICULAR INTEGRAL

TRY $y = P\sin 2x + Q\cos 2x$ (SINCE $\frac{dy}{dx}$ IS MISSING)

$\frac{dy}{dx} = 2P\cos 2x$

$\frac{d^2y}{dx^2} = -4P\sin 2x$

SUB INTO THE O.D.E.

$-4P\sin 2x + P\sin 2x \equiv \sin 2x$

$-3P\sin 2x = \sin 2x$

$P = -\frac{1}{3}$

$\therefore$ GENERAL SOLUTION IS

$y = A\cos x + B\sin x - \frac{1}{3}\sin 2x$

TO APPLY CONDITIONS FIND $\frac{dy}{dx} = -A\sin x + B\cos x - \frac{2}{3}\cos 2x$

NOW

• $x = \frac{\pi}{2}, y = 0$ $\quad 0 = 0 + B - 0$

$\therefore B = 0$

• $x = \frac{\pi}{2}, \frac{dy}{dx} = 0$ $\quad 0 = -A + \frac{2}{3}$

$A = \frac{2}{3}$

$\therefore y = \frac{2}{3}\cos x - \frac{1}{3}\sin 2x$

$y = \frac{2}{3}\cos x - \frac{2}{3}\sin x\cos x$

$y = \frac{2}{3}\cos x(1 - \sin x)$ // AS REQUIRED

Question 8 (**+)

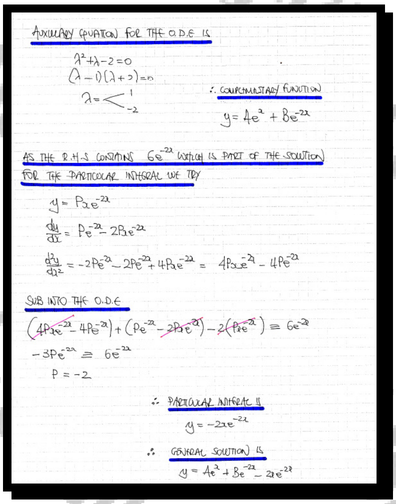
$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=6e^{-2x},$$

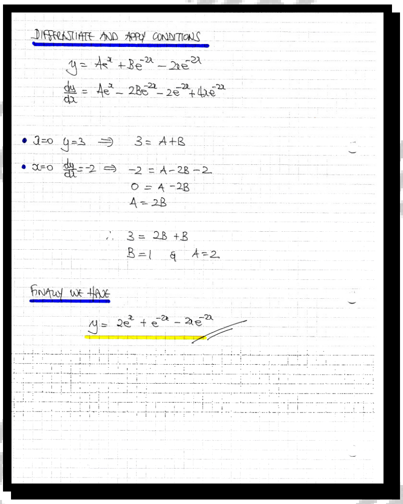
with $y=3$ and $\frac{dy}{dx}=-2$ at $x=0$.

Show that the solution of the above differential equation is

$$y=2e^{x}+(1-2x)e^{-2x}.$$

[ ], [proof]



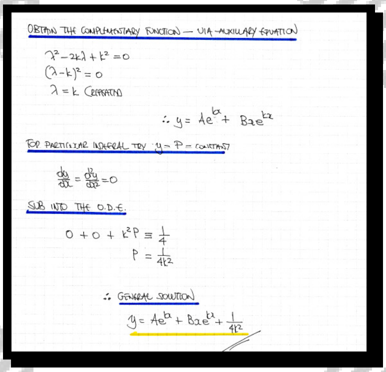


Question 9 (**+)

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2 y = \frac{1}{4}, \ k > 0 .$$

$$y = A\mathrm{e}^{kx} + Bx\mathrm{e}^{kx} + \frac{1}{4k^2}$$


OBTAIN THE COMPLEMENTARY FUNCTION — VIA AUXILIARY EQUATION

$\lambda^2 - 2k\lambda + k^2 = 0$

$(\lambda - k)^2 = 0$

$\lambda = k$ (REPEATED)

$\therefore y = Ae^{kx} + Bxe^{kx}$

FOR PARTICULAR INTEGRAL TRY $y = P$ = CONSTANT

$\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$

SUB INTO THE O.D.E.

$0 + 0 + k^2 P \equiv \frac{1}{4}$

$P = \frac{1}{4k^2}$

$\therefore$ GENERAL SOLUTION

$y = Ae^{kx} + Bxe^{kx} + \frac{1}{4k^2}$

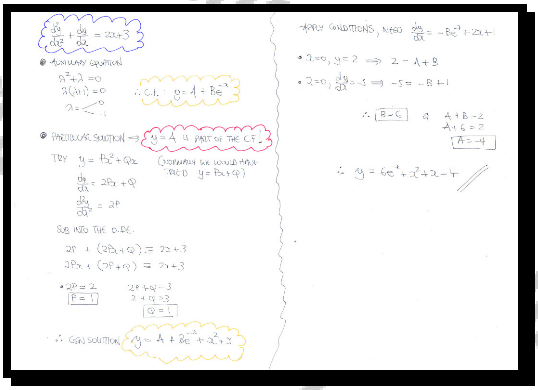
Question 10 (+)**

Find the solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}=2x+3,$$

subject to the conditions $y=2$, $\frac{dy}{dx}=-5$ at $x=0$.

$$\boxed{y=x^2+x-4+6e^{-x}}$$



$\frac{d^2y}{dx^2}+\frac{dy}{dx}=2x+3$

• AUXILIARY EQUATION

$\lambda^2+\lambda=0$

$\lambda(\lambda+1)=0$

$\lambda=0, -1$

$\therefore$ C.F.: $y=A+Be^{-x}$

• PARTICULAR SOLUTION $\Rightarrow$ $y=A$ IS PART OF THE C.F.!

TRY $y=Px^2+Qx$ (NORMALLY WE WOULD HAVE TRIED $y=Px+Q$)

$\frac{dy}{dx}=2Px+Q$

$\frac{d^2y}{dx^2}=2P$

SUB INTO THE O.D.E.

$2P+(2Px+Q)\equiv 2x+3$

$2Px+(2P+Q)\equiv 2x+3$

• $2P=2$, $\boxed{P=1}$ $\quad$ $2P+Q=3$, $2+Q=3$, $\boxed{Q=1}$

$\therefore$ GEN SOLUTION $y=A+Be^{-x}+x^2+x$

APPLY CONDITIONS, NEED $\frac{dy}{dx}=-Be^{-x}+2x+1$

• $x=0, y=2 \Rightarrow 2=A+B$

• $x=0, \frac{dy}{dx}=-5 \Rightarrow -5=-B+1$

$\therefore \boxed{B=6}$ & $A+B=2$, $A+6=2$, $\boxed{A=-4}$

$\therefore y=6e^{-x}+x^2+x-4$

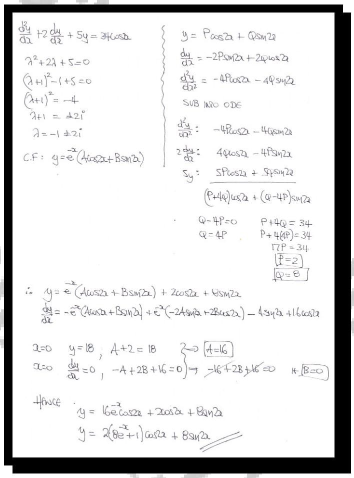
Question 11 (**+)

Find a solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 34\cos 2x,$$

subject to the boundary conditions $y = 18$ and $\frac{dy}{dx} = 0$ at $x = 0$.

$$\boxed{y = 2\left(8e^{-x} + 1\right)\cos 2x + 8\sin 2x}$$



$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 34\cos 2x$

$\lambda^2 + 2\lambda + 5 = 0$

$(\lambda+1)^2 - 1 + 5 = 0$

$(\lambda+1)^2 = -4$

$\lambda + 1 = \pm 2i$

$\lambda = -1 \pm 2i$

C.F: $y = e^{-x}(A\cos 2x + B\sin 2x)$

$y = P\cos 2x + Q\sin 2x$

$\frac{dy}{dx} = -2P\sin 2x + 2Q\cos 2x$

$\frac{d^2y}{dx^2} = -4P\cos 2x - 4Q\sin 2x$

SUB INTO ODE

$\frac{d^2y}{dx^2}$: $-4P\cos 2x - 4Q\sin 2x$

$2\frac{dy}{dx}$: $4Q\cos 2x - 4P\sin 2x$

$5y$: $5P\cos 2x + 5Q\sin 2x$

$(P+4Q)\cos 2x + (Q-4P)\sin 2x$

$Q - 4P = 0$, $Q = 4P$

$P + 4Q = 34$, $P + 4(4P) = 34$, $17P = 34$, $\boxed{P=2}$, $\boxed{Q=8}$

$\therefore y = e^{-x}(A\cos 2x + B\sin 2x) + 2\cos 2x + 8\sin 2x$

$\frac{dy}{dx} = -e^{-x}(A\cos 2x + B\sin 2x) + e^{-x}(-2A\sin 2x + 2B\cos 2x) - 4\sin 2x + 16\cos 2x$

$x=0$, $y = 18$, $A + 2 = 18$ $\Rightarrow \boxed{A=16}$

$x=0$, $\frac{dy}{dx} = 0$, $-A + 2B + 16 = 0$ $\Rightarrow -16 + 2B + 16 = 0$ $\boxed{B=0}$

HENCE

$y = 16e^{-x}\cos 2x + 2\cos 2x + 8\sin 2x$

$y = 2(8e^{-x}+1)\cos 2x + 8\sin 2x$

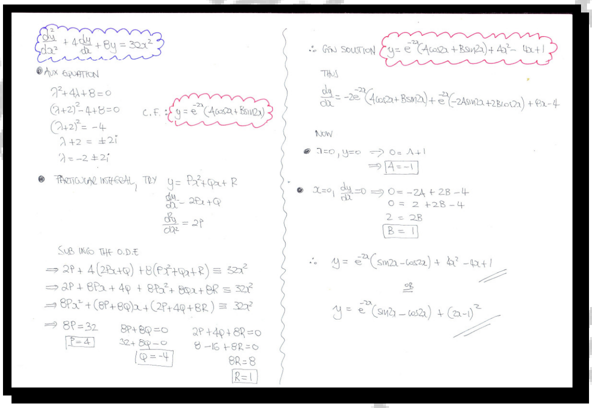
Question 12 (**+)

The curve C has a local minimum at the origin and satisfies the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 32x^2 .$$

Find an equation for C.

$$\boxed{y = e^{x}\left(\sin 2x + \cos 2x\right) + \left(2x-1\right)^2}$$



$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 32x^2$

- AUX EQUATION

$\lambda^2 + 4\lambda + 8 = 0$

$(\lambda+2)^2 - 4 + 8 = 0$

$(\lambda+2)^2 = -4$

$\lambda + 2 = \pm 2i$

$\lambda = -2 \pm 2i$

C.F. : $y = e^{-2x}(A\cos 2x + B\sin 2x)$

- PARTICULAR INTEGRAL, TRY $y = Px^2 + Qx + R$

$\frac{dy}{dx} = 2Px + Q$

$\frac{d^2y}{dx^2} = 2P$

SUB INTO THE O.D.E

$\Rightarrow 2P + 4(2Px+Q) + 8(Px^2+Qx+R) \equiv 32x^2$

$\Rightarrow 2P + 8Px + 4Q + 8Px^2 + 8Qx + 8R \equiv 32x^2$

$\Rightarrow 8Px^2 + (8P+8Q)x + (2P+4Q+8R) \equiv 32x^2$

$\Rightarrow 8P = 32$, $\boxed{P=4}$; $8P+8Q=0$, $32+8Q=0$, $\boxed{Q=-4}$; $2P+4Q+8R=0$, $8-16+8R=0$, $8R=8$, $\boxed{R=1}$

$\therefore$ GEN SOLUTION $y = e^{-2x}(A\cos 2x + B\sin 2x) + 4x^2 - 4x + 1$

THUS

$\frac{dy}{dx} = -2e^{-2x}(A\cos 2x + B\sin 2x) + e^{-2x}(-2A\sin 2x + 2B\cos 2x) + 8x - 4$

NOW

- $x=0, y=0 \Rightarrow 0 = A + 1 \Rightarrow \boxed{A=-1}$

- $x=0, \frac{dy}{dx}=0 \Rightarrow 0 = -2A + 2B - 4$

$0 = 2 + 2B - 4$

$2 = 2B$

$\boxed{B=1}$

$\therefore y = e^{-2x}(\sin 2x - \cos 2x) + 4x^2 - 4x + 1$

OR

$y = e^{-2x}(\sin 2x - \cos 2x) + (2x-1)^2$

Question 13 (**+)

$$\frac{d^2x}{dt^2} + 9x + 12\sin 3t = 0, \; t \ge 0,$$

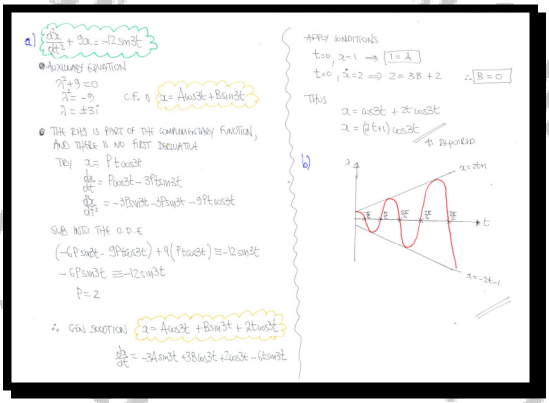
with $x = 1$, $\dfrac{dx}{dt} = 2$ at $t = 0$.

a) Show that a solution of the differential equation is

$$x = (2t+1)\cos 3t.$$

b) Sketch the graph of x.

proof



Question 14 (**+)

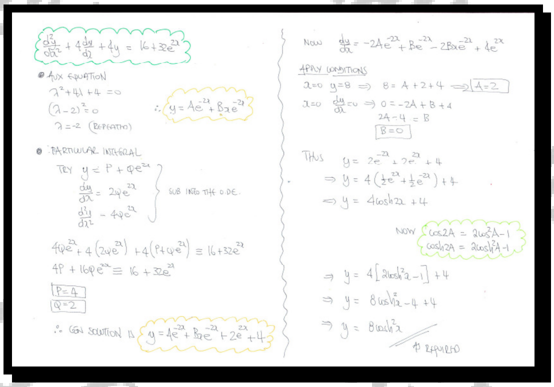
$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=16+32e^{2x},$$

with $y=8$ and $\frac{dy}{dx}=0$ at $x=0$.

Show that the solution of the above differential equation is

$$y=8\cosh^2 x.$$

proof



Question 15 (**+)

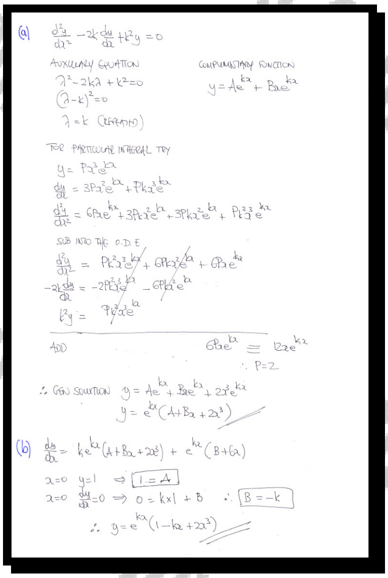
$$\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2 y = 12x\mathrm{e}^{kx}, \ k > 0$$

a) Find a general solution of the differential equation given that $y = Px^3\mathrm{e}^{kx}$, where P is a constant, is part of the solution.

b) Given further that $y = 1$, $\frac{dy}{dx} = 0$ at $x = 0$ show that

$$y = \mathrm{e}^{kx}\left(2x^3 - kx + 1\right).$$

$$\boxed{y = \mathrm{e}^{kx}\left(2x^3 + Ax + B\right)}$$



Question 16 (**+)

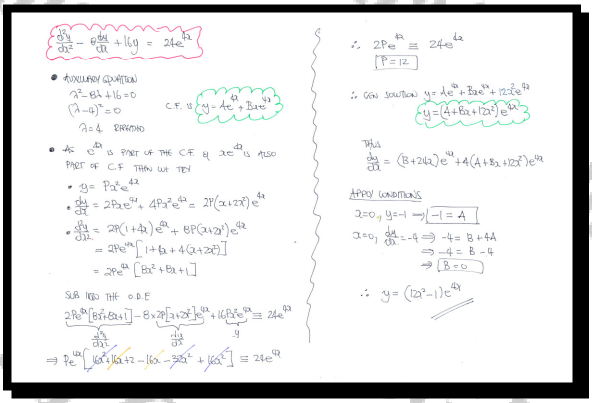
Show that the solution of the differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 24e^{4x},$$

subject to the boundary conditions $y = -1$, $\frac{dy}{dx} = -4$ at $x = 0$, can be written as

$$y = \left(12x^2 - 1\right)e^{4x}.$$

proof



Question 17 (***)

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}.$$

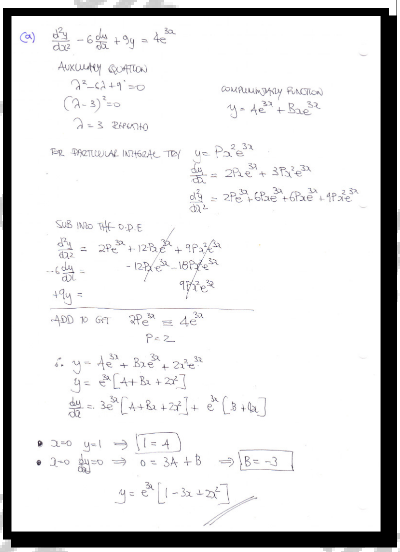
a) Find a solution of the differential equation given that $y = 1$, $\frac{dy}{dx} = 0$ at $x = 0$.

b) Sketch the graph of y.

The sketch must include …

- the coordinates of any points where the graph meets the coordinate axes.
- the coordinates of any stationary points of the curve.
- clear indications of how the graph looks for large positive or negative values of x.

$$\boxed{y = e^{3x}\left(2x^2 - 3x + 1\right)}$$



(a) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}$

AUXILIARY EQUATION

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$

$\lambda = 3$ REPEATED

COMPLEMENTARY FUNCTION

$y = Ae^{3x} + Bxe^{3x}$

FOR PARTICULAR INTEGRAL TRY $y = Px^2e^{3x}$

$\frac{dy}{dx} = 2Pxe^{3x} + 3Px^2e^{3x}$

$\frac{d^2y}{dx^2} = 2Pe^{3x} + 6Pxe^{3x} + 6Pxe^{3x} + 9Px^2e^{3x}$

SUB INTO THE O.D.E

$\frac{d^2y}{dx^2} = 2Pe^{3x} + 12Pxe^{3x} + 9Px^2e^{3x}$

$-6\frac{dy}{dx} = -12Pxe^{3x} - 18Px^2e^{3x}$

$+9y = 9Px^2e^{3x}$

ADD TO GET $2Pe^{3x} \equiv 4e^{3x}$

$P = 2$

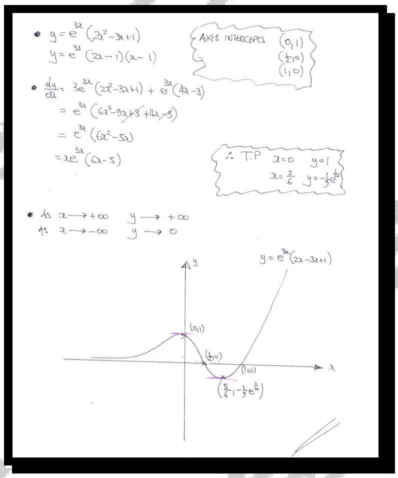
$\therefore y = Ae^{3x} + Bxe^{3x} + 2x^2e^{3x}$

$y = e^{3x}[A + Bx + 2x^2]$

$\frac{dy}{dx} = 3e^{3x}[A + Bx + 2x^2] + e^{3x}[B + 4x]$

- $x = 0$ $y = 1$ $\Rightarrow$ $\boxed{1 = A}$
- $x = 0$ $\frac{dy}{dx} = 0$ $\Rightarrow$ $0 = 3A + B$ $\Rightarrow$ $\boxed{B = -3}$

$y = e^{3x}[1 - 3x + 2x^2]$



- $y = e^{3x}(2x^2 - 3x + 1)$

$y = e^{3x}(2x - 1)(x - 1)$

AXES INTERCEPTS $(0,1)$ $(\frac{1}{2},0)$ $(1,0)$

- $\frac{dy}{dx} = 3e^{3x}(2x^2 - 3x + 1) + e^{3x}(4x - 3)$

$= e^{3x}(6x^2 - 9x + 3 + 4x - 3)$

$= e^{3x}(6x^2 - 5x)$

$= xe^{3x}(6x - 5)$

$\therefore$ T.P $x = 0$ $y = 1$; $x = \frac{5}{6}$ $y = -\frac{1}{9}e^{\frac{5}{2}}$

- As $x \to +\infty$ $y \to +\infty$

As $x \to -\infty$ $y \to 0$

Question 18 (***+)

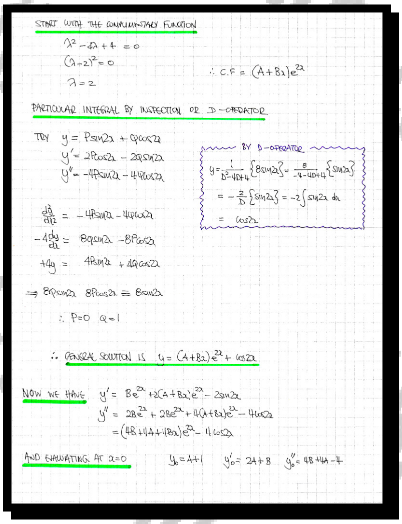
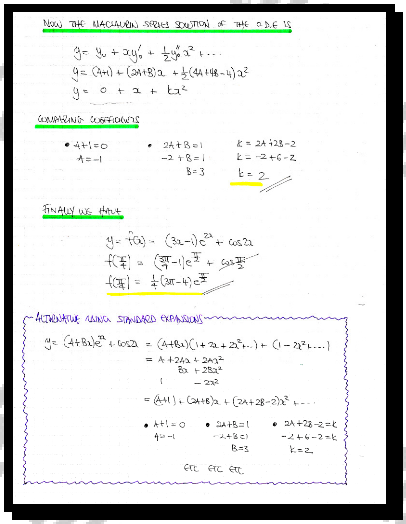
The curve with equation $y = f(x)$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8\sin 2x .$$

The first two non zero terms in Maclaurin series expansion of $f(x)$ are $x + kx^2$, where k is a constant.

Determine in any order the value of k and the exact value of $f\left(\frac{1}{4}\pi\right)$.

$\boxed{}$, $\boxed{k = 2}$, $\boxed{f\left(\frac{1}{4}\pi\right) = \frac{1}{2}(3\pi - 4)e^{\frac{1}{2}\pi}}$

Question 19 (***+)

The function $y = f(x)$ satisfies the following differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 2\mathrm{e}^{-x}(\sin 2x - 2\cos 2x),$$

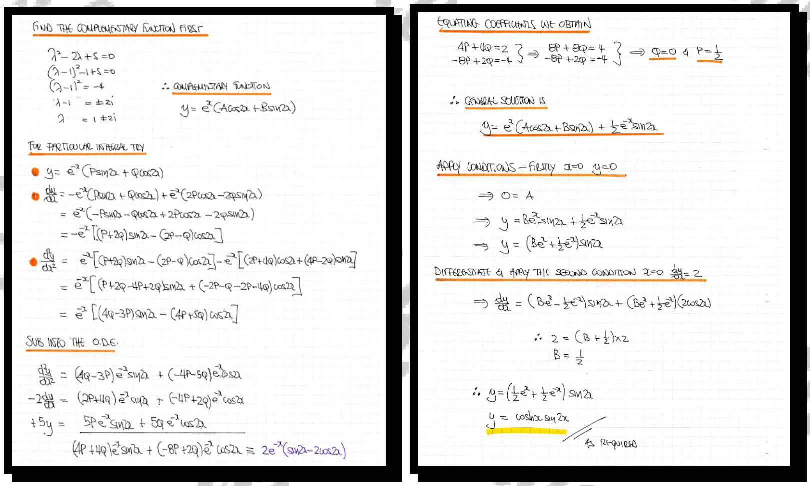
subject to the boundary conditions $y = 0$, $\frac{dy}{dx} = 2$ at $x = 0$.

Solve the differential equation to show that

$$y = \cosh x \sinh 2x .$$

No credit will be given for verification methods.

proof



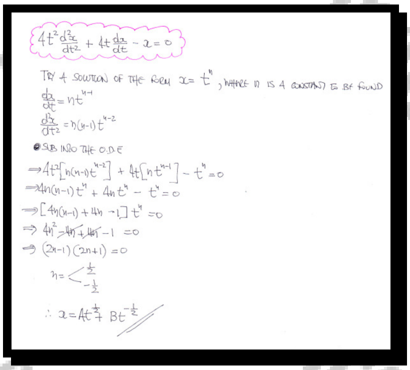
2^{ND} ORDER or HIGHER CAUCHY EULER TYPE

Question 1 **(**)**

Find the general solution of the following differential equation.

$$4t^2 \frac{d^2x}{dt^2} + 4t\frac{dx}{dt} - x = 0 .$$

$$\boxed{x = At^{\frac{1}{2}} + Bt^{-\frac{1}{2}}}$$



$4t^2\frac{d^2x}{dt^2} + 4t\frac{dx}{dt} - x = 0$

TRY A SOLUTION OF THE FORM $x = t^n$, WHERE n IS A CONSTANT TO BE FOUND

$\frac{dx}{dt} = nt^{n-1}$

$\frac{d^2x}{dt^2} = n(n-1)t^{n-2}$

• SUB INTO THE O.D.E

$\Rightarrow 4t^2\left[n(n-1)t^{n-2}\right] + 4t\left[nt^{n-1}\right] - t^n = 0$

$\Rightarrow 4n(n-1)t^n + 4nt^n - t^n = 0$

$\Rightarrow \left[4n(n-1) + 4n - 1\right]t^n = 0$

$\Rightarrow 4n^2 - 4n + 4n - 1 = 0$

$\Rightarrow (2n-1)(2n+1) = 0$

$n = \frac{1}{2}, -\frac{1}{2}$

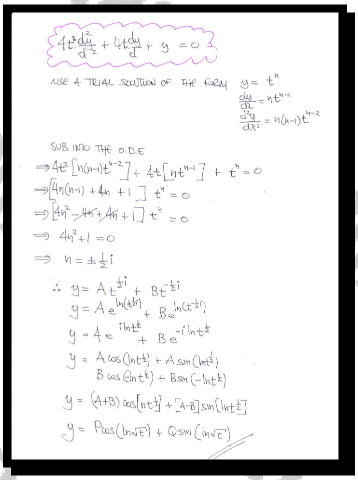
$\therefore x = At^{\frac{1}{2}} + Bt^{-\frac{1}{2}}$

Question 2 **(**+)**

Find the general solution of the following differential equation.

$$4t^2 \frac{d^2y}{dt^2} + 4t\frac{dy}{dt} + y = 0 .$$

$$\boxed{y = P\cos\left[\ln\sqrt{t}\right] + P\sin\left[\ln\sqrt{t}\right]}$$



$4t^2\frac{d^2y}{dt^2} + 4t\frac{dy}{dt} + y = 0$

USE A TRIAL SOLUTION OF THE FORM $y = t^n$

$\frac{dy}{dx} = nt^{n-1}$

$\frac{d^2y}{dx^2} = n(n-1)t^{n-2}$

SUB INTO THE O.D.E

$\Rightarrow 4t^2\left[n(n-1)t^{n-2}\right] + 4t\left[nt^{n-1}\right] + t^n = 0$

$\Rightarrow \left[4n(n-1) + 4n + 1\right]t^n = 0$

$\Rightarrow \left[4n^2 - 4n + 4n + 1\right]t^n = 0$

$\Rightarrow 4n^2 + 1 = 0$

$\Rightarrow n = \pm\frac{1}{2}i$

$\therefore y = At^{\frac{1}{2}i} + Bt^{-\frac{1}{2}i}$

$y = Ae^{\ln(t^{\frac{1}{2}i})} + Be^{\ln(t^{-\frac{1}{2}i})}$

$y = Ae^{i\ln t^{\frac{1}{2}}} + Be^{-i\ln t^{\frac{1}{2}}}$

$y = A\cos(\ln t^{\frac{1}{2}}) + A\sin(\ln t^{\frac{1}{2}})$
$B\cos(-\ln t^{\frac{1}{2}}) + B\sin(-\ln t^{\frac{1}{2}})$

$y = (A+B)\cos[\ln t^{\frac{1}{2}}] + [A-B]\sin[\ln t^{\frac{1}{2}}]$

$y = P\cos(\ln\sqrt{t}) + Q\sin(\ln\sqrt{t})$

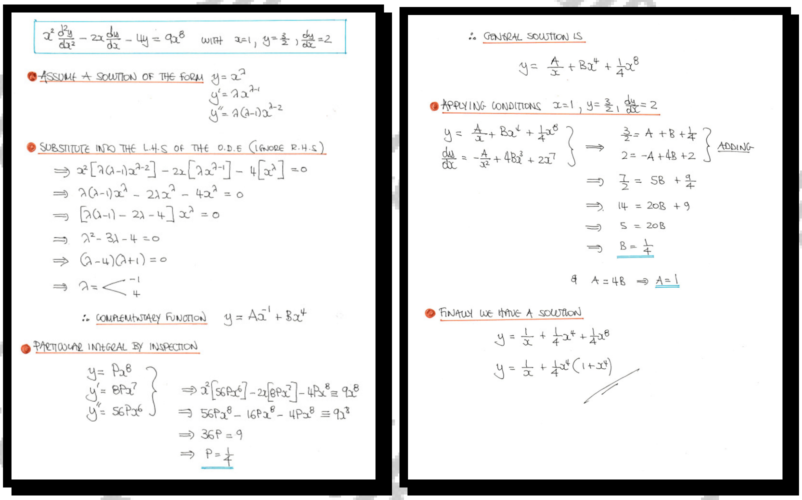
Question 3 (***)

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 9x^8.$$

Determine the solution of the above differential equation subject to the boundary conditions

$$y = \frac{3}{2},\ \frac{dy}{dx} = 2 \text{ at } x = 1.$$

$$y = \frac{1}{4}x^4\left(x^4+1\right) + \frac{1}{x}$$



$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 9x^8 \quad \text{WITH } x=1,\ y=\frac{3}{2},\ \frac{dy}{dx}=2$$

- ASSUME A SOLUTION OF THE FORM $y = x^\lambda$, $y' = \lambda x^{\lambda-1}$, $y'' = \lambda(\lambda-1)x^{\lambda-2}$
- SUBSTITUTE INTO THE L.H.S OF THE O.D.E (IGNORE R.H.S)

$\Rightarrow x^2\left[\lambda(\lambda-1)x^{\lambda-2}\right] - 2x\left[\lambda x^{\lambda-1}\right] - 4\left[x^\lambda\right] = 0$

$\Rightarrow \lambda(\lambda-1)x^\lambda - 2\lambda x^\lambda - 4x^\lambda = 0$

$\Rightarrow \left[\lambda(\lambda-1) - 2\lambda - 4\right]x^\lambda = 0$

$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow (\lambda-4)(\lambda+1) = 0$

$\Rightarrow \lambda = -1, 4$

$\therefore$ COMPLEMENTARY FUNCTION $y = Ax^{-1} + Bx^4$

- PARTICULAR INTEGRAL BY INSPECTION

$y = Px^8$, $y' = 8Px^7$, $y'' = 56Px^6$

$\Rightarrow x^2\left[56Px^6\right] - 2x\left[8Px^7\right] - 4Px^8 \equiv 9x^8$

$\Rightarrow 56Px^8 - 16Px^8 - 4Px^8 \equiv 9x^8$

$\Rightarrow 36P = 9$

$\Rightarrow P = \frac{1}{4}$

$\therefore$ GENERAL SOLUTION IS

$$y = \frac{A}{x} + Bx^4 + \frac{1}{4}x^8$$

- APPLYING CONDITIONS $x=1,\ y=\frac{3}{2},\ \frac{dy}{dx}=2$

$y = \frac{A}{x} + Bx^4 + \frac{1}{4}x^8$, $\frac{dy}{dx} = -\frac{A}{x^2} + 4Bx^3 + 2x^7$ $\Longrightarrow$ $\frac{3}{2} = A + B + \frac{1}{4}$, $2 = -A + 4B + 2$ ADDING

$\Rightarrow \frac{7}{2} = 5B + \frac{9}{4}$

$\Rightarrow 14 = 20B + 9$

$\Rightarrow 5 = 20B$

$\Rightarrow B = \frac{1}{4}$

& $A = 4B \Rightarrow A = 1$

- FINALLY WE HAVE A SOLUTION

$y = \frac{1}{x} + \frac{1}{4}x^4 + \frac{1}{4}x^8$

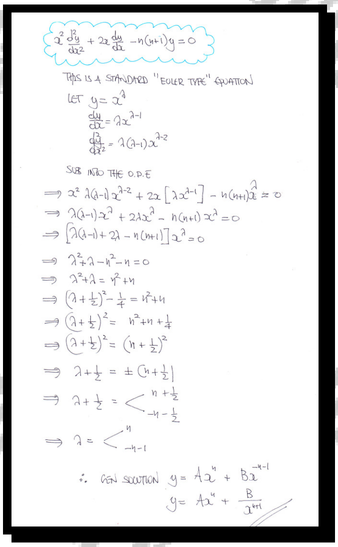
$y = \frac{1}{x} + \frac{1}{4}x^4(1+x^4)$

Question 4 (***)

Find the general solution of the following differential equation

$$x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)\,y = 0\,.$$

$$\boxed{y = Ax^n + \frac{B}{x^{n+1}}}$$



Question 5 (***+)

Given that if $x = e^{t}$ and $y = f(x)$, show clearly that …

a) … $x\dfrac{dy}{dx} = \dfrac{dy}{dt}$.

b) … $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$.

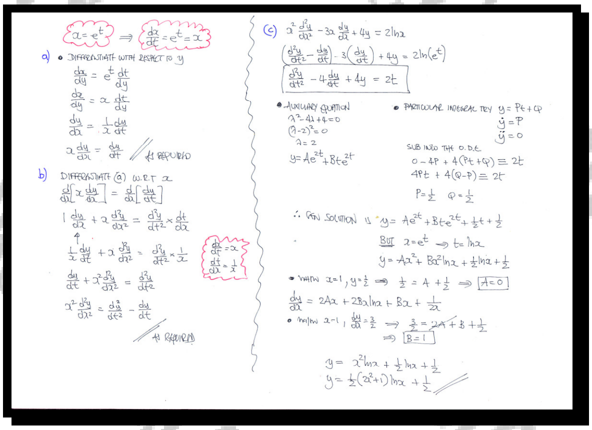
The following differential equation is to be solved

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2\ln x$$

subject to the boundary conditions $y = \dfrac{1}{2}, \dfrac{dy}{dx} = \dfrac{3}{2}$ at $x = 1$.

c) Use the substitution $x = e^{t}$ to solve the above differential equation.

$$\boxed{y = \frac{1}{2} + \frac{1}{2}\left(2x^2 + 1\right)\ln x}$$

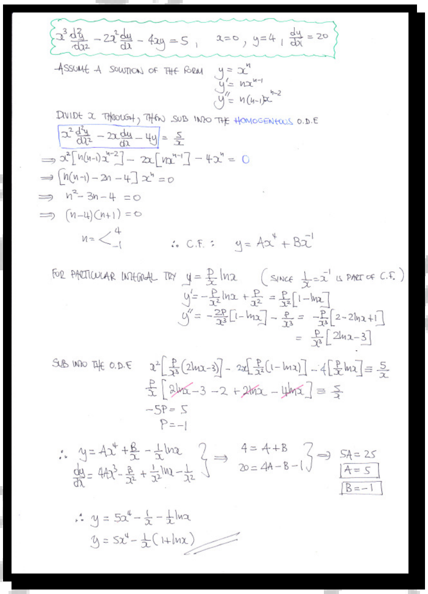


Question 6 (***+)

$$x^3\frac{d^2y}{dx^2}-2x^2\frac{dy}{dx}-4xy=5.$$

Find the solution of the above differential equation subject to the boundary conditions $y=4$, $\frac{dy}{dx}=20$ at $x=0$.

$$\boxed{y=5x^4-\frac{1}{x}(1+\ln x)}$$

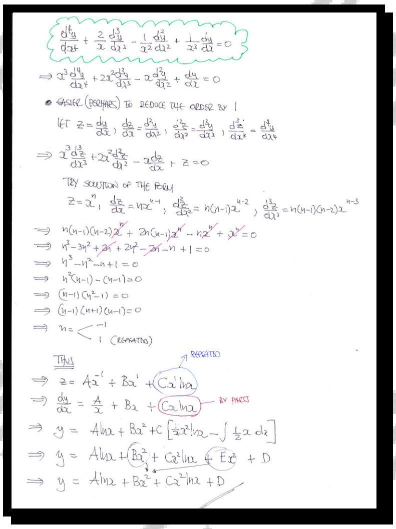


Question 7 (***+)

Find the general solution of the following differential equation.

$$\frac{d^4y}{dx^4}+\frac{2}{x}\frac{d^3y}{dx^3}-\frac{1}{x^2}\frac{d^2y}{dx^2}+\frac{1}{x^3}\frac{dy}{dx}=0,\ x>0.$$

$\boxed{}$, $\boxed{y=A\ln x+Bx^2+Cx^2\ln x+D}$



$$\frac{d^4y}{dx^4}+\frac{2}{x}\frac{d^3y}{dx^3}-\frac{1}{x^2}\frac{d^2y}{dx^2}+\frac{1}{x^3}\frac{dy}{dx}=0$$

$\Rightarrow x^3\frac{d^4y}{dx^4}+2x^2\frac{d^3y}{dx^3}-x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$

• EASIER (PERHAPS) TO REDUCE THE ORDER BY 1

LET $z=\frac{dy}{dx}$, $\frac{dz}{dx}=\frac{d^2y}{dx^2}$, $\frac{d^2z}{dx^2}=\frac{d^3y}{dx^3}$, $\frac{d^3z}{dx^3}=\frac{d^4y}{dx^4}$

$\Rightarrow x^3\frac{d^3z}{dx^3}+2x^2\frac{d^2z}{dx^2}-x\frac{dz}{dx}+z=0$

TRY SOLUTION OF THE FORM

$z=x^n$, $\frac{dz}{dx}=nx^{n-1}$, $\frac{d^2z}{dx^2}=n(n-1)x^{n-2}$, $\frac{d^3z}{dx^3}=n(n-1)(n-2)x^{n-3}$

$\Rightarrow n(n-1)(n-2)x^n+2n(n-1)x^n-nx^n+x^n=0$

$\Rightarrow n^3-3n^2+2n+2n^2-2n-n+1=0$

$\Rightarrow n^3-n^2-n+1=0$

$\Rightarrow n^2(n-1)-(n-1)=0$

$\Rightarrow (n-1)(n^2-1)=0$

$\Rightarrow (n-1)(n+1)(n-1)=0$

$\Rightarrow n=-1,\ 1$ (REPEATED)

THUS

$\Rightarrow z=Ax^{-1}+Bx^1+Cx^1\ln x$ (REPEATED)

$\Rightarrow \frac{dy}{dx}=\frac{A}{x}+Bx+Cx\ln x$ (BY PARTS)

$\Rightarrow y=A\ln x+Bx^2+C\left[\frac{1}{2}x^2\ln x-\int\frac{1}{2}x\,dx\right]$

$\Rightarrow y=A\ln x+Bx^2+Cx^2\ln x+Ex^2+D$

$\Rightarrow y=A\ln x+Bx^2+Cx^2\ln x+D$

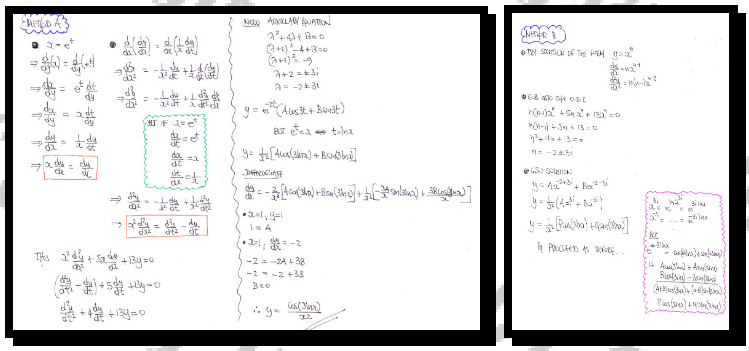
Question 8 (***+)

The curve with equation $y = f(x)$ satisfies

$$x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 13y = 0, \quad x > 0.$$

By using the substitution $x = e^t$, or otherwise, determine an equation for $y = f(x)$, given further that $y = 1$ and $\frac{dy}{dx} = -2$ at $x = 1$.

$$\boxed{y = \frac{\cos(3\ln x)}{x^2}}$$

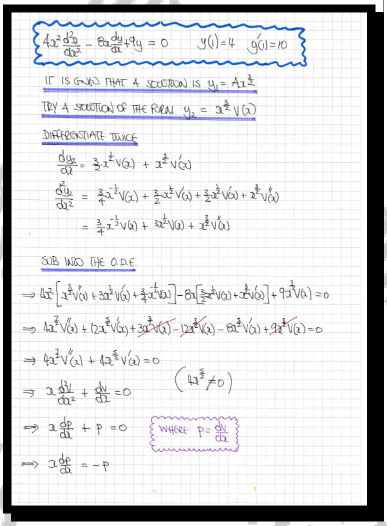


Question 9 (***+)

$$x^2\frac{d^2y}{dx^2}-8x\frac{dy}{dx}+9y=0,\ x>0.$$

Use the fact that $y=Ax^{\frac{3}{2}}$ satisfies the above differential equation, to find the full solution subject to $y=4$ and $\frac{dy}{dx}=10$ at $x=1$.

$y=4x^{\frac{3}{2}}(1+\ln x)$



$4x^2\frac{d^2y}{dx^2}-8x\frac{dy}{dx}+9y=0 \qquad y(1)=4 \quad y'(1)=10$

IT IS GIVEN THAT A SOLUTION IS $y_1=Ax^{\frac{3}{2}}$

TRY A SOLUTION OF THE FORM $y_2=x^{\frac{3}{2}}v(x)$

DIFFERENTIATE TWICE

$\frac{dy_2}{dx}=\frac{3}{2}x^{\frac{1}{2}}v(x)+x^{\frac{3}{2}}v'(x)$

$\frac{d^2y_2}{dx^2}=\frac{3}{4}x^{-\frac{1}{2}}v(x)+\frac{3}{2}x^{\frac{1}{2}}v'(x)+\frac{3}{2}x^{\frac{1}{2}}v'(x)+x^{\frac{3}{2}}v''(x)$

$=\frac{3}{4}x^{-\frac{1}{2}}v(x)+3x^{\frac{1}{2}}v'(x)+x^{\frac{3}{2}}v''(x)$

SUB INTO THE O.D.E.

$\Rightarrow 4x^2\left[x^{\frac{3}{2}}v''(x)+3x^{\frac{1}{2}}v'(x)+\frac{3}{4}x^{-\frac{1}{2}}v(x)\right]-8x\left[\frac{3}{2}x^{\frac{1}{2}}v(x)+x^{\frac{3}{2}}v'(x)\right]+9x^{\frac{3}{2}}v(x)=0$

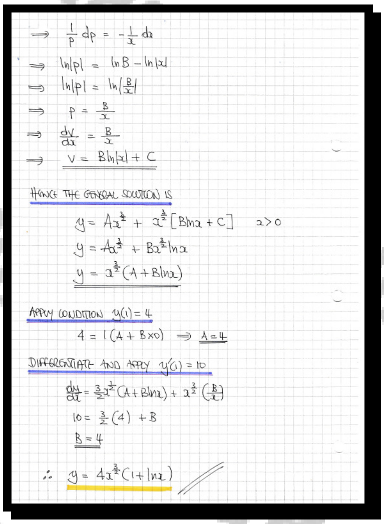
$\Rightarrow 4x^{\frac{7}{2}}v''(x)+12x^{\frac{5}{2}}v'(x)+3x^{\frac{3}{2}}v(x)-12x^{\frac{3}{2}}v(x)-8x^{\frac{5}{2}}v'(x)+9x^{\frac{3}{2}}v(x)=0$

$\Rightarrow 4x^{\frac{7}{2}}v''(x)+4x^{\frac{5}{2}}v'(x)=0 \qquad (4x^{\frac{5}{2}}\neq 0)$

$\Rightarrow x\frac{d^2v}{dx^2}+\frac{dv}{dx}=0$

$\Rightarrow x\frac{dp}{dx}+p=0 \qquad$ WHERE $p=\frac{dv}{dx}$

$\Rightarrow x\frac{dp}{dx}=-p$



$\Rightarrow \frac{1}{p}\,dp=-\frac{1}{x}\,dx$

$\Rightarrow \ln|p|=\ln B-\ln|x|$

$\Rightarrow \ln|p|=\ln\left|\frac{B}{x}\right|$

$\Rightarrow p=\frac{B}{x}$

$\Rightarrow \frac{dv}{dx}=\frac{B}{x}$

$\Rightarrow v=B\ln|x|+C$

HENCE THE GENERAL SOLUTION IS

$y=Ax^{\frac{3}{2}}+x^{\frac{3}{2}}[B\ln x+C] \qquad x>0$

$y=Ax^{\frac{3}{2}}+Bx^{\frac{3}{2}}\ln x$

$y=x^{\frac{3}{2}}(A+B\ln x)$

APPLY CONDITION $y(1)=4$

$4=1(A+B\times 0) \Rightarrow A=4$

DIFFERENTIATE AND APPLY $y'(1)=10$

$\frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}(A+B\ln x)+x^{\frac{3}{2}}\left(\frac{B}{x}\right)$

$10=\frac{3}{2}(4)+B$

$B=4$

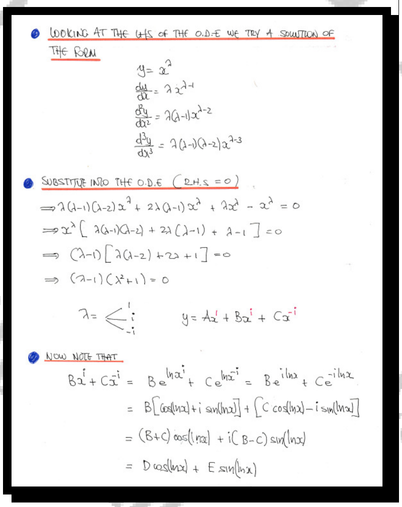
$\therefore\ y=4x^{\frac{3}{2}}(1+\ln x)$

Question 10 (****)

Find the general solution of the following differential equation.

$$x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=2x\,,\quad x>0\,.$$

$\boxed{}$, $\boxed{y=Ax+B\cos(\ln x)+C\sin(\ln x)+x\ln x}$



LOOKING AT THE LHS OF THE O.D.E WE TRY A SOLUTION OF THE FORM

$y=x^{\lambda}$

$\frac{dy}{dx}=\lambda x^{\lambda-1}$

$\frac{d^2y}{dx^2}=\lambda(\lambda-1)x^{\lambda-2}$

$\frac{d^3y}{dx^3}=\lambda(\lambda-1)(\lambda-2)x^{\lambda-3}$

SUBSTITUTE INTO THE O.D.E (R.H.S = 0)

$\Rightarrow \lambda(\lambda-1)(\lambda-2)x^{\lambda}+2\lambda(\lambda-1)x^{\lambda}+\lambda x^{\lambda}-x^{\lambda}=0$

$\Rightarrow x^{\lambda}\left[\lambda(\lambda-1)(\lambda-2)+2\lambda(\lambda-1)+\lambda-1\right]=0$

$\Rightarrow (\lambda-1)\left[\lambda(\lambda-2)+2\lambda+1\right]=0$

$\Rightarrow (\lambda-1)(\lambda^2+1)=0$

$\lambda = 1,\ i,\ -i \qquad y=Ax^{1}+Bx^{i}+Cx^{-i}$

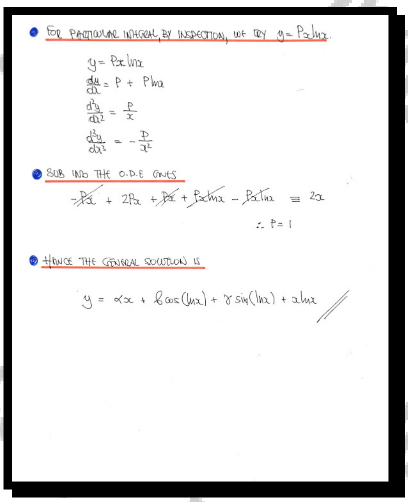
NOW NOTE THAT

$Bx^{i}+Cx^{-i}=Be^{\ln x^{i}}+Ce^{\ln x^{-i}}=Be^{i\ln x}+Ce^{-i\ln x}$

$=B\left[\cos(\ln x)+i\sin(\ln x)\right]+\left[C\cos(\ln x)-i\sin(\ln x)\right]$

$=(B+C)\cos(\ln x)+i(B-C)\sin(\ln x)$

$=D\cos(\ln x)+E\sin(\ln x)$



FOR PARTICULAR INTEGRAL, BY INSPECTION, WE TRY $y=Px\ln x$.

$y=Px\ln x$

$\frac{dy}{dx}=P+P\ln x$

$\frac{d^2y}{dx^2}=\frac{P}{x}$

$\frac{d^3y}{dx^3}=-\frac{P}{x^2}$

SUB INTO THE O.D.E GIVES

$-Px+2Px+Px+Px\ln x-Px\ln x\equiv 2x$

$\therefore P=1$

HENCE THE GENERAL SOLUTION IS

$y=\alpha x+\beta\cos(\ln x)+\gamma\sin(\ln x)+x\ln x$

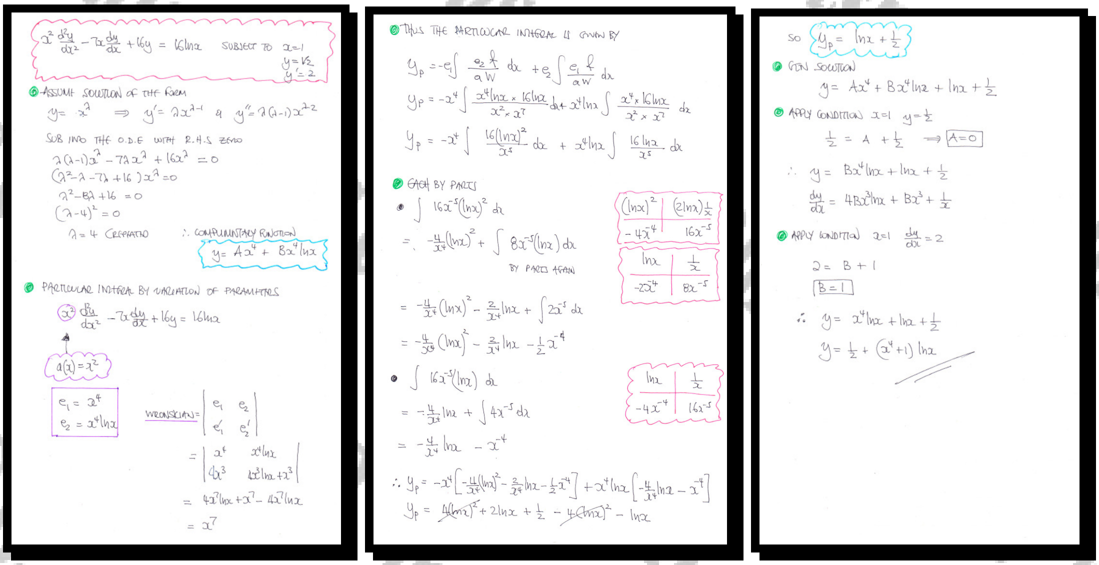
Question 11 (****)

Use variation of parameters to determine the specific solution of the following differential equation

$$x^2 \frac{d^2 y}{dx^2} - 7x\frac{dy}{dx} + 16y = 16\ln x ,$$

given further that $y = \frac{1}{2}$, $\frac{dy}{dx} = 2$ at $x = 1$.

$$\boxed{y = \frac{1}{2} + \left(1 + x^4\right)\ln x}$$



2^{ND} ORDER ODEs WITH MISSING INDEPENDENT VARIABLE

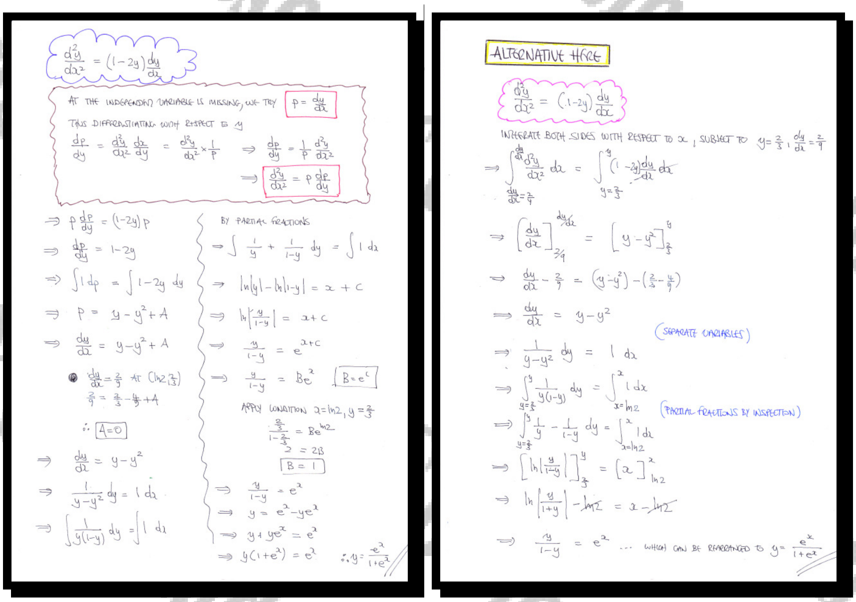
Question 1 (****+)

The curve C, has gradient $\frac{2}{9}$ at the point with coordinates $\left(\ln 2, \frac{2}{3}\right)$, and satisfies the differential relationship

$$\frac{d^2y}{dx^2} = (1-2y)\frac{dy}{dx}, \quad y < \frac{1}{2}.$$

Find an equation for C, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{e^x}{1+e^x} = \frac{1}{e^x + e^{-x}} = \frac{1}{2}\operatorname{sech} x}$$

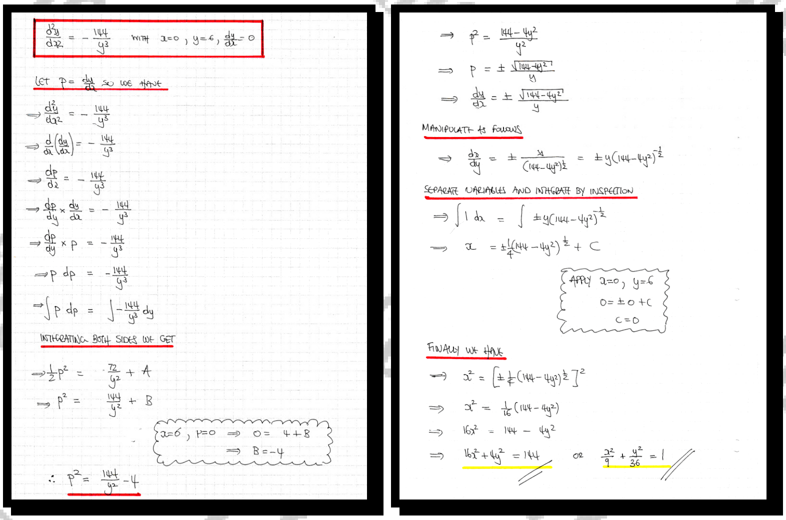


Question 2 (****)

Use appropriate techniques to solve the following differential equation.

$$\frac{d^2y}{dx^2} = -\frac{144}{y^3}, \qquad y(0) = 6, \qquad \left.\frac{dy}{dx}\right|_{x=0} = 0.$$

$\boxed{}$, $\boxed{\frac{x^2}{9} + \frac{y^2}{36} = 1}$



$\boxed{\frac{d^2y}{dx^2} = -\frac{144}{y^3} \quad \text{WITH } x=0,\ y=6,\ \frac{dy}{dx}=0}$

LET $p = \frac{dy}{dx}$ SO WE HAVE

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{144}{y^3}$

$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{144}{y^3}$

$\Rightarrow \frac{dp}{dx} = -\frac{144}{y^3}$

$\Rightarrow \frac{dp}{dy} \times \frac{dy}{dx} = -\frac{144}{y^3}$

$\Rightarrow \frac{dp}{dy} \times p = -\frac{144}{y^3}$

$\Rightarrow p\,dp = -\frac{144}{y^3}$

$\Rightarrow \int p\,dp = \int -\frac{144}{y^3}\,dy$

INTEGRATING BOTH SIDES WE GET

$\Rightarrow \frac{1}{2}p^2 = \frac{72}{y^2} + A$

$\Rightarrow p^2 = \frac{144}{y^2} + B$

$x=6,\ p=0 \Rightarrow 0 = 4 + B \Rightarrow B = -4$

$\therefore p^2 = \frac{144}{y^2} - 4$

$\Rightarrow p^2 = \frac{144-4y^2}{y^2}$

$\Rightarrow p = \pm\frac{\sqrt{144-4y^2}}{y}$

$\Rightarrow \frac{dy}{dx} = \pm\frac{\sqrt{144-4y^2}}{y}$

MANIPULATE AS FOLLOWS

$\Rightarrow \frac{dx}{dy} = \pm\frac{y}{(144-4y^2)^{\frac{1}{2}}} = \pm y(144-4y^2)^{-\frac{1}{2}}$

SEPARATE VARIABLES AND INTEGRATE BY INSPECTION

$\Rightarrow \int 1\,dx = \int \pm y(144-4y^2)^{-\frac{1}{2}}$

$\Rightarrow x = \pm\frac{1}{4}(144-4y^2)^{\frac{1}{2}} + C$

APPLY $x=0,\ y=6$: $0 = \pm 0 + C$, $C = 0$

FINALLY WE HAVE

$\Rightarrow x^2 = \left[\pm\frac{1}{4}(144-4y^2)^{\frac{1}{2}}\right]^2$

$\Rightarrow x^2 = \frac{1}{16}(144-4y^2)$

$\Rightarrow 16x^2 = 144 - 4y^2$

$\Rightarrow 16x^2 + 4y^2 = 144$ OR $\frac{x^2}{9} + \frac{y^2}{36} = 1$

Question 3 (****+)

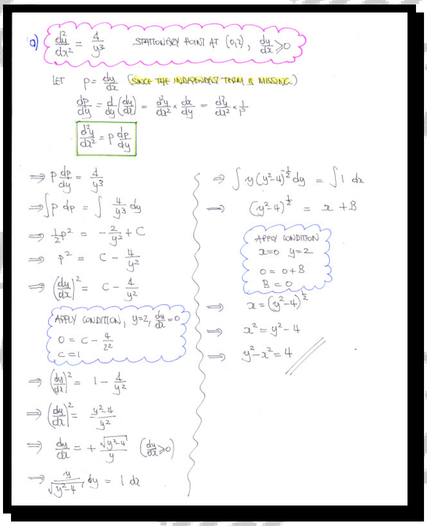
The curve C, has a stationary point at $(0,2)$ and satisfies the differential relationship

$$\frac{d^2y}{dx^2} = \frac{4}{y^3}, \quad y \neq 0.$$

a) Given further that $\frac{dy}{dx} \geq 0$ along C, determine a simplified expression for the Cartesian equation of C.

b) Verify by differentiation the answer to part **(a)**.

$$\boxed{y^2 - x^2 = 4}$$



a) $\frac{d^2y}{dx^2} = \frac{4}{y^3}$ STATIONARY POINT AT $(0,2)$, $\frac{dy}{dx} \geq 0$

LET $p = \frac{dy}{dx}$ (SINCE THE INDEPENDENT TERM IS MISSING)

$\frac{dp}{dy} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \times \frac{dx}{dy} = \frac{d^2y}{dx^2} \times \frac{1}{p}$

$\frac{d^2y}{dx^2} = p\frac{dp}{dy}$

$\Rightarrow p\frac{dp}{dy} = \frac{4}{y^3}$

$\Rightarrow \int p\, dp = \int \frac{4}{y^3}\, dy$

$\Rightarrow \frac{1}{2}p^2 = -\frac{2}{y^2} + C$

$\Rightarrow p^2 = C - \frac{4}{y^2}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = C - \frac{4}{y^2}$

APPLY CONDITION, $y=2$, $\frac{dy}{dx} = 0$

$0 = C - \frac{4}{2^2}$

$C = 1$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 1 - \frac{4}{y^2}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{y^2-4}{y^2}$

$\Rightarrow \frac{dy}{dx} = +\frac{\sqrt{y^2-4}}{y}$ $\left(\frac{dy}{dx} \geq 0\right)$

$\Rightarrow \frac{y}{\sqrt{y^2-4}}\, dy = 1\, dx$

$\Rightarrow \int y(y^2-4)^{-\frac{1}{2}}\, dy = \int 1\, dx$

$\Rightarrow (y^2-4)^{\frac{1}{2}} = x + B$

APPLY CONDITION $x=0$ $y=2$

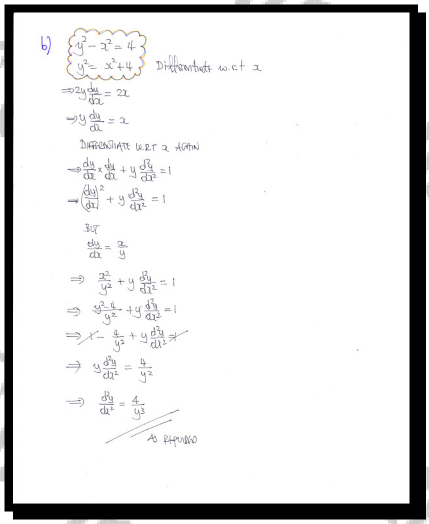
$0 = 0 + B$

$B = 0$

$\Rightarrow x = (y^2-4)^{\frac{1}{2}}$

$\Rightarrow x^2 = y^2 - 4$

$\Rightarrow y^2 - x^2 = 4$



b) $y^2 - x^2 = 4$

$y^2 = x^2 + 4$ Differentiate w.r.t x

$\Rightarrow 2y\frac{dy}{dx} = 2x$

$\Rightarrow y\frac{dy}{dx} = x$

DIFFERENTIATE W.R.T x AGAIN

$\Rightarrow \frac{dy}{dx} \times \frac{dy}{dx} + y\frac{d^2y}{dx^2} = 1$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 1$

BUT $\frac{dy}{dx} = \frac{x}{y}$

$\Rightarrow \frac{x^2}{y^2} + y\frac{d^2y}{dx^2} = 1$

$\Rightarrow \frac{y^2-4}{y^2} + y\frac{d^2y}{dx^2} = 1$

$\Rightarrow 1 - \frac{4}{y^2} + y\frac{d^2y}{dx^2} = 1$

$\Rightarrow y\frac{d^2y}{dx^2} = \frac{4}{y^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{4}{y^3}$

AS REQUIRED

Question 4 (****+)

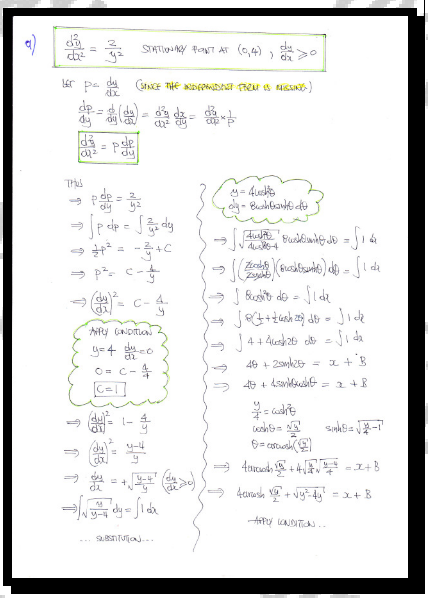
The curve C, has a stationary point at $(0,4)$ and satisfies the differential equation

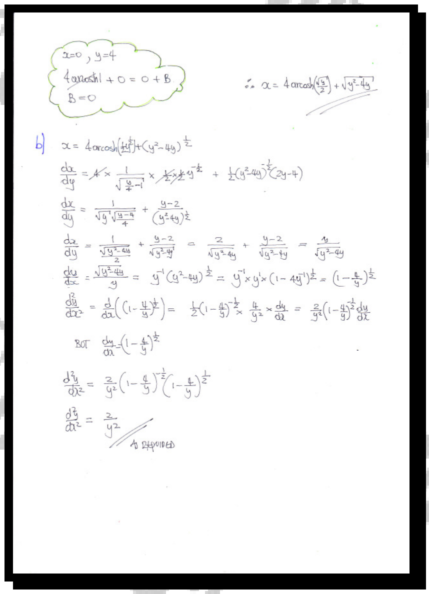
$$\frac{d^2y}{dx^2} = \frac{2}{y^2}, \quad y \neq 0.$$

a) Given further that $\frac{dy}{dx} \geq 0$ along C, determine a simplified expression for the Cartesian equation of C, giving the answer in the form $x = f(y)$.

b) Verify by differentiation the answer to part (**a**).

$$x = 4\,\text{arcosh}\left(\frac{1}{2}\sqrt{y}\right) + \sqrt{y^2 - 4y}$$





Question 5 (****+)

The curve C with Cartesian equation $f(x,y)=0$, satisfies the differential equation

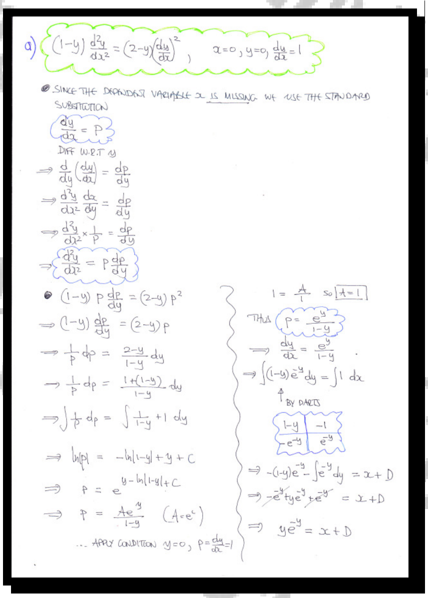
$$(1-y)\,y''=(2-y)(y')^2 .$$

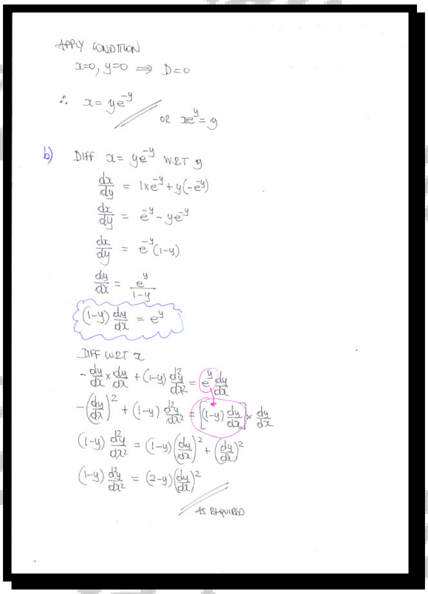
It is further given that $y(0)=0$ and $y'(0)=1$

a) Determine a simplified expression for the Cartesian equation of C.

b) Verify by differentiation the answer to part (**a**).

$$\boxed{x = y\,\mathrm{e}^{-y}}$$





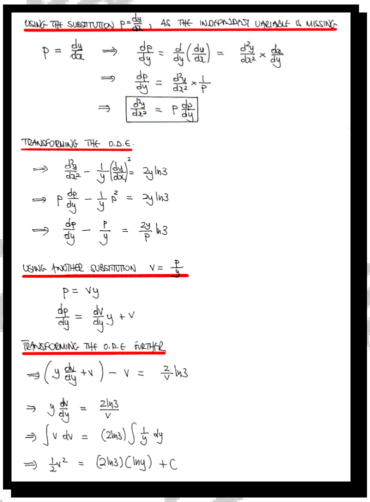
Question 6 (*****)

The function with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 = 2y\ln 3, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 2\ln 3.$$

Solve the above differential equation to show that $y = 3^{x^2+2x}$.

[ ] , [proof]



USING THE SUBSTITUTION $p = \frac{dy}{dx}$, AS THE INDEPENDENT VARIABLE IS MISSING

$p = \frac{dy}{dx} \Rightarrow \frac{dp}{dy} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \times \frac{dx}{dy}$

$\Rightarrow \frac{dp}{dy} = \frac{d^2y}{dx^2} \times \frac{1}{p}$

$\Rightarrow \frac{d^2y}{dx^2} = p\frac{dp}{dy}$

TRANSFORMING THE O.D.E.

$\Rightarrow \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 = 2y\ln 3$

$\Rightarrow p\frac{dp}{dy} - \frac{1}{y}p^2 = 2y\ln 3$

$\Rightarrow \frac{dp}{dy} - \frac{p}{y} = \frac{2y}{p}\ln 3$

USING ANOTHER SUBSTITUTION $v = \frac{p}{y}$

$p = vy$

$\frac{dp}{dy} = \frac{dv}{dy}y + v$

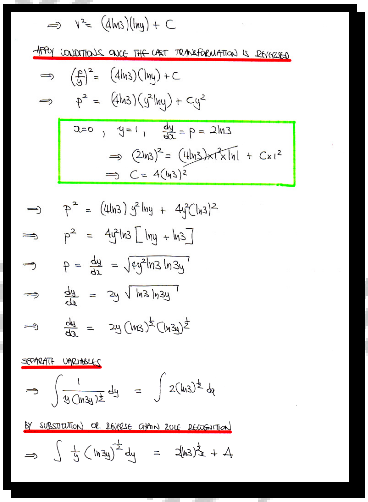
TRANSFORMING THE O.D.E FURTHER

$\Rightarrow \left(y\frac{dv}{dy} + v\right) - v = \frac{2}{v}\ln 3$

$\Rightarrow y\frac{dv}{dy} = \frac{2\ln 3}{v}$

$\Rightarrow \int v\,dv = (2\ln 3)\int \frac{1}{y}\,dy$

$\Rightarrow \frac{1}{2}v^2 = (2\ln 3)(\ln y) + C$



$\Rightarrow v^2 = (4\ln 3)(\ln y) + C$

APPLY CONDITIONS ONCE THE LAST TRANSFORMATION IS REVERSED

$\Rightarrow \left(\frac{p}{y}\right)^2 = (4\ln 3)(\ln y) + C$

$\Rightarrow p^2 = (4\ln 3)(y^2\ln y) + Cy^2$

$x = 0,\ y = 1,\ \frac{dy}{dx} = p = 2\ln 3$

$\Rightarrow (2\ln 3)^2 = (4\ln 3)\times 1^2 \times \ln 1 + C \times 1^2$

$\Rightarrow C = 4(\ln 3)^2$

$\Rightarrow p^2 = (4\ln 3)y^2\ln y + 4y^2(\ln 3)^2$

$\Rightarrow p^2 = 4y^2\ln 3\left[\ln y + \ln 3\right]$

$\Rightarrow p = \frac{dy}{dx} = \sqrt{4y^2\ln 3\ln 3y}$

$\Rightarrow \frac{dy}{dx} = 2y\sqrt{\ln 3\ln 3y}$

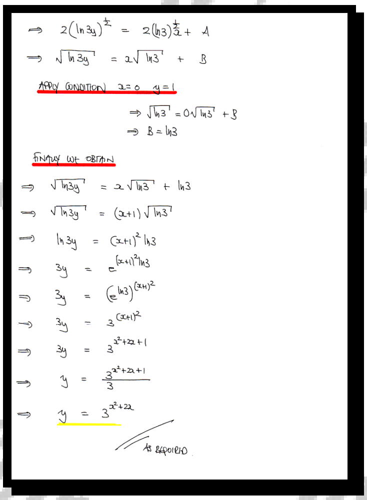
$\Rightarrow \frac{dy}{dx} = 2y(\ln 3)^{\frac{1}{2}}(\ln 3y)^{\frac{1}{2}}$

SEPARATE VARIABLES

$\Rightarrow \int \frac{1}{y(\ln 3y)^{\frac{1}{2}}}\,dy = \int 2(\ln 3)^{\frac{1}{2}}\,dx$

BY SUBSTITUTION OR REVERSE CHAIN RULE RECOGNITION

$\Rightarrow \int \frac{1}{y}(\ln 3y)^{-\frac{1}{2}}\,dy = 2(\ln 3)^{\frac{1}{2}}x + A$



$\Rightarrow 2(\ln 3y)^{\frac{1}{2}} = 2(\ln 3)^{\frac{1}{2}}x + A$

$\Rightarrow \sqrt{\ln 3y} = x\sqrt{\ln 3} + B$

APPLY CONDITION $x = 0$ $y = 1$

$\Rightarrow \sqrt{\ln 3} = 0\sqrt{\ln 3} + B$

$\Rightarrow B = \ln 3$

FINALLY WE OBTAIN

$\Rightarrow \sqrt{\ln 3y} = x\sqrt{\ln 3} + \ln 3$

$\Rightarrow \sqrt{\ln 3y} = (x+1)\sqrt{\ln 3}$

$\Rightarrow \ln 3y = (x+1)^2\ln 3$

$\Rightarrow 3y = e^{(x+1)^2\ln 3}$

$\Rightarrow 3y = \left(e^{\ln 3}\right)^{(x+1)^2}$

$\Rightarrow 3y = 3^{(x+1)^2}$

$\Rightarrow 3y = 3^{x^2+2x+1}$

$\Rightarrow y = \frac{3^{x^2+2x+1}}{3}$

$\Rightarrow y = 3^{x^2+2x}$

AS REQUIRED

Question 7 (*****)

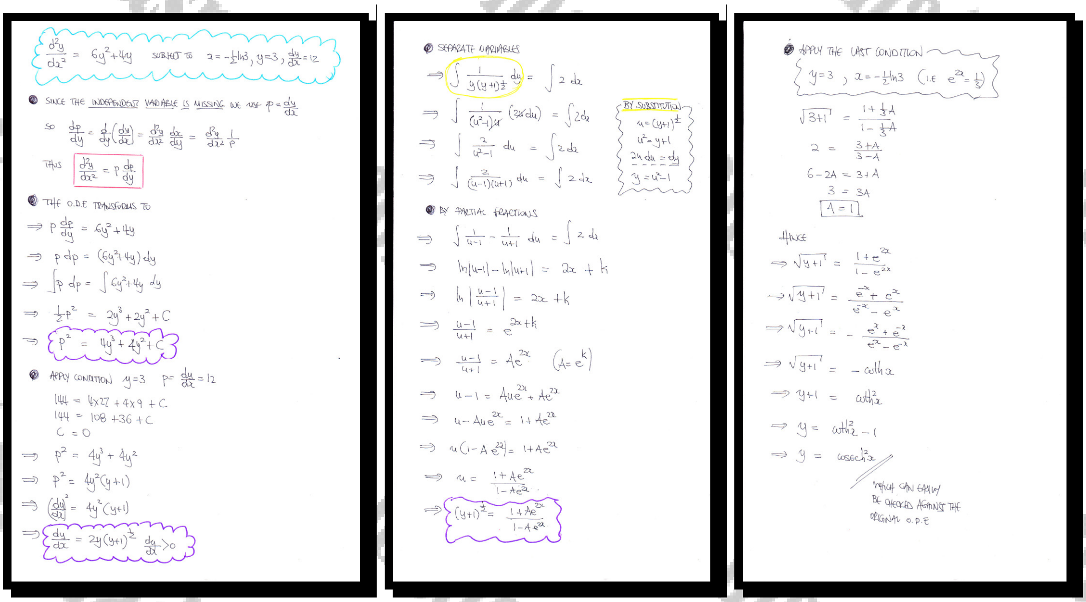
The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} = 6y^2 + 4y, \quad \frac{dy}{dx} \geq 0.$$

If $y = 3$, $\frac{dy}{dx} = 12$ at $x = -\frac{1}{2}\ln 3$, solve the differential equation to show that

$$y = \operatorname{cosech}^2 x.$$

proof



$\frac{d^2y}{dx^2} = 6y^2+4y$ SUBJECT TO $x=-\frac{1}{2}\ln 3$, $y=3$, $\frac{dy}{dx}=12$

• SINCE THE INDEPENDENT VARIABLE IS MISSING WE USE $p=\frac{dy}{dx}$

so $\frac{dp}{dy} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}\frac{dx}{dy} = \frac{d^2y}{dx^2}\frac{1}{p}$

THUS $\frac{d^2y}{dx^2} = p\frac{dp}{dy}$

• THE O.D.E TRANSFORMS TO

$\Rightarrow p\frac{dp}{dy} = 6y^2+4y$

$\Rightarrow p\,dp = (6y^2+4y)\,dy$

$\Rightarrow \int p\,dp = \int 6y^2+4y\,dy$

$\Rightarrow \frac{1}{2}p^2 = 2y^3+2y^2+C$

$\Rightarrow p^2 = 4y^3+4y^2+C$

• APPLY CONDITION $y=3$ $p=\frac{dy}{dx}=12$

$144 = 4\times27+4\times9+C$

$144 = 108+36+C$

$C=0$

$\Rightarrow p^2 = 4y^3+4y^2$

$\Rightarrow p^2 = 4y^2(y+1)$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 4y^2(y+1)$

$\Rightarrow \frac{dy}{dx} = 2y(y+1)^{\frac{1}{2}}$ $\frac{dy}{dx}>0$

• SEPARATE VARIABLES

$\Rightarrow \int \frac{1}{y(y+1)^{\frac{1}{2}}}\,dy = \int 2\,dx$

$\Rightarrow \int \frac{1}{(u^2-1)u}(2u\,du) = \int 2\,dx$

$\Rightarrow \int \frac{2}{u^2-1}\,du = \int 2\,dx$

$\Rightarrow \int \frac{2}{(u-1)(u+1)}\,du = \int 2\,dx$

BY SUBSTITUTION: $u=(y+1)^{\frac{1}{2}}$, $u^2=y+1$, $2u\,du=dy$, $y=u^2-1$

• BY PARTIAL FRACTIONS

$\Rightarrow \int \frac{1}{u-1}-\frac{1}{u+1}\,du = \int 2\,dx$

$\Rightarrow \ln|u-1|-\ln|u+1| = 2x+k$

$\Rightarrow \ln\left|\frac{u-1}{u+1}\right| = 2x+k$

$\Rightarrow \frac{u-1}{u+1} = e^{2x+k}$

$\Rightarrow \frac{u-1}{u+1} = Ae^{2x}$ $(A=e^k)$

$\Rightarrow u-1 = Aue^{2x}+Ae^{2x}$

$\Rightarrow u-Aue^{2x} = 1+Ae^{2x}$

$\Rightarrow u(1-Ae^{2x}) = 1+Ae^{2x}$

$\Rightarrow u = \frac{1+Ae^{2x}}{1-Ae^{2x}}$

$\Rightarrow (y+1)^{\frac{1}{2}} = \frac{1+Ae^{2x}}{1-Ae^{2x}}$

• APPLY THE LAST CONDITION

$y=3$, $x=-\frac{1}{2}\ln 3$ (i.e. $e^{2x}=\frac{1}{3}$)

$\sqrt{3+1} = \frac{1+\frac{1}{3}A}{1-\frac{1}{3}A}$

$2 = \frac{3+A}{3-A}$

$6-2A = 3+A$

$3 = 3A$

$A=1$

HENCE

$\Rightarrow \sqrt{y+1} = \frac{1+e^{2x}}{1-e^{2x}}$

$\Rightarrow \sqrt{y+1} = \frac{e^{-x}+e^{x}}{e^{-x}-e^{x}}$

$\Rightarrow \sqrt{y+1} = -\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

$\Rightarrow \sqrt{y+1} = -\coth x$

$\Rightarrow y+1 = \coth^2 x$

$\Rightarrow y = \coth^2 x - 1$

$\Rightarrow y = \operatorname{cosech}^2 x$

WHICH CAN EASILY BE CHECKED AGAINST THE ORIGINAL O.D.E

Question 8 (*****)

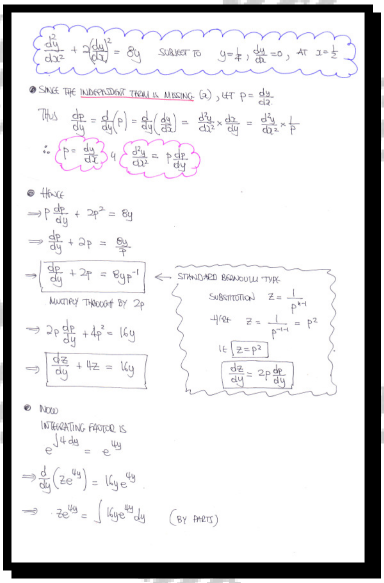
The curve with equation $y = f(x)$ satisfies the differential equation

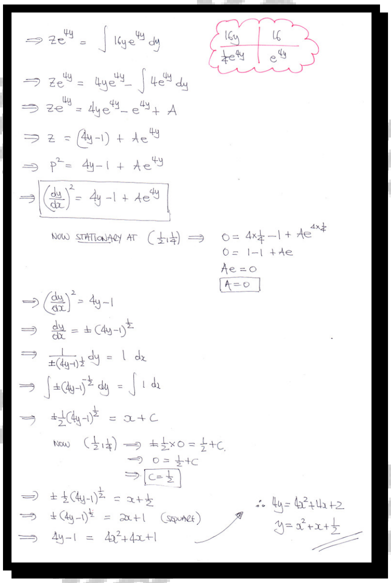
$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 8y .$$

Given further that the curve has a stationary point at $\left(\frac{1}{2}, \frac{1}{4}\right)$, solve the differential equation to show that

$$y = x^2 + x + \frac{1}{2} .$$

proof





Question 9 (*****)

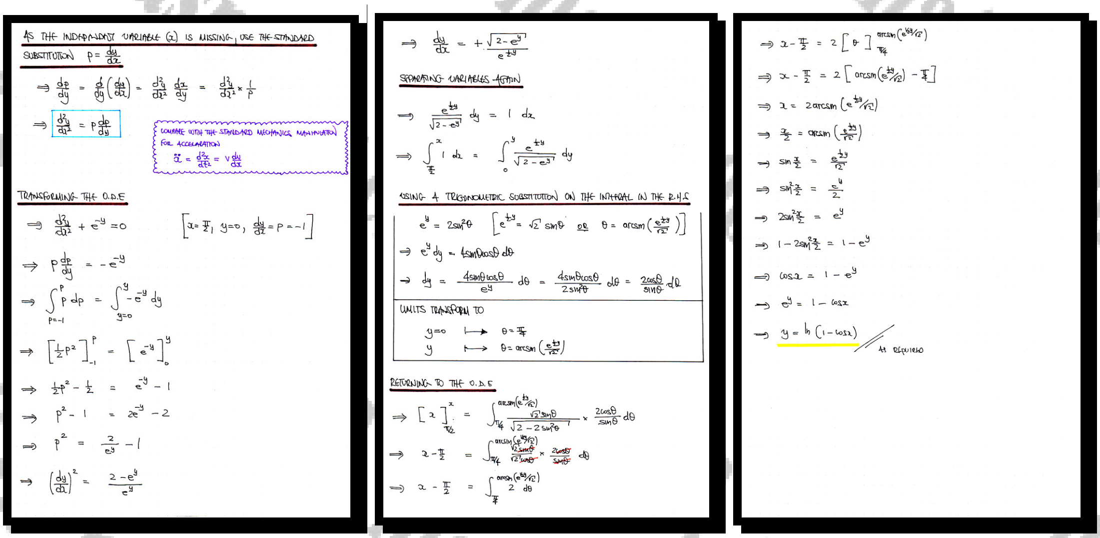
The curve with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + e^{-y} = 0, \quad \frac{dy}{dx} \geq 0.$$

If $y = 0$, $\frac{dy}{dx} = -1$ at $x = \frac{1}{2}\pi$, solve the differential equation to show that

$$y = \ln(1 - \cos x).$$

[ ], proof



AS THE INDEPENDENT VARIABLE (x) IS MISSING, USE THE STANDARD SUBSTITUTION $p = \frac{dy}{dx}$

$\Rightarrow \frac{dp}{dy} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}\frac{dx}{dy} = \frac{d^2y}{dx^2} \times \frac{1}{p}$

$\Rightarrow \frac{d^2y}{dx^2} = p\frac{dp}{dy}$

COMPARE WITH THE STANDARD MECHANICS NOTATION FOR ACCELERATION $\ddot{x} = \frac{d^2x}{dt^2} = v\frac{dv}{dx}$

TRANSFORMING THE O.D.E

$\Rightarrow \frac{d^2y}{dx^2} + e^{-y} = 0 \qquad \left[x = \frac{\pi}{2}, y = 0, \frac{dy}{dx} = p = -1\right]$

$\Rightarrow p\frac{dp}{dy} = -e^{-y}$

$\Rightarrow \int_{p=-1}^{p} p \, dp = \int_{y=0}^{y} -e^{-y} \, dy$

$\Rightarrow \left[\frac{1}{2}p^2\right]_{-1}^{p} = \left[e^{-y}\right]_0^y$

$\Rightarrow \frac{1}{2}p^2 - \frac{1}{2} = e^{-y} - 1$

$\Rightarrow p^2 - 1 = 2e^{-y} - 2$

$\Rightarrow p^2 = \frac{2}{e^y} - 1$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{2 - e^y}{e^y}$

$\Rightarrow \frac{dy}{dx} = +\frac{\sqrt{2 - e^y}}{e^{\frac{1}{2}y}}$

SEPARATING VARIABLES AGAIN

$\Rightarrow \frac{e^{\frac{1}{2}y}}{\sqrt{2 - e^y}} dy = 1 \, dx$

$\Rightarrow \int_{\frac{\pi}{2}}^{x} 1 \, dx = \int_0^y \frac{e^{\frac{1}{2}y}}{\sqrt{2 - e^y}} dy$

USING A TRIGONOMETRIC SUBSTITUTION ON THE INTEGRAL IN THE R.H.S

$e^y = 2\sin^2\theta \qquad \left[e^{\frac{1}{2}y} = \sqrt{2}\sin\theta \text{ or } \theta = \arcsin\left(\frac{e^{\frac{1}{2}y}}{\sqrt{2}}\right)\right]$

$\Rightarrow e^y dy = 4\sin\theta\cos\theta \, d\theta$

$\Rightarrow dy = \frac{4\sin\theta\cos\theta}{e^y} d\theta = \frac{4\sin\theta\cos\theta}{2\sin^2\theta} d\theta = \frac{2\cos\theta}{\sin\theta} d\theta$

LIMITS TRANSFORM TO

$y = 0 \mapsto \theta = \frac{\pi}{4}$

$y \mapsto \theta = \arcsin\left(\frac{e^{\frac{1}{2}y}}{\sqrt{2}}\right)$

RETURNING TO THE O.D.E

$\Rightarrow [x]_{\frac{\pi}{2}}^{x} = \int_{\frac{\pi}{4}}^{\arcsin(e^{\frac{1}{2}y}/\sqrt{2})} \frac{\sqrt{2}\sin\theta}{\sqrt{2 - 2\sin^2\theta}} \times \frac{2\cos\theta}{\sin\theta} d\theta$

$\Rightarrow x - \frac{\pi}{2} = \int_{\frac{\pi}{4}}^{\arcsin(e^{\frac{1}{2}y}/\sqrt{2})} \frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta} \times \frac{2\cos\theta}{\sin\theta} d\theta$

$\Rightarrow x - \frac{\pi}{2} = \int_{\frac{\pi}{4}}^{\arcsin(e^{\frac{1}{2}y}/\sqrt{2})} 2 \, d\theta$

$\Rightarrow x - \frac{\pi}{2} = 2[\theta]_{\frac{\pi}{4}}^{\arcsin(e^{\frac{1}{2}y}/\sqrt{2})}$

$\Rightarrow x - \frac{\pi}{2} = 2\left[\arcsin\left(e^{\frac{1}{2}y}/\sqrt{2}\right) - \frac{\pi}{4}\right]$

$\Rightarrow x = 2\arcsin\left(e^{\frac{1}{2}y}/\sqrt{2}\right)$

$\Rightarrow \frac{x}{2} = \arcsin\left(\frac{e^{\frac{1}{2}y}}{\sqrt{2}}\right)$

$\Rightarrow \sin\frac{x}{2} = \frac{e^{\frac{1}{2}y}}{\sqrt{2}}$

$\Rightarrow \sin^2\frac{x}{2} = \frac{e^y}{2}$

$\Rightarrow 2\sin^2\frac{x}{2} = e^y$

$\Rightarrow 1 - 2\sin^2\frac{x}{2} = 1 - e^y$

$\Rightarrow \cos x = 1 - e^y$

$\Rightarrow e^y = 1 - \cos x$

$\Rightarrow y = \ln(1 - \cos x)$ // AS REQUIRED

Question 10 (*****)

The curve C, has gradient 1 at the origin and satisfies the differential relationship

$$\frac{d^2y}{dx^2}\sqrt{1-2y} = \frac{dy}{dx}(3y-2), \qquad y < \frac{1}{2}.$$

Find an equation for C, giving the answer in the form $y = f(x)$.

$$y = \frac{\sin x}{1+\sin x} = (\sec x - \tan x)\tan x$$

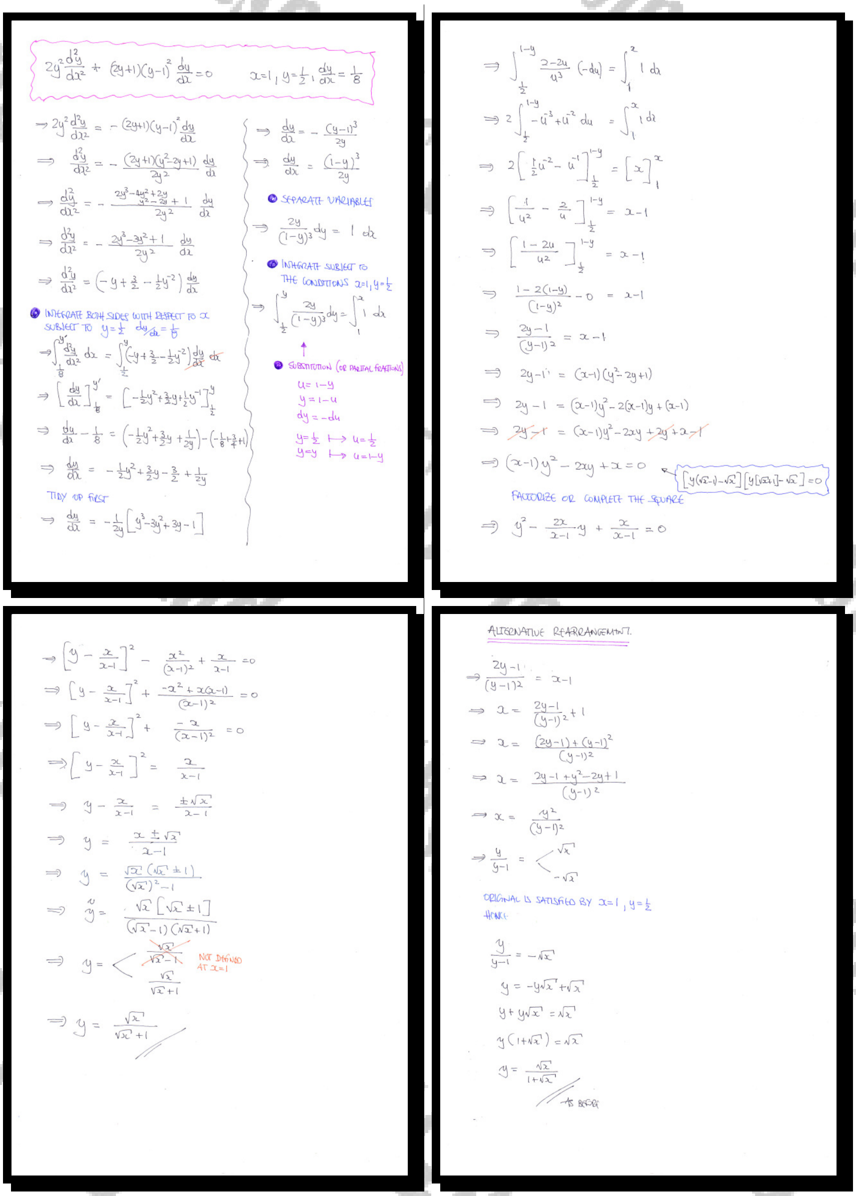
Question 11 (*****)

The curve C, has gradient $\frac{1}{8}$ at the point with coordinates $\left(1, \frac{1}{2}\right)$ and further satisfies the differential relationship

$$2y^2 \frac{d^2 y}{dx^2} + (2y+1)(y-1)^2 \frac{dy}{dx} = 0, \quad y \neq 0.$$

Find an equation for C, giving the answer in the form $y = f(x)$.

$$\boxed{y = \frac{\sqrt{x}}{1+\sqrt{x}}}$$



2^{ND} ORDER BY SUBSTITUTIONS

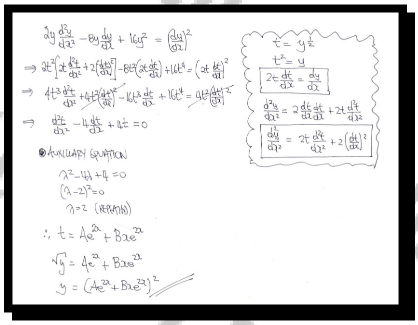
Question 1 (***)

$$2y\frac{d^2y}{dx^2} - 8y\frac{dy}{dx} + 16y^2 = \left(\frac{dy}{dx}\right)^2, \quad y \neq 0,$$

Find the general solution of the above differential equation by using the transformation equation $t = \sqrt{y}$.

Give the answer in the form $y = f(x)$.

$$\boxed{y = \left(Ae^{2x} + Bxe^{2x}\right)^2}$$



Question 2 (***)

The differential equation

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3x\,,\;\;x\neq 0\,,$$

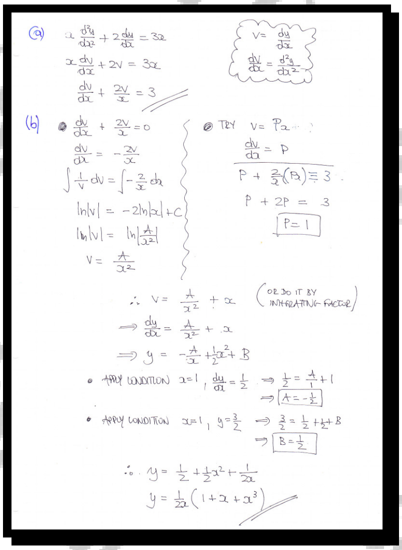
is to be solved subject to the boundary conditions $y=\frac{3}{2}$, $\frac{dy}{dx}=\frac{1}{2}$ at $x=1$.

a) Show that the substitution $v=\frac{dy}{dx}$, transforms the above differential equation into

$$\frac{dv}{dx}+\frac{2v}{x}=3\,.$$

b) Hence find the solution of the original differential equation, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{1}{2}\left(x^2+\frac{1}{x}+1\right)}$$



(a) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3x$

$x\frac{dv}{dx}+2v=3x$

$\frac{dv}{dx}+\frac{2v}{x}=3$

$v=\frac{dy}{dx}$, $\frac{dv}{dx}=\frac{d^2y}{dx^2}$

(b) • $\frac{dv}{dx}+\frac{2v}{x}=0$

$\frac{dv}{dx}=-\frac{2v}{x}$

$\int\frac{1}{v}\,dv=\int-\frac{2}{x}\,dx$

$\ln|v|=-2\ln|x|+C$

$\ln|v|=\ln\left|\frac{A}{x^2}\right|$

$v=\frac{A}{x^2}$

• TRY $v=Px$

$\frac{dv}{dx}=P$

$P+\frac{2}{x}(Px)\equiv 3$

$P+2P=3$

$\boxed{P=1}$

$\therefore\; v=\frac{A}{x^2}+x$ (OR DO IT BY INTEGRATING FACTOR)

$\Rightarrow \frac{dy}{dx}=\frac{A}{x^2}+x$

$\Rightarrow y=-\frac{A}{x}+\frac{1}{2}x^2+B$

• APPLY CONDITION $x=1$, $\frac{dy}{dx}=\frac{1}{2}$ $\Rightarrow \frac{1}{2}=\frac{A}{1}+1$ $\Rightarrow \boxed{A=-\frac{1}{2}}$

• APPLY CONDITION $x=1$, $y=\frac{3}{2}$ $\Rightarrow \frac{3}{2}=\frac{1}{2}+\frac{1}{2}+B$ $\Rightarrow \boxed{B=\frac{1}{2}}$

$\therefore\; y=\frac{1}{2}+\frac{1}{2}x^2+\frac{1}{2x}$

$y=\frac{1}{2x}\left(1+x+x^3\right)$

Question 3 (***)

The curve C has equation $y = f(x)$ and satisfies the differential equation

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2y\left(2x^2 - 1\right) = 3x^3 \mathrm{e}^x, \quad x \neq 0$$

is to be solved subject to the boundary conditions $y = \frac{3}{2}$, $\frac{dy}{dx} = \frac{1}{2}$ at $x = 1$.

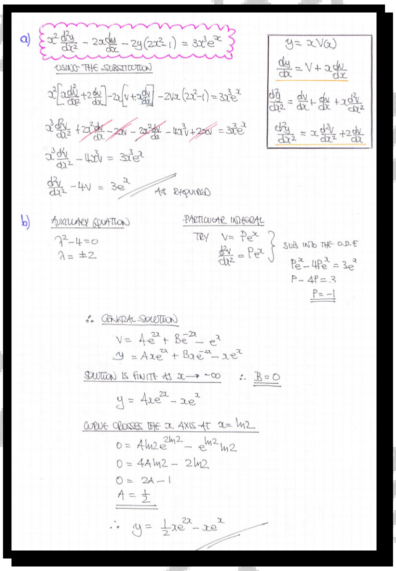
a) Show that the substitution $y = xv$, where v is a function of x transforms the above differential equation into

$$\frac{d^2v}{dx^2} - 4v = 3\mathrm{e}^x.$$

It is further given that C meets the x axis at $x = \ln 2$ and has a finite value for y as x gets infinitely negatively large.

b) Express the equation of C in the form $y = f(x)$.

$$\boxed{y = \frac{1}{2}x\mathrm{e}^{2x} - x\mathrm{e}^x}$$



Question 4 (***+)

The differential equation

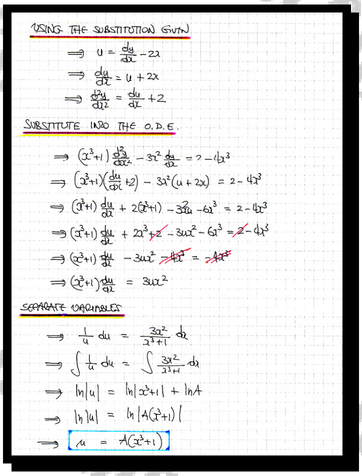
$$\left(x^3+1\right)\frac{d^2y}{dx^2}-3x^2\frac{dy}{dx}=2-4x^3,$$

is to be solved subject to the boundary conditions $y=0$, $\frac{dy}{dx}=4$ at $x=0$.

Use the substitution $u=\frac{dy}{dx}-2x$, where u is a function of x, to show that the solution of the above differential equation is

$$y=x^4+x^2+4x.$$

, proof



USING THE SUBSTITUTION GIVEN

$\Rightarrow u = \frac{dy}{dx} - 2x$

$\Rightarrow \frac{dy}{dx} = u + 2x$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{du}{dx} + 2$

SUBSTITUTE INTO THE O.D.E.

$\Rightarrow (x^3+1)\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} = 2 - 4x^3$

$\Rightarrow (x^3+1)\left(\frac{du}{dx}+2\right) - 3x^2(u+2x) = 2-4x^3$

$\Rightarrow (x^3+1)\frac{du}{dx} + 2(x^3+1) - 3x^2u - 6x^3 = 2-4x^3$

$\Rightarrow (x^3+1)\frac{du}{dx} + 2x^3 + 2 - 3x^2u - 6x^3 = 2 - 4x^3$

$\Rightarrow (x^3+1)\frac{du}{dx} - 3x^2u - 4x^3 = -4x^3$

$\Rightarrow (x^3+1)\frac{du}{dx} = 3x^2u$

SEPARATE VARIABLES

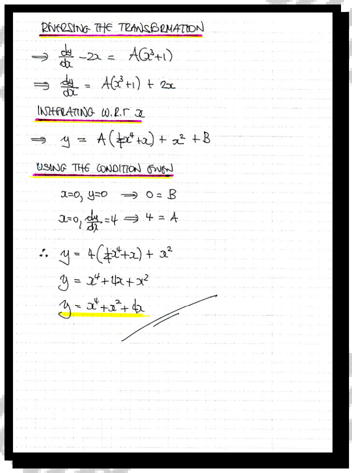
$\Rightarrow \frac{1}{u}\,du = \frac{3x^2}{x^3+1}\,dx$

$\Rightarrow \int \frac{1}{u}\,du = \int \frac{3x^2}{x^3+1}\,dx$

$\Rightarrow \ln|u| = \ln|x^3+1| + \ln A$

$\Rightarrow \ln|u| = \ln|A(x^3+1)|$

$\Rightarrow u = A(x^3+1)$



REVERSING THE TRANSFORMATION

$\Rightarrow \frac{dy}{dx} - 2x = A(x^3+1)$

$\Rightarrow \frac{dy}{dx} = A(x^3+1) + 2x$

INTEGRATING W.R.T. x

$\Rightarrow y = A\left(\frac{1}{4}x^4 + x\right) + x^2 + B$

USING THE CONDITION GIVEN

$x=0,\ y=0 \Rightarrow 0 = B$

$x=0,\ \frac{dy}{dx}=4 \Rightarrow 4 = A$

$\therefore\ y = 4\left(\frac{1}{4}x^4 + x\right) + x^2$

$y = x^4 + 4x + x^2$

$y = x^4 + x^2 + 4x$

Question 5 (****)

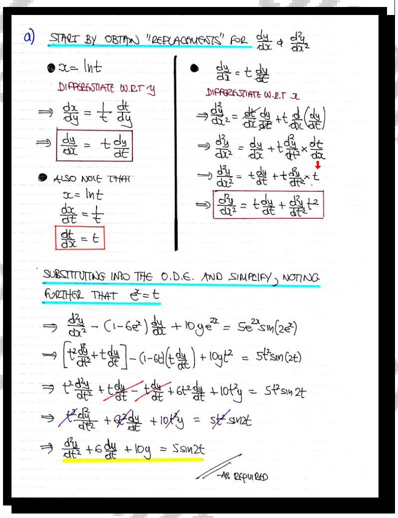
$$\frac{d^2y}{dx^2}-\left(1-6e^{x}\right)\frac{dy}{dx}+10ye^{2x}=5e^{2x}\sin\left(2e^{x}\right).$$

a) By using the substitution $x=\ln t$ or otherwise, show that the above differential equation can be transformed to

$$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+10y=5\sin 2t\,.$$

b) Hence find a general solution for the original differential equation.

$$y=e^{-3e^{x}}\left[A\cos\left(e^{x}\right)+B\sin\left(e^{x}\right)\right]+\tfrac{1}{6}\sin\left(2e^{x}\right)-\tfrac{1}{3}\cos\left(2e^{x}\right)$$


a) START BY OBTAIN "REPLACEMENTS" FOR $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$

- $x=\ln t$
  DIFFERENTIATE W.R.T y
  $\Rightarrow \frac{dx}{dy}=\frac{1}{t}\frac{dt}{dy}$
  $\Rightarrow \boxed{\frac{dy}{dx}=t\frac{dy}{dt}}$
- ALSO NOTE THAT
  $x=\ln t$
  $\frac{dx}{dt}=\frac{1}{t}$
  $\boxed{\frac{dt}{dx}=t}$
- $\frac{dy}{dx}=t\frac{dy}{dt}$
  DIFFERENTIATE W.R.T x
  $\Rightarrow \frac{d^2y}{dx^2}=\frac{dt}{dx}\frac{dy}{dt}+t\frac{d}{dx}\left(\frac{dy}{dt}\right)$
  $\Rightarrow \frac{d^2y}{dx^2}=\frac{dy}{dx}+t\frac{d^2y}{dt^2}\times\frac{dt}{dx}$
  $\Rightarrow \frac{d^2y}{dx^2}=t\frac{dy}{dt}+t\frac{d^2y}{dt^2}\times t$
  $\Rightarrow \boxed{\frac{d^2y}{dx^2}=t\frac{dy}{dt}+\frac{d^2y}{dt^2}t^2}$

SUBSTITUTING INTO THE O.D.E. AND SIMPLIFY, NOTING FURTHER THAT $e^{x}=t$

$\Rightarrow \frac{d^2y}{dx^2}-(1-6e^{x})\frac{dy}{dx}+10ye^{2x}=5e^{2x}\sin(2e^{x})$

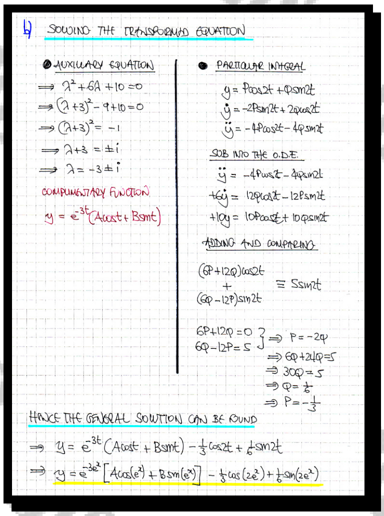
$\Rightarrow \left[t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}\right]-(1-6t)\left(t\frac{dy}{dt}\right)+10yt^2=5t^2\sin(2t)$

$\Rightarrow t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}-t\frac{dy}{dt}+6t^2\frac{dy}{dt}+10t^2y=5t^2\sin 2t$

$\Rightarrow t^2\frac{d^2y}{dt^2}+6t^2\frac{dy}{dt}+10t^2y=5t^2\sin 2t$

$\Rightarrow \frac{d^2y}{dt^2}+6\frac{dy}{dt}+10y=5\sin 2t$

AS REQUIRED


b) SOLVING THE TRANSFORMED EQUATION

- AUXILIARY EQUATION
  $\Rightarrow \lambda^2+6\lambda+10=0$
  $\Rightarrow (\lambda+3)^2-9+10=0$
  $\Rightarrow (\lambda+3)^2=-1$
  $\Rightarrow \lambda+3=\pm i$
  $\Rightarrow \lambda=-3\pm i$
  COMPLEMENTARY FUNCTION
  $y=e^{-3t}(A\cos t+B\sin t)$
- PARTICULAR INTEGRAL
  $y=P\cos 2t+Q\sin 2t$
  $\dot{y}=-2P\sin 2t+2Q\cos 2t$
  $\ddot{y}=-4P\cos 2t-4Q\sin 2t$
  SUB INTO THE O.D.E.
  $\ddot{y}=-4P\cos 2t-4Q\sin 2t$
  $+6\dot{y}=12Q\cos 2t-12P\sin 2t$
  $+10y=10P\cos 2t+10Q\sin 2t$
  ADDING AND COMPARING
  $(6P+12Q)\cos 2t+(6Q-12P)\sin 2t\equiv 5\sin 2t$
  $6P+12Q=0$, $6Q-12P=5$ $\Rightarrow P=-2Q$
  $\Rightarrow 6Q+24Q=5$
  $\Rightarrow 30Q=5$
  $\Rightarrow Q=\frac{1}{6}$
  $\Rightarrow P=-\frac{1}{3}$

HENCE THE GENERAL SOLUTION CAN BE FOUND

$\Rightarrow y=e^{-3t}(A\cos t+B\sin t)-\frac{1}{3}\cos 2t+\frac{1}{6}\sin 2t$

$\Rightarrow y=e^{-3e^{x}}\left[A\cos(e^{x})+B\sin(e^{x})\right]-\frac{1}{3}\cos(2e^{x})+\frac{1}{6}\sin(2e^{x})$

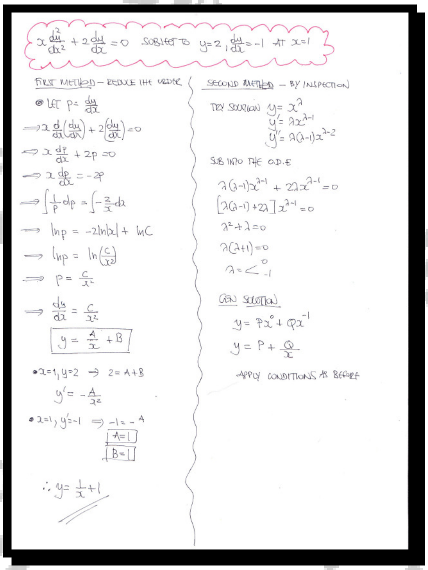
Question 6 (***+)

Solve the differential equation

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0,$$

subject to the boundary conditions $y=2$, $\frac{dy}{dx}=-1$ at $x=1$.

$$\boxed{y=\frac{2e^{2x}}{e^{2x}+x}}$$

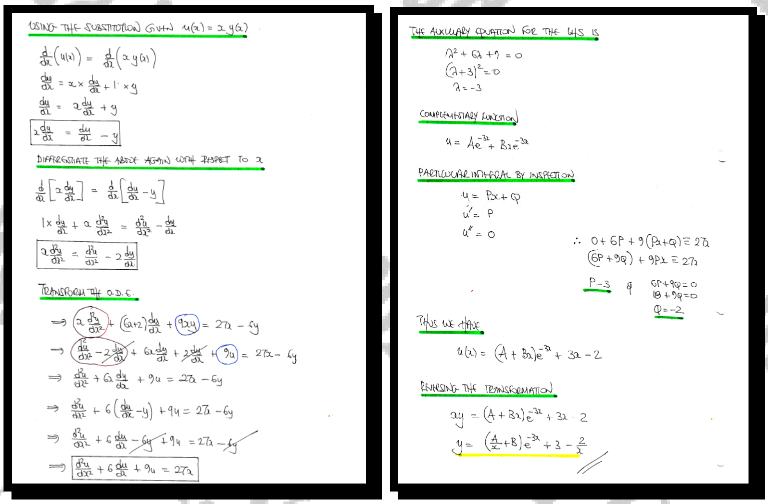


Question 7 (***+)

$$x\frac{d^2y}{dx^2}+(6x+2)\frac{dy}{dx}+9xy=27x-6y.$$

Use the substitution $u=xy$, where u is a function of x, to find a general solution of the above differential equation.

$$\boxed{y=\frac{A}{x}e^{-3x}+Be^{-3x}+3-\frac{2}{x}}$$



USING THE SUBSTITUTION GIVEN $u(x)=xy(x)$

$\frac{d}{dx}(u(x))=\frac{d}{dx}(xy(x))$

$\frac{du}{dx}=x\times\frac{dy}{dx}+1\times y$

$\frac{du}{dx}=x\frac{dy}{dx}+y$

$x\frac{dy}{dx}=\frac{du}{dx}-y$

DIFFERENTIATE THE ABOVE AGAIN WITH RESPECT TO x

$\frac{d}{dx}\left[x\frac{dy}{dx}\right]=\frac{d}{dx}\left[\frac{du}{dx}-y\right]$

$1\times\frac{dy}{dx}+x\frac{d^2y}{dx^2}=\frac{d^2u}{dx^2}-\frac{dy}{dx}$

$x\frac{d^2y}{dx^2}=\frac{d^2u}{dx^2}-2\frac{dy}{dx}$

TRANSFORM THE O.D.E.

$\Rightarrow x\frac{d^2y}{dx^2}+(6x+2)\frac{dy}{dx}+9xy=27x-6y$

$\Rightarrow \frac{d^2u}{dx^2}-2\frac{dy}{dx}+6x\frac{dy}{dx}+2\frac{dy}{dx}+9u=27x-6y$

$\Rightarrow \frac{d^2u}{dx^2}+6x\frac{dy}{dx}+9u=27x-6y$

$\Rightarrow \frac{d^2u}{dx^2}+6\left(\frac{du}{dx}-y\right)+9u=27x-6y$

$\Rightarrow \frac{d^2u}{dx^2}+6\frac{du}{dx}-6y+9u=27x-6y$

$\Rightarrow \frac{d^2u}{dx^2}+6\frac{du}{dx}+9u=27x$

THE AUXILIARY EQUATION FOR THE LHS IS

$\lambda^2+6\lambda+9=0$

$(\lambda+3)^2=0$

$\lambda=-3$

COMPLEMENTARY FUNCTION

$u=Ae^{-3x}+Bxe^{-3x}$

PARTICULAR INTEGRAL BY INSPECTION

$u=Px+Q$

$u'=P$

$u''=0$

$\therefore 0+6P+9(Px+Q)\equiv 27x$

$(6P+9Q)+9Px\equiv 27x$

$P=3$ & $6P+9Q=0$, $18+9Q=0$, $Q=-2$

THUS WE HAVE

$u(x)=(A+Bx)e^{-3x}+3x-2$

REVERSING THE TRANSFORMATION

$xy=(A+Bx)e^{-3x}+3x-2$

$y=\left(\frac{A}{x}+B\right)e^{-3x}+3-\frac{2}{x}$

Question 8 (***+)

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\tan x - y\sec^4 x = 0.$$

The above differential equation is to be solved by a substitution.

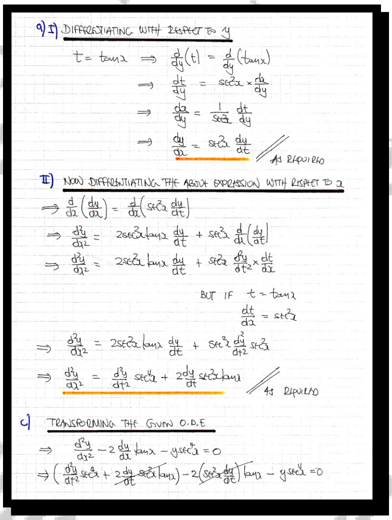
a) If $t = \tan x$ show that …

i. … $\dfrac{dy}{dx} = \dfrac{dy}{dt}\sec^2 x$

ii. … $\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2}\sec^4 x + 2\dfrac{dy}{dt}\sec^2 x\tan x$

b) Use the results obtained in part **(a)** to find a general solution of the differential equation in the form $y = f(x)$.

$\boxed{y = A\mathrm{e}^{\tan x} + B\mathrm{e}^{-\tan x}}$



a) I) DIFFERENTIATING WITH RESPECT TO y

$t = \tan x \implies \frac{d}{dy}(t) = \frac{d}{dy}(\tan x)$

$\implies \frac{dt}{dy} = \sec^2 x \times \frac{dx}{dy}$

$\implies \frac{dx}{dy} = \frac{1}{\sec^2 x}\frac{dt}{dy}$

$\implies \frac{dy}{dx} = \sec^2 x \frac{dy}{dt}$ // AS REQUIRED

II) NOW DIFFERENTIATING THE ABOVE EXPRESSION WITH RESPECT TO x

$\implies \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\sec^2 x \frac{dy}{dt}\right)$

$\implies \frac{d^2y}{dx^2} = 2\sec^2 x\tan x\frac{dy}{dt} + \sec^2 x\frac{d}{dx}\left(\frac{dy}{dt}\right)$

$\implies \frac{d^2y}{dx^2} = 2\sec^2 x\tan x\frac{dy}{dt} + \sec^2 x\frac{d^2y}{dt^2} \times \frac{dt}{dx}$

BUT IF $t = \tan x$, $\frac{dt}{dx} = \sec^2 x$

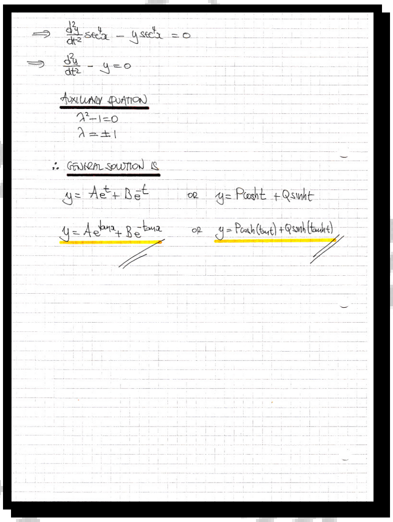
$\implies \frac{d^2y}{dx^2} = 2\sec^2 x\tan x\frac{dy}{dt} + \sec^2 x\frac{d^2y}{dt^2}\sec^2 x$

$\implies \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\sec^4 x + 2\frac{dy}{dt}\sec^2 x\tan x$ // AS REQUIRED

c) TRANSFORMING THE GIVEN O.D.E

$\implies \frac{d^2y}{dx^2} - 2\frac{dy}{dx}\tan x - y\sec^4 x = 0$

$\implies \left(\frac{d^2y}{dt^2}\sec^4 x + 2\frac{dy}{dt}\sec^2 x\tan x\right) - 2\left(\sec^2 x\frac{dy}{dt}\right)\tan x - y\sec^4 x = 0$



$\implies \frac{d^2y}{dt^2}\sec^4 x - y\sec^4 x = 0$

$\implies \frac{d^2y}{dt^2} - y = 0$

AUXILIARY EQUATION

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

$\therefore$ GENERAL SOLUTION IS

$y = Ae^{t} + Be^{-t}$ OR $y = P\cosh t + Q\sinh t$

$y = Ae^{\tan x} + Be^{-\tan x}$ OR $y = P\cosh(\tan t) + Q\sinh(\tan t)$ //

Question 9 (***+)

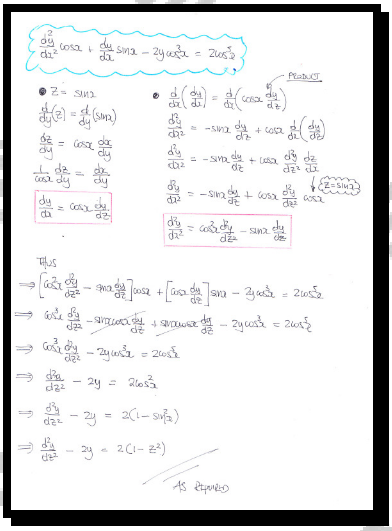
Show clearly that the substitution $z = \sin x$, transforms the differential equation

$$\frac{d^2y}{dx^2}\cos x + \frac{dy}{dx}\sin x - 2y\cos^3 x = 2\cos^5 x,$$

into the differential equation

$$\frac{d^2y}{dz^2} - 2y = 2\left(1 - z^2\right)$$

proof



$$\frac{d^2y}{dx^2}\cos x + \frac{dy}{dx}\sin x - 2y\cos^3 x = 2\cos^5 x$$

- $z = \sin x$

$$\frac{d}{dy}(z) = \frac{d}{dy}(\sin x)$$

$$\frac{dz}{dy} = \cos x \frac{dx}{dy}$$

$$\frac{1}{\cos x}\frac{dz}{dy} = \frac{dx}{dy}$$

$$\frac{dy}{dx} = \cos x \frac{dy}{dz}$$

- $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\cos x \frac{dy}{dz}\right)$ PRODUCT

$$\frac{d^2y}{dx^2} = -\sin x \frac{dy}{dz} + \cos x \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$\frac{d^2y}{dx^2} = -\sin x \frac{dy}{dz} + \cos x \frac{d^2y}{dz^2}\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = -\sin x \frac{dy}{dz} + \cos x \frac{d^2y}{dz^2}\cos x \quad (z = \sin x)$$

$$\frac{d^2y}{dx^2} = \cos^2 x \frac{d^2y}{dz^2} - \sin x \frac{dy}{dz}$$

Thus

$$\Rightarrow \left[\cos^2 x \frac{d^2y}{dz^2} - \sin x \frac{dy}{dz}\right]\cos x + \left[\cos x \frac{dy}{dz}\right]\sin x - 2y\cos^3 x = 2\cos^5 x$$

$$\Rightarrow \cos^3 x \frac{d^2y}{dz^2} - \sin x\cos x \frac{dy}{dz} + \sin x\cos x \frac{dy}{dz} - 2y\cos^3 x = 2\cos^5 x$$

$$\Rightarrow \cos^3 x \frac{d^2y}{dz^2} - 2y\cos^3 x = 2\cos^5 x$$

$$\Rightarrow \frac{d^2y}{dz^2} - 2y = 2\cos^2 x$$

$$\Rightarrow \frac{d^2y}{dz^2} - 2y = 2(1 - \sin^2 x)$$

$$\Rightarrow \frac{d^2y}{dz^2} - 2y = 2(1 - z^2)$$

AS REQUIRED

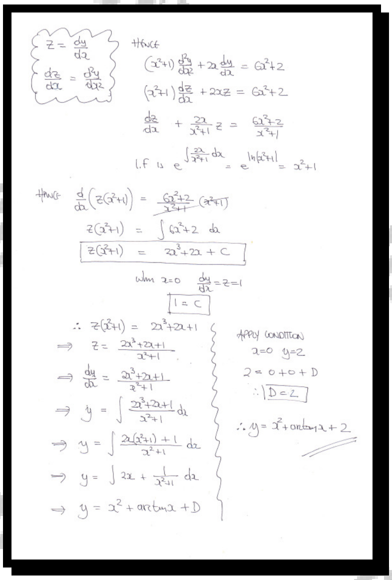
Question 10 **(***+)**

By using the substitution $z = \frac{dy}{dx}$, or otherwise, solve the differential equation

$$\left(x^2+1\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 6x^2 + 2,$$

subject to the conditions $x = 0$, $y = 2$, $\frac{dy}{dx} = 1$

$$\boxed{y = x^2 + 2 + \arctan x}$$


$z = \frac{dy}{dx}$, $\frac{dz}{dx} = \frac{d^2y}{dx^2}$

Hence

$(x^2+1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 6x^2+2$

$(x^2+1)\frac{dz}{dx} + 2xz = 6x^2+2$

$\frac{dz}{dx} + \frac{2x}{x^2+1}z = \frac{6x^2+2}{x^2+1}$

I.F is $e^{\int \frac{2x}{x^2+1}dx} = e^{\ln|x^2+1|} = x^2+1$

Hence $\frac{d}{dx}\left(z(x^2+1)\right) = \frac{6x^2+2}{x^2+1}(x^2+1)$

$z(x^2+1) = \int 6x^2+2 \, dx$

$\boxed{z(x^2+1) = 2x^3+2x+C}$

when $x=0$ $\frac{dy}{dx}=z=1$

$\boxed{1=C}$

$\therefore z(x^2+1) = 2x^3+2x+1$

$\Rightarrow z = \frac{2x^3+2x+1}{x^2+1}$

$\Rightarrow \frac{dy}{dx} = \frac{2x^3+2x+1}{x^2+1}$

$\Rightarrow y = \int \frac{2x^3+2x+1}{x^2+1}dx$

$\Rightarrow y = \int \frac{2x(x^2+1)+1}{x^2+1}dx$

$\Rightarrow y = \int 2x + \frac{1}{x^2+1}dx$

$\Rightarrow y = x^2 + \arctan x + D$

APPLY CONDITION

$x=0$ $y=2$

$2 = 0+0+D$

$\therefore \boxed{D=2}$

$\therefore y = x^2 + \arctan x + 2$

Question 11 (****)

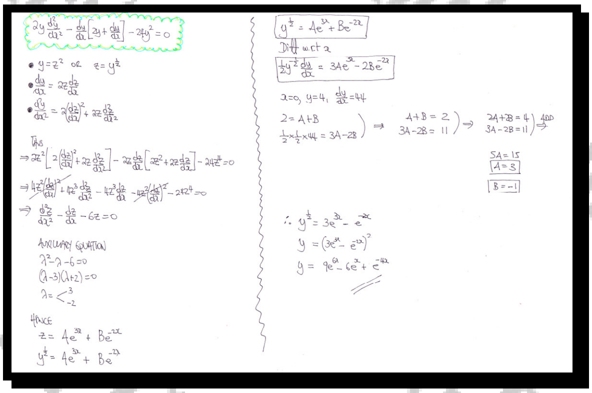
Use the substitution $z=\sqrt{y}$, where $y=f(x)$, to solve the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{y}\left(\frac{dy}{dx}\right)^2-5\frac{dy}{dx}+2y=0,$$

subject to the boundary conditions $y=4$, $\frac{dy}{dx}=44$ at $x=0$.

Give the answer in the form $y=f(x)$.

$$\boxed{y=9\mathrm{e}^{6x}-6\mathrm{e}^{x}+\mathrm{e}^{-4x}}$$



Question 12 (****)

$$2x\frac{d^2y}{dx^2}+\left(1-3x^{\frac{1}{2}}\right)\frac{dy}{dx}+y=0.$$

The above differential equation is to be solved by a substitution.

a) Given that $y=f(x)$ and $t=x^{\frac{1}{2}}$, show clearly that …

i. … $\dfrac{dy}{dx}=\dfrac{1}{2t}\dfrac{dy}{dt}$.

ii. … $\dfrac{d^2y}{dx^2}=\dfrac{1}{4t^2}\dfrac{d^2y}{dt^2}-\dfrac{1}{4t^3}\dfrac{dy}{dt}$.

b) Hence show further that the differential equation

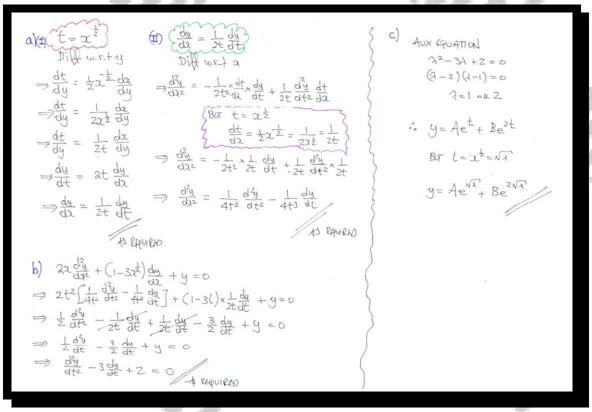
$$2x\frac{d^2y}{dx^2}+\left(1-3x^{\frac{1}{2}}\right)\frac{dy}{dx}+y=0,$$

can be transformed to the differential equation

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=0.$$

c) Find a general solution of the **original** differential equation, giving the answer in the form $y=f(x)$.

$$\boxed{y=A\mathrm{e}^{\sqrt{x}}+B\mathrm{e}^{2\sqrt{x}}}$$



Question 13 (****)

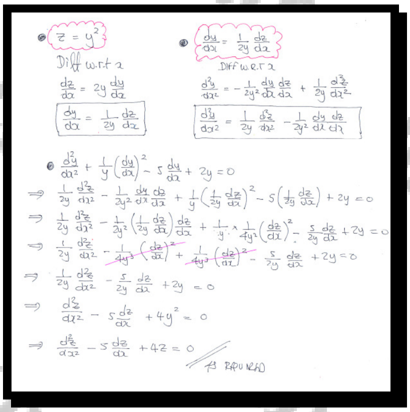
Show clearly that the substitution $z = y^2$, where $y = f(x)$, transforms the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 2y = 0,$$

into the differential equation

$$\frac{d^2z}{dx^2} - 5\frac{dz}{dx} + 4z = 0$$

proof



Question 14 (****)

Given that if $x = t^{\frac{1}{2}}$, where $y = f(x)$, show clearly that

a) $\frac{dy}{dx} = 2t^{\frac{1}{2}}\frac{dy}{dt}$.

b) $\frac{d^2y}{dx^2} = 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$.

The following differential equation is to be solved

$$x\frac{d^2y}{dx^2} - \left(8x^2+1\right)\frac{dy}{dx} + 12x^3y = 12x^5,$$

subject to the boundary conditions $y = \frac{10}{3}$, $\frac{d^2y}{dx^2} = 10$ at $x = 0$.

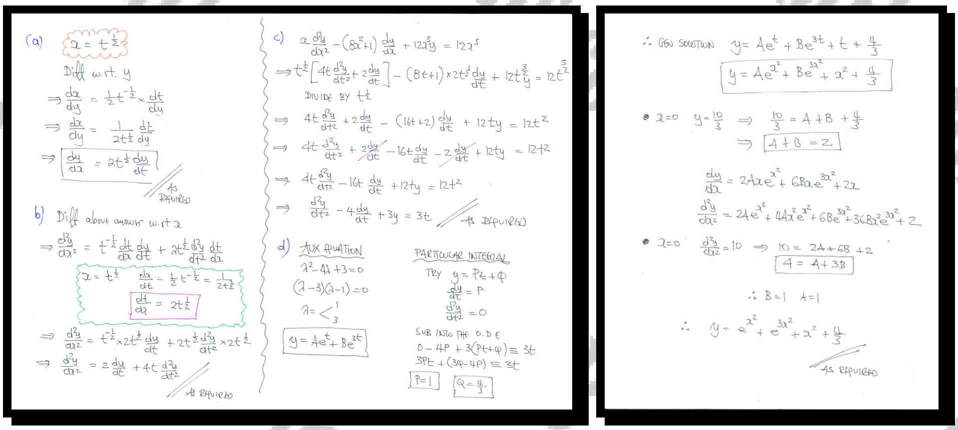
c) Show further that the substitution $x = t^{\frac{1}{2}}$, where $y = f(x)$, transforms the above differential equation into the differential equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 3t.$$

d) Show that a solution of the **original** differential equation is

$$y = e^{3x^2} + e^{x^2} + x^2 + \frac{4}{3}.$$

proof



Question 15 (****)

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx}\cot x + 2y\,\text{cosec}^2 x = 2\cos x - 2\cos^3 x.$$

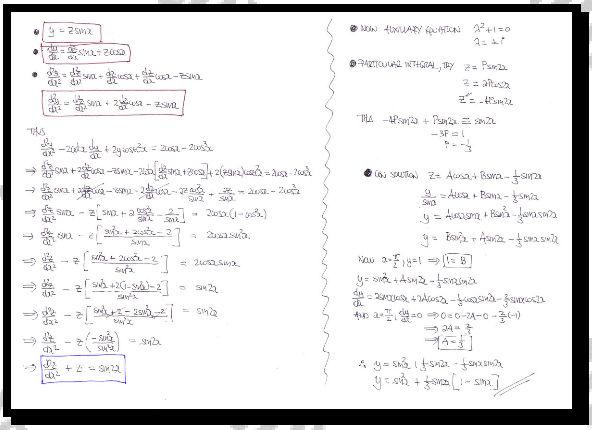
Use the substitution $y = z\sin x$, where z is a function of x, to solve the above differential equation subject to the boundary conditions $y = 1$, $\frac{dy}{dx} = 0$ at $x = \frac{\pi}{2}$.

Give the answer in the form

$$y = a\sin^2 x + b(1 - \sin x)\sin 2x,$$

where a and b are constants to be found.

$\boxed{a = 1}$, $\boxed{b = \frac{1}{3}}$

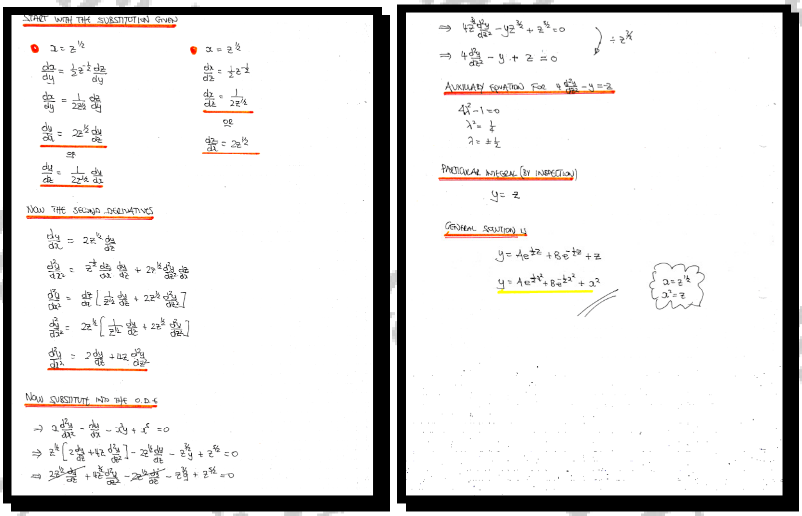


Question 16 (****)

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - x^3 y + x^5 = 0 .$$

Use the substitution $x = z^{\frac{1}{2}}$, where $y = f(x)$, to find a general solution of the above differential equation.

V, , $y = A e^{\frac{1}{2}x^2} + B e^{-\frac{1}{2}x^2} + x^2$



Question 17 (****)

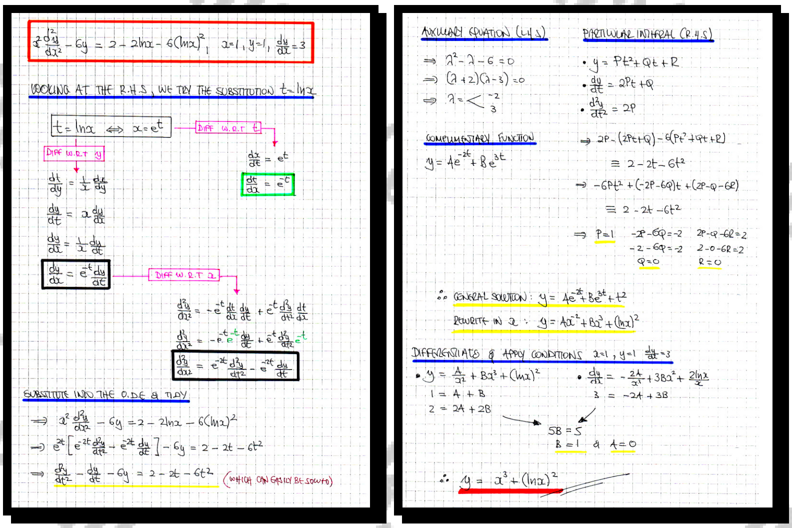
Use a suitable substitution to solve the differential equation

$$x^2 \frac{d^2y}{dx^2} - 6y = 2 - 2\ln x - 6(\ln x)^2 ,$$

subject to the boundary conditions $y(1) = 1$, $\frac{dy}{dx}(1) = 3$

Give a simplified answer in the form $y = f(x)$.

$y = x^3 + (\ln x)^2$


$x^2\frac{d^2y}{dx^2} - 6y = 2 - 2\ln x - 6(\ln x)^2, \quad x=1, y=1, \frac{dy}{dx}=3$

LOOKING AT THE R.H.S, WE TRY THE SUBSTITUTION $t = \ln x$

$t = \ln x \Leftrightarrow x = e^t$

DIFF W.R.T y

$\frac{dt}{dy} = \frac{1}{x}\frac{dx}{dy}$

$\frac{dy}{dt} = x\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$

$\frac{dy}{dx} = e^{-t}\frac{dy}{dt}$

DIFF W.R.T t

$\frac{dx}{dt} = e^t$

$\frac{dt}{dx} = e^{-t}$

DIFF W.R.T x

$\frac{d^2y}{dx^2} = -e^{-t}\frac{dt}{dx}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\frac{dt}{dx}$

$\frac{d^2y}{dx^2} = -e^{-t}e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}e^{-t}$

$\frac{d^2y}{dx^2} = e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}$

SUBSTITUTE INTO THE O.D.E & TIDY

$\Rightarrow x^2\frac{d^2y}{dx^2} - 6y = 2 - 2\ln x - 6(\ln x)^2$

$\Rightarrow e^{2t}\left[e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}\right] - 6y = 2 - 2t - 6t^2$

$\Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 2 - 2t - 6t^2$ (WHICH CAN EASILY BE SOLVED)

AUXILIARY EQUATION (L.H.S)

$\Rightarrow \lambda^2 - \lambda - 6 = 0$

$\Rightarrow (\lambda+2)(\lambda-3) = 0$

$\Rightarrow \lambda = -2, 3$

COMPLEMENTARY FUNCTION

$y = Ae^{-2t} + Be^{3t}$

PARTICULAR INTEGRAL (R.H.S)

• $y = Pt^2 + Qt + R$

• $\frac{dy}{dt} = 2Pt + Q$

• $\frac{d^2y}{dt^2} = 2P$

$\Rightarrow 2P - (2Pt+Q) - 6(Pt^2+Qt+R) \equiv 2 - 2t - 6t^2$

$\Rightarrow -6Pt^2 + (-2P-6Q)t + (2P-Q-6R) \equiv 2 - 2t - 6t^2$

$\Rightarrow P = 1 \qquad -2P-6Q=-2 \qquad 2P-Q-6R=2$

$-2-6Q=-2 \qquad 2-0-6R=2$

$Q=0 \qquad R=0$

$\therefore$ GENERAL SOLUTION: $y = Ae^{-2t} + Be^{3t} + t^2$

REWRITE IN x: $y = Ax^{-2} + Bx^3 + (\ln x)^2$

DIFFERENTIATE & APPLY CONDITIONS $x=1, y=1, \frac{dy}{dx}=3$

• $y = \frac{A}{x^2} + Bx^3 + (\ln x)^2$

$1 = A + B$

$2 = 2A + 2B$

• $\frac{dy}{dx} = -\frac{2A}{x^3} + 3Bx^2 + \frac{2\ln x}{x}$

$3 = -2A + 3B$

$5B = 5$

$B = 1$ & $A = 0$

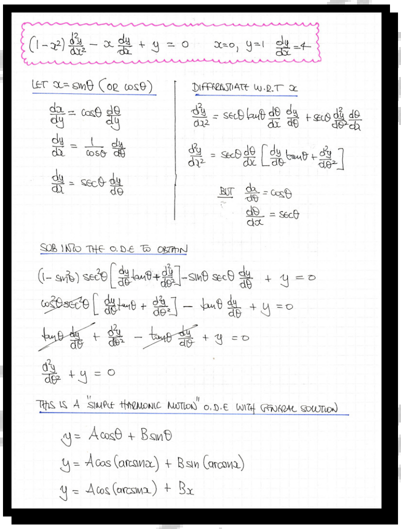
$\therefore y = x^3 + (\ln x)^2$

Question 18 (****)

Use a suitable trigonometric substitution to solve the following differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0, \quad y(0) = 1 \quad \frac{dy}{dx}(0) = 4.$$

$$\boxed{y = 3x - \cos(\arcsin x)}$$



$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \qquad x=0,\ y=1\ \frac{dy}{dx}=4$

LET $x = \sin\theta$ (OR $\cos\theta$)

$\frac{dx}{dy} = \cos\theta\frac{d\theta}{dy}$

$\frac{dy}{dx} = \frac{1}{\cos\theta}\frac{dy}{d\theta}$

$\frac{dy}{dx} = \sec\theta\frac{dy}{d\theta}$

DIFFERENTIATE W.R.T x

$\frac{d^2y}{dx^2} = \sec\theta\tan\theta\frac{d\theta}{dx}\frac{dy}{d\theta} + \sec\theta\frac{d^2y}{d\theta^2}\frac{d\theta}{dx}$

$\frac{d^2y}{dx^2} = \sec\theta\frac{d\theta}{dx}\left[\frac{dy}{d\theta}\tan\theta + \frac{d^2y}{d\theta^2}\right]$

BUT $\frac{dx}{d\theta} = \cos\theta$

$\frac{d\theta}{dx} = \sec\theta$

SUB INTO THE O.D.E TO OBTAIN

$(1-\sin^2\theta)\sec^2\theta\left[\frac{dy}{d\theta}\tan\theta + \frac{d^2y}{d\theta^2}\right] - \sin\theta\sec\theta\frac{dy}{d\theta} + y = 0$

$\cos^2\theta\sec^2\theta\left[\frac{dy}{d\theta}\tan\theta + \frac{d^2y}{d\theta^2}\right] - \tan\theta\frac{dy}{d\theta} + y = 0$

$\tan\theta\frac{dy}{d\theta} + \frac{d^2y}{d\theta^2} - \tan\theta\frac{dy}{d\theta} + y = 0$

$\frac{d^2y}{d\theta^2} + y = 0$

THIS IS A "SIMPLE HARMONIC MOTION" O.D.E WITH GENERAL SOLUTION

$y = A\cos\theta + B\sin\theta$

$y = A\cos(\arcsin x) + B\sin(\arcsin x)$

$y = A\cos(\arcsin x) + Bx$

Question 19 (****)

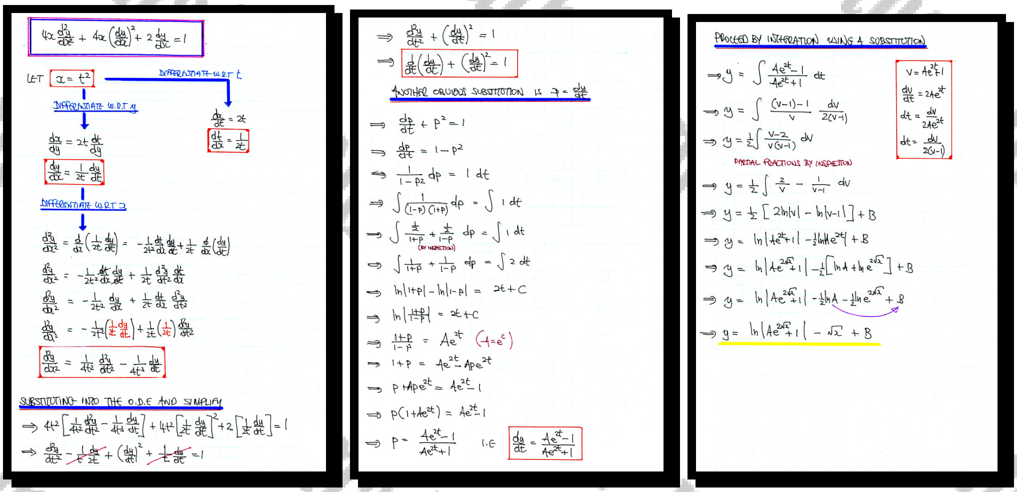
$$4x\frac{d^2y}{dx^2}+4x\left(\frac{dy}{dx}\right)^2+2\frac{dy}{dx}=1.$$

By using the substitution $t=\sqrt{x}$, or otherwise, show that the general solution of the above differential equation is

$$y=A-\sqrt{x}+\ln\left[1+B\mathrm{e}^{2\sqrt{x}}\right],$$

where A and B are arbitrary constants.

[ ], proof



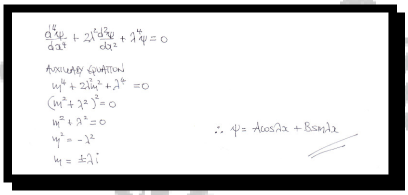
VARIOUS TYPES

Question 1 (**+)

Find the general solution of the following differential equation.

$$\frac{d^4\psi}{dx^4}+2\lambda\frac{d^2\psi}{dx^2}+\lambda^4\psi=0.$$

$$\boxed{\psi=A\cos\lambda x+B\sin\lambda x}$$



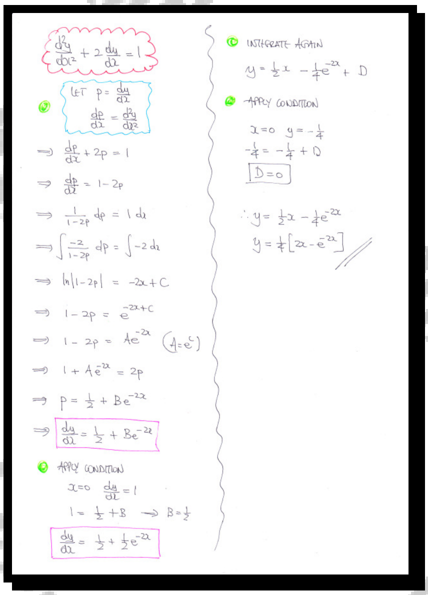
Question 2 (***)

Solve the differential equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}=1,$$

given that $y=-\frac{1}{4}$ and $\frac{dy}{dx}=1$ at $x=0$, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{1}{2}\left[2x-\mathrm{e}^{-2x}\right]}$$



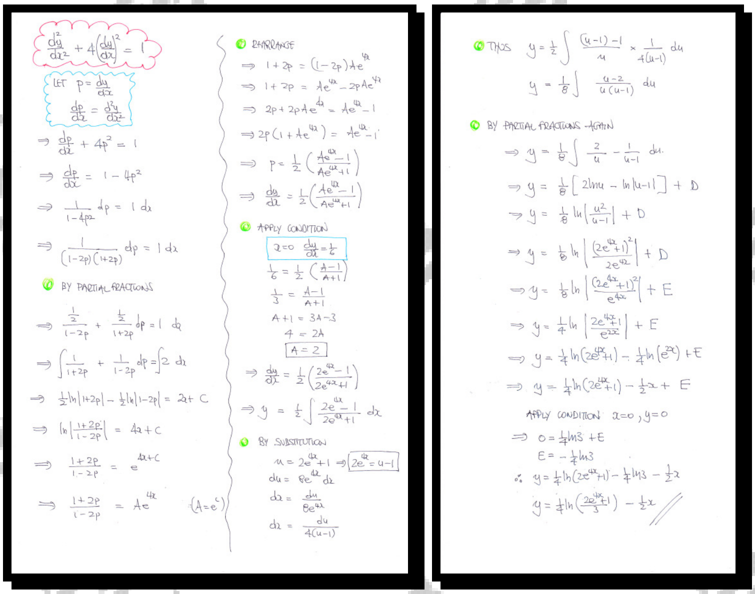
Question 3 (***+)

Solve the differential equation

$$\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)^2=1,$$

given that $y=0$ and $\frac{dy}{dx}=\frac{1}{6}$ at $x=0$, giving the answer in the form $y=f(x)$.

$$\boxed{y=\frac{1}{4}\ln\left[\frac{1+2e^{4x}}{3}\right]-\frac{1}{2}x}$$



$\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)^2=1$

LET $p=\frac{dy}{dx}$, $\frac{dp}{dx}=\frac{d^2y}{dx^2}$

$\Rightarrow \frac{dp}{dx}+4p^2=1$

$\Rightarrow \frac{dp}{dx}=1-4p^2$

$\Rightarrow \frac{1}{1-4p^2}\,dp=1\,dx$

$\Rightarrow \frac{1}{(1-2p)(1+2p)}\,dp=1\,dx$

BY PARTIAL FRACTIONS

$\Rightarrow \frac{\frac{1}{2}}{1-2p}+\frac{\frac{1}{2}}{1+2p}\,dp=1\,dx$

$\Rightarrow \int\frac{1}{1+2p}+\frac{1}{1-2p}\,dp=\int 2\,dx$

$\Rightarrow \frac{1}{2}\ln|1+2p|-\frac{1}{2}\ln|1-2p|=2x+C$

$\Rightarrow \ln\left|\frac{1+2p}{1-2p}\right|=4x+C$

$\Rightarrow \frac{1+2p}{1-2p}=e^{4x+C}$

$\Rightarrow \frac{1+2p}{1-2p}=Ae^{4x}$ $(A=e^C)$

REARRANGE

$\Rightarrow 1+2p=(1-2p)Ae^{4x}$

$\Rightarrow 1+2p=Ae^{4x}-2pAe^{4x}$

$\Rightarrow 2p+2pAe^{4x}=Ae^{4x}-1$

$\Rightarrow 2p(1+Ae^{4x})=Ae^{4x}-1$

$\Rightarrow p=\frac{1}{2}\left(\frac{Ae^{4x}-1}{Ae^{4x}+1}\right)$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left(\frac{Ae^{4x}-1}{Ae^{4x}+1}\right)$

APPLY CONDITION

$x=0$, $\frac{dy}{dx}=\frac{1}{6}$

$\frac{1}{6}=\frac{1}{2}\left(\frac{A-1}{A+1}\right)$

$\frac{1}{3}=\frac{A-1}{A+1}$

$A+1=3A-3$

$4=2A$

$A=2$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left(\frac{2e^{4x}-1}{2e^{4x}+1}\right)$

$\Rightarrow y=\frac{1}{2}\int\frac{2e^{4x}-1}{2e^{4x}+1}\,dx$

BY SUBSTITUTION

$u=2e^{4x}+1 \Rightarrow 2e^{4x}=u-1$

$du=8e^{4x}\,dx$

$dx=\frac{du}{8e^{4x}}$

$dx=\frac{du}{4(u-1)}$

THUS $y=\frac{1}{2}\int\frac{(u-1)-1}{u}\times\frac{1}{4(u-1)}\,du$

$y=\frac{1}{8}\int\frac{u-2}{u(u-1)}\,du$

BY PARTIAL FRACTIONS AGAIN

$\Rightarrow y=\frac{1}{8}\int\frac{2}{u}-\frac{1}{u-1}\,du$

$\Rightarrow y=\frac{1}{8}\left[2\ln u-\ln|u-1|\right]+D$

$\Rightarrow y=\frac{1}{8}\ln\left|\frac{u^2}{u-1}\right|+D$

$\Rightarrow y=\frac{1}{8}\ln\left|\frac{(2e^{4x}+1)^2}{2e^{4x}}\right|+D$

$\Rightarrow y=\frac{1}{8}\ln\left|\frac{(2e^{4x}+1)^2}{e^{4x}}\right|+E$

$\Rightarrow y=\frac{1}{4}\ln\left|\frac{2e^{4x}+1}{e^{2x}}\right|+E$

$\Rightarrow y=\frac{1}{4}\ln(2e^{4x}+1)-\frac{1}{4}\ln(e^{2x})+E$

$\Rightarrow y=\frac{1}{4}\ln(2e^{4x}+1)-\frac{1}{2}x+E$

APPLY CONDITION $x=0$, $y=0$

$\Rightarrow 0=\frac{1}{4}\ln 3+E$

$E=-\frac{1}{4}\ln 3$

$\therefore y=\frac{1}{4}\ln(2e^{4x}+1)-\frac{1}{4}\ln 3-\frac{1}{2}x$

$y=\frac{1}{4}\ln\left(\frac{2e^{4x}+1}{3}\right)-\frac{1}{2}x$

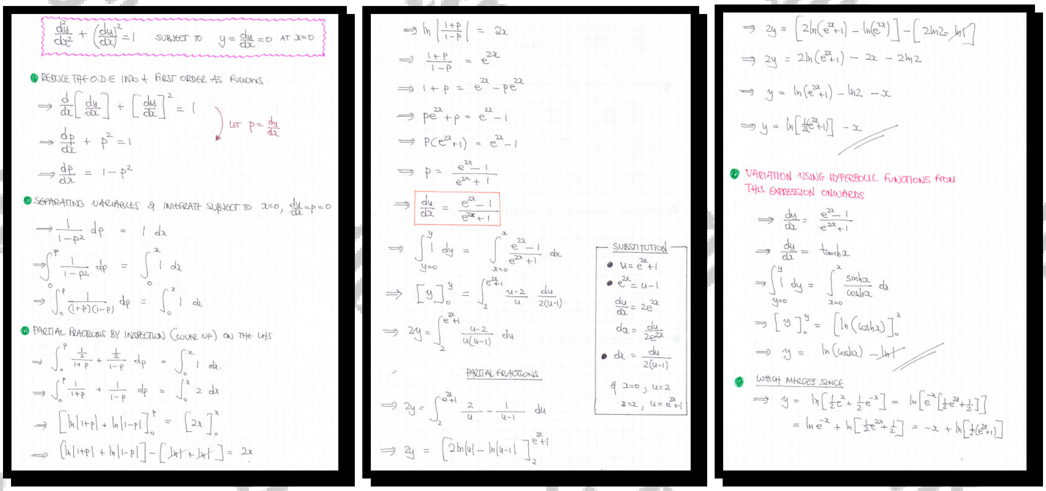
Question 4 (***+)

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=1.$$

Given that $y=\frac{dy}{dx}=0$ at $x=0$, show that

$$y=-x+\ln\left[\tfrac{1}{2}\left(1+e^{2x}\right)\right].$$

proof



$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=1$ SUBJECT TO $y=\frac{dy}{dx}=0$ AT $x=0$

- REDUCE THE O.D.E INTO A FIRST ORDER AS FOLLOWS

$\Rightarrow \frac{d}{dx}\left[\frac{dy}{dx}\right]+\left[\frac{dy}{dx}\right]^2=1$ LET $p=\frac{dy}{dx}$

$\Rightarrow \frac{dp}{dx}+p^2=1$

$\Rightarrow \frac{dp}{dx}=1-p^2$

- SEPARATING VARIABLES & INTEGRATE SUBJECT TO $x=0$, $\frac{dy}{dx}=p=0$

$\Rightarrow \frac{1}{1-p^2}\,dp=1\,dx$

$\Rightarrow \int_0^p \frac{1}{1-p^2}\,dp=\int_0^x 1\,dx$

$\Rightarrow \int_0^p \frac{1}{(1+p)(1-p)}\,dp=\int_0^x 1\,dx$

- PARTIAL FRACTIONS BY INSPECTION (COVER UP) ON THE LHS

$\Rightarrow \int_0^p \frac{\frac{1}{2}}{1+p}+\frac{\frac{1}{2}}{1-p}\,dp=\int_0^x 1\,dx.$

$\Rightarrow \int_0^p \frac{1}{1+p}+\frac{1}{1-p}\,dp=\int_0^x 2\,dx$

$\Rightarrow \left[\ln|1+p|+\ln|1-p|\right]_0^p=\left[2x\right]_0^x$

$\Rightarrow \left(\ln|1+p|+\ln|1-p|\right]-\left[\ln 1+\ln 1\right]=2x$

$\Rightarrow \ln\left|\frac{1+p}{1-p}\right|=2x$

$\Rightarrow \frac{1+p}{1-p}=e^{2x}$

$\Rightarrow 1+p=e^{2x}-pe^{2x}$

$\Rightarrow pe^{2x}+p=e^{2x}-1$

$\Rightarrow p(e^{2x}+1)=e^{2x}-1$

$\Rightarrow p=\frac{e^{2x}-1}{e^{2x}+1}$

$\Rightarrow \frac{dy}{dx}=\frac{e^{2x}-1}{e^{2x}+1}$

$\Rightarrow \int_{y=0}^{y} 1\,dy=\int_{x=0}^{x}\frac{e^{2x}-1}{e^{2x}+1}\,dx$

$\Rightarrow \left[y\right]_0^y=\int_2^{e^{2x}+1}\frac{u-2}{u}\,\frac{du}{2(u-1)}$

$\Rightarrow 2y=\int_2^{e^{2x}+1}\frac{u-2}{u(u-1)}\,du$

SUBSTITUTION
- $u=e^{2x}+1$
- $e^{2x}=u-1$
- $\frac{du}{dx}=2e^{2x}$
- $dx=\frac{du}{2e^{2x}}$
- $dx=\frac{du}{2(u-1)}$
- $x=0,\ u=2$
- $x=x,\ u=e^{2x}+1$

PARTIAL FRACTIONS

$\Rightarrow 2y=\int_2^{e^{2x}+1}\frac{2}{u}-\frac{1}{u-1}\,du$

$\Rightarrow 2y=\left[2\ln|u|-\ln|u-1|\right]_2^{e^{2x}+1}$

$\Rightarrow 2y=\left[2\ln(e^{2x}+1)-\ln(e^{2x})\right]-\left[2\ln 2-\ln 1\right]$

$\Rightarrow 2y=2\ln(e^{2x}+1)-2x-2\ln 2$

$\Rightarrow y=\ln(e^{2x}+1)-\ln 2-x$

$\Rightarrow y=\ln\left[\tfrac{1}{2}(e^{2x}+1)\right]-x$

- VARIATION USING HYPERBOLIC FUNCTIONS FROM THIS EXPRESSION ONWARDS

$\Rightarrow \frac{dy}{dx}=\frac{e^{2x}-1}{e^{2x}+1}$

$\Rightarrow \frac{dy}{dx}=\tanh x$

$\Rightarrow \int_{y=0}^{y}1\,dy=\int_{x=0}^{x}\frac{\sinh x}{\cosh x}\,dx$

$\Rightarrow \left[y\right]_0^y=\left[\ln(\cosh x)\right]_0^x$

$\Rightarrow y=\ln(\cosh x)-\ln 1$

- WHICH MERGES SINCE

$\Rightarrow y=\ln\left[\tfrac{1}{2}e^{x}+\tfrac{1}{2}e^{-x}\right]=\ln\left[e^{-x}\left[\tfrac{1}{2}e^{2x}+\tfrac{1}{2}\right]\right]$

$=\ln e^{-x}+\ln\left[\tfrac{1}{2}e^{2x}+\tfrac{1}{2}\right]=-x+\ln\left[\tfrac{1}{2}(e^{2x}+1)\right]$

Question 5 (***+)

The function with equation $y = f(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{2}{2x-1}\left(1-\frac{dy}{dx}\right), \quad y(0)=1, \quad \frac{dy}{dx}(0) = -1.$$

Solve the above differential equation giving the answer in the form $y = f(x)$.

$$\boxed{y = x + 1 + \ln|2x-1|}$$

- START BY REARRANGING THE O.D.E AS FOLLOWS

$\Rightarrow \frac{d^2y}{dx^2} = \frac{2}{2x-1}\left(1-\frac{dy}{dx}\right)$

$\Rightarrow (2x-1)\frac{d^2y}{dx^2} = 2\left(1-\frac{dy}{dx}\right)$

$\Rightarrow (2x-1)\frac{d^2y}{dx^2} = 2 - 2\frac{dy}{dx}$

$\Rightarrow (2x-1)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2$

- BY INSPECTION THE L.H.S IS A PERFECT DIFFERENTIAL

$\Rightarrow \frac{d}{dx}\left[(2x-1)\frac{dy}{dx}\right] = 2$

$\Rightarrow (2x-1)\frac{dy}{dx} = 2x + A$

$\Rightarrow \frac{dy}{dx} = \frac{2x+A}{2x-1}$

- NOW $\frac{dy}{dx} = -1$ AT $x=0$, GIVES $A=1$

$\Rightarrow \frac{dy}{dx} = \frac{2x+1}{2x-1}$

$\Rightarrow y = \int \frac{2x+1}{2x-1}\,dx$

$\Rightarrow y = \int \frac{(2x-1)+2}{2x-1}\,dx$

$\Rightarrow y = \int 1 + \frac{2}{2x-1}\,dx$

$\Rightarrow y = x + \ln|2x-1| + B$

NOW IF $x=0, y=1 \Rightarrow B=1$

$\therefore y = x + \ln|2x-1| + 1$

Question 6 (****)

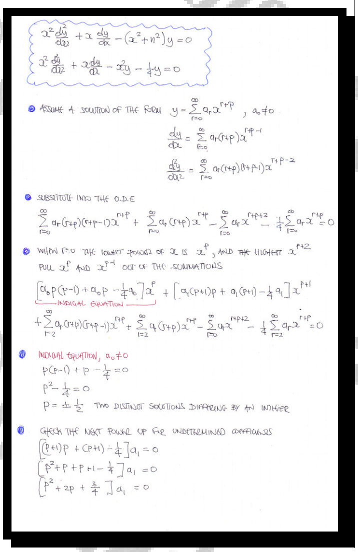
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\left(x^2+n^2\right)y=0.$$

The above differential equation is known as modified Bessel's Equation.

Use the Frobenius method to show that the general solution of this differential equation, for $n=\frac{1}{2}$, is

$$y=x^{-\frac{1}{2}}\left[A\cosh x+B\sinh x\right].$$

proof



$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+n^2)y=0$$
$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-x^2y-\frac{1}{4}y=0$$

ASSUME A SOLUTION OF THE FORM $y=\sum_{r=0}^{\infty}a_rx^{r+p}$, $a_0\neq0$

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(r+p)x^{r+p-1}$$
$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(r+p)(r+p-1)x^{r+p-2}$$

SUBSTITUTE INTO THE O.D.E

$$\sum_{r=0}^{\infty}a_r(r+p)(r+p-1)x^{r+p}+\sum_{r=0}^{\infty}a_r(r+p)x^{r+p}-\sum_{r=0}^{\infty}a_rx^{r+p+2}-\frac{1}{4}\sum_{r=0}^{\infty}a_rx^{r+p}=0$$

WHEN $r=0$ THE LOWEST POWER OF x IS x^p, AND THE HIGHEST x^{p+2}
PULL x^p AND x^{p+1} OUT OF THE SUMMATIONS

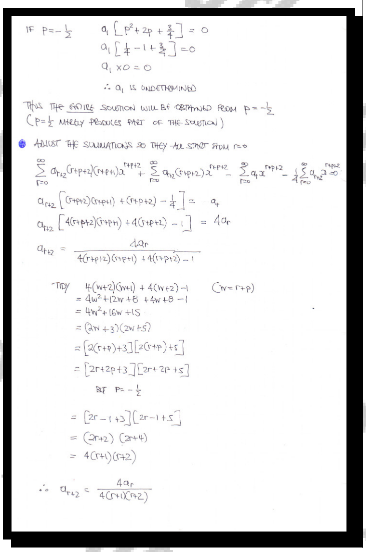
$$\left[a_0p(p-1)+a_0p-\frac{1}{4}a_0\right]x^p+\left[a_1(p+1)p+a_1(p+1)-\frac{1}{4}a_1\right]x^{p+1}$$
INDICIAL EQUATION
$$+\sum_{r=2}^{\infty}a_r(r+p)(r+p-1)x^{r+p}+\sum_{r=2}^{\infty}a_r(r+p)x^{r+p}-\sum_{r=0}^{\infty}a_rx^{r+p+2}-\frac{1}{4}\sum_{r=2}^{\infty}a_rx^{r+p}=0$$

INDICIAL EQUATION, $a_0\neq0$

$$p(p-1)+p-\frac{1}{4}=0$$
$$p^2-\frac{1}{4}=0$$
$p=\pm\frac{1}{2}$ TWO DISTINCT SOLUTIONS DIFFERING BY AN INTEGER

CHECK THE NEXT POWER UP FOR UNDETERMINED COEFFICIENTS

$$\left[(p+1)p+(p+1)-\frac{1}{4}\right]a_1=0$$
$$\left[p^2+p+p+1-\frac{1}{4}\right]a_1=0$$
$$\left[p^2+2p+\frac{3}{4}\right]a_1=0$$



IF $p=-\frac{1}{2}$
$$a_1\left[p^2+2p+\frac{3}{4}\right]=0$$
$$a_1\left[\frac{1}{4}-1+\frac{3}{4}\right]=0$$
$$a_1\times0=0$$
$\therefore$ a_1 IS UNDETERMINED

THUS THE ENTIRE SOLUTION WILL BE OBTAINED FROM $p=-\frac{1}{2}$
($p=\frac{1}{2}$ MERELY PRODUCES PART OF THE SOLUTION)

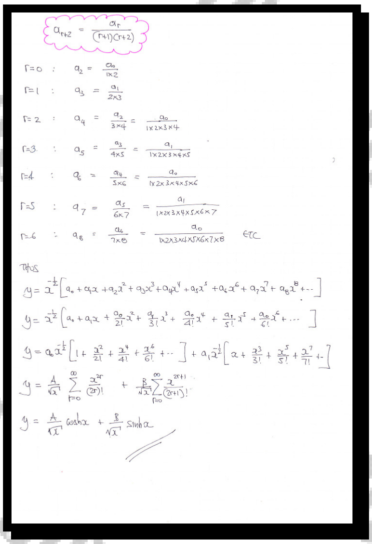
ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=0$

$$\sum_{r=0}^{\infty}a_{r+2}(r+p+2)(r+p+1)x^{r+p+2}+\sum_{r=0}^{\infty}a_{r+2}(r+p+2)x^{r+p+2}-\sum_{r=0}^{\infty}a_rx^{r+p+2}-\frac{1}{4}\sum_{r=0}^{\infty}a_{r+2}x^{r+p+2}=0$$

$$a_{r+2}\left[(r+p+2)(r+p+1)+(r+p+2)-\frac{1}{4}\right]=a_r$$
$$a_{r+2}\left[4(r+p+2)(r+p+1)+4(r+p+2)-1\right]=4a_r$$
$$a_{r+2}=\frac{4a_r}{4(r+p+2)(r+p+1)+4(r+p+2)-1}$$

TIDY $4(w+2)(w+1)+4(w+2)-1$ ($w=r+p$)
$$=4w^2+12w+8+4w+8-1$$
$$=4w^2+16w+15$$
$$=(2w+3)(2w+5)$$
$$=\left[2(r+p)+3\right]\left[2(r+p)+5\right]$$
$$=\left[2r+2p+3\right]\left[2r+2p+5\right]$$
BUT $p=-\frac{1}{2}$
$$=\left[2r-1+3\right]\left[2r-1+5\right]$$
$$=(2r+2)(2r+4)$$
$$=4(r+1)(r+2)$$

$$\therefore\ a_{r+2}=\frac{4a_r}{4(r+1)(r+2)}$$



$$a_{r+2}=\frac{a_r}{(r+1)(r+2)}$$

$r=0$: $a_2=\frac{a_0}{1\times2}$

$r=1$: $a_3=\frac{a_1}{2\times3}$

$r=2$: $a_4=\frac{a_2}{3\times4}=\frac{a_0}{1\times2\times3\times4}$

$r=3$: $a_5=\frac{a_3}{4\times5}=\frac{a_1}{1\times2\times3\times4\times5}$

$r=4$: $a_6=\frac{a_4}{5\times6}=\frac{a_0}{1\times2\times3\times4\times5\times6}$

$r=5$: $a_7=\frac{a_5}{6\times7}=\frac{a_1}{1\times2\times3\times4\times5\times6\times7}$

$r=6$: $a_8=\frac{a_6}{7\times8}=\frac{a_0}{1\times2\times3\times4\times5\times6\times7\times8}$ ETC

THUS

$$y=x^{-\frac{1}{2}}\left[a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7+a_8x^8+\dots\right]$$
$$y=x^{-\frac{1}{2}}\left[a_0+a_1x+\frac{a_0}{2!}x^2+\frac{a_1}{3!}x^3+\frac{a_0}{4!}x^4+\frac{a_1}{5!}x^5+\frac{a_0}{6!}x^6+\dots\right]$$
$$y=a_0x^{-\frac{1}{2}}\left[1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots\right]+a_1x^{-\frac{1}{2}}\left[x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\dots\right]$$
$$y=\frac{A}{\sqrt{x}}\sum_{r=0}^{\infty}\frac{x^{2r}}{(2r)!}+\frac{B}{\sqrt{x}}\sum_{r=0}^{\infty}\frac{x^{2r+1}}{(2r+1)!}$$
$$y=\frac{A}{\sqrt{x}}\cosh x+\frac{B}{\sqrt{x}}\sinh x$$

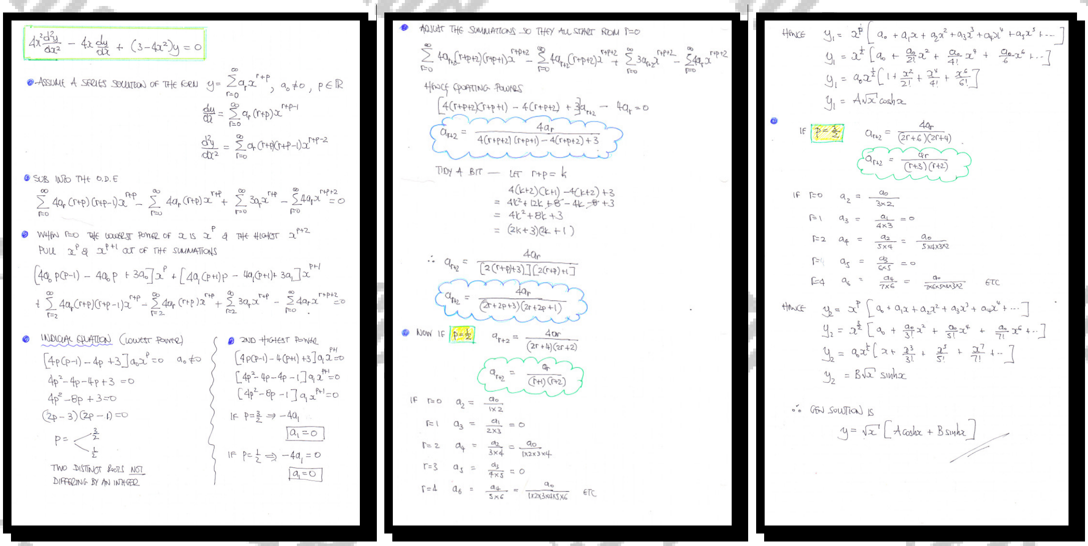
Question 7 (****)

Use the Frobenius method to find a general solution, as an infinite series, for the following differential equation

$$4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + \left(3 - 4x^2\right)y = 0 .$$

Give the final answer in terms of elementary functions.

$$\boxed{y = \sqrt{x}\left(A\cosh x + B\sinh x\right)}$$



Question 8 (****)

Find the solution of following differential equation

$$\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3},$$

subject to the boundary conditions.

$$y\left(-\tfrac{1}{2}\pi\right) = y'\left(-\tfrac{1}{2}\pi\right) = 0, \qquad y''\left(-\tfrac{1}{2}\pi\right) = \tfrac{1}{2}.$$

Given the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = 2\ln\left|\sec\left(\frac{1}{2}x + \frac{1}{4}\pi\right)\right|}$

BY SUBSTITUTION — LET $p = \frac{dy}{dx}$ & SEPARATE VARIABLES

$\Rightarrow \frac{dy}{dx} \times \frac{d^2y}{dx^2} = \frac{d^3y}{dx^3}$

$\Rightarrow p\frac{dp}{dx} = \frac{d^2p}{dx^2}$

$\Rightarrow \int p\,dp = \int \frac{d^2p}{dx^2}\,dx$

$\Rightarrow \frac{1}{2}p^2 = \frac{dp}{dx} + A$

$\Rightarrow p^2 = 2\frac{dp}{dx} + A$

APPLY CONDITION $x = -\frac{\pi}{2}$, $\frac{dy}{dx} = p = 0$, $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{1}{2}$

$\Rightarrow 0 = 2 \times \frac{1}{2} + A$

$\Rightarrow A = -1$

$\Rightarrow p^2 = 2\frac{dp}{dx} - 1$

REARRANGE & SEPARATE VARIABLES AGAIN

$\Rightarrow p^2 + 1 = 2\frac{dp}{dx}$

$\Rightarrow 1\,dx = \frac{2}{p^2+1}\,dp$

$\Rightarrow \int \frac{2}{p^2+1}\,dp = \int 1\,dx$

$\Rightarrow 2\arctan p = x + B$

$\Rightarrow \arctan p = \frac{1}{2}x + B$

$\Rightarrow p = \tan\left(\frac{1}{2}x + B\right)$

$\Rightarrow \frac{dy}{dx} = \tan\left(\frac{1}{2}x + B\right)$

APPLY THE BOUNDARY CONDITION $x = -\frac{\pi}{2}$, $\frac{dy}{dx} = 0$

$\Rightarrow 0 = \tan\left(-\frac{\pi}{4} + B\right)$

$\Rightarrow B = \frac{\pi}{4}$ ONLY AS THIS IN FACT CAME FROM ARCTAN

$\Rightarrow \frac{dy}{dx} = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

FINALLY WE HAVE BY DIRECT INTEGRATION

$\frac{dy}{dx} = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

$y = 2\ln\left|\sec\left(\frac{1}{2}x + \frac{\pi}{4}\right)\right| + C$

APPLY THE LAST CONDITION, $x = -\frac{\pi}{2}$, $y = 0$

$\Rightarrow 0 = 2\ln\left|\sec\left(-\frac{\pi}{4} + \frac{\pi}{4}\right)\right| + C$

$\Rightarrow 0 = 2\ln(\sec 0) + C$

$\Rightarrow 0 = 2\ln 1 + C$

$\Rightarrow C = 0$

$\therefore y = 2\ln\left|\sec\left(\frac{1}{2}x + \frac{\pi}{4}\right)\right|$

Question 9 **(****+)**

A curve has a stationary point at $\left(-\frac{1}{2},-\frac{1}{2}\right)$.

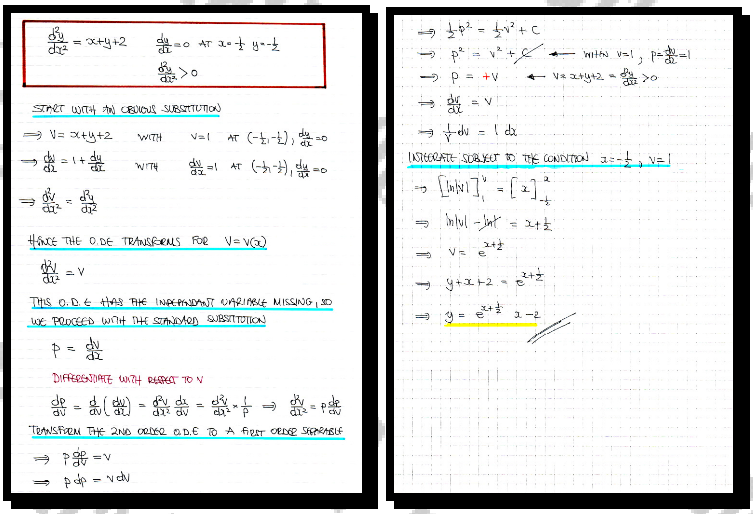
The rate of change of the gradient function of the curve is given by

$$x+y+2,$$

where $x+y+2>0$.

Determine the equation of the curve, giving the answer in the form $y=f(x)$.

$$y=e^{x+\frac{1}{2}}-x-2$$



Question 10 (****+)

Solve the following differential equation

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} = 0, \quad y(0) = 2, \quad \frac{dy}{dx}(0) = -\frac{1}{2}.$$

Give the answer in the form $y^2 = f(x)$.

$\boxed{}$, $\boxed{y^2 = 3 + e^{-2x}}$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y\frac{dy}{dx} = 0 \qquad x=0,\ y=2,\ \frac{dy}{dx} = -\frac{1}{2}$$

BEFORE WE ATTEMPT A SUBSTITUTION, WE OBSERVE THAT THE TERMS OF THIS O.D.E. RESEMBLE DIFFERENTIALS

$$\frac{d}{dx}\left(y\frac{dy}{dx}\right) = \frac{dy}{dx}\times\frac{dy}{dx} + y\times\frac{d^2y}{dx^2} = y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2$$

$$\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$$

HENCE WE MAY REWRITE AS

$$\Rightarrow \frac{d}{dx}\left[y\frac{dy}{dx} + y^2\right] = 0$$

$$\Rightarrow y\frac{dy}{dx} + y^2 = C$$

APPLY CONDITION $y=2$, $\frac{dy}{dx} = -\frac{1}{2}$

$$\Rightarrow 2\left(-\tfrac{1}{2}\right) + 2^2 = C$$

$$\Rightarrow C = 3$$

$$\Rightarrow y\frac{dy}{dx} + y^2 = 3$$

PROCEED BY SEPARATION OF VARIABLES

$$\Rightarrow y\frac{dy}{dx} = 3 - y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3-y^2}{y}$$

$$\Rightarrow \frac{y}{3-y^2}\,dy = 1\,dx$$

$$\Rightarrow \int \frac{-2y}{3-y^2}\,dy = \int -2\,dx$$

$$\Rightarrow \ln|3-y^2| = -2x + A$$

$$\Rightarrow 3 - y^2 = e^{-2x+A}$$

$$\Rightarrow 3 - y^2 = e^{-2x}\times e^{A}$$

$$\Rightarrow 3 - y^2 = Be^{-2x}$$

$$\Rightarrow y^2 = 3 + Be^{-2x}$$

APPLY FINAL CONDITION, $x=0$ $y=2$

$$\Rightarrow 4 = 3 + Be^{0}$$

$$\Rightarrow B = 1$$

$$\therefore\ y^2 = 3 + e^{-2x}$$

NOTE THAT THE SUBSTITUTION $y = \frac{dp}{dx} \Rightarrow \frac{d^2y}{dx^2} = y\frac{dp}{dy}$ YIELDS

$$yp\frac{dp}{dy} + p^2 + 2yp = 0$$

$$\frac{dp}{dy} + \frac{p}{y} = -2$$

BY INTEGRATING FACTOR & CONDITION $y=2$ $p = \frac{dy}{dx} = -\frac{1}{2}$ WE OBTAIN

$$\frac{dy}{dx} = -y + \frac{3}{y}$$

$$\frac{dy}{dx} = \frac{3-y^2}{y}$$

WHICH MERGES WITH THIS SOLUTION

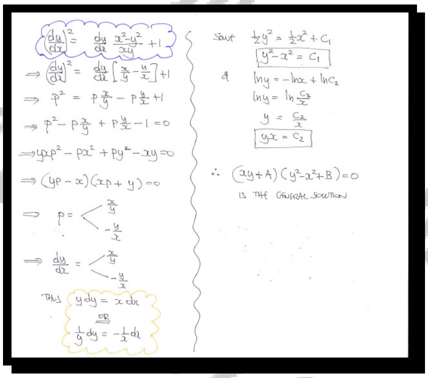
Question 11 (****+)

By writing $\frac{dy}{dx} = p$ and seeking a suitable factorization find a general solution for the non linear differential equation

$$\left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}\left(\frac{x^2 - y^2}{xy}\right) + 1.$$

Give the solution in the form $F(x, y)G(x, y) = 0$.

$$\boxed{(xy + A)\left(x^2 - y^2 + B\right) = 0}$$



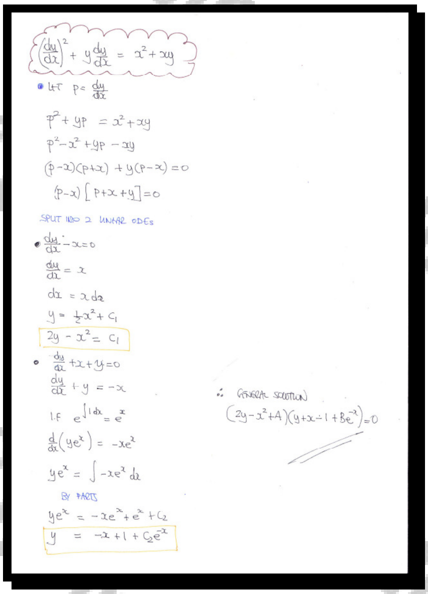
Question 12 (****+)

By writing $\frac{dy}{dx} = p$ and seeking a suitable factorization find a general solution for the non linear differential equation

$$\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = x^2 + xy.$$

Give the solution in the form $F(x, y)G(x, y) = 0$.

$$\boxed{\left(2y - x^2 + A\right)\left(x + y - 1 + B\mathrm{e}^{-x}\right) = 0}$$



$\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = x^2 + xy$

- LET $p = \frac{dy}{dx}$

$p^2 + yp = x^2 + xy$

$p^2 - x^2 + yp - xy$

$(p-x)(p+x) + y(p-x) = 0$

$(p-x)[p+x+y] = 0$

SPLIT INTO 2 LINEAR ODEs

- $\frac{dy}{dx} - x = 0$

$\frac{dy}{dx} = x$

$dx = x\,dx$

$y = \frac{1}{2}x^2 + C_1$

$2y - x^2 = C_1$

- $\frac{dy}{dx} + x + y = 0$

$\frac{dy}{dx} + y = -x$

I.F. $e^{\int 1\,dx} = e^x$

$\frac{d}{dx}\left(ye^x\right) = -xe^x$

$ye^x = \int -xe^x\,dx$

BY PARTS

$ye^x = -xe^x + e^x + C_2$

$y = -x + 1 + C_2e^{-x}$

$\therefore$ GENERAL SOLUTION

$\left(2y - x^2 + A\right)\left(y + x - 1 + Be^{-x}\right) = 0$

Question 13 (*****)

A curve C is described implicitly by the equation

$$xy^2 = e^y .$$

a) Show, by a detailed method, that

$$\left(y^2 - 2y\right)\frac{d^2y}{dx^2} + \left(y^2 - 2\right)\left(\frac{dy}{dx}\right)^2 - 4y^3\frac{dy}{dx}e^{-y} = 0 .$$

b) Use an analytical method, with suitable boundary conditions, to obtain the equation of C by solving the above differential equation.

V , , proof

a) DIFFERENTIATE WITH RESPECT TO x

$\Rightarrow \frac{d}{dx}(xy^2) = \frac{d}{dx}(e^y)$

$\Rightarrow 1 \times y^2 + x\left(2y\frac{dy}{dx}\right) = e^y\frac{dy}{dx}$

$\Rightarrow y^2 + 2xy\frac{dy}{dx} = e^y\frac{dy}{dx}$

DIFFERENTIATE AGAIN WITH RESPECT TO x — TRIPLE PRODUCT RULE IS NEEDED

$\Rightarrow 2y\frac{dy}{dx} + 2y\frac{dy}{dx} + 2x\frac{dy}{dx}\frac{dy}{dx} + 2xy\frac{d^2y}{dx^2} = e^y\frac{dy}{dx}\frac{dy}{dx} + e^y\frac{d^2y}{dx^2}$

$\Rightarrow 4y\frac{dy}{dx} + 2x\left(\frac{dy}{dx}\right)^2 + 2xy\frac{d^2y}{dx^2} = e^y\left[\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2}\right]$

$\Rightarrow 4y\frac{dy}{dx} + 2xy^2\left[\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\frac{d^2y}{dx^2}\right] = e^y\left[\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2}\right]$

$\Rightarrow 4y\frac{dy}{dx} + 2e^y\left[\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\frac{d^2y}{dx^2}\right] = e^y\left[\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2}\right]$

MULTIPLY THROUGH BY y^2e^{-y} & REARRANGE

$\Rightarrow 4y^3e^{-y}\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = y^2\left(\frac{dy}{dx}\right)^2 + y^2\frac{d^2y}{dx^2}$

$\Rightarrow 0 = (y^2 - 2y)\frac{d^2y}{dx^2} + (y^2 - 2)\left(\frac{dy}{dx}\right)^2 - 4y^3e^{-y}\frac{dy}{dx}$

$\Rightarrow (y^2 - 2y)\frac{d^2y}{dx^2} + (y^2 - 2)\left(\frac{dy}{dx}\right)^2 - 4y^3e^{-y}\frac{dy}{dx} = 0$ // AS REQUIRED

b) INTRODUCE FIRSTLY SOME BOUNDARY CONDITIONS BASED ON THE EQUATION OF PART (a), SAY

$x = e,\ y = 1,\ \frac{dy}{dx} = -\frac{1}{e}$

- $e \times 1^2 = e^1 \rightarrow e = e$
- $1^2 + 2 \times e \times 1\frac{dy}{dx} = e\frac{dy}{dx}$

  $1 + 2e\frac{dy}{dx} = e\frac{dy}{dx}$

  $\frac{dy}{dx} = -\frac{1}{e}$

AS THE O.D.E HAS NO x PRESENT TRY THE STANDARD SUBSTITUTION

$p = \frac{dy}{dx}$

$\frac{dp}{dy} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \times \frac{dx}{dy} = \frac{d^2y}{dx^2} \times \frac{1}{p}$

$\therefore \frac{d^2y}{dx^2} = \frac{dp}{dy}p$

TRANSFORM THE O.D.E

$\Rightarrow (y^2 - 2y)\frac{d^2y}{dx^2} + (y^2 - 2)\left(\frac{dy}{dx}\right)^2 - 4y^3e^{-y}\frac{dy}{dx} = 0$

$\Rightarrow (y^2 - 2y)p\frac{dp}{dy} + (y^2 - 2)p^2 - 4y^3e^{-y}p = 0$

$\Rightarrow (y^2 - 2y)\frac{dp}{dy} + (y^2 - 2)p - 4y^3e^{-y} = 0$

$\Rightarrow \frac{dp}{dy} + \frac{y^2 - 2}{y^2 - 2y}p = \frac{4y^3e^{-y}}{y^2 - 2y}$

NEXT LOOK FOR AN INTEGRATING FACTOR — PARTIAL FRACTIONS BY INSPECTION

$e^{\int \frac{y^2 - 2}{y^2 - 2y}dy} = e^{\int \frac{y^2 - 2y + 2y - 2}{y^2 - 2y}dy} = e^{\int 1 + \frac{2y - 2}{y(y-2)}dy}$

$= e^{\int 1 + \frac{1}{y} + \frac{1}{y-2}dy} = e^{y + \ln y + \ln(y-2)}$

$= e^y \times e^{\ln y} \times e^{\ln(y-2)} = e^y[y(y-2)] = e^y(y^2 - 2y)$

$\Rightarrow \frac{d}{dy}\left[pe^y(y^2 - 2y)\right] = \frac{4y^3e^{-y}}{y^2 - 2y} \times e^y(y^2 - 2y)$

$\Rightarrow \frac{d}{dy}\left[pe^y(y^2 - 2y)\right] = 4y^3$

$\Rightarrow pe^y(y^2 - 2y) = \int 4y^3\,dy$

$\Rightarrow pe^y(y^2 - 2y) = y^4 + A$

$\Rightarrow p = \frac{y^4 + A}{y^2 - 2y}e^{-y}$

$\Rightarrow \frac{dy}{dx} = \frac{(y^4 + A)e^{-y}}{y^2 - 2y}$

APPLY THE CONDITION $y = 1,\ \frac{dy}{dx} = -\frac{1}{e}$

$-\frac{1}{e} = \frac{(1 + A)e^{-1}}{1 - 2}$

$-e^{-1} = -(1 + A)e^{-1}$

$-1 = -1 + A$

$A = 0$

$\therefore \frac{dy}{dx} = \frac{y^4e^{-y}}{y^2 - 2y}$

SEPARATING VARIABLES

$\Rightarrow \left(\frac{y^2 - 2y}{y^4e^{-y}}\right)dy = 1\,dx$

$\Rightarrow \left(\frac{y^2 - 2y}{y^4}\right)e^y\,dy = 1\,dx$

$\Rightarrow \int\left(\frac{1}{y^2} - \frac{2}{y^3}\right)e^y\,dy = \int 1\,dx$

$\Rightarrow \int \frac{1}{y^2}e^y\,dy - \int \frac{2}{y^3}e^y\,dy = x + B$

NOW INTEGRATION BY PARTS — ONLY ON ONE OF THE INTEGRALS OF THE L.H.S, SAY THE FIRST ONE

$\frac{1}{y^2}$	$-\frac{2}{y^3}$
e^y	e^y

$\Rightarrow \left[\frac{1}{y^2}e^y + \int \frac{2e^y}{y^3}dy\right] - \int \frac{2e^y}{y^3}dy = x + B$

$\Rightarrow \frac{e^y}{y^2} = x + B$

APPLY THE FINAL CONDITION $x = e,\ y = 1$

$\Rightarrow \frac{e^1}{1^2} = e + B$

$\Rightarrow e = e + B$

$\Rightarrow B = 0$

$\therefore \frac{e^y}{y^2} = x$

$xy^2 = e^y$ // AS EXPECTED

Question 14 (*****)

Find a general solution of the following differential equation.

$$y = x\frac{dy}{dx} + e^{\frac{dy}{dx}}.$$

$\boxed{}$, $\boxed{\left(y + Ax + B\right)\left(y - x\ln x + Cx\right) = 0}$