SERIES

EXAM QUESTIONS

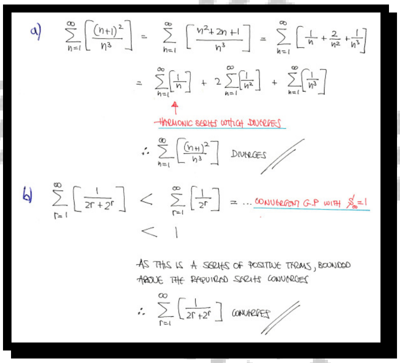
Question 1 (**)

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3}.$$

b) $$\sum_{r=1}^{\infty} \frac{1}{2r+2^r}.$$

[ ] , [divergent] , [convergent]



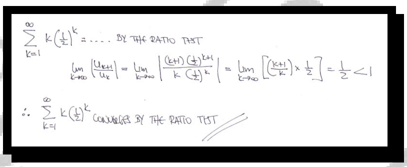
Question 2 (**)

Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k.$$

Show a full method, justifying every step in the workings.

[convergent]



Question 3 (**)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\displaystyle\sum_{r=3}^{\infty} \frac{\sqrt{r}}{r-2}$.

b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\left(k^4+2\right)\sqrt{k}}$.

divergent, convergent

(a) $\displaystyle\sum_{r=3}^{\infty} \frac{\sqrt{r}}{r-2} > \sum_{r=3}^{\infty} \frac{\sqrt{r}}{r} = \sum_{r=3}^{\infty} \frac{1}{\sqrt{r}}$ WHICH DIVERGES BY THE P-TEST

$\therefore \displaystyle\sum_{r=1}^{\infty} \frac{\sqrt{r}}{r-2}$ IS DIVERGENT BY COMPARISON

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\left(k^4+2\right)} < \sum_{k=1}^{\infty} \frac{1}{\sqrt{k} \times k^4} = \sum_{k=1}^{\infty} \frac{1}{k^{9/2}}$ WHICH CONVERGES BY THE P-TEST

$\therefore \displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\left(k^4+2\right)}$ IS CONVERGENT BY COMPARISON

P-TEST $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ CONVERGES IF $p>1$, DIVERGES IF $p \leq 1$

Question 4 (**)

Determine whether the following series converges or diverges.

$$\sum_{r=1}^{\infty} \frac{1}{r^2+4r}.$$

Show a full method, justifying every step in the workings.

convergent

$\displaystyle\sum_{r=1}^{\infty} \left(\frac{1}{r^2+4r}\right) < \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$

$\therefore \displaystyle\sum_{r=1}^{\infty} \frac{1}{r^2+4r}$ CONVERGES BY COMPARISON

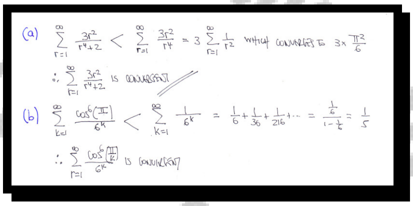
Question 5 (**)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $$\sum_{r=1}^{\infty} \frac{3r^2}{r^4+2}.$$

b) $$\sum_{k=1}^{\infty} \left[\frac{\cos^6\left(\frac{\pi}{k}\right)}{6^k} \right].$$

convergent , convergent



(a) $\sum_{r=1}^{\infty} \frac{3r^2}{r^4+2} < \sum_{r=1}^{\infty} \frac{3r^2}{r^4} = 3\sum_{r=1}^{\infty} \frac{1}{r^2}$ WHICH CONVERGES TO $3 \times \frac{\pi^2}{6}$

$\therefore \sum_{r=1}^{\infty} \frac{3r^2}{r^4+2}$ IS CONVERGENT

(b) $\sum_{k=1}^{\infty} \frac{\cos^6\left(\frac{\pi}{k}\right)}{6^k} < \sum_{k=1}^{\infty} \frac{1}{6^k} = \frac{1}{6}+\frac{1}{36}+\frac{1}{216}+\dots = \frac{\frac{1}{6}}{1-\frac{1}{6}} = \frac{1}{5}$

$\therefore \sum_{r=1}^{\infty} \frac{\cos^6\left(\frac{\pi}{k}\right)}{6^k}$ IS CONVERGENT

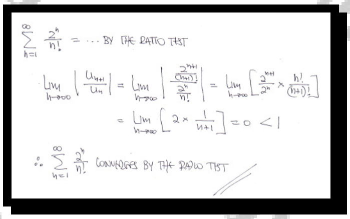
Question 6 (**)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}.$$

Show a full method, justifying every step in the workings.

convergent



$\sum_{n=1}^{\infty} \frac{2^n}{n!} = \dots$ BY THE RATIO TEST

$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right| = \lim_{n\to\infty} \left[\frac{2^{n+1}}{2^n} \times \frac{n!}{(n+1)!} \right]$

$= \lim_{n\to\infty} \left[2 \times \frac{1}{n+1} \right] = 0 < 1$

$\therefore \sum_{n=1}^{\infty} \frac{2^n}{n!}$ CONVERGES BY THE RATIO TEST

Question 7 **(**)**

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} n^2 \left(\tfrac{1}{2}\right)^n .$$

Show a full method, justifying every step in the workings,

convergent

$\sum_{n=1}^{\infty} n^2\left(\frac{1}{2}\right)^n$ BY THE RATIO TEST

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2\left(\frac{1}{2}\right)^{n+1}}{n^2\left(\frac{1}{2}\right)^n}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^2 \times \frac{1}{2}\right|$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = \frac{1}{2} < 1$$

$\therefore \sum_{n=1}^{\infty} n^2\left(\frac{1}{2}\right)^n$ CONVERGES BY THE RATIO TEST

Question 8 **(**)**

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} .$$

Show a full method, justifying every step in the workings.

divergent

$$\sum_{n=1}^{\infty}\frac{(n+1)^n}{n^{n+1}} > \sum_{n=1}^{\infty}\frac{n^n}{n^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{n} \text{ WHICH DIVERGES}$$

$\therefore \sum_{n=1}^{\infty}\frac{(n+1)^n}{n^{n+1}}$ DIVERGES BY COMPARISON

Question 9 (**)

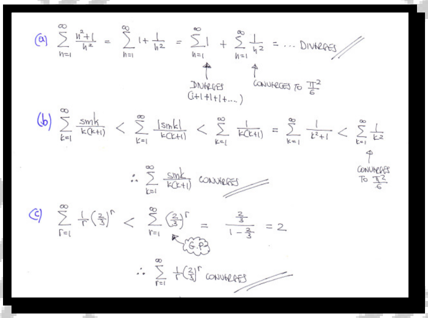
By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^2}$.

b) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k(k+1)}$.

c) $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r}\left(\frac{2}{3}\right)^r$.

divergent, convergent, convergent



(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^2} = \sum_{n=1}^{\infty} 1+\frac{1}{n^2} = \sum_{n=1}^{\infty} 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} = \dots$ DIVERGES

DIVERGES $(1+1+1+1+\dots)$; CONVERGES TO $\frac{\pi^2}{6}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k(k+1)} < \sum_{k=1}^{\infty} \frac{|\sin k|}{k(k+1)} < \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k^2+1} < \sum_{k=1}^{\infty} \frac{1}{k^2}$

CONVERGES TO $\frac{\pi^2}{6}$

$\therefore \displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k(k+1)}$ CONVERGES

(c) $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r}\left(\frac{2}{3}\right)^r < \sum_{r=1}^{\infty} \left(\frac{2}{3}\right)^r = \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2$

G.P.

$\therefore \displaystyle\sum_{r=1}^{\infty} \frac{1}{r}\left(\frac{2}{3}\right)^r$ CONVERGES

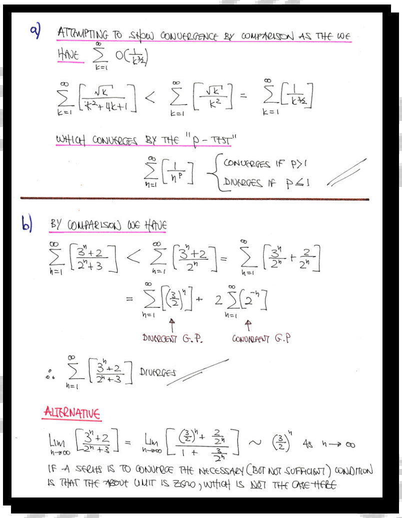
Question 10 (**)

Investigate the convergence or divergence of each of the following series justifying every step in the workings.

a) $\displaystyle\sum_{k=1}^{\infty}\left[\frac{\sqrt{k}}{k^2+4k+1}\right]$.

b) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{3^n+2}{2^n+3}\right]$.

[ ] , [convergent] , [divergent]



a) ATTEMPTING TO SHOW CONVERGENCE BY COMPARISON AS THE WE HAVE $\sum_{k=1}^{\infty} O\left(\frac{1}{k^{3/2}}\right)$

$$\sum_{k=1}^{\infty}\left[\frac{\sqrt{k}}{k^2+4k+1}\right] < \sum_{k=1}^{\infty}\left[\frac{\sqrt{k}}{k^2}\right] = \sum_{k=1}^{\infty}\left[\frac{1}{k^{3/2}}\right]$$

WHICH CONVERGES BY THE "p – TEST"

$$\sum_{n=1}^{\infty}\left[\frac{1}{n^p}\right] \begin{cases} \text{CONVERGES IF } p>1 \\ \text{DIVERGES IF } p\le 1 \end{cases}$$

b) BY COMPARISON WE HAVE

$$\sum_{n=1}^{\infty}\left[\frac{3^n+2}{2^n+3}\right] < \sum_{n=1}^{\infty}\left[\frac{3^n+2}{2^n}\right] = \sum_{n=1}^{\infty}\left[\frac{3^n}{2^n}+\frac{2}{2^n}\right]$$

$$= \sum_{n=1}^{\infty}\left[\left(\tfrac{3}{2}\right)^n\right] + 2\sum_{n=1}^{\infty}\left[2^{-n}\right]$$

DIVERGENT G.P. CONVERGENT G.P

$$\therefore \sum_{n=1}^{\infty}\left[\frac{3^n+2}{2^n+3}\right] \text{ DIVERGES}$$

ALTERNATIVE

$$\lim_{n\to\infty}\left[\frac{3^n+2}{2^n+3}\right] = \lim_{n\to\infty}\left[\frac{\left(\frac{3}{2}\right)^n+\frac{2}{2^n}}{1+\frac{3}{2^n}}\right] \sim \left(\tfrac{3}{2}\right)^n \text{ as } n\to\infty$$

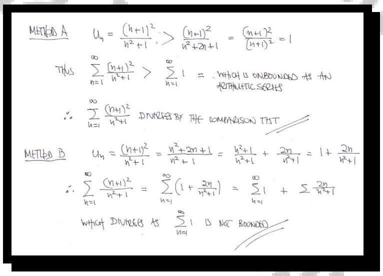
IF A SERIES IS TO CONVERGE THE NECESSARY (BUT NOT SUFFICIENT) CONDITION IS THAT THE ABOVE LIMIT IS ZERO, WHICH IS NOT THE CASE HERE

Question 11 **(**)**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^2+1}.$$

divergent


METHOD A $u_n = \frac{(n+1)^2}{n^2+1} > \frac{(n+1)^2}{n^2+2n+1} = \frac{(n+1)^2}{(n+1)^2} = 1$

THUS $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^2+1} > \sum_{n=1}^{\infty} 1$ = WHICH IS UNBOUNDED AS AN ARITHMETIC SERIES

$\therefore \sum_{n=1}^{\infty} \frac{(n+1)^2}{n^2+1}$ DIVERGES BY THE COMPARISON TEST

METHOD B $u_n = \frac{(n+1)^2}{n^2+1} = \frac{n^2+2n+1}{n^2+1} = \frac{n^2+1}{n^2+1} + \frac{2n}{n^2+1} = 1 + \frac{2n}{n^2+1}$

$\therefore \sum_{n=1}^{\infty} \frac{(n+1)^2}{n^2+1} = \sum_{n=1}^{\infty}\left(1+\frac{2n}{n^2+1}\right) = \sum_{n=1}^{\infty} 1 + \sum \frac{2n}{n^2+1}$

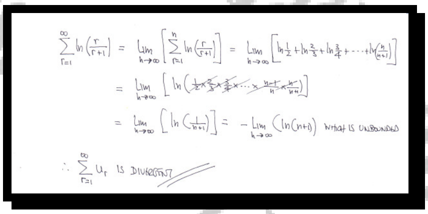
WHICH DIVERGES AS $\sum_{n=1}^{\infty} 1$ IS NOT BOUNDED

Question 12 **(**+)**

$$u_r = \ln\left(\frac{r}{r+1}\right), \quad r \in \mathbb{N}.$$

Show clearly that $\sum_{r=1}^{\infty} u_r$ is divergent.

proof


$\sum_{r=1}^{\infty} \ln\left(\frac{r}{r+1}\right) = \lim_{n\to\infty}\left[\sum_{r=1}^{n} \ln\left(\frac{r}{r+1}\right)\right] = \lim_{n\to\infty}\left[\ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \dots + \ln\left(\frac{n}{n+1}\right)\right]$

$= \lim_{n\to\infty}\left[\ln\left(\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\dots\times\frac{n-1}{n}\times\frac{n}{n+1}\right)\right]$

$= \lim_{n\to\infty}\left[\ln\left(\frac{1}{n+1}\right)\right] = -\lim_{n\to\infty}\left(\ln(n+1)\right)$ WHICH IS UNBOUNDED

$\therefore \sum_{r=1}^{\infty} u_r$ IS DIVERGENT

Question 13 (**+)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=2}^{\infty} \frac{(n-1)^2}{n^2-1}.$$

divergent

$$\sum_{n=2}^{\infty} \frac{(n-1)^2}{n^2-1} = \sum_{n=2}^{\infty} \frac{n^2-2n+1}{n^2-1} > \sum_{n=2}^{\infty} \frac{n^2-2n+1}{n^2+1} = \sum_{n=2}^{\infty}\left(\frac{n^2+1}{n^2+1} - \frac{2n}{n^2+1}\right)$$

$$= \sum_{n=2}^{\infty}\left(1 - \frac{2n}{n^2+1}\right) = \sum_{n=2}^{\infty} 1 - \sum_{n=2}^{\infty} \frac{2n}{n^2+1}$$

WHICH DIVERGES AS THE FIRST SERIES IS AN ARITHMETIC PROGRESSION

Question 14 (***)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $$\sum_{n=1}^{\infty} \left[\frac{\sin^2 n}{n(n+1)}\right].$$

b) $$\sum_{n=1}^{\infty} \left[\frac{2n}{3n^2-4}\right].$$

convergent, divergent

(a) $u_n = \frac{\sin^2 n}{n(n+1)} < \frac{1}{n^2+n} < \frac{1}{n^2}$

NOW $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ IE IT CONVERGES (STANDARD RESULT)

$\therefore \sum_{n=1}^{\infty} \frac{\sin^2 n}{n(n+1)}$ CONVERGES BY THE COMPARISON TEST

(b) $u_n = \frac{2n}{3n^2-4} = \frac{2}{3n-\frac{4}{n}} > \frac{2}{3n} = \frac{2}{3}\left(\frac{1}{n}\right)$

NOW $\sum_{n=1}^{\infty} \frac{2}{3n} = \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n}$ DIVERGES (HARMONIC SERIES)

$\therefore \sum_{n=1}^{\infty}\left(\frac{2n}{3n^2-4}\right)$ DIVERGES BY THE COMPARISON TEST

Question 15 (***)

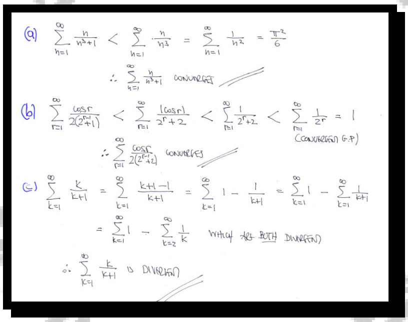
By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$.

b) $\displaystyle\sum_{r=1}^{\infty} \frac{\cos r}{2\left(2^{r-1}+1\right)}$.

c) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k+1}$.

convergent, convergent, divergent



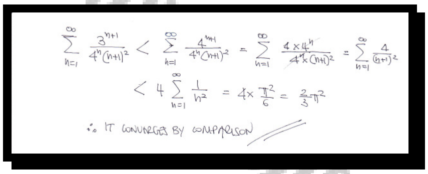
Question 16 (***)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n (n+1)^2} .$$

Show a full method, justifying every step in the workings.

convergent



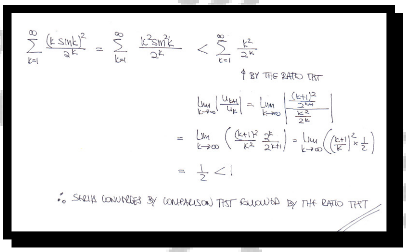
Question 17 (***)

Determine whether the following series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{(k \sin k)^2}{2^k} .$$

Show a full method, justifying every step in the workings,

convergent

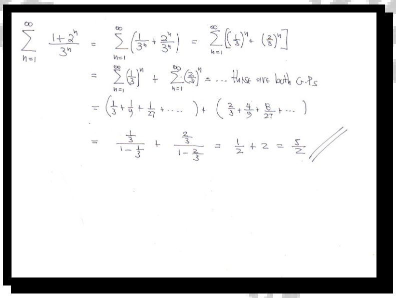


Question 18 **(***)**

Evaluate showing clearly your method

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}.$$

$\boxed{\frac{5}{2}}$



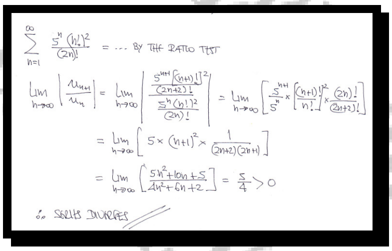
Question 19 **(***)**

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5^n (n!)^2}{(2n)!}.$$

Show a full method, justifying every step in the workings.

$\boxed{\text{divergent}}$



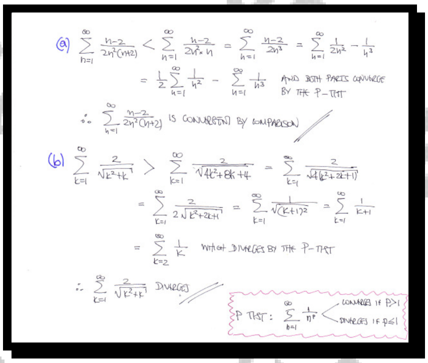
Question 20 (***)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $$\sum_{n=1}^{\infty} \frac{n-2}{2n^2(n+2)}.$$

b) $$\sum_{k=1}^{\infty} \frac{2}{\sqrt{k^2+k}}.$$

convergent , divergent



Question 21 (***)

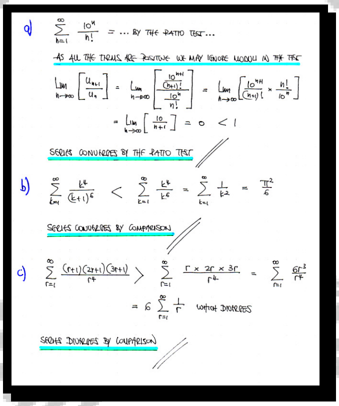
By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$.

b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^4}{(k+1)^6}$.

c) $\displaystyle\sum_{r=1}^{\infty} \frac{(r+1)(2r+1)(3r+1)}{r^4}$.

[ ], convergent, convergent, divergent



a) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!} = \ldots$ BY THE RATIO TEST...

AS ALL THE TERMS ARE POSITIVE WE MAY IGNORE MODULI IN THE TEST

$$\lim_{n\to\infty}\left[\frac{u_{n+1}}{u_n}\right] = \lim_{n\to\infty}\left[\frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}}\right] = \lim_{n\to\infty}\left[\frac{10^{n+1}}{(n+1)!}\times\frac{n!}{10^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{10}{n+1}\right] = 0 < 1$$

SERIES CONVERGES BY THE RATIO TEST

b) $\displaystyle\sum_{k=1}^{\infty}\frac{k^4}{(k+1)^6} < \sum_{k=1}^{\infty}\frac{k^4}{k^6} = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$

SERIES CONVERGES BY COMPARISON

c) $\displaystyle\sum_{r=1}^{\infty}\frac{(r+1)(2r+1)(3r+1)}{r^4} > \sum_{r=1}^{\infty}\frac{r\times 2r\times 3r}{r^4} = \sum_{r=1}^{\infty}\frac{6r^3}{r^4}$

$\displaystyle = 6\sum_{r=1}^{\infty}\frac{1}{r}$ WHICH DIVERGES

SERIES DIVERGES BY COMPARISON

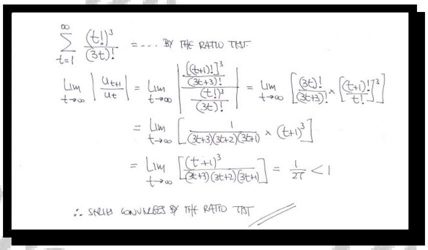
Question 22 (***)

Determine whether the following series converges or diverges.

$$\sum_{t=1}^{\infty} \frac{(t!)^3}{(3t)!}.$$

Show a full method, justifying every step in the workings.

convergent



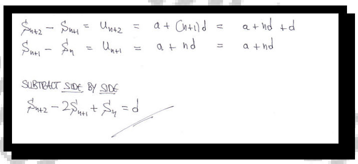
Question 23 (***)

The sum of the first n terms of an arithmetic series with first term a and common difference d, is denoted by S_n.

Simplify fully

$$S_n - 2S_{n+1} + S_{n+2}.$$

$S_n - 2S_{n+1} + S_{n+2} = d$



Question 24 (***)

It is given that

$$\frac{1}{n}\sum_{r=1}^{n} x_r = 2 \qquad \text{and} \qquad \sqrt{\frac{1}{n}\sum_{r=1}^{n}(x_r)^2 - \frac{1}{n^2}\left(\sum_{r=1}^{n} x_r\right)^2} = 3 .$$

Determine, in terms of n, the value of

$$\sum_{r=1}^{n}(x_r + 1)^2 .$$

$$\boxed{}, \quad \boxed{\sum_{r=1}^{n}(x_r+1)^2 = 18n}$$

$$\frac{\sum_{r=1}^{n} x_r}{n} = 2 \implies \sum_{r=1}^{n} x_r = 2n$$

$$\sqrt{\frac{\sum_{r=1}^{n}(x_r)^2}{n} - \left(\frac{\sum_{r=1}^{n} x_r}{n}\right)^2} = 3$$

$$\implies \frac{\sum_{r=1}^{n}(x_r)^2}{n} - \frac{\left(\sum_{r=1}^{n} x_r\right)^2}{n^2} = 9$$

$$\implies \frac{\sum_{r=1}^{n}(x_r)^2}{n} - \frac{(2n)^2}{n^2} = 9$$

$$\implies \frac{\sum_{r=1}^{n}(x_r)^2}{n} - 4 = 9$$

$$\implies \sum_{r=1}^{n}(x_r)^2 = 13n$$

Thus

$$\sum_{r=1}^{n}(x_r+1)^2 = \sum_{r=1}^{n}\left[(x_r)^2 + 2x_r + 1\right]$$

$$= \sum_{r=1}^{n}(x_r)^2 + 2\sum_{r=1}^{n}(x_r) + \sum_{r=1}^{n} 1$$

$$= 13n + 2(2n) + n$$

$$= 18n$$

Question 25 (***)

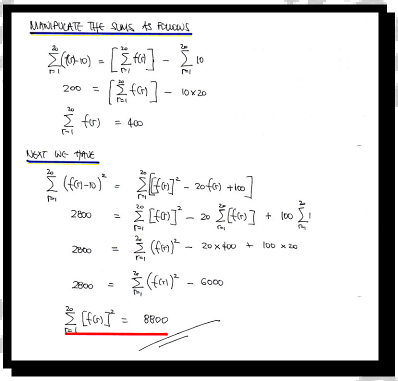
It is given that

$$\sum_{r=1}^{20}\left[f(r)-10\right]=200 \quad \text{and} \quad \sum_{r=1}^{20}\left[f(r)-10\right]^2=2800.$$

Find the value of

$$\sum_{r=1}^{20}\left[f(r)\right]^2.$$

$\boxed{}$, $\boxed{\sum_{r=1}^{20}\left[f(r)\right]^2=8800}$


MANIPULATE THE SUMS AS FOLLOWS

$$\sum_{r=1}^{20}(f(r)-10)=\left[\sum_{r=1}^{20}f(r)\right]-\sum_{r=1}^{20}10$$
$$200=\left[\sum_{r=1}^{20}f(r)\right]-10\times 20$$
$$\sum_{r=1}^{20}f(r)=400$$

NEXT WE HAVE

$$\sum_{r=1}^{20}(f(r)-10)^2=\sum_{r=1}^{20}\left[[f(r)]^2-20f(r)+100\right]$$
$$2800=\sum_{r=1}^{20}[f(r)]^2-20\sum_{r=1}^{20}[f(r)]+100\sum_{r=1}^{20}1$$
$$2800=\sum_{r=1}^{20}(f(r))^2-20\times 400+100\times 20$$
$$2800=\sum_{r=1}^{20}(f(r))^2-6000$$
$$\sum_{r=1}^{20}[f(r)]^2=8800$$

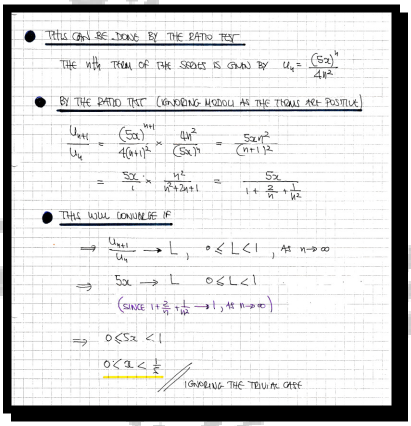
Question 26 (***)

It is given that the following series converges.

$$\sum_{n=1}^{\infty} \frac{(5x)^n}{4n^2}, \quad x \in \mathbb{R}, \; x > 0 .$$

Determine the range of possible values of x.

$\boxed{}$, $\boxed{0 < x < \frac{1}{5}}$



- THIS CAN BE DONE BY THE RATIO TEST

THE nth TERM OF THE SERIES IS GIVEN BY $u_n = \frac{(5x)^n}{4n^2}$

- BY THE RATIO TEST (IGNORING MODULI AS THE TERMS ARE POSITIVE)

$$\frac{u_{n+1}}{u_n} = \frac{(5x)^{n+1}}{4(n+1)^2} \times \frac{4n^2}{(5x)^n} = \frac{5xn^2}{(n+1)^2}$$

$$= \frac{5x}{1} \times \frac{n^2}{n^2+2n+1} = \frac{5x}{1+\frac{2}{n}+\frac{1}{n^2}}$$

- THIS WILL CONVERGE IF

$\Rightarrow \frac{u_{n+1}}{u_n} \to L, \quad 0 \leqslant L < 1, \text{ AS } n \to \infty$

$\Rightarrow 5x \to L \quad 0 \leqslant L < 1$

(SINCE $1 + \frac{2}{n} + \frac{1}{n^2} \to 1$, AS $n \to \infty$)

$\Rightarrow 0 \leqslant 5x < 1$

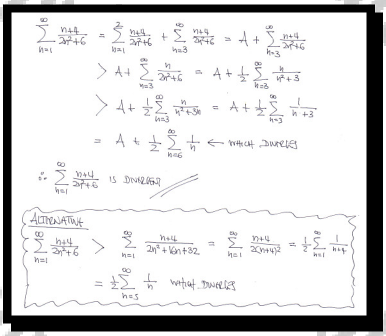
$0 < x < \frac{1}{5}$ // IGNORING THE TRIVIAL CASE

Question 27 **(***+)**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{n+4}{2n^2+6}.$$

divergent



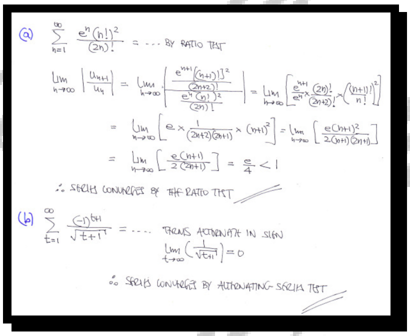
Question 28 (***+)

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $$\sum_{n=1}^{\infty} \frac{e^n (n!)^2}{(2n)!}.$$

b) $$\sum_{t=1}^{\infty} \frac{(-1)^{t+1}}{\sqrt{t+1}}.$$

convergent, convergent



(a) $\sum_{n=1}^{\infty} \frac{e^n (n!)^2}{(2n)!} = \dots$ BY RATIO TEST

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{e^{n+1}[(n+1)!]^2}{(2n+2)!}}{\frac{e^n (n!)^2}{(2n)!}}\right| = \lim_{n\to\infty}\left[\frac{e^{n+1}}{e^n}\times\frac{(2n)!}{(2n+2)!}\times\left(\frac{(n+1)!}{n!}\right)^2\right]$$

$$= \lim_{n\to\infty}\left[e\times\frac{1}{(2n+2)(2n+1)}\times(n+1)^2\right] = \lim_{n\to\infty}\left[\frac{e(n+1)^2}{2(n+1)(2n+1)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{e(n+1)}{2(2n+1)}\right] = \frac{e}{4} < 1$$

∴ SERIES CONVERGES BY THE RATIO TEST

(b) $\sum_{t=1}^{\infty} \frac{(-1)^{t+1}}{\sqrt{t+1}} = \dots$ TERMS ALTERNATE IN SIGN

$$\lim_{t\to\infty}\left(\frac{1}{\sqrt{t+1}}\right) = 0$$

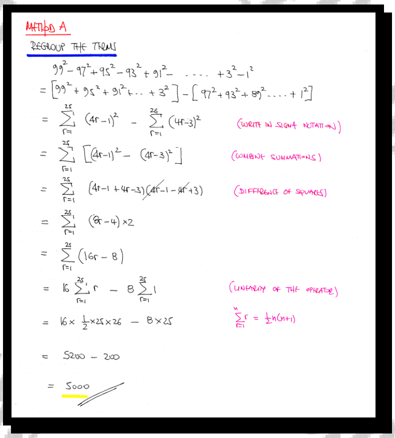
∴ SERIES CONVERGES BY ALTERNATING SERIES TEST

Question 29 (*+)**

By using an algebraic method, find the value of

$$99^2 - 97^2 + 95^2 - 93^2 + \ldots + 3^2 - 1^2$$

, $\boxed{5000}$



METHOD A

REGROUP THE TERMS

$99^2 - 97^2 + 95^2 - 93^2 + 91^2 - \ldots + 3^2 - 1^2$

$= [99^2 + 95^2 + 91^2 + \ldots + 3^2] - [97^2 + 93^2 + 89^2 \ldots + 1^2]$

$= \sum_{r=1}^{25} (4r-1)^2 - \sum_{r=1}^{25} (4r-3)^2$ (WRITE IN SIGMA NOTATION)

$= \sum_{r=1}^{25} [(4r-1)^2 - (4r-3)^2]$ (COMBINE SUMMATIONS)

$= \sum_{r=1}^{25} (4r-1+4r-3)(4r-1-4r+3)$ (DIFFERENCE OF SQUARES)

$= \sum_{r=1}^{25} (8r-4) \times 2$

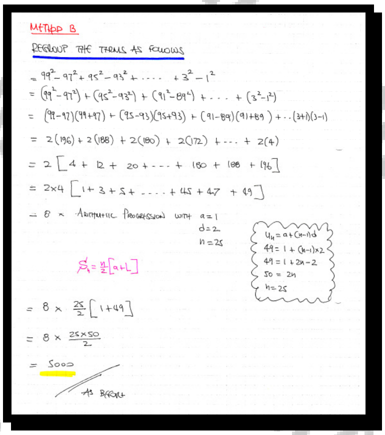
$= \sum_{r=1}^{25} (16r-8)$

$= 16 \sum_{r=1}^{25} r - 8 \sum_{r=1}^{25} 1$ (LINEARITY OF THE OPERATOR)

$= 16 \times \frac{1}{2} \times 25 \times 26 - 8 \times 25$ $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

$= 5200 - 200$

$= 5000$



METHOD B

REGROUP THE TERMS AS FOLLOWS

$= 99^2 - 97^2 + 95^2 - 93^2 + \ldots + 3^2 - 1^2$

$= (99^2 - 97^2) + (95^2 - 93^2) + (91^2 - 89^2) + \ldots + (3^2 - 1^2)$

$= (99-97)(99+97) + (95-93)(95+93) + (91-89)(91+89) + \ldots (3+1)(3-1)$

$= 2(196) + 2(188) + 2(180) + 2(172) + \ldots + 2(4)$

$= 2[4 + 12 + 20 + \ldots + 180 + 188 + 196]$

$= 2 \times 4 [1 + 3 + 5 + \ldots + 45 + 47 + 49]$

$= 8 \times$ ARITHMETIC PROGRESSION WITH $a=1$, $d=2$, $n=25$

$u_n = a + (n-1)d$

$49 = 1 + (n-1) \times 2$

$49 = 1 + 2n - 2$

$50 = 2n$

$n = 25$

$S_n = \frac{n}{2}[a+L]$

$= 8 \times \frac{25}{2}[1+49]$

$= 8 \times \frac{25 \times 50}{2}$

$= 5000$

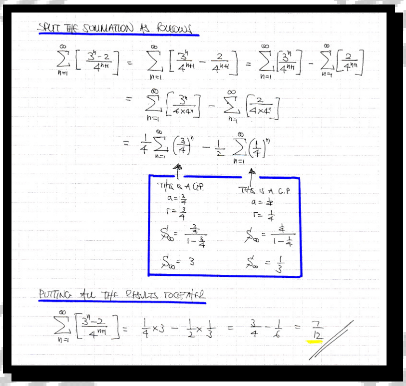
AS BEFORE

Question 30 (***+)

Evaluate, showing clearly your method

$$\sum_{n=1}^{\infty} \frac{3^n - 2}{4^{n+1}}.$$

$\boxed{}$, $\boxed{\frac{7}{12}}$

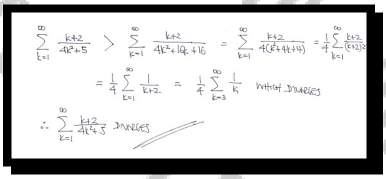


Question 31 (***+)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{k=1}^{\infty} \frac{k+2}{4k^2+5}.$$

$\boxed{\text{divergent}}$

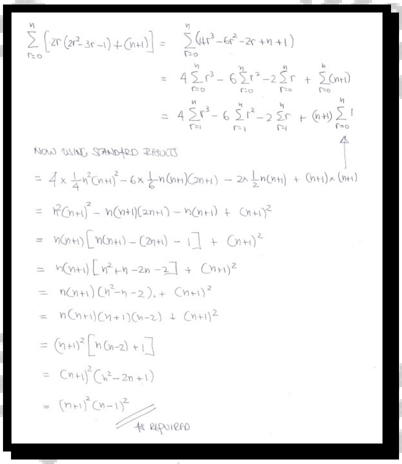


Question 32 (***+)

Show clearly that

$$\sum_{r=1}^{n}\left[2r\left(2r^2-3r-1\right)+n+1\right]=(n+1)^2(n-1)^2 .$$

proof



Question 33 **(***+)**

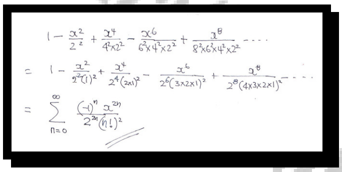
Consider the infinite series

$$1-\frac{x^2}{2^2}+\frac{x^4}{4^2\times 2^2}-\frac{x^6}{6^2\times 4^2\times 2^2}+\frac{x^8}{8^2\times 6^2\times 4^2\times 2^2}-\dots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}x^{2n}}$$


$1-\frac{x^2}{2^2}+\frac{x^4}{4^2\times2^2}-\frac{x^6}{6^2\times4^2\times2^2}+\frac{x^8}{8^2\times6^2\times4^2\times2^2}-\dots$

$=1-\frac{x^2}{2^2(1)^2}+\frac{x^4}{2^4(2\times1)^2}-\frac{x^6}{2^6(3\times2\times1)^2}+\frac{x^8}{2^8(4\times3\times2\times1)^2}-\dots$

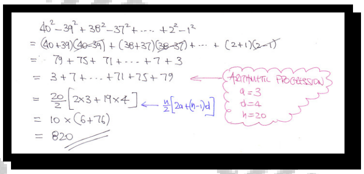
$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

Question 34 **(***+)**

Show clearly by an algebraic method that

$$40^2-39^2+38^2-37^2+\dots+2^2-1^2=820.$$

proof


$40^2-39^2+38^2-37^2+\dots+2^2-1^2$

$=(40+39)(40-39)+(38+37)(38-37)+\dots+(2+1)(2-1)$

$=79+75+71+\dots+7+3$

$=3+7+\dots+71+75+79$ ← ARITHMETIC PROGRESSION $a=3$, $d=4$, $n=20$

$=\frac{20}{2}[2\times3+19\times4]$ ← $\frac{n}{2}[2a+(n-1)d]$

$=10\times(6+76)$

$=820$

Question 35 (****)

By justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[3]{n^2}}.$$

convergent

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[3]{n^2}} = \sum_{n=1}^{\infty} \frac{\left[\sqrt{n+1}-\sqrt{n}\right]\left[\sqrt{n+1}+\sqrt{n}\right]}{\sqrt[3]{n^2}\left[\sqrt{n+1}+\sqrt{n}\right]}$$

$$= \sum_{n=1}^{\infty} \frac{(n+1)-n}{n^{\frac{2}{3}}\left[(n+1)^{\frac{1}{2}}+n^{\frac{1}{2}}\right]}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}\left[(n+1)^{\frac{1}{2}}+n^{\frac{1}{2}}\right]}$$

$$< \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}\left[n^{\frac{1}{2}}+n^{\frac{1}{2}}\right]}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}\left(2n^{\frac{1}{2}}\right)}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^{7/6}}$$

WHICH CONVERGES SINCE

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{CONVERGES IF } p>1, \quad \text{DIVERGES IF } p\le 1$$

Question 36 (****)

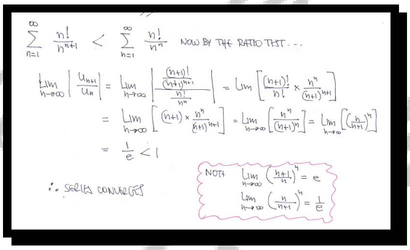
Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n!}{n^{n+1}}.$$

Show a full method, justifying every step in the workings.

You may assume without proof the value of $\lim_{n \to \infty} \left[\left(\frac{n}{n+1} \right)^n \right]$.

convergent



$\sum_{n=1}^{\infty} \frac{n!}{n^{n+1}} < \sum_{n=1}^{\infty} \frac{n!}{n^n}$ NOW BY THE RATIO TEST...

$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \right| = \lim \left[\frac{(n+1)!}{n!} \times \frac{n^n}{(n+1)^{n+1}} \right]$

$= \lim_{n \to \infty} \left[(n+1) \times \frac{n^n}{(n+1)^{n+1}} \right] = \lim_{n \to \infty} \left[\frac{n^n}{(n+1)^n} \right] = \lim_{n \to \infty} \left[\left(\frac{n}{n+1} \right)^n \right]$

$= \frac{1}{e} < 1$

∴ SERIES CONVERGES

NOTE $\lim_{n \to \infty} \left(\frac{n+1}{n} \right)^n = e$

$\lim_{n \to \infty} \left(\frac{n}{n+1} \right)^n = \frac{1}{e}$

Question 37 (****)

The sum of the first n terms of a series with general term u_n is given by the expression

$$S_n = n^2(n+1)(n+2).$$

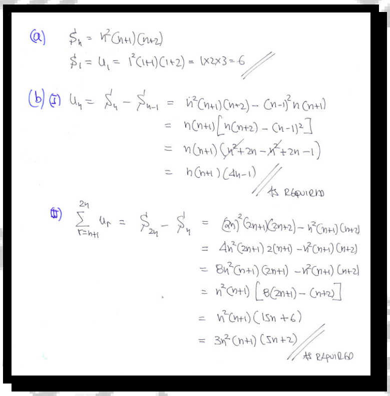
a) Find the first term of the series.

b) Show clearly that …

i. … $u_n = n(n+1)(4n-1)$

ii. … $\displaystyle\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2)$.

$\boxed{u_1 = 6}$


(a) $S_n = n^2(n+1)(n+2)$

$S_1 = u_1 = 1^2(1+1)(1+2) = 1\times2\times3 = 6$

(b) (i) $u_n = S_n - S_{n-1} = n^2(n+1)(n+2) - (n-1)^2 n(n+1)$

$= n(n+1)\left[n(n+2) - (n-1)^2\right]$

$= n(n+1)(n^2+2n-n^2+2n-1)$

$= n(n+1)(4n-1)$ AS REQUIRED

(ii) $\displaystyle\sum_{r=n+1}^{2n} u_r = S_{2n} - S_n = (2n)^2(2n+1)(2n+2) - n^2(n+1)(n+2)$

$= 4n^2(2n+1)2(n+1) - n^2(n+1)(n+2)$

$= 8n^2(n+1)(2n+1) - n^2(n+1)(n+2)$

$= n^2(n+1)\left[8(2n+1) - (n+2)\right]$

$= n^2(n+1)(15n+6)$

$= 3n^2(n+1)(5n+2)$ AS REQUIRED

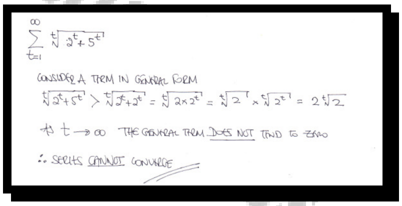
Question 38 (****)

Determine whether the following series converges or diverges.

$$\sum_{t=1}^{\infty} \sqrt[t]{2^t + 5^t} \, .$$

Show a full method, justifying every step in the workings.

divergent


$\sum_{t=1}^{\infty} \sqrt[t]{2^t+5^t}$

CONSIDER A TERM IN GENERAL FORM

$\sqrt[t]{2^t+5^t} > \sqrt[t]{2^t+2^t} = \sqrt[t]{2\times 2^t} = \sqrt[t]{2} \times \sqrt[t]{2^t} = 2\sqrt[t]{2}$

AS $t \to \infty$ THE GENERAL TERM DOES NOT TEND TO ZERO

∴ SERIES CANNOT CONVERGE

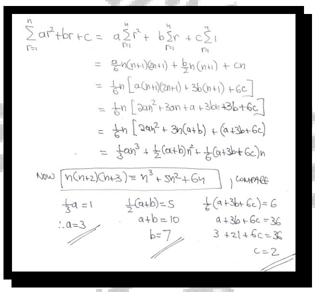
Question 39 (****)

$$\sum_{r=1}^{n} \left(ar^2 + br + c\right) \equiv n^3 + 5n^2 + 6n \, ,$$

where a, b and c are integer constants.

Determine the value of a, b and c.

$a = 3$, $b = 7$, $c = 2$


$\sum_{r=1}^{n} ar^2+br+c = a\sum_{r=1}^{n} r^2 + b\sum_{r=1}^{n} r + c\sum_{r=1}^{n} 1$

$= \frac{a}{6}n(n+1)(2n+1) + \frac{b}{2}n(n+1) + cn$

$= \frac{1}{6}n\left[a(n+1)(2n+1) + 3b(n+1) + 6c\right]$

$= \frac{1}{6}n\left[2an^2 + 3an + a + 3bn + 3b + 6c\right]$

$= \frac{1}{6}n\left[2an^2 + 3n(a+b) + (a+3b+6c)\right]$

$= \frac{1}{3}an^3 + \frac{1}{2}(a+b)n^2 + \frac{1}{6}(a+3b+6c)n$

NOW $n(n+2)(n+3) \equiv n^3 + 5n^2 + 6n$, COMPARE

$\frac{1}{3}a = 1$, ∴ $a = 3$

$\frac{1}{2}(a+b) = 5$, $a+b = 10$, $b = 7$

$\frac{1}{6}(a+3b+6c) = 6$, $a+3b+6c = 36$, $3+21+6c = 36$, $c = 2$

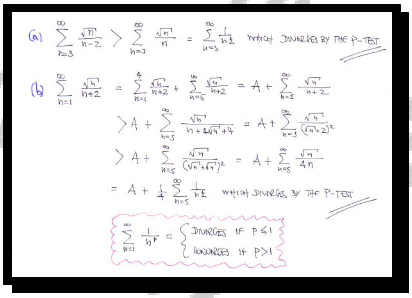
Question 40 (****)

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $$\sum_{n=3}^{\infty}\left[\frac{\sqrt{n}}{n-2}\right].$$

b) $$\sum_{n=1}^{\infty}\left[\frac{\sqrt{n}}{n+2}\right].$$

$\boxed{\text{divergent}}$, $\boxed{\text{divergent}}$

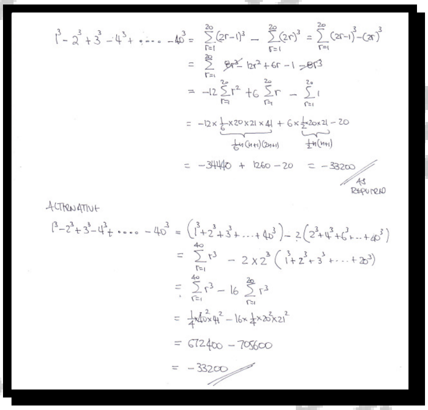


Question 41 **(****)**

Show clearly that

$$1^3 - 2^3 + 3^3 - 4^3 + \ldots - 40^3 = -33200 .$$

proof

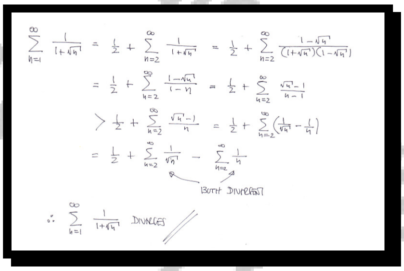


Question 42 **(****)**

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}} .$$

divergent



Question 43 (****)

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

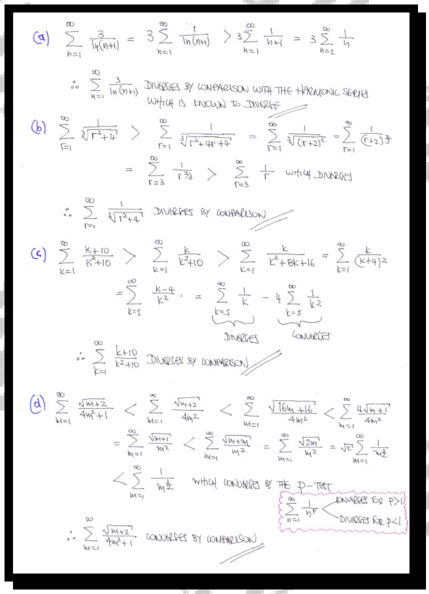
a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$.

b) $\displaystyle\sum_{r=1}^{\infty} \frac{1}{\sqrt[3]{r^2+4}}$.

c) $\displaystyle\sum_{k=1}^{\infty} \frac{k+10}{k^2+10}$.

d) $\displaystyle\sum_{m=1}^{\infty} \frac{\sqrt{m+2}}{4m^2+1}$

divergent, divergent, divergent, convergent



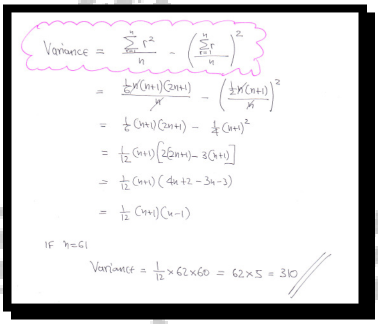
Question 44 (****)

The variance $\text{Var}(n)$ of the first n natural numbers is given by

$$\text{Var}(n) = \frac{1}{n}\sum_{r=1}^{n} r^2 - \left[\frac{1}{n}\sum_{r=1}^{n} r\right]^2 .$$

Determine a simplified expression $\text{Var}(n)$ and hence evaluate $\text{Var}(61)$.

$\boxed{\text{Var}(n) = \frac{1}{12}(n^2 - 1)}$, $\boxed{\text{Var}(61) = 310}$



$$\text{Variance} = \frac{\sum_{r=1}^{n} r^2}{n} - \left(\frac{\sum_{r=1}^{n} r}{n}\right)^2$$

$$= \frac{\frac{1}{6}n(n+1)(2n+1)}{n} - \left(\frac{\frac{1}{2}n(n+1)}{n}\right)^2$$

$$= \frac{1}{6}(n+1)(2n+1) - \frac{1}{4}(n+1)^2$$

$$= \frac{1}{12}(n+1)\left[2(2n+1) - 3(n+1)\right]$$

$$= \frac{1}{12}(n+1)(4n+2-3n-3)$$

$$= \frac{1}{12}(n+1)(n-1)$$

IF $n=61$

$$\text{Variance} = \frac{1}{12} \times 62 \times 60 = 62 \times 5 = 310$$

Question 45 (****)

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

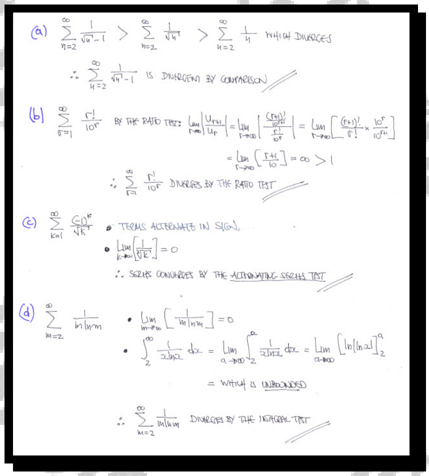
a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$.

b) $\displaystyle\sum_{r=1}^{\infty} \frac{r!}{10^r}$.

c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$.

d) $\displaystyle\sum_{m=2}^{\infty} \frac{1}{m \ln m}$

divergent , divergent , convergent , divergent


(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} > \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} > \sum_{n=2}^{\infty} \frac{1}{n}$ WHICH DIVERGES

$\therefore \displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ IS DIVERGENT BY COMPARISON

(b) $\displaystyle\sum_{r=1}^{\infty} \frac{r!}{10^r}$ BY THE RATIO TEST: $\displaystyle\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right| = \lim_{r\to\infty}\left|\frac{\frac{(r+1)!}{10^{r+1}}}{\frac{r!}{10^r}}\right| = \lim_{r\to\infty}\left[\frac{(r+1)!}{r!}\times\frac{10^r}{10^{r+1}}\right]$

$\displaystyle = \lim_{r\to\infty}\left[\frac{r+1}{10}\right] = \infty > 1$

$\therefore \displaystyle\sum_{r=1}^{\infty} \frac{r!}{10^r}$ DIVERGES BY THE RATIO TEST

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

- TERMS ALTERNATE IN SIGN.
- $\displaystyle\lim_{k\to\infty}\left[\frac{1}{\sqrt{k}}\right] = 0$

$\therefore$ SERIES CONVERGES BY THE ALTERNATING SERIES TEST

(d) $\displaystyle\sum_{m=2}^{\infty} \frac{1}{m\ln m}$

- $\displaystyle\lim_{m\to\infty}\left[\frac{1}{m\ln m}\right] = 0$
- $\displaystyle\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{a\to\infty}\int_2^{a} \frac{1}{x\ln x}\,dx = \lim_{a\to\infty}\Big[\ln(\ln x)\Big]_2^{a}$

$=$ WHICH IS UNBOUNDED

$\therefore \displaystyle\sum_{m=2}^{\infty} \frac{1}{m\ln m}$ DIVERGES BY THE INTEGRAL TEST

Question 46 (****)

Consider the infinite series

$$1-\frac{x^2}{2}+\frac{x^4}{4\times 2}-\frac{x^6}{6\times 4\times 2}+\frac{x^8}{8\times 6\times 4\times 2}-\ldots$$

a) Write the above series in Sigma notation, in its simplest form.

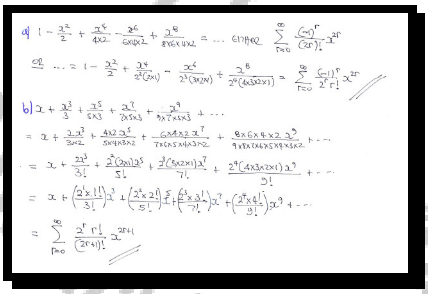
Next consider another infinite series

$$x+\frac{x^3}{3}+\frac{x^5}{5\times 3}+\frac{x^6}{7\times 5\times 3}+\frac{x^9}{9\times 7\times 5\times 3}+\ldots$$

b) Also, write this series in Sigma notation, in its simplest form.

[*You are not required to investigate the convergence or the sum of these series.*]

$$\boxed{\sum_{r=0}^{\infty}\frac{(-1)^r}{2^r\,r!}x^{2r}}\,,\ \boxed{\sum_{r=0}^{\infty}\frac{2^r\,r!}{(2r+1)!}x^{2r}}$$



a) $1-\frac{x^2}{2}+\frac{x^4}{4\times2}-\frac{x^6}{6\times4\times2}+\frac{x^8}{8\times6\times4\times2}=\ldots$ EITHER $\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r)!}x^{2r}$

OR $\ldots=1-\frac{x^2}{2}+\frac{x^4}{2^2(2\times1)}-\frac{x^6}{2^3(3\times2\times1)}+\frac{x^8}{2^4(4\times3\times2\times1)}=\sum_{r=0}^{\infty}\frac{(-1)^r}{2^r r!}x^{2r}$

b) $x+\frac{x^3}{3}+\frac{x^5}{5\times3}+\frac{x^7}{7\times5\times3}+\frac{x^9}{9\times7\times5\times3}+\ldots$

$=x+\frac{2x^3}{3\times2}+\frac{4\times2\,x^5}{5\times4\times3\times2}+\frac{6\times4\times2\,x^7}{7\times6\times5\times4\times3\times2}+\frac{8\times6\times4\times2\,x^9}{9\times8\times7\times6\times5\times4\times3\times2}+\ldots$

$=x+\frac{2x^3}{3!}+\frac{2^2(2\times1)x^5}{5!}+\frac{2^3(3\times2\times1)x^7}{7!}+\frac{2^4(4\times3\times2\times1)x^9}{9!}+\ldots$

$=x+\left(\frac{2^1\times1!}{3!}\right)x^3+\left(\frac{2^2\times2!}{5!}\right)x^5+\left(\frac{2^3\times3!}{7!}\right)x^7+\left(\frac{2^4\times4!}{9!}\right)x^9+\ldots$

$=\sum_{r=0}^{\infty}\frac{2^r r!}{(2r+1)!}x^{2r+1}$

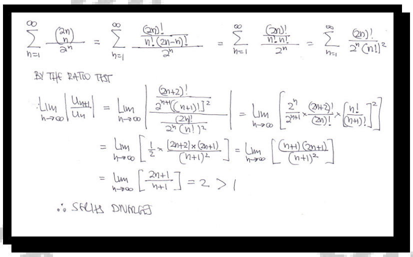
Question 47 (****)

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^n}$$

Show a full method, justifying every step in the workings.

divergent

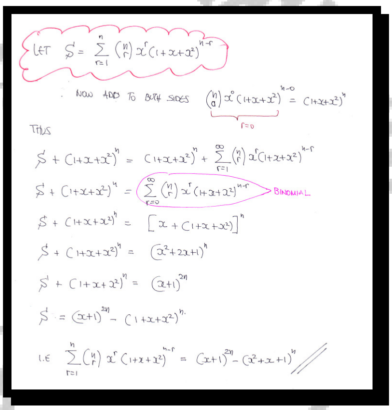


Question 48 (****)

$$\sum_{r=1}^{n} \left[\binom{n}{r} x^r \left(1 + x + x^2\right)^{n-r} \right].$$

Simplify fully the above sum, into a summation free expression

$$\left(x+1\right)^{2n} - \left(x^2 + x - 1\right)^n$$



Question 49 (****+)

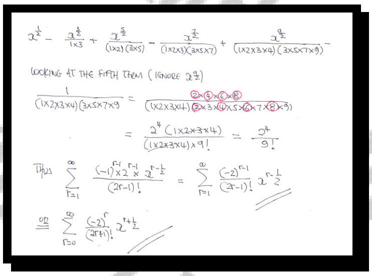
Consider the infinite series

$$x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{1\times 3} + \frac{x^{\frac{5}{2}}}{(1\times 2)(3\times 5)} - \frac{x^{\frac{7}{2}}}{(1\times 2\times 3)(3\times 5\times 7)} + \frac{x^{\frac{9}{2}}}{(1\times 2\times 3\times 4)(3\times 5\times 7\times 9)} - \dots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\boxed{\sum_{r=0}^{\infty}\left[\frac{(-2)^r}{(2r+1)!}x^{\frac{1}{2}+r}\right]}$$



Question 50 (****+)

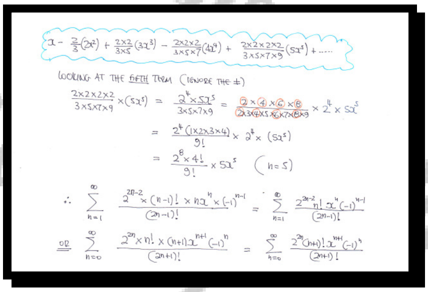
Consider the infinite series

$$x-\frac{2}{3}\left(2x^{2}\right)+\frac{2\times2}{3\times5}\left(3x^{3}\right)-\frac{2\times2\times2}{3\times5\times7}\left(4x^{4}\right)+\frac{2\times2\times2\times2}{3\times5\times7\times9}\left(5x^{5}\right)-\ldots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\sum_{n=0}^{\infty}\left[\frac{2^{2n}\times(-1)^{n}\times(n+1)!\times x^{n+1}}{(2n+1)!}\right]$$

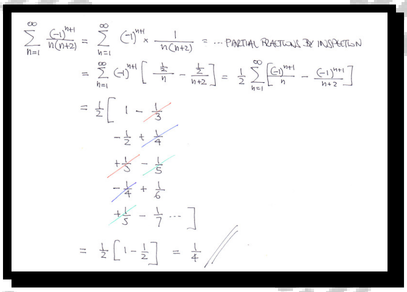


Question 51 (****+)

Use a suitable method to sum the following series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)}.$$

$\boxed{\frac{1}{4}}$



Question 52 (****+)

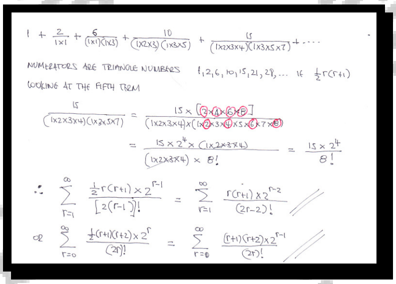
Consider the infinite series

$$1+\frac{2}{1\times 1}+\frac{6}{(1\times 2)(1\times 3)}+\frac{10}{(1\times 2\times 3)(1\times 3\times 5)}+\frac{15}{(1\times 2\times 3\times 4)(1\times 3\times 5\times 7)}+\ldots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\boxed{\sum_{r=0}^{\infty} \frac{(r+1)(r+2)2^{r-1}}{(2r)!}}$$



Question 53 (****+)

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of the following series.

$$\sum_{n=1}^{\infty} \frac{n(2n+3)^{\frac{1}{2}}}{n^3+4}.$$

convergent

$$\sum_{n=1}^{\infty} \frac{n(2n+3)^{\frac{1}{2}}}{n^3+4} < \sum_{n=1}^{\infty} \frac{n(2n+3)^{\frac{1}{2}}}{n^3} = \sum_{n=1}^{\infty} \frac{(2n+3)^{\frac{1}{2}}}{n^2}$$

$$< \sum_{n=1}^{\infty} \frac{(4n+4)^{\frac{1}{2}}}{n^2} = 2\sum_{n=1}^{\infty} \frac{(n+1)^{\frac{1}{2}}}{n^2}$$

$$< 2\sum_{n=1}^{\infty} \frac{(n+n)^{\frac{1}{2}}}{n^2} = 2\sum_{n=1}^{\infty} \frac{(2n)^{\frac{1}{2}}}{n^2}$$

$$= 2\sqrt{2}\sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}}}{n^2} = 2\sqrt{2}\sum \frac{1}{n^{\frac{3}{2}}}$$

WHICH CONVERGES BY THE P-TEST

$$\sum \frac{1}{n^p} = \begin{cases} \text{CONVERGES IF } p>1 \\ \text{DIVERGES IF } p \le 1 \end{cases}$$

Question 54 (****+)

A sequence is generated by the function

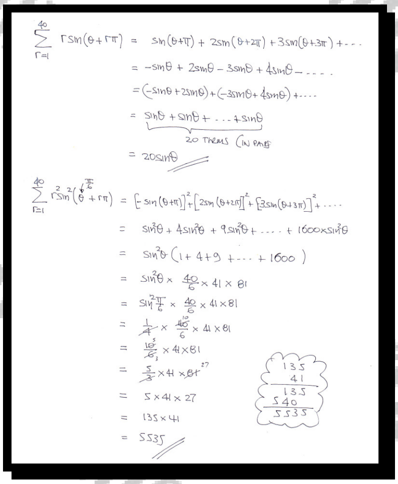
$$u_r(\theta) \equiv r\sin(\theta + r\pi), \; r \in \mathbb{N}.$$

Find an expression or the value, whichever is appropriate, for each of the series

a) $\displaystyle\sum_{r=1}^{40} u_r(\theta)$.

b) $\displaystyle\sum_{r=1}^{40} \left[u_r\left(\frac{\pi}{6}\right)\right]^2$.

$\boxed{20\sin\theta}$, $\boxed{5535}$



Question 55 (****+)

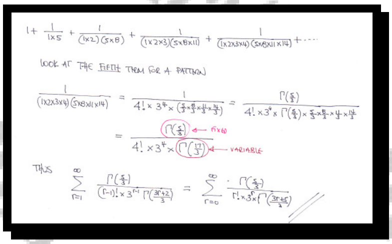
Consider the infinite series

$$1+\frac{1}{1\times5}+\frac{1}{(1\times2)(5\times8)}+\frac{1}{(1\times2\times3)(5\times8\times11)}+\frac{1}{(1\times2\times3\times4)(5\times8\times11\times14)}+\ldots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\boxed{\sum_{r=1}^{\infty}\frac{\Gamma\left(\frac{5}{3}\right)}{3^{r-1}\times(r-1)!\times\Gamma\left(\frac{3r+2}{3}\right)}=\sum_{r=0}^{\infty}\frac{\Gamma\left(\frac{5}{3}\right)}{3^{r}\times r!\times\Gamma\left(\frac{3r+5}{3}\right)}}$$



Question 56 (****+)

By using a suitable test and justifying every step in the workings, determine the convergence or divergence of the following series.

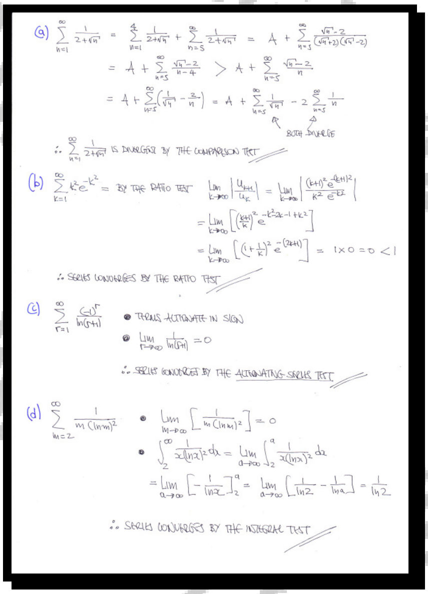
a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$.

b) $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k^2}$.

c) $\displaystyle\sum_{r=1}^{\infty} \frac{(-1)^r}{\ln(r+1)}$.

d) $\displaystyle\sum_{m=2}^{\infty} \frac{1}{m(\ln m)^2}$

divergent, convergent, convergent, convergent



(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}} = \sum_{n=1}^{4}\frac{1}{2+\sqrt{n}} + \sum_{n=5}^{\infty}\frac{1}{2+\sqrt{n}} = A + \sum_{n=5}^{\infty}\frac{\sqrt{n}-2}{(\sqrt{n}+2)(\sqrt{n}-2)}$

$\displaystyle = A + \sum_{n=5}^{\infty}\frac{\sqrt{n}-2}{n-4} > A + \sum_{n=5}^{\infty}\frac{\sqrt{n}-2}{n}$

$\displaystyle = A + \sum_{n=5}^{\infty}\left(\frac{1}{\sqrt{n}} - \frac{2}{n}\right) = A + \sum_{n=5}^{\infty}\frac{1}{\sqrt{n}} - 2\sum_{n=5}^{\infty}\frac{1}{n}$

BOTH DIVERGE

$\therefore \displaystyle\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$ IS DIVERGENT BY THE COMPARISON TEST

(b) $\displaystyle\sum_{k=1}^{\infty}k^2e^{-k^2}$ = BY THE RATIO TEST $\displaystyle\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)^2e^{-(k+1)^2}}{k^2e^{-k^2}}\right|$

$\displaystyle = \lim_{k\to\infty}\left[\left(\frac{k+1}{k}\right)^2 e^{-k^2-2k-1+k^2}\right]$

$\displaystyle = \lim_{k\to\infty}\left[\left(1+\frac{1}{k}\right)^2 e^{-(2k+1)}\right] = 1\times 0 = 0 < 1$

$\therefore$ SERIES CONVERGES BY THE RATIO TEST

(c) $\displaystyle\sum_{r=1}^{\infty}\frac{(-1)^r}{\ln(r+1)}$

- TERMS ALTERNATE IN SIGN
- $\displaystyle\lim_{r\to\infty}\frac{1}{\ln(r+1)} = 0$

$\therefore$ SERIES CONVERGES BY THE ALTERNATING SERIES TEST

(d) $\displaystyle\sum_{m=2}^{\infty}\frac{1}{m(\ln m)^2}$

- $\displaystyle\lim_{m\to\infty}\left[\frac{1}{m(\ln m)^2}\right] = 0$
- $\displaystyle\int_2^{\infty}\frac{1}{x(\ln x)^2}\,dx = \lim_{a\to\infty}\int_2^{a}\frac{1}{x(\ln x)^2}\,dx$

$\displaystyle = \lim_{a\to\infty}\left[-\frac{1}{\ln x}\right]_2^a = \lim_{a\to\infty}\left[\frac{1}{\ln 2} - \frac{1}{\ln a}\right] = \frac{1}{\ln 2}$

$\therefore$ SERIES CONVERGES BY THE INTEGRAL TEST

Question 57 (****+)

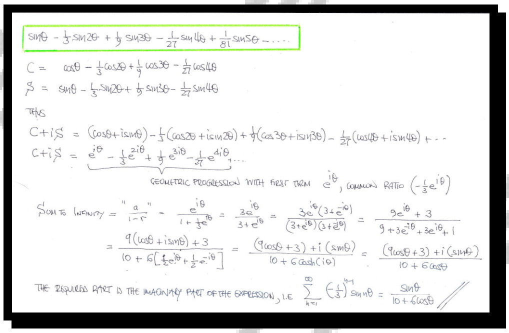
The following convergent series S is given below

$$S = \sin\theta - \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta - \frac{1}{27}\sin 4\theta \ldots$$

By considering the sum to infinity of a suitable geometric series involving the complex exponential function, show that

$$S = \frac{\sin\theta}{10 + 6\cos\theta}.$$

proof



$\sin\theta - \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta - \frac{1}{27}\sin 4\theta + \frac{1}{81}\sin 5\theta - \ldots$

$C = \cos\theta - \frac{1}{3}\cos 2\theta + \frac{1}{9}\cos 3\theta - \frac{1}{27}\cos 4\theta$

$S = \sin\theta - \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta - \frac{1}{27}\sin 4\theta$

THUS

$C + iS = (\cos\theta + i\sin\theta) - \frac{1}{3}(\cos 2\theta + i\sin 2\theta) + \frac{1}{9}(\cos 3\theta + i\sin 3\theta) - \frac{1}{27}(\cos 4\theta + i\sin 4\theta) + \ldots$

$C + iS = e^{i\theta} - \frac{1}{3}e^{2i\theta} + \frac{1}{9}e^{3i\theta} - \frac{1}{27}e^{4i\theta} + \ldots$

GEOMETRIC PROGRESSION WITH FIRST TERM $e^{i\theta}$, COMMON RATIO $\left(-\frac{1}{3}e^{i\theta}\right)$

SUM TO INFINITY $= \text{"}\frac{a}{1-r}\text{"} = \frac{e^{i\theta}}{1+\frac{1}{3}e^{i\theta}} = \frac{3e^{i\theta}}{3+e^{i\theta}} = \frac{3e^{i\theta}(3+e^{-i\theta})}{(3+e^{i\theta})(3+e^{-i\theta})} = \frac{9e^{i\theta}+3}{9+3e^{-i\theta}+3e^{i\theta}+1}$

$= \frac{9(\cos\theta + i\sin\theta)+3}{10+6\left[\frac{1}{2}e^{i\theta}+\frac{1}{2}e^{-i\theta}\right]} = \frac{(9\cos\theta+3)+i(\sin\theta)}{10+6\cosh(i\theta)} = \frac{(9\cos\theta+3)+i(\sin\theta)}{10+6\cos\theta}$

THE REQUIRED PART IS THE IMAGINARY PART OF THE EXPRESSION, I.E. $\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^{n-1}\sin n\theta = \frac{\sin\theta}{10+6\cos\theta}$

Question 58 (****+)

A sequence of positive integers is generated by

$$u_n = 3^n - 1, \quad n = 1, 2, 3, 4, \ldots$$

a) Write down the first seven terms of this sequence.

b) Verify that

$$u_{n+1} = 3u_n + 2.$$

c) Show clearly that …

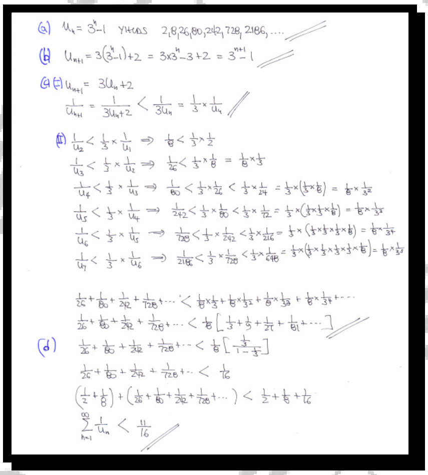
i. … $\dfrac{1}{u_{n+1}} < \dfrac{1}{3} \times \dfrac{1}{u_n}$.

ii. … $\dfrac{1}{26} + \dfrac{1}{80} + \dfrac{1}{242} + \dfrac{1}{728} + \dfrac{1}{2186} + \ldots < \dfrac{1}{8}\left[\dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \dfrac{1}{81} + \ldots\right]$

d) Deduce that

$$\sum_{n=1}^{\infty} u_n < \frac{11}{16}.$$

2, 8, 26, 80, 242, 728, 2186,...



(a) $u_n = 3^n - 1$ YIELDS $2, 8, 26, 80, 242, 728, 2186, \ldots$

(b) $u_{n+1} = 3(3^n - 1) + 2 = 3 \times 3^n - 3 + 2 = 3^{n+1} - 1$

(c)(i) $u_{n+1} = 3u_n + 2$

$\frac{1}{u_{n+1}} = \frac{1}{3u_n + 2} < \frac{1}{3u_n} = \frac{1}{3} \times \frac{1}{u_n}$

(ii) $\frac{1}{u_2} < \frac{1}{3} \times \frac{1}{u_1} \Rightarrow \frac{1}{8} < \frac{1}{3} \times \frac{1}{2}$

$\frac{1}{u_3} < \frac{1}{3} \times \frac{1}{u_2} \Rightarrow \frac{1}{26} < \frac{1}{3} \times \frac{1}{8} = \frac{1}{8} \times \frac{1}{3}$

$\frac{1}{u_4} < \frac{1}{3} \times \frac{1}{u_3} \Rightarrow \frac{1}{80} < \frac{1}{3} \times \frac{1}{26} < \frac{1}{3} \times \frac{1}{24} = \frac{1}{3} \times \left(\frac{1}{3} \times \frac{1}{8}\right) = \frac{1}{8} \times \frac{1}{3^2}$

$\frac{1}{u_5} < \frac{1}{3} \times \frac{1}{u_4} \Rightarrow \frac{1}{242} < \frac{1}{3} \times \frac{1}{80} < \frac{1}{3} \times \frac{1}{72} = \frac{1}{3} \times \left(\frac{1}{3} \times \frac{1}{3} \times \frac{1}{8}\right) = \frac{1}{8} \times \frac{1}{3^3}$

$\frac{1}{u_6} < \frac{1}{3} \times \frac{1}{u_5} \Rightarrow \frac{1}{728} < \frac{1}{3} \times \frac{1}{242} < \frac{1}{3} \times \frac{1}{216} = \frac{1}{3} \times \left(\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{8}\right) = \frac{1}{8} \times \frac{1}{3^4}$

$\frac{1}{u_7} < \frac{1}{3} \times \frac{1}{u_6} \Rightarrow \frac{1}{2186} < \frac{1}{3} \times \frac{1}{728} < \frac{1}{3} \times \frac{1}{648} = \frac{1}{3} \times \left(\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{8}\right) = \frac{1}{8} \times \frac{1}{3^5}$

$\frac{1}{26} + \frac{1}{80} + \frac{1}{242} + \frac{1}{728} + \ldots < \frac{1}{8} \times \frac{1}{3} + \frac{1}{8} \times \frac{1}{3^2} + \frac{1}{8} \times \frac{1}{3^3} + \frac{1}{8} \times \frac{1}{3^4} + \ldots$

$\frac{1}{26} + \frac{1}{80} + \frac{1}{242} + \frac{1}{728} + \ldots < \frac{1}{8}\left[\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots\right]$

(d) $\frac{1}{26} + \frac{1}{80} + \frac{1}{242} + \frac{1}{728} + \ldots < \frac{1}{8}\left[\frac{\frac{1}{3}}{1 - \frac{1}{3}}\right]$

$\frac{1}{26} + \frac{1}{80} + \frac{1}{242} + \frac{1}{728} + \ldots < \frac{1}{16}$

$\left(\frac{1}{2} + \frac{1}{8}\right) + \left(\frac{1}{26} + \frac{1}{80} + \frac{1}{242} + \frac{1}{728} + \ldots\right) < \frac{1}{2} + \frac{1}{8} + \frac{1}{16}$

$\sum_{n=1}^{\infty} \frac{1}{u_n} < \frac{11}{16}$

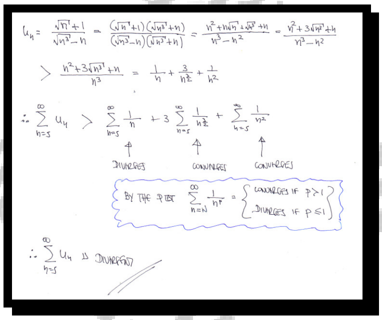
Question 59 (****+)

$$u_n = \frac{\sqrt{n}+1}{\sqrt{n^3}-n}, \quad n \in \mathbb{N},\ n \geq 5.$$

By using the comparison test and justifying every step in the workings, determine the convergence or divergence of

$$\sum_{n=5}^{\infty} u_n .$$

convergent

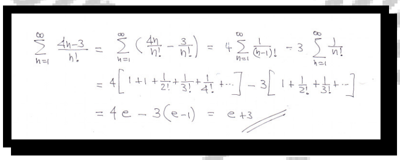


Question 60 (****+)

Show clearly that

$$\sum_{n=1}^{\infty} \frac{4n-3}{n!} = \mathrm{e} + 3 .$$

proof



Question 61 (****+)

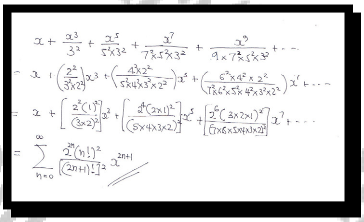
Consider the infinite series

$$x+\frac{x^3}{3^2}+\frac{x^5}{5^2\times 3^2}+\frac{x^7}{7^2\times 5^2\times 3^2}+\frac{x^9}{9^2\times 7^2\times 5^2\times 3^2}+\dots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\boxed{\sum_{n=0}^{\infty}\left[\frac{2^{2n}(n!)^2}{[(2n+1)!]^2}x^{2n+1}\right]}$$

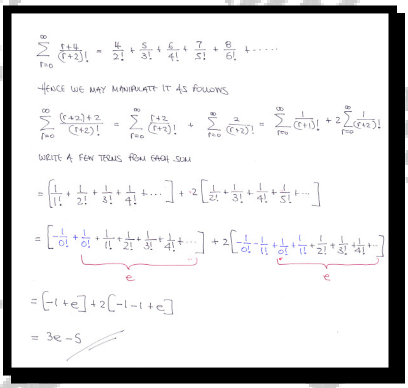


Question 62 (****+)

Show clearly that

$$\sum_{r=0}^{\infty}\frac{r+4}{(r+2)!}=3\mathrm{e}-5.$$

proof



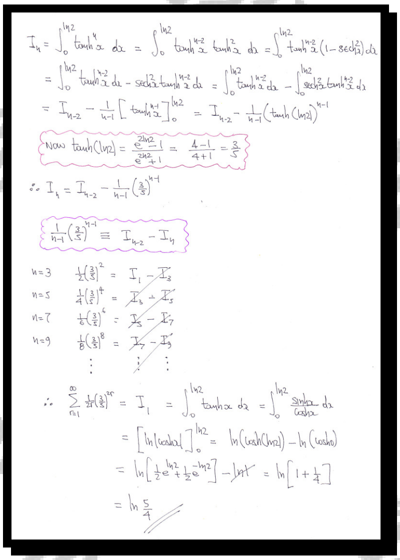
Question 63 (****+)

$$I_n = \int_0^{\ln 2} \tanh^n x \ dx, \ n \in \mathbb{N}.$$

By considering a reduction formula for I_n, or otherwise, show clearly that

$$\sum_{r=1}^{\infty} \frac{1}{2r}\left(\frac{3}{5}\right)^{2r} = \ln\left(\frac{5}{4}\right).$$

proof


$I_n = \int_0^{\ln 2} \tanh^n x \, dx = \int_0^{\ln 2} \tanh^{n-2} x \tanh^2 x \, dx = \int_0^{\ln 2} \tanh^{n-2} x (1 - \text{sech}^2 x) \, dx$

$= \int_0^{\ln 2} \tanh^{n-2} x \, dx - \text{sech}^2 x \tanh^{n-2} x \, dx = \int_0^{\ln 2} \tanh^{n-2} x \, dx - \int_0^{\ln 2} \text{sech}^2 x \tanh^{n-2} x \, dx$

$= I_{n-2} - \frac{1}{n-1}\left[\tanh^{n-1} x\right]_0^{\ln 2} = I_{n-2} - \frac{1}{n-1}\left(\tanh(\ln 2)\right)^{n-1}$

Now $\tanh(\ln 2) = \frac{e^{2\ln 2} - 1}{e^{2\ln 2} + 1} = \frac{4-1}{4+1} = \frac{3}{5}$

$\therefore I_n = I_{n-2} - \frac{1}{n-1}\left(\frac{3}{5}\right)^{n-1}$

$\frac{1}{n-1}\left(\frac{3}{5}\right)^{n-1} \equiv I_{n-2} - I_n$

$n=3 \quad \frac{1}{2}\left(\frac{3}{5}\right)^2 = I_1 - I_3$

$n=5 \quad \frac{1}{4}\left(\frac{3}{5}\right)^4 = I_3 - I_5$

$n=7 \quad \frac{1}{6}\left(\frac{3}{5}\right)^6 = I_5 - I_7$

$n=9 \quad \frac{1}{8}\left(\frac{3}{5}\right)^8 = I_7 - I_9$

$\vdots$

$\therefore \sum_{r=1}^{\infty} \frac{1}{2r}\left(\frac{3}{5}\right)^{2r} = I_1 = \int_0^{\ln 2} \tanh x \, dx = \int_0^{\ln 2} \frac{\sinh x}{\cosh x} \, dx$

$= \left[\ln|\cosh x|\right]_0^{\ln 2} = \ln(\cosh(\ln 2)) - \ln(\cosh 0)$

$= \ln\left[\frac{1}{2}e^{\ln 2} + \frac{1}{2}e^{-\ln 2}\right] - \ln 1 = \ln\left[1 + \frac{1}{4}\right]$

$= \ln \frac{5}{4}$

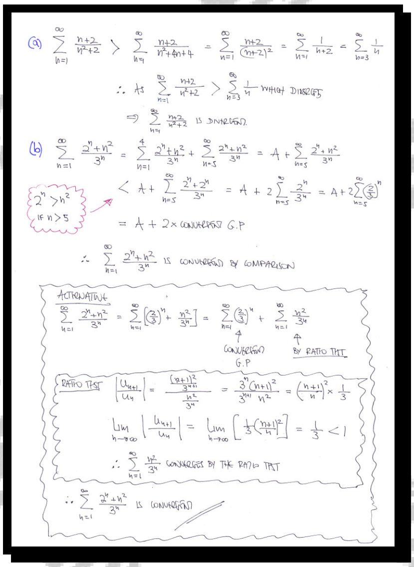
Question 64 (****+)

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{n+2}{n^2+2}\right]$.

b) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{2^n+n^2}{3^n}\right]$.

divergent, convergent



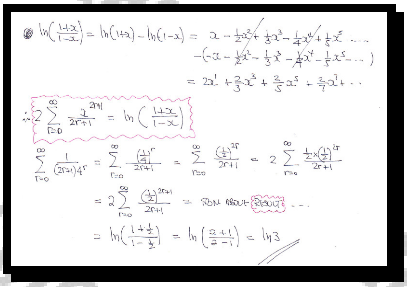
Question 65 (****+)

By considering the Mclaurin expansion of $\ln\left(\frac{1+x}{1-x}\right)$ find the value of

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r},$$

giving the final answer as the natural logarithm of an integer.

$\boxed{\ln 3}$



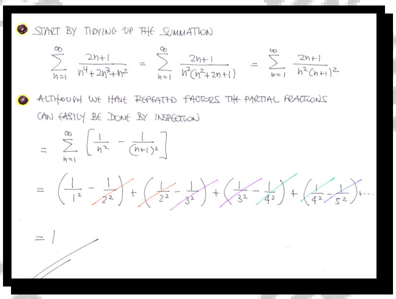
Question 66 (****+)

Use partial fractions to sum the following series.

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^4+2n^3+n^2}.$$

You may assume the series converges.

$\boxed{}$, $\boxed{1}$



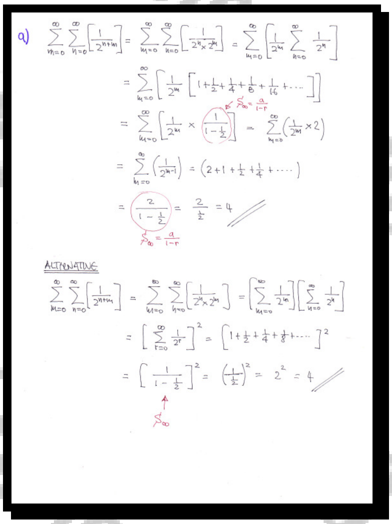
Question 67 (****+)

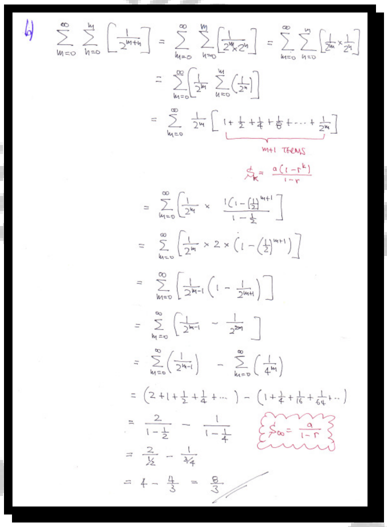
Sum each of the following double series.

a) $\displaystyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{1}{2^{m+n}}\right]$.

b) $\displaystyle\sum_{m=0}^{\infty}\sum_{n=0}^{m}\left[\frac{1}{2^{m+n}}\right]$.

$\boxed{}$, $\boxed{4}$, $\boxed{\frac{8}{3}}$





Question 68 (****+)

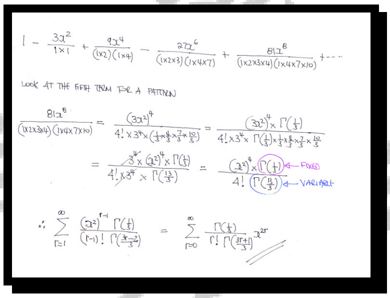
Consider the infinite series

$$1-\frac{3x^2}{1\times 1}+\frac{9x^4}{(1\times 2)(1\times 4)}-\frac{27x^6}{(1\times 2\times 3)(1\times 4\times 7)}-\frac{81x^8}{(1\times 2\times 3\times 4)(1\times 4\times 7\times 10)}+\dots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}\Gamma\left(\frac{1}{3}\right)}{(r-1)!\times\Gamma\left(\frac{3r-2}{3}\right)}\left(x^2\right)^{r-1}=\sum_{r=0}^{\infty}\frac{(-1)^{r}\Gamma\left(\frac{1}{3}\right)}{r!\times\Gamma\left(\frac{3r+1}{3}\right)}x^{2r}$$

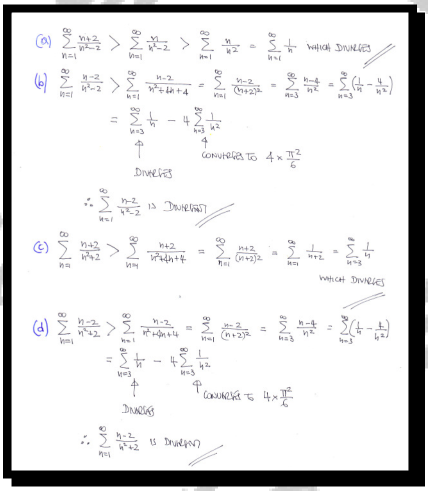


Question 69 (****+)

$$\sum_{n=1}^{\infty}\left[\frac{n\pm 2}{n^2\pm 2}\right].$$

Use a comparison test to show that all four series described by the above expression are divergent.

proof

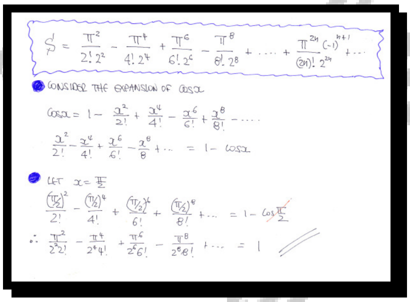


Question 70 **(****+)**

By showing a detailed method, sum the following series.

$$\frac{\pi^2}{2^2\,2!}-\frac{\pi^4}{2^4\,4!}+\frac{\pi^6}{2^6\,6!}-\frac{\pi^8}{2^8\,8!}+\ldots+\frac{(-1)^{n+1}\pi^{2n}}{2^{2n}(2n)!}+\ldots$$

$\boxed{1}$

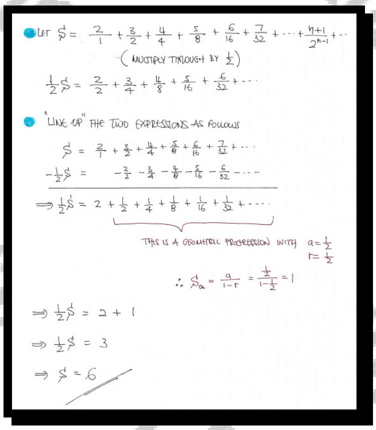


Question 71 **(****+)**

By showing a detailed method, sum the following series.

$$\frac{2}{1}+\frac{3}{2}+\frac{4}{4}+\frac{5}{8}+\frac{6}{16}+\frac{7}{32}\ldots$$

$\boxed{}$, $\boxed{6}$



Question 72 (****+)

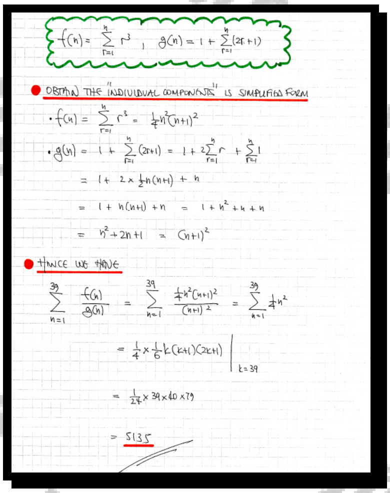
The positive integer functions f and g are defined as

$$f(n) = \sum_{r=1}^{n} r^3 \qquad \text{and} \qquad g(n) = 1 + \sum_{r=1}^{n} (2r+1).$$

Evaluate

$$\sum_{n=1}^{39} \left[\frac{f(n)}{g(n)} \right].$$

$\boxed{}$, $\boxed{5135}$


$$f(n) = \sum_{r=1}^{n} r^3 \,, \quad g(n) = 1 + \sum_{r=1}^{n} (2r+1)$$

• OBTAIN THE "INDIVIDUAL COMPONENTS" IS SIMPLIFIED FORM

$$\bullet\ f(n) = \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

$$\bullet\ g(n) = 1 + \sum_{r=1}^{n}(2r+1) = 1 + 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$= 1 + 2 \times \frac{1}{2}n(n+1) + n$$

$$= 1 + n(n+1) + n = 1 + n^2 + n + n$$

$$= n^2 + 2n + 1 = (n+1)^2$$

• HENCE WE HAVE

$$\sum_{n=1}^{39} \frac{f(n)}{g(n)} = \sum_{n=1}^{39} \frac{\frac{1}{4}n^2(n+1)^2}{(n+1)^2} = \sum_{n=1}^{39} \frac{1}{4}n^2$$

$$= \frac{1}{4} \times \frac{1}{6}k(k+1)(2k+1) \Big|_{k=39}$$

$$= \frac{1}{24} \times 39 \times 40 \times 79$$

$$= 5135$$

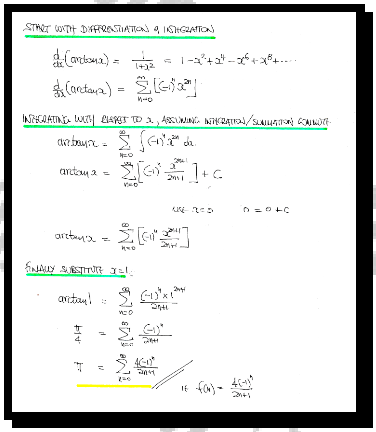
Question 73 (****+)

Find the Maclaurin expansion of $\arctan x$, and use it to show that

$$\pi = \sum_{n=0}^{\infty} f(n),$$

for some suitable function f.

$$\pi = \sum_{n=0}^{\infty} \frac{4(-1)n}{2n+1}$$


START WITH DIFFERENTIATION & INTEGRATION

$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \dots$

$\frac{d}{dx}(\arctan x) = \sum_{n=0}^{\infty} \left[(-1)^n x^{2n}\right]$

INTEGRATING WITH RESPECT TO x, ASSUMING INTEGRATION/SUMMATION COMMUTE

$\arctan x = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\, dx.$

$\arctan x = \sum_{n=0}^{\infty} \left[(-1)^n \frac{x^{2n+1}}{2n+1}\right] + C$

USE $x=0$ $\quad 0 = 0 + C$

$\arctan x = \sum_{n=0}^{\infty} \left[(-1)^n \frac{x^{2n+1}}{2n+1}\right]$

FINALLY SUBSTITUTE $x=1$

$\arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n \times 1^{2n+1}}{2n+1}$

$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

$\pi = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$ // IF $f(n) = \frac{4(-1)^n}{2n+1}$

Question 74 (*****)

It is given that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n-1}+\binom{n}{n}=k^{n},$$

where n and k are positive integer constants.

a) By considering the binomial expansion of $(1+x)^n$, determine the value of k.

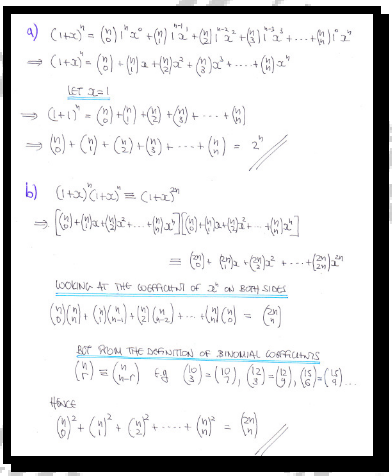
b) By considering the coefficient of x^n in

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n},$$

simplify fully

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\ldots+\binom{n}{n-1}^2+\binom{n}{n}^2.$$

$\boxed{}$, $\boxed{k=2}$



a) $(1+x)^n = \binom{n}{0}1^n x^0 + \binom{n}{1}1^{n-1}x^1 + \binom{n}{2}1^{n-2}x^2 + \binom{n}{3}1^{n-3}x^3 + \ldots + \binom{n}{n}1^0 x^n$

$\Rightarrow (1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n$

LET $x=1$

$\Rightarrow (1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n}$

$\Rightarrow \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} = 2^n$

b) $(1+x)^n(1+x)^n \equiv (1+x)^{2n}$

$\Rightarrow \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n\right]\left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n\right]$

$\equiv \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \ldots + \binom{2n}{2n}x^{2n}$

LOOKING AT THE COEFFICIENT OF x^n ON BOTH SIDES

$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \ldots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}$

BUT FROM THE DEFINITION OF BINOMIAL COEFFICIENTS

$\binom{n}{r} \equiv \binom{n}{n-r}$ e.g. $\binom{10}{3} = \binom{10}{7}$, $\binom{12}{3} = \binom{12}{9}$, $\binom{15}{6} = \binom{15}{9}$ ...

HENCE

$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$

Question 75 (*****)

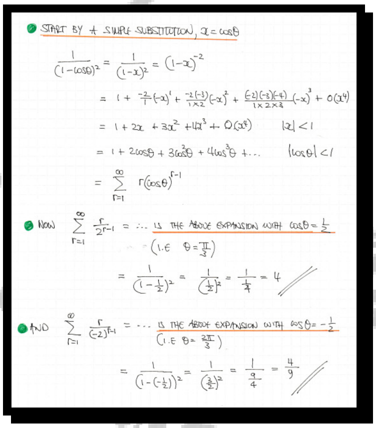
By considering the binomial expansion of

$$\frac{1}{(1-\cos\theta)^2},$$

sum each of the following series.

- $\displaystyle\sum_{r=1}^{\infty}\left[\frac{r}{2^{r-1}}\right]$.
- $\displaystyle\sum_{r=1}^{\infty}\left[\frac{r}{(-2)^{r-1}}\right]$.

$$\boxed{\quad}, \quad \boxed{\sum_{r=1}^{\infty}\left[\frac{r}{2^{r-1}}\right]=4}, \quad \boxed{\sum_{r=1}^{\infty}\left[\frac{r}{(-2)^{r-1}}\right]=\frac{4}{9}}$$



Question 76 (*****)

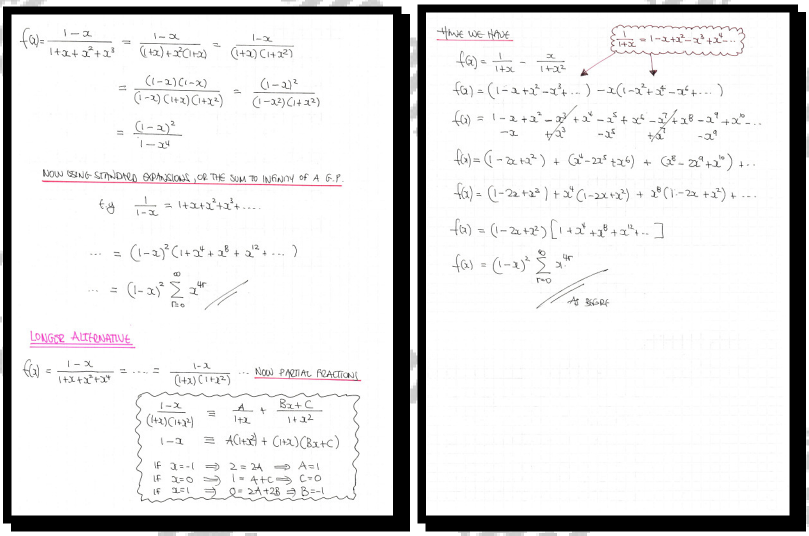
$$f(x) \equiv \frac{1-x}{1+x+x^2+x^3}, \quad -1 < x < 1.$$

Show that $f(x)$ can be written in the form

$$f(x) = g(x)\sum_{r=0}^{\infty}\left(x^{4r}\right),$$

where $g(x)$ is a simplified function to be found.

$\boxed{}$, $\boxed{g(x) = (1-x)^2}$



Question 77 (*****)

The sum to infinity S of the convergent geometric series is given by

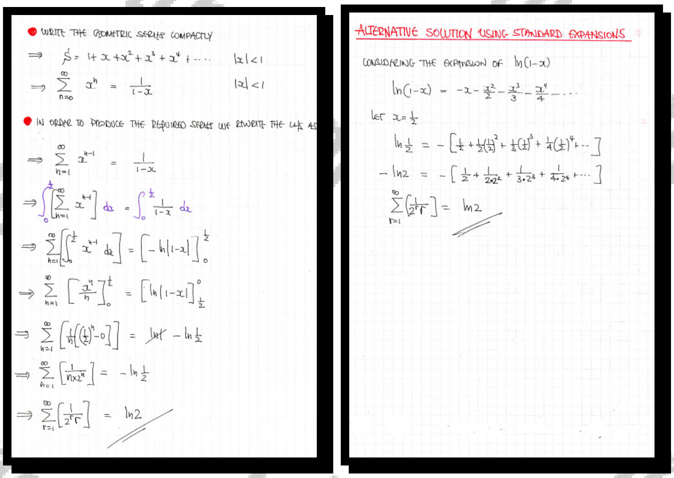
$$S = 1 + x + x^2 + x^3 + x^4 + \ldots, \quad |x| < 1,$$

By integrating the above equation between suitable limits, or otherwise, find

$$\sum_{r=1}^{\infty}\left[\frac{1}{r \times 2^r}\right].$$

You may assume that integration and summation commute.

$\boxed{}$, $\boxed{\ln 2}$



Question 78 (*****)

Show clearly that

$$\sum_{r=1}^{\infty} \frac{r^2}{r!} = 2\mathrm{e} \ .$$

proof

$$\sum_{r=1}^{\infty} \frac{r^2}{r!} = \frac{1^2}{1!} + \sum_{r=2}^{\infty} \frac{r^2}{r!} = 1 + \sum_{r=2}^{\infty} \frac{r}{(r-1)!} = 1 + \sum_{r=1}^{\infty} \frac{r+1}{r!}$$

$$= 1 + \sum_{r=1}^{\infty} \left(\frac{r}{r!} + \frac{1}{r!} \right) = 1 + \sum_{r=1}^{\infty} \frac{r}{r!} + \sum_{r=1}^{\infty} \frac{1}{r!}$$

$$= 1 + \sum_{r=1}^{\infty} \frac{1}{(r-1)!} + \sum_{r=1}^{\infty} \frac{1}{r!} = 1 + \sum_{r=0}^{\infty} \frac{1}{r!} + \sum_{r=1}^{\infty} \frac{1}{r!}$$

BUT

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$e = \sum_{r=0}^{\infty} \frac{1}{r!} = 1 + \sum_{r=1}^{\infty} \frac{1}{r!}$$

$$= \left[1 + \sum_{r=1}^{\infty} \frac{1}{r!} \right] + \sum_{r=0}^{\infty} \frac{1}{r!} = e + e = 2e$$

Question 79 (*****)

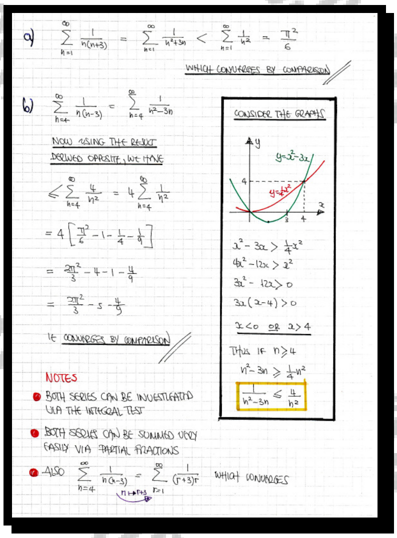
Investigate the convergence or divergence of each of the following two series using standard tests and justifying every step in the workings.

a) $$\sum_{n=1}^{\infty}\left[\frac{1}{n(n+3)}\right].$$

b) $$\sum_{n=4}^{\infty}\left[\frac{1}{n(n-3)}\right].$$

You may not conclude simply by summing each the series.

, convergent , convergent



a) $\sum_{n=1}^{\infty}\frac{1}{n(n+3)} = \sum_{n=1}^{\infty}\frac{1}{n^2+3n} < \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

WHICH CONVERGES BY COMPARISON

b) $\sum_{n=4}^{\infty}\frac{1}{n(n-3)} = \sum_{n=4}^{\infty}\frac{1}{n^2-3n}$

NOW USING THE RESULT DERIVED OPPOSITE, WE HAVE

$\leqslant \sum_{n=4}^{\infty}\frac{4}{n^2} = 4\sum_{n=4}^{\infty}\frac{1}{n^2}$

$= 4\left[\frac{\pi^2}{6} - 1 - \frac{1}{4} - \frac{1}{9}\right]$

$= \frac{2\pi^2}{3} - 4 - 1 - \frac{4}{9}$

$= \frac{2\pi^2}{3} - 5 - \frac{4}{9}$

IT CONVERGES BY COMPARISON

CONSIDER THE GRAPHS

$x^2 - 3x > \frac{1}{4}x^2$

$4x^2 - 12x > x^2$

$3x^2 - 12x > 0$

$3x(x-4) > 0$

$x<0$ OR $x>4$

THUS IF $n \geqslant 4$

$n^2 - 3n \geqslant \frac{1}{4}n^2$

$\frac{1}{n^2-3n} \leqslant \frac{4}{n^2}$

NOTES

- BOTH SERIES CAN BE INVESTIGATED VIA THE INTEGRAL TEST
- BOTH SERIES CAN BE SUMMED VERY EASILY VIA PARTIAL FRACTIONS
- ALSO $\sum_{n=4}^{\infty}\frac{1}{n(n-3)} = \sum_{r=1}^{\infty}\frac{1}{(r+3)r}$ WHICH CONVERGES ($n \mapsto r+3$)

Question 80 (*****)

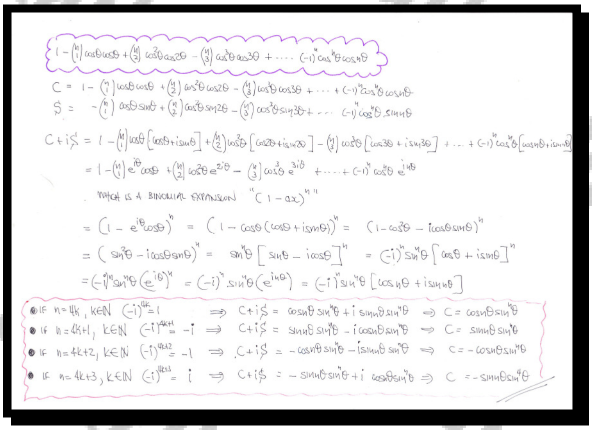
The finite sum C is given below.

$$C = \sum_{r=0}^{n} \left[\binom{n}{r} (-1)^n \cos^n \theta \cos n\theta \right].$$

Given that $n \in \mathbb{N}$ determine the 4 possible expressions for C.

Give the answers in exact fully simplified form.

$\boxed{n = 4k,\ k \in \mathbb{N} : C = \cos n\theta \sin^n \theta}$, $\boxed{n = 4k+1,\ k \in \mathbb{N} : C = \sin n\theta \sin^n \theta}$, $\boxed{n = 4k+2,\ k \in \mathbb{N} : C = -\cos n\theta \sin^n \theta}$, $\boxed{n = 4k+3,\ k \in \mathbb{N} : C = -\sin n\theta \sin^n \theta}$



Question 81 (*****)

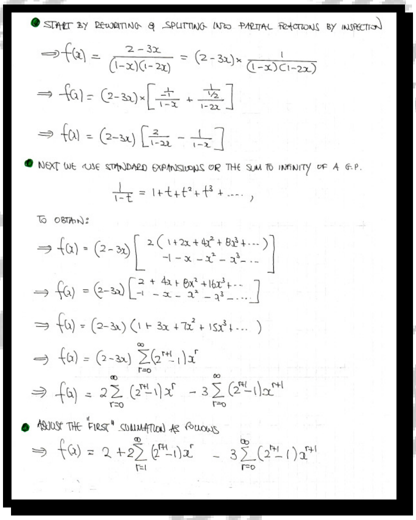
$$f(x) \equiv \frac{2-3x}{(1-x)(1-2x)}, \quad -\tfrac{1}{2} < x < \tfrac{1}{2}.$$

Show that $f(x)$ can be written in the form

$$f(x) = \sum_{r=0}^{\infty} \left[x^r g(r) \right],$$

where $g(r)$ is a simplified function to be found.

[ ], $g(r) = 2^r + 1$



START BY REWRITING & SPLITTING INTO PARTIAL FRACTIONS BY INSPECTION

$$\Rightarrow f(x) = \frac{2-3x}{(1-x)(1-2x)} = (2-3x) \times \frac{1}{(1-x)(1-2x)}$$

$$\Rightarrow f(x) = (2-3x) \times \left[\frac{-1}{1-x} + \frac{\frac{1}{1/2}}{1-2x} \right]$$

$$\Rightarrow f(x) = (2-3x) \left[\frac{2}{1-2x} - \frac{1}{1-x} \right]$$

NEXT WE USE STANDARD EXPANSIONS OR THE SUM TO INFINITY OF A G.P.

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots,$$

TO OBTAIN:

$$\Rightarrow f(x) = (2-3x) \left[\begin{array}{l} 2(1 + 2x + 4x^2 + 8x^3 + \dots) \\ -1 - x - x^2 - x^3 - \dots \end{array} \right]$$

$$\Rightarrow f(x) = (2-3x) \left[\begin{array}{l} 2 + 4x + 8x^2 + 16x^3 + \dots \\ -1 - x - x^2 - x^3 - \dots \end{array} \right]$$

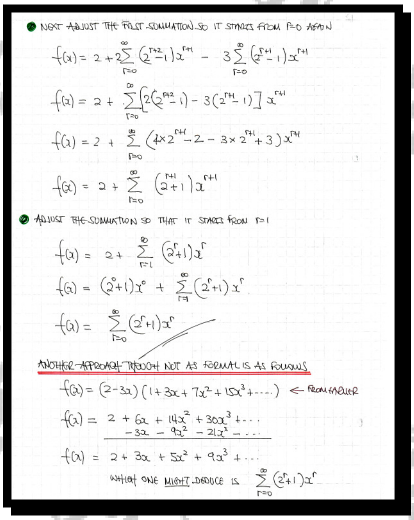
$$\Rightarrow f(x) = (2-3x)(1 + 3x + 7x^2 + 15x^3 + \dots)$$

$$\Rightarrow f(x) = (2-3x) \sum_{r=0}^{\infty} (2^{r+1} - 1) x^r$$

$$\Rightarrow f(x) = 2 \sum_{r=0}^{\infty} (2^{r+1} - 1) x^r - 3 \sum_{r=0}^{\infty} (2^{r+1} - 1) x^{r+1}$$

ADJUST THE "FIRST" SUMMATION AS FOLLOWS

$$\Rightarrow f(x) = 2 + 2 \sum_{r=1}^{\infty} (2^{r+1} - 1) x^r - 3 \sum_{r=0}^{\infty} (2^{r+1} - 1) x^{r+1}$$



NEXT ADJUST THE FIRST SUMMATION SO IT STARTS FROM r=0 AGAIN

$$f(x) = 2 + 2 \sum_{r=0}^{\infty} (2^{r+2} - 1) x^{r+1} - 3 \sum_{r=0}^{\infty} (2^{r+1} - 1) x^{r+1}$$

$$f(x) = 2 + \sum_{r=0}^{\infty} \left[2(2^{r+2} - 1) - 3(2^{r+1} - 1) \right] x^{r+1}$$

$$f(x) = 2 + \sum_{r=0}^{\infty} (4 \times 2^{r+1} - 2 - 3 \times 2^{r+1} + 3) x^{r+1}$$

$$f(x) = 2 + \sum_{r=0}^{\infty} (2^{r+1} + 1) x^{r+1}$$

ADJUST THE SUMMATION SO THAT IT STARTS FROM r=1

$$f(x) = 2 + \sum_{r=1}^{\infty} (2^r + 1) x^r$$

$$f(x) = (2^0 + 1) x^0 + \sum_{r=1}^{\infty} (2^r + 1) x^r$$

$$f(x) = \sum_{r=0}^{\infty} (2^r + 1) x^r$$

ANOTHER APPROACH THOUGH NOT AS FORMAL IS AS FOLLOWS

$$f(x) = (2-3x)(1 + 3x + 7x^2 + 15x^3 + \dots) \leftarrow \text{FROM EARLIER}$$

$$f(x) = \begin{array}{l} 2 + 6x + 14x^2 + 30x^3 + \dots \\ \quad - 3x - 9x^2 - 21x^3 - \dots \end{array}$$

$$f(x) = 2 + 3x + 5x^2 + 9x^3 + \dots$$

WHICH ONE MIGHT DEDUCE IS $\sum_{r=0}^{\infty} (2^r + 1) x^r$

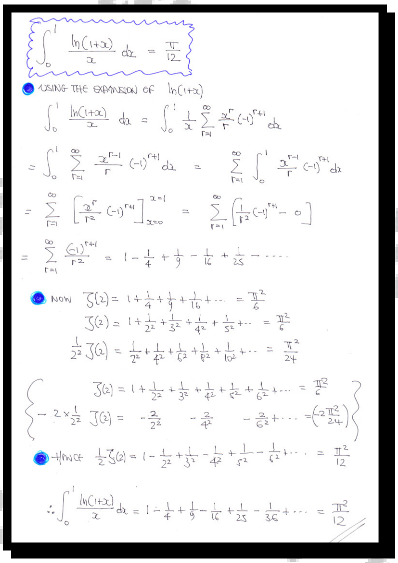
Question 82 (*****)

Show, by considering standard series, that

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}.$$

You may assume without proof that $\displaystyle\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right] = \frac{1}{6}\pi^2$

[ ], [proof]



$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi}{12}$$

USING THE EXPANSION OF $\ln(1+x)$

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \int_0^1 \frac{1}{x}\sum_{r=1}^{\infty}\frac{x^r}{r}(-1)^{r+1}\,dx$$

$$= \int_0^1 \sum_{r=1}^{\infty}\frac{x^{r-1}}{r}(-1)^{r+1}\,dx = \sum_{r=1}^{\infty}\int_0^1 \frac{x^{r-1}}{r}(-1)^{r+1}\,dx$$

$$= \sum_{r=1}^{\infty}\left[\frac{x^r}{r^2}(-1)^{r+1}\right]_{x=0}^{x=1} = \sum_{r=1}^{\infty}\left[\frac{1}{r^2}(-1)^{r+1} - 0\right]$$

$$= \sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots$$

NOW $\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}$

$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6}$

$\frac{1}{2^2}\zeta(2) = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots = \frac{\pi^2}{24}$

$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots = \frac{\pi^2}{6}$

$-2\times\frac{1}{2^2}\zeta(2) = -\frac{2}{2^2} - \frac{2}{4^2} - \frac{2}{6^2} + \dots = \left(-2\frac{\pi^2}{24}\right)$

HENCE $\frac{1}{2}\zeta(2) = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots = \frac{\pi^2}{12}$

$$\therefore \int_0^1 \frac{\ln(1+x)}{x}\,dx = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \dots = \frac{\pi^2}{12}$$

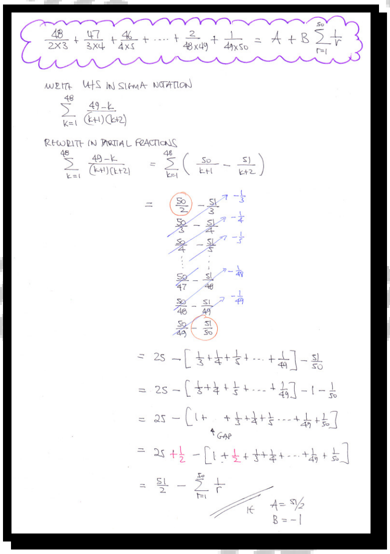
Question 83 (*****)

Show, by a detailed method, that

$$\frac{48}{2\times 3}+\frac{47}{3\times 4}+\frac{46}{4\times 5}\ ...+\frac{2}{48\times 49}+\frac{1}{49\times 50}=A+B\sum_{r=1}^{50}\frac{1}{r},$$

where A and B are constants to be found.

$\boxed{}$, $\boxed{A=\frac{51}{2},\quad B=-1}$

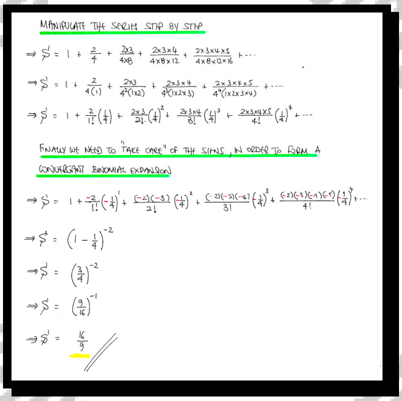


Question 84 (*****)

$$S = 1 + \frac{2}{4} + \frac{2\cdot 3}{4\cdot 8} + \frac{2\cdot 3\cdot 4}{4\cdot 8\cdot 12} + \frac{2\cdot 3\cdot 4\cdot 5}{4\cdot 8\cdot 12\cdot 16} + \ldots$$

By considering a suitable binomial series, or other wise, find the sum to infinity of S.

$$S_\infty = \frac{16}{9}$$



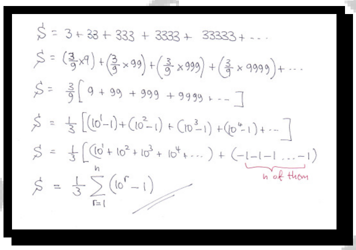
Question 85 (*****)

$$3 + 33 + 333 + 3333 + 33333 + \ldots$$

Express the sum of the first n terms of the above series in sigma notation.

You are not required to sum the series.

$$S_n = \sum_{r=1}^{n} \left[\frac{1}{3}\left(10^r - 1\right) \right]$$



Question 86 (*****)

$$S_n = (2\times1!)+(5\times2!)+(10\times3!)+(17\times4!)+\ldots+\left(n^2+1\right)n!$$

Use an appropriate method to show that

$$S_n = n(n+1)!$$

[ ] , [proof]

START BY WRITING THE SERIES IN SIGMA NOTATION

$$(2\times1!)+(5\times2!)+(10\times3!)+(17\times4!)+\ldots+\left[(n^2+1)\times n!\right]=\sum_{r=1}^{n}\left[(r^2+1)r!\right]$$

TRY SOME DIFFERENCES INVOLVING FACTORIALS, TRYING TO OBTAIN THE SUMMAND

$$(r+1)!-r! = (r+1)r!-r! = r\times r!$$

AS THIS DOES NOT PRODUCE A QUADRATIC TERM IN r WE MAY TRY

$$(r+2)!-r! = (r+2)(r+1)r!-r!$$
$$(r+2)!-r! = (r^2+3r+2)r!-r!$$
$$(r+2)!-r! = (r^2+3r+1)r!$$
$$(r+2)!-r! = (r^2+1)r!+3r\times r!$$

FROM ABOVE $r\times r! \equiv (r+1)!-r!$

$$(r+2)!-r! = (r^2+1)r!+3\left[(r+1)!-r!\right]$$
$$(r+2)!-r! = (r^2+1)r!+3(r+1)!-3r!$$
$$(r+2)! = (r^2+1)r!+3(r+1)!-2r!$$
$$(r+2)!-3(r+1)!+2r! = (r^2+1)r!$$

HENCE WE HAVE

$$(r^2+1)r! \equiv (r+2)!-3(r+1)!+2r!$$

WRITING THE IDENTITY JUST OBTAINED

$$(r^2+1)r! \equiv (r+2)! - 3(r+1)! + 2r!$$

$r=1$ $\quad 2\times1! = 3! - 3\times2! + 2\times1!$

$r=2$ $\quad 5\times2! = 4! - 3\times3! + 2\times2!$

$r=3$ $\quad 10\times3! = 5! - 3\times4! + 2\times3!$

$r=4$ $\quad 17\times4! = 6! - 3\times5! + 2\times4!$

$\vdots$

$r=n-1$ $\quad \left[(n-1)^2+1\right](n-1)! = (n+1)! - 3\times n! + 2\times(n-1)!$

$r=n$ $\quad (n^2+1)n! = (n+2)! - 3\times(n+1)! + 2\times n!$

$$\sum_{r=1}^{n}\left[(r^2+1)r!\right] = (n+2)!-2(n+1)!-3\times2!+2\times1!+2\times2!$$
$$= (n+2)(n+1)!-2(n+1)!-6+2+4$$
$$= (n+2-2)(n+1)!$$
$$= n(n+1)!$$

Question 87 (*****)

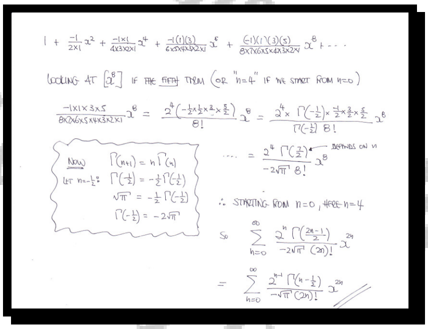
Consider the infinite series

$$1+\frac{-1}{2\times1}x^2+\frac{-1\times1}{4\times3\times2\times1}x^4+\frac{-1\times1\times3}{6\times5\times4\times3\times2\times1}x^6+\frac{-1\times1\times3\times5}{8\times7\times6\times5\times4\times3\times2\times1}x^8+\ldots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\boxed{\sum_{n=0}^{\infty}\left[\frac{2^{n-1}\Gamma\left(n-\frac{1}{2}\right)}{-\sqrt{\pi}\times(2n)!}x^{2n}\right]}$$


$$1+\frac{-1}{2\times1}x^2+\frac{-1\times1}{4\times3\times2\times1}x^4+\frac{-1(1)(3)}{6\times5\times4\times3\times2\times1}x^6+\frac{(-1)(1)(3)(5)}{8\times7\times6\times5\times4\times3\times2\times1}x^8+\ldots$$

LOOKING AT $[x^8]$ IF THE FIFTH TERM (OR "$n=4$" IF WE START FROM $n=0$)

$$\frac{-1\times1\times3\times5}{8\times7\times6\times5\times4\times3\times2\times1}x^8=\frac{2^4\left(-\frac{1}{2}\times\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\right)}{8!}x^8=\frac{2^4\times\Gamma\left(-\frac{1}{2}\right)\times\frac{-1}{2}\times\frac{3}{2}\times\frac{5}{2}}{\Gamma\left(-\frac{1}{2}\right)8!}x^8$$

NOW $\Gamma(n+1)=n\Gamma(n)$

LET $n=-\frac{1}{2}$: $\Gamma\left(\frac{1}{2}\right)=-\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$

$\sqrt{\pi}=-\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$

$\Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi}$

$$\ldots=\frac{2^4\Gamma\left(\frac{7}{2}\right)}{-2\sqrt{\pi}\,8!}x^8$$ ← DEPENDS ON n

$\therefore$ STARTING FROM $n=0$, HERE $n=4$

$$\text{So}\quad\sum_{n=0}^{\infty}\frac{2^n\Gamma\left(\frac{2n-1}{2}\right)}{-2\sqrt{\pi}\,(2n)!}x^{2n}$$

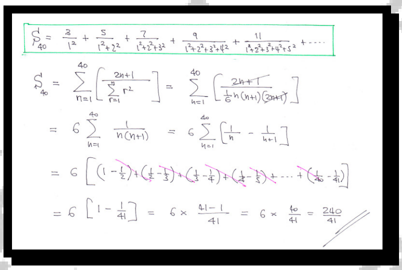
$$=\sum_{n=0}^{\infty}\frac{2^{n-1}\Gamma\left(n-\frac{1}{2}\right)}{-\sqrt{\pi}\,(2n)!}x^{2n}$$

Question 88 (*****)

$$\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\frac{9}{1^2+2^2+3^2+4^2}+\frac{11}{1^2+2^2+3^2+4^2+5^2}+\ldots,$$

Show, by a detailed method, that the sum of the first 40 terms of the series shown above is $\frac{240}{41}$.

[ ] , proof


$$S_{40}=\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\frac{9}{1^2+2^2+3^2+4^2}+\frac{11}{1^2+2^2+3^2+4^2+5^2}+\ldots$$

$$S_{40}=\sum_{n=1}^{40}\left[\frac{2n+1}{\sum_{r=1}^{n}r^2}\right]=\sum_{n=1}^{40}\left[\frac{2n+1}{\frac{1}{6}n(n+1)(2n+1)}\right]$$

$$=6\sum_{n=1}^{40}\frac{1}{n(n+1)}=6\sum_{n=1}^{40}\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

$$=6\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{40}-\frac{1}{41}\right)\right]$$

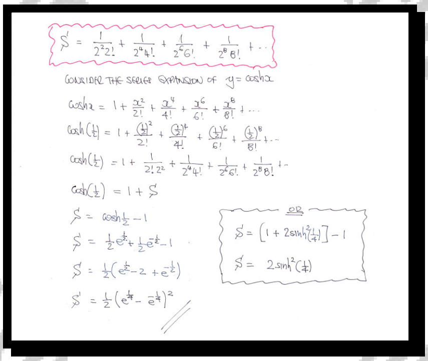
$$=6\left[1-\frac{1}{41}\right]=6\times\frac{41-1}{41}=6\times\frac{40}{41}=\frac{240}{41}$$

Question 89 (*****)

By showing a detailed method, sum the following series.

$$\frac{1}{2^2\,2!}+\frac{1}{2^4\,4!}+\frac{1}{2^6\,6!}+\frac{1}{2^8\,8!}+\ldots+\frac{1}{2^{2r}\,(2r)!}+\ldots$$

$$\boxed{\frac{1}{2}\left[e^{\frac{1}{4}}-e^{-\frac{1}{4}}\right]^2 = 2\sinh^2\left(\frac{1}{4}\right)}$$


$S = \frac{1}{2^2 2!} + \frac{1}{2^4 4!} + \frac{1}{2^6 6!} + \frac{1}{2^8 8!} + \ldots$

CONSIDER THE SERIES EXPANSION OF $y = \cosh x$

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots$

$\cosh\left(\frac{1}{2}\right) = 1 + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^4}{4!} + \frac{\left(\frac{1}{2}\right)^6}{6!} + \frac{\left(\frac{1}{2}\right)^8}{8!} + \ldots$

$\cosh\left(\frac{1}{2}\right) = 1 + \frac{1}{2!\,2^2} + \frac{1}{2^4 4!} + \frac{1}{2^6 6!} + \frac{1}{2^8 8!} + \ldots$

$\cosh\left(\frac{1}{2}\right) = 1 + S$

$S = \cosh\frac{1}{2} - 1$

$S = \frac{1}{2}e^{\frac{1}{2}} + \frac{1}{2}e^{-\frac{1}{2}} - 1$

$S = \frac{1}{2}\left(e^{\frac{1}{2}} - 2 + e^{-\frac{1}{2}}\right)$

$S = \frac{1}{2}\left(e^{\frac{1}{4}} - e^{-\frac{1}{4}}\right)^2$

OR

$S = \left[1 + 2\sinh^2\left(\frac{1}{4}\right)\right] - 1$

$S = 2\sinh^2\left(\frac{1}{4}\right)$

Question 90 (*****)

A function is defined as

$$[x] \equiv \{\text{the greatest integer less or equal to } x\}.$$

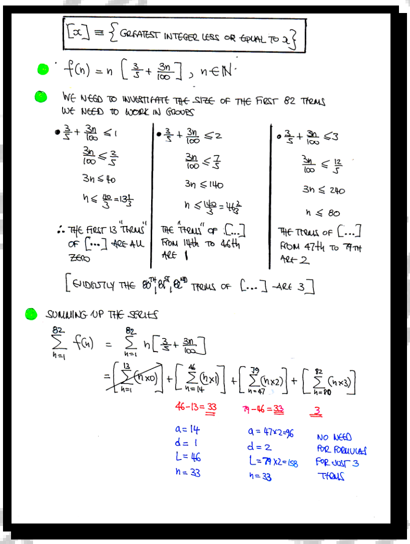
The function f is defined as

$$f(n) = n\left[\frac{3}{5} + \frac{3n}{100}\right], \quad n \in \mathbb{N}.$$

Determine the value of

$$\sum_{n=1}^{82} f(n).$$

$\boxed{}$, $\boxed{5877}$



$[x] \equiv \{\text{GREATEST INTEGER LESS OR EQUAL TO } x\}$

- $f(n) = n\left[\frac{3}{5} + \frac{3n}{100}\right]$, $n \in \mathbb{N}$.
- WE NEED TO INVESTIGATE THE SIZE OF THE FIRST 82 TERMS
WE NEED TO WORK IN GROUPS

- $\frac{3}{5} + \frac{3n}{100} \leq 1$, $\frac{3n}{100} \leq \frac{2}{5}$, $3n \leq 40$, $n \leq \frac{40}{3} = 13\frac{1}{3}$
∴ THE FIRST 13 "TERMS" OF [...] ARE ALL ZERO
- $\frac{3}{5} + \frac{3n}{100} \leq 2$, $\frac{3n}{100} \leq \frac{7}{5}$, $3n \leq 140$, $n \leq \frac{140}{3} = 46\frac{2}{3}$
THE "TERMS" OF [...] FROM 14th TO 46th ARE 1
- $\frac{3}{5} + \frac{3n}{100} \leq 3$, $\frac{3n}{100} \leq \frac{12}{5}$, $3n \leq 240$, $n \leq 80$
THE TERMS OF [...] FROM 47th TO 79TH ARE 2

[EVIDENTLY THE $80^{TH}, 81^{ST}, 82^{ND}$ TERMS OF [...] ARE 3]

- SUMMING UP THE SERIES

$$\sum_{n=1}^{82} f(n) = \sum_{n=1}^{82} n\left[\frac{3}{5} + \frac{3n}{100}\right]$$

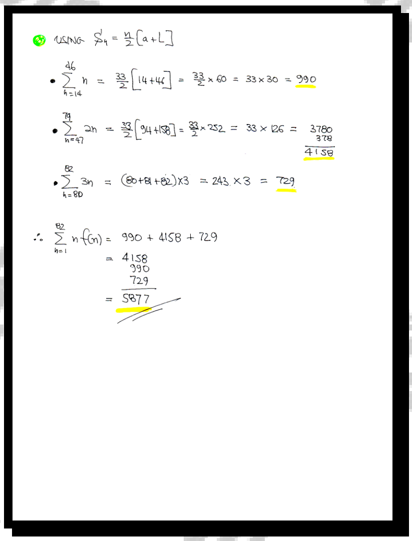
$$= \left[\sum_{n=1}^{13}(n\times 0)\right] + \left[\sum_{n=14}^{46}(n\times 1)\right] + \left[\sum_{n=47}^{79}(n\times 2)\right] + \left[\sum_{n=80}^{82}(n\times 3)\right]$$

$46-13 = 33$, $79-46 = 33$, 3

$a = 14$, $d = 1$, $L = 46$, $n = 33$

$a = 47\times 2 = 94$, $d = 2$, $L = 79\times 2 = 158$, $n = 33$

NO NEED FOR FORMULAS FOR JUST 3 TERMS



- USING $S_n = \frac{n}{2}[a + L]$

- $\sum_{n=14}^{46} n = \frac{33}{2}[14+46] = \frac{33}{2}\times 60 = 33\times 30 = 990$
- $\sum_{n=47}^{79} 2n = \frac{33}{2}[94+158] = \frac{33}{2}\times 252 = 33\times 126 = 3780 + 378 = 4158$
- $\sum_{n=80}^{82} 3n = (80+81+82)\times 3 = 243\times 3 = 729$

$\therefore \sum_{n=1}^{82} nf(n) = 990 + 4158 + 729$
$= 4158 + 990 + 729$
$= 5877$

Question 91 (*****)

Use partial fractions and a suitable Mclaurin expansion to sum the following series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+3)}.$$

$$\boxed{\frac{2}{3}\ln 2 - \frac{5}{18}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n(n+3)} = \sum_{n=1}^{\infty} (-1)^{n+1} \times \frac{1}{n(n+3)} = \dots \text{ PARTIAL FRACTIONS BY INSPECTION} \dots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \times \left[\frac{\frac{1}{3}}{n} - \frac{\frac{1}{3}}{n+3}\right] = \frac{1}{3}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \frac{(-1)^{n+1}}{n+3}$$

$$= \frac{1}{3}\Big[1 - \frac{1}{4}$$
$$-\frac{1}{2} + \frac{1}{5}$$
$$+\frac{1}{3} - \frac{1}{6}$$
$$-\frac{1}{4} + \frac{1}{7}$$
$$+\frac{1}{5} - \frac{1}{8}$$
$$-\frac{1}{6} + \dots\Big]$$

$$= \frac{1}{3}\left[\left(1 - \frac{1}{2} + \frac{1}{3}\right) + 2\left(-\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\right)\right]$$

$$= \frac{5}{18} + \frac{2}{3}\left(-\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots\right)$$

NOW $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \dots$

$\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

$\ln 2 = \left(1 - \frac{1}{2} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots\right)$

$\ln 2 - \frac{5}{6} = -\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$

$$= \frac{5}{18} + \frac{2}{3}\left(\ln 2 - \frac{5}{6}\right)$$

$$= \frac{2}{3}\ln 2 - \frac{5}{18}$$

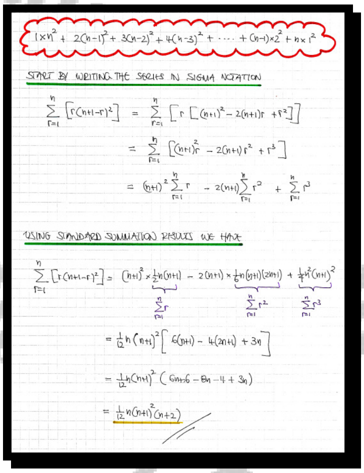
Question 92 (*****)

The function f is defined for $n \in \mathbb{N}$ as

$$f(n) \equiv 1 \times n^2 + 2(n-1)^2 + 3(n-2)^2 + 4(n-3)^2 + \ldots + (n-1) \times 2^2 + n \times 1^2 .$$

Determine a simplified expression for the sum of $f(n)$, giving the final answer in fully factorized form.

$$f(n) = \frac{1}{12} n(n+2)(n+1)^2$$

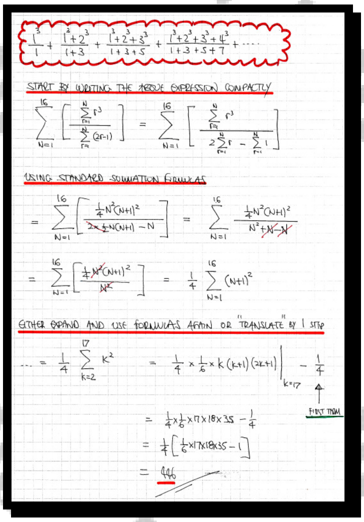


Question 93 (*****)

Find the sum of the first 16 terms of the following series.

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\frac{1^3+2^3+3^3+4^3}{1+3+5+7}+\ldots$$

SP-M , $\boxed{446}$



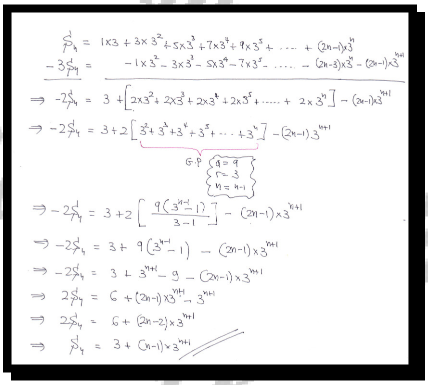
Question 94 (*****)

$$S_n = 1\times 3 + 3\times 3^2 + 5\times 3^3 + 7\times 3^4 + \ldots + (2n-1)\times 3^n$$

Find a simplified expression for S_n, giving the answer in the form $A + f(n)\times 3^{n+1}$, where A is an integer and $f(n)$ a linear function of n.

[*The standard techniques used for the summation of a geometric series are useful in this question*]

$$\boxed{S_n = 3 + (n-1)\times 3^{n+1}}$$

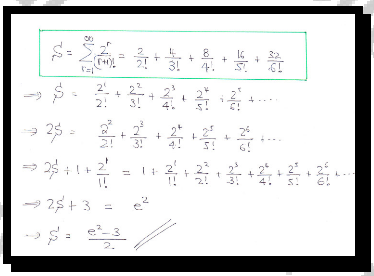


Question 95 (*****)

By showing a detailed method, sum the following series.

$$\sum_{r=1}^{\infty}\left[\frac{2^r}{(r+1)!}\right].$$

$$\boxed{\tfrac{1}{2}\left(e^2-3\right)}$$



Question 96 (*****)

The r^{th} term of a progression is given by

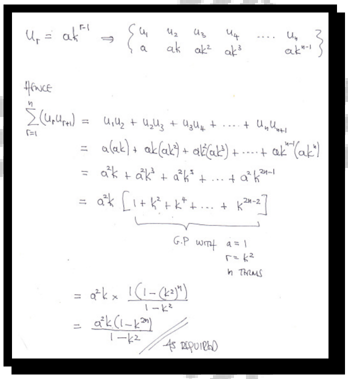
$$u_r = ak^{r-1},$$

where a and k are constants with $k \neq \pm 1$.

Show clearly that

$$\sum_{r=1}^{n} \left(u_r \times u_{r+1}\right) = \frac{a^2 k\left(1-k^{2n}\right)}{1-k^2}.$$

proof



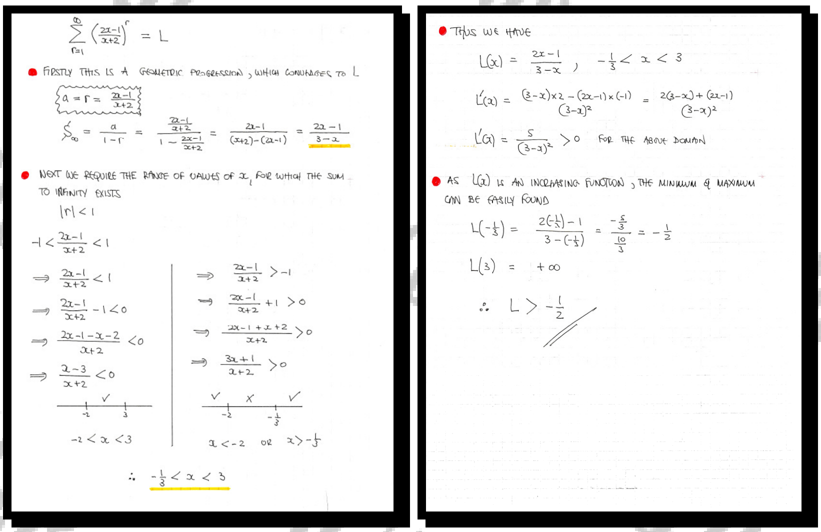
Question 97 (*****)

It is given that the following series converges to a limit L.

$$\sum_{r=1}^{\infty}\left[\frac{2x-1}{x+2}\right]^{r}.$$

Determine with full justification the range of possible values of L.

$\boxed{}$, $\boxed{L > -\frac{1}{2}}$



$\sum_{r=1}^{\infty}\left(\frac{2x-1}{x+2}\right)^r = L$

- FIRSTLY THIS IS A GEOMETRIC PROGRESSION, WHICH CONVERGES TO L

$a = r = \frac{2x-1}{x+2}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{2x-1}{x+2}}{1-\frac{2x-1}{x+2}} = \frac{2x-1}{(x+2)-(2x-1)} = \frac{2x-1}{3-x}$

- NEXT WE REQUIRE THE RANGE OF VALUES OF x, FOR WHICH THE SUM TO INFINITY EXISTS

$|r| < 1$

$-1 < \frac{2x-1}{x+2} < 1$

$\Rightarrow \frac{2x-1}{x+2} < 1$ $\Rightarrow \frac{2x-1}{x+2} - 1 < 0$ $\Rightarrow \frac{2x-1-x-2}{x+2} < 0$ $\Rightarrow \frac{x-3}{x+2} < 0$, giving $-2 < x < 3$

$\Rightarrow \frac{2x-1}{x+2} > -1$ $\Rightarrow \frac{2x-1}{x+2} + 1 > 0$ $\Rightarrow \frac{2x-1+x+2}{x+2} > 0$ $\Rightarrow \frac{3x+1}{x+2} > 0$, giving $x < -2$ or $x > -\frac{1}{3}$

$\therefore -\frac{1}{3} < x < 3$

- THUS WE HAVE

$L(x) = \frac{2x-1}{3-x}, \quad -\frac{1}{3} < x < 3$

$L'(x) = \frac{(3-x)\times 2 - (2x-1)\times(-1)}{(3-x)^2} = \frac{2(3-x)+(2x-1)}{(3-x)^2}$

$L'(x) = \frac{5}{(3-x)^2} > 0$ FOR THE ABOVE DOMAIN

- AS $L(x)$ IS AN INCREASING FUNCTION, THE MINIMUM & MAXIMUM CAN BE EASILY FOUND

$L(-\frac{1}{3}) = \frac{2(-\frac{1}{3})-1}{3-(-\frac{1}{3})} = \frac{-\frac{5}{3}}{\frac{10}{3}} = -\frac{1}{2}$

$L(3) = +\infty$

$\therefore L > -\frac{1}{2}$

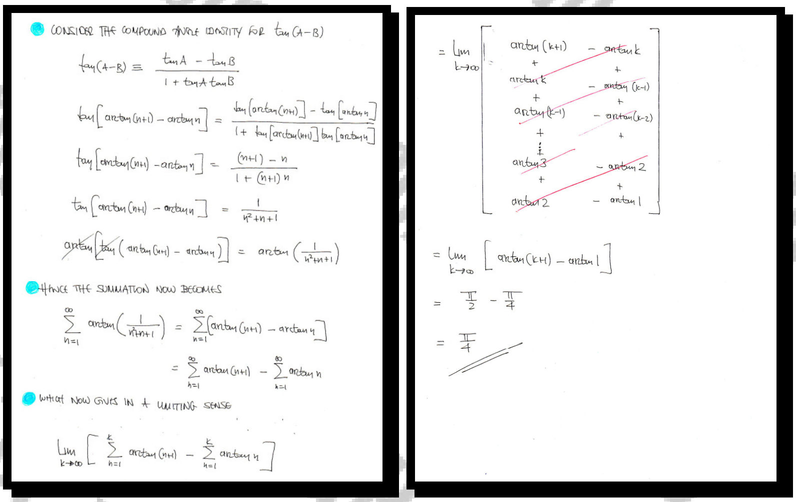
Question 98 (*****)

By considering the trigonometric identity for $\tan(A-B)$, with $A=\arctan(n+1)$ and $B=\arctan(n)$, sum the following series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right).$$

You may assume the series converges.

$\boxed{}$, $\boxed{\frac{\pi}{4}}$



CONSIDER THE COMPOUND ANGLE IDENTITY FOR $\tan(A-B)$

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1+\tan A \tan B}$$

$$\tan[\arctan(n+1) - \arctan n] = \frac{\tan[\arctan(n+1)] - \tan[\arctan n]}{1+\tan[\arctan(n+1)]\tan[\arctan n]}$$

$$\tan[\arctan(n+1) - \arctan n] = \frac{(n+1)-n}{1+(n+1)n}$$

$$\tan[\arctan(n+1) - \arctan n] = \frac{1}{n^2+n+1}$$

$$\arctan\left[\tan(\arctan(n+1) - \arctan n)\right] = \arctan\left(\frac{1}{n^2+n+1}\right)$$

HENCE THE SUMMATION NOW BECOMES

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right) = \sum_{n=1}^{\infty}[\arctan(n+1) - \arctan n]$$

$$= \sum_{n=1}^{\infty} \arctan(n+1) - \sum_{n=1}^{\infty} \arctan n$$

WHICH NOW GIVES IN A LIMITING SENSE

$$\lim_{k\to\infty}\left[\sum_{n=1}^{k} \arctan(n+1) - \sum_{n=1}^{k} \arctan n\right]$$

$$= \lim_{k\to\infty}\left[\begin{matrix} \arctan(k+1) & -\arctan k \\ +\arctan k & -\arctan(k-1) \\ +\arctan(k-1) & -\arctan(k-2) \\ \vdots & \\ +\arctan 3 & -\arctan 2 \\ +\arctan 2 & -\arctan 1 \end{matrix}\right]$$

$$= \lim_{k\to\infty}[\arctan(k+1) - \arctan 1]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Question 99 (*****)

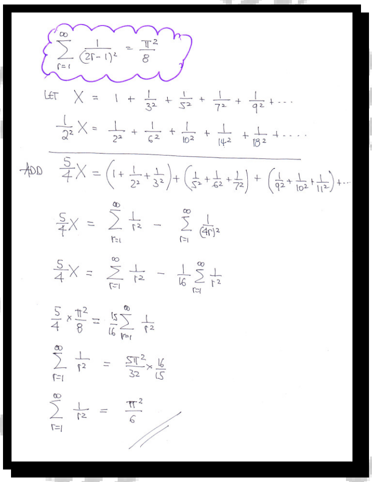
It is given that

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}.$$

By using this fact alone find the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{r^2}.$$

$$\boxed{\frac{\pi^2}{6}}$$



$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$

LET $X = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$

$\frac{1}{2^2}X = \frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{18^2} + \dots$

ADD $\frac{5}{4}X = \left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) + \left(\frac{1}{9^2} + \frac{1}{10^2} + \frac{1}{11^2}\right) + \dots$

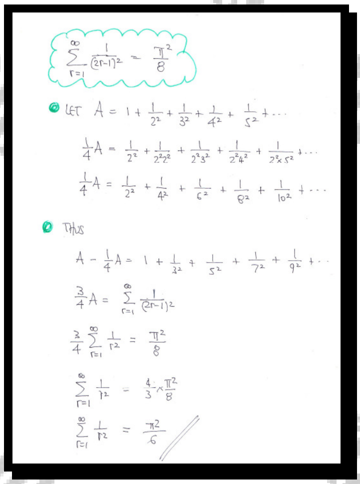
$\frac{5}{4}X = \sum_{r=1}^{\infty} \frac{1}{r^2} - \sum_{r=1}^{\infty} \frac{1}{(4r)^2}$

$\frac{5}{4}X = \sum_{r=1}^{\infty} \frac{1}{r^2} - \frac{1}{16}\sum_{r=1}^{\infty} \frac{1}{r^2}$

$\frac{5}{4} \times \frac{\pi^2}{8} = \frac{15}{16}\sum_{r=1}^{\infty} \frac{1}{r^2}$

$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{5\pi^2}{32} \times \frac{16}{15}$

$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$



$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2} = \frac{\pi^2}{8}$$

- LET $A = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$

$\frac{1}{4}A = \frac{1}{2^2} + \frac{1}{2^2 2^2} + \frac{1}{2^2 3^2} + \frac{1}{2^2 4^2} + \frac{1}{2^2 \times 5^2} + \dots$

$\frac{1}{4}A = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots$

- THUS

$A - \frac{1}{4}A = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$

$\frac{3}{4}A = \sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}$

$\frac{3}{4}\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{8}$

$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{4}{3} \times \frac{\pi^2}{8}$

$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$

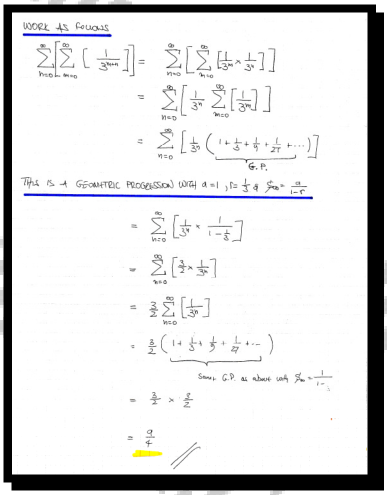
Question 100 (*****)

Evaluate the following expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{1}{3^{m+n}}\right].$$

Detailed workings must be shown.

$\boxed{}$, $\boxed{\frac{9}{4}}$


WORK AS FOLLOWS

$$\sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\left[\frac{1}{3^{m+n}}\right]\right] = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\left[\frac{1}{3^m}\times\frac{1}{3^n}\right]\right] = \sum_{n=0}^{\infty}\left[\frac{1}{3^n}\sum_{m=0}^{\infty}\left[\frac{1}{3^m}\right]\right] = \sum_{n=0}^{\infty}\left[\frac{1}{3^n}\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots\right)\right]$$

G.P.

THIS IS A GEOMETRIC PROGRESSION WITH $a=1$, $r=\frac{1}{3}$ & $S_\infty = \frac{a}{1-r}$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{3^n}\times\frac{1}{1-\frac{1}{3}}\right] = \sum_{n=0}^{\infty}\left[\frac{3}{2}\times\frac{1}{3^n}\right] = \frac{3}{2}\sum_{n=0}^{\infty}\left[\frac{1}{3^n}\right] = \frac{3}{2}\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots\right)$$

Same G.P. as above with $S_\infty = \frac{1}{1-\frac{1}{3}}$

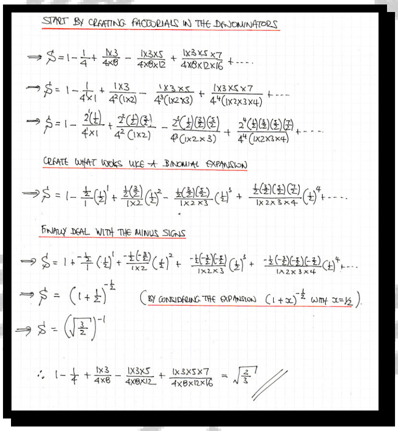
$$= \frac{3}{2}\times\frac{3}{2} = \frac{9}{4}$$

Question 101 (*****)

$$S = 1 - \frac{1}{4} + \frac{1\cdot 3}{4\cdot 8} - \frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12} + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 8\cdot 12\cdot 16} - \dots$$

Find the sum to infinity of S, by considering the binomial series expansion of $(1+x)^n$ for suitable values of x and n.

$\boxed{}$, $\boxed{S_\infty = \sqrt{\frac{2}{3}}}$

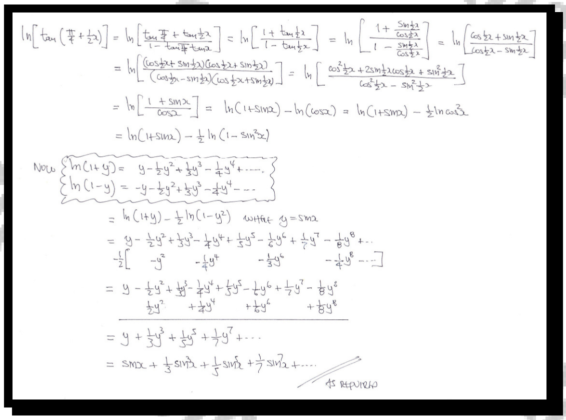


Question 102 (*****)

Show clearly that

$$\ln\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]=\sum_{n=1}^{\infty}\frac{\sin^{2n-1}x}{2n-1}.$$

proof



$$\ln\left[\tan\left(\tfrac{\pi}{4}+\tfrac{1}{2}x\right)\right]=\ln\left[\frac{\tan\frac{\pi}{4}+\tan\frac{1}{2}x}{1-\tan\frac{\pi}{4}\tan\frac{1}{2}x}\right]=\ln\left[\frac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x}\right]=\ln\left[\frac{1+\frac{\sin\frac{1}{2}x}{\cos\frac{1}{2}x}}{1-\frac{\sin\frac{1}{2}x}{\cos\frac{1}{2}x}}\right]=\ln\left[\frac{\cos\frac{1}{2}x+\sin\frac{1}{2}x}{\cos\frac{1}{2}x-\sin\frac{1}{2}x}\right]$$

$$=\ln\left[\frac{(\cos\frac{1}{2}x+\sin\frac{1}{2}x)(\cos\frac{1}{2}x+\sin\frac{1}{2}x)}{(\cos\frac{1}{2}x-\sin\frac{1}{2}x)(\cos\frac{1}{2}x+\sin\frac{1}{2}x)}\right]=\ln\left[\frac{\cos^2\frac{1}{2}x+2\sin\frac{1}{2}x\cos\frac{1}{2}x+\sin^2\frac{1}{2}x}{\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x}\right]$$

$$=\ln\left[\frac{1+\sin x}{\cos x}\right]=\ln(1+\sin x)-\ln(\cos x)=\ln(1+\sin x)-\tfrac{1}{2}\ln\cos^2 x$$

$$=\ln(1+\sin x)-\tfrac{1}{2}\ln(1-\sin^2 x)$$

Now

$$\ln(1+y)=y-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3-\tfrac{1}{4}y^4+\dots$$
$$\ln(1-y)=-y-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3-\tfrac{1}{4}y^4-\dots$$

$$=\ln(1+y)-\tfrac{1}{2}\ln(1-y^2)\quad\text{with } y=\sin x$$

$$=y-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3-\tfrac{1}{4}y^4+\tfrac{1}{5}y^5-\tfrac{1}{6}y^6+\tfrac{1}{7}y^7-\tfrac{1}{8}y^8+\dots$$
$$-\tfrac{1}{2}\left[-y^2-\tfrac{1}{4}y^4-\tfrac{1}{3}y^6-\tfrac{1}{4}y^8-\dots\right]$$

$$=y-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3-\tfrac{1}{4}y^4+\tfrac{1}{5}y^5-\tfrac{1}{6}y^6+\tfrac{1}{7}y^7-\tfrac{1}{8}y^8$$
$$\tfrac{1}{2}y^2+\tfrac{1}{4}y^4+\tfrac{1}{6}y^6+\tfrac{1}{8}y^8$$

$$=y+\tfrac{1}{3}y^3+\tfrac{1}{5}y^5+\tfrac{1}{7}y^7+\dots$$

$$=\sin x+\tfrac{1}{3}\sin^3 x+\tfrac{1}{5}\sin^5 x+\tfrac{1}{7}\sin^7 x+\dots$$

AS REQUIRED

Question 103 (*****)

The function f is defined as

$$f(n) = \frac{e^{-\lambda}\lambda^{n}}{n!},$$

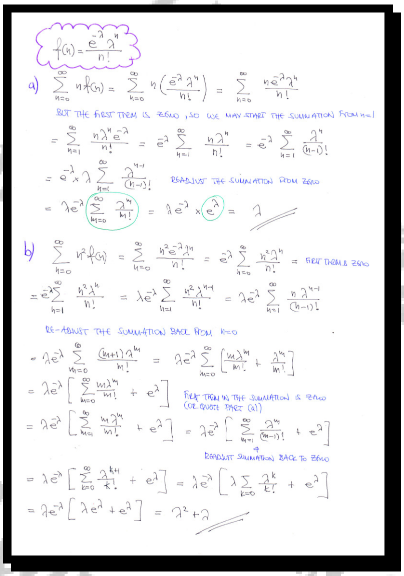
where $n = 0, 1, 2, 3, 4, \ldots$ and λ is a positive constant.

By showing a detailed method, prove that …

a) … $\displaystyle\sum_{n=0}^{\infty}\left[n f(n)\right] = \lambda$.

b) … $\displaystyle\sum_{n=0}^{\infty}\left[n^{2} f(n)\right] = \lambda^{2} + \lambda$.

proof

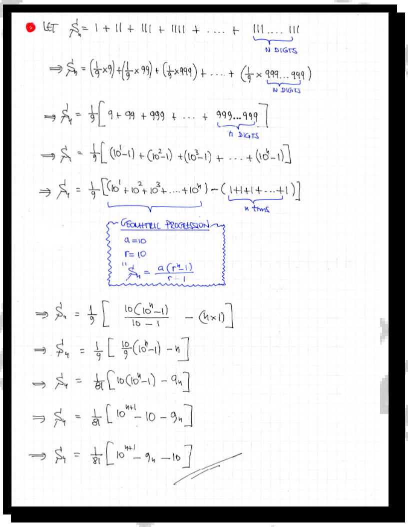


Question 104 (*****)

Find in exact simplified form an exact expression for the sum of the first n terms of the following series

$$1 + 11 + 111 + 1111 + 11111 + \dots$$

$\boxed{}$, $\boxed{S_n = \frac{1}{81}\left[10^{n+1} - 10 - 9n\right]}$



- LET $S_n = 1 + 11 + 111 + 1111 + \dots + \underbrace{111\dots111}_{\text{N DIGITS}}$

$\Rightarrow S_n = \left(\frac{1}{9}\times 9\right) + \left(\frac{1}{9}\times 99\right) + \left(\frac{1}{9}\times 999\right) + \dots + \left(\frac{1}{9}\times \underbrace{999\dots999}_{\text{N DIGITS}}\right)$

$\Rightarrow S_n = \frac{1}{9}\left[9 + 99 + 999 + \dots + \underbrace{999\dots999}_{n\text{ DIGITS}}\right]$

$\Rightarrow S_n = \frac{1}{9}\left[(10^1-1) + (10^2-1) + (10^3-1) + \dots + (10^n-1)\right]$

$\Rightarrow S_n = \frac{1}{9}\left[(10^1+10^2+10^3+\dots+10^n) - (\underbrace{1+1+1+\dots+1}_{n\text{ terms}})\right]$

GEOMETRIC PROGRESSION: $a=10$, $r=10$, "$S_n = \frac{a(r^n-1)}{r-1}$"

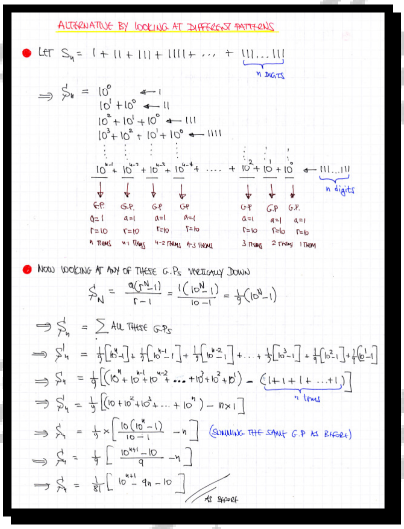
$\Rightarrow S_n = \frac{1}{9}\left[\frac{10(10^n-1)}{10-1} - (n\times 1)\right]$

$\Rightarrow S_n = \frac{1}{9}\left[\frac{10}{9}(10^n-1) - n\right]$

$\Rightarrow S_n = \frac{1}{81}\left[10(10^n-1) - 9n\right]$

$\Rightarrow S_n = \frac{1}{81}\left[10^{n+1} - 10 - 9n\right]$

$\Rightarrow S_n = \frac{1}{81}\left[10^{n+1} - 9n - 10\right]$



ALTERNATIVE BY LOOKING AT DIFFERENT PATTERNS

- LET $S_n = 1 + 11 + 111 + 1111 + \dots + \underbrace{111\dots111}_{n\text{ DIGITS}}$

$\Rightarrow S_n = 10^0 \leftarrow 1$
$10^1 + 10^0 \leftarrow 11$
$10^2 + 10^1 + 10^0 \leftarrow 111$
$10^3 + 10^2 + 10^1 + 10^0 \leftarrow 1111$
$\vdots$
$10^{n-1} + 10^{n-2} + 10^{n-3} + 10^{n-4} + \dots + 10^2 + 10^1 + 10^0 \leftarrow \underbrace{111\dots111}_{n\text{ digits}}$

G.P. $a=1$, $r=10$, n TERMS; G.P. $a=1$, $r=10$, $n-1$ TERMS; G.P. $a=1$, $r=10$, $n-2$ TERMS; G.P. $a=1$, $r=10$, $n-3$ TERMS; … G.P. $a=1$, $r=10$, 3 TERMS; G.P. $a=1$, $r=10$, 2 TERMS; G.P. $a=1$, $r=10$, 1 TERM

- NOW LOOKING AT ANY OF THESE G.Ps VERTICALLY DOWN

$S_N = \frac{a(r^N-1)}{r-1} = \frac{1(10^N-1)}{10-1} = \frac{1}{9}(10^N-1)$

$\Rightarrow S_n = \sum$ ALL THESE G.Ps

$\Rightarrow S_n = \frac{1}{9}[10^n-1] + \frac{1}{9}[10^{n-1}-1] + \frac{1}{9}[10^{n-2}-1] + \dots + \frac{1}{9}[10^3-1] + \frac{1}{9}[10^2-1] + \frac{1}{9}[10^1-1]$

$\Rightarrow S_n = \frac{1}{9}\left[(10^n + 10^{n-1} + 10^{n-2} + \dots + 10^3 + 10^2 + 10^1) - (\underbrace{1+1+1+\dots+1}_{n\text{ terms}})\right]$

$\Rightarrow S_n = \frac{1}{9}\left[(10 + 10^2 + 10^3 + \dots + 10^n) - n\times 1\right]$

$\Rightarrow S_n = \frac{1}{9}\times\left[\frac{10(10^n-1)}{10-1} - n\right]$ (SUMMING THE SAME G.P AS BEFORE)

$\Rightarrow S_n = \frac{1}{9}\left[\frac{10^{n+1}-10}{9} - n\right]$

$\Rightarrow S_n = \frac{1}{81}\left[10^{n+1} - 9n - 10\right]$ AS BEFORE

Question 105 (*****)

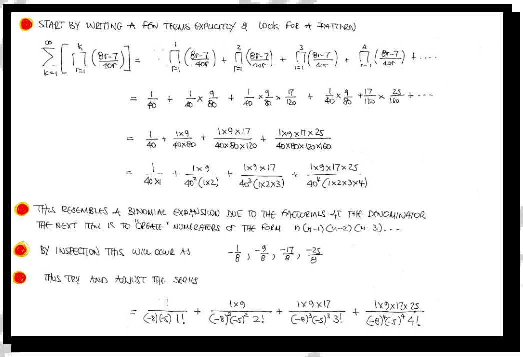
The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Find the sum to infinity of the following expression

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right].$$

SP-K , $\sqrt[8]{\frac{5}{4}} - 1$


START BY WRITING A FEW TERMS EXPLICITLY & LOOK FOR A PATTERN

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right] = \prod_{r=1}^{1}\left(\frac{8r-7}{40r}\right) + \prod_{r=1}^{2}\left(\frac{8r-7}{40r}\right) + \prod_{r=1}^{3}\left(\frac{8r-7}{40r}\right) + \prod_{r=1}^{4}\left(\frac{8r-7}{40r}\right) + \ldots$$

$$= \frac{1}{40} + \frac{1}{40}\times\frac{9}{80} + \frac{1}{40}\times\frac{9}{80}\times\frac{17}{120} + \frac{1}{40}\times\frac{9}{80}\times\frac{17}{120}\times\frac{25}{160} + \ldots$$

$$= \frac{1}{40} + \frac{1\times9}{40\times80} + \frac{1\times9\times17}{40\times80\times120} + \frac{1\times9\times17\times25}{40\times80\times120\times160}$$

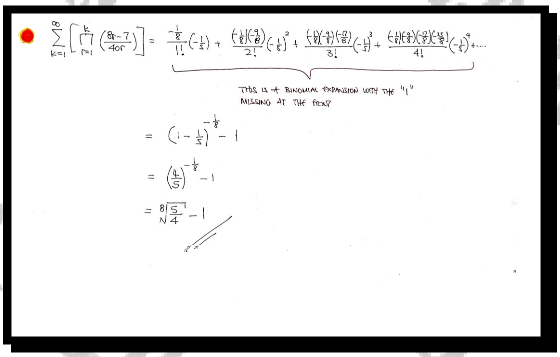
$$= \frac{1}{40\times1} + \frac{1\times9}{40^2(1\times2)} + \frac{1\times9\times17}{40^3(1\times2\times3)} + \frac{1\times9\times17\times25}{40^4(1\times2\times3\times4)}$$

THIS RESEMBLES A BINOMIAL EXPANSION DUE TO THE FACTORIALS AT THE DENOMINATOR
THE NEXT ITEM IS TO "CREATE" NUMERATORS OF THE FORM $n(n-1)(n-2)(n-3)\ldots$

BY INSPECTION THIS WILL OCCUR AS $-\frac{1}{8}, -\frac{9}{8}, \frac{-17}{8}, \frac{-25}{8}$

THUS TRY AND ADJUST THE SERIES

$$= \frac{1}{(-8)(-5)\,1!} + \frac{1\times9}{(-8)^2(-5)^2\,2!} + \frac{1\times9\times17}{(-8)^3(-5)^3\,3!} + \frac{1\times9\times17\times25}{(-8)^4(-5)^4\,4!}$$


$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right] = \frac{-\frac{1}{8}}{1!}\left(-\frac{1}{5}\right) + \frac{\left(-\frac{1}{8}\right)\left(-\frac{9}{8}\right)}{2!}\left(-\frac{1}{5}\right)^2 + \frac{\left(-\frac{1}{8}\right)\left(-\frac{9}{8}\right)\left(-\frac{17}{8}\right)}{3!}\left(-\frac{1}{5}\right)^3 + \frac{\left(-\frac{1}{8}\right)\left(-\frac{9}{8}\right)\left(-\frac{17}{8}\right)\left(-\frac{25}{8}\right)}{4!}\left(-\frac{1}{5}\right)^4 + \ldots$$

THIS IS A BINOMIAL EXPANSION WITH THE "1" MISSING AT THE FRONT

$$= \left(1-\frac{1}{5}\right)^{-\frac{1}{8}} - 1$$

$$= \left(\frac{4}{5}\right)^{-\frac{1}{8}} - 1$$

$$= \sqrt[8]{\frac{5}{4}} - 1$$

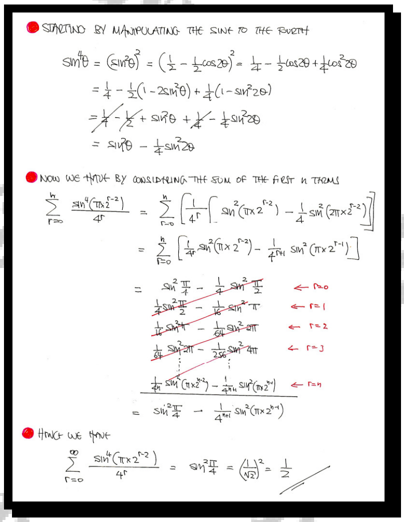
Question 106 (*****)

Find the value of

$$\sum_{r=0}^{\infty}\left[\frac{\sin^4\left(\pi\times 2^{r-2}\right)}{4^r}\right].$$

Hint: Express $\sin^4\theta$ in terms of $\sin^2\theta$ and $\sin^2 2\theta$ only.

$\boxed{}$, $\boxed{\frac{1}{2}}$



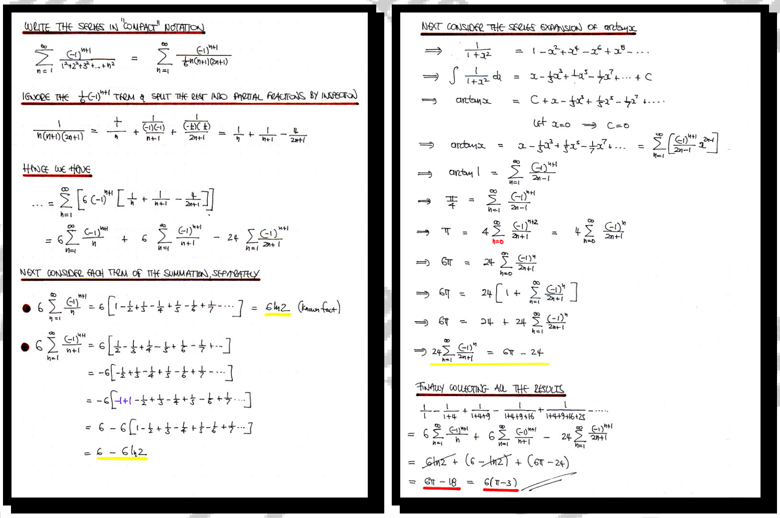
Question 107 (*****)

Find the sum to infinity of the following series.

$$\frac{1}{1}-\frac{1}{1+4}+\frac{1}{1+4+9}-\frac{1}{1+4+9+16}+\frac{1}{1+4+9+16+25}+\ldots$$

You may find the series expansion of $\arctan x$ useful in this question.

$\boxed{}$, $\boxed{6(\pi-3)}$



WRITE THE SERIES IN "COMPACT" NOTATION

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{1^2+2^2+3^2+\ldots+n^2}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\frac{1}{6}n(n+1)(2n+1)}$$

IGNORE THE $\frac{1}{6}(-1)^{n+1}$ TERM & SPLIT THE REST INTO PARTIAL FRACTIONS BY INSPECTION

$$\frac{1}{n(n+1)(2n+1)}=\frac{\frac{1}{1}}{n}+\frac{\frac{1}{(-1)(-1)}}{n+1}+\frac{\frac{1}{(-\frac{1}{2})(\frac{1}{2})}}{2n+1}=\frac{1}{n}+\frac{1}{n+1}-\frac{4}{2n+1}$$

HENCE WE HAVE

$$\ldots=\sum_{n=1}^{\infty}\left[6(-1)^{n+1}\left[\frac{1}{n}+\frac{1}{n+1}-\frac{4}{2n+1}\right]\right]$$

$$=6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}+6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}-24\sum_{n=1}\frac{(-1)^{n+1}}{2n+1}$$

NEXT CONSIDER EACH TERM OF THE SUMMATION, SEPARATELY

- $6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=6\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots\right]=6\ln 2$ (known fact)
- $6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}=6\left[\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\ldots\right]$

$$=-6\left[-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots\right]$$

$$=-6\left[-1+1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}\ldots\right]$$

$$=6-6\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}\ldots\right]$$

$$=6-6\ln 2$$

NEXT CONSIDER THE SERIES EXPANSION OF $\arctan x$

$$\Rightarrow \frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8-\ldots$$

$$\Rightarrow \int\frac{1}{1+x^2}\,dx=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\ldots+C$$

$$\Rightarrow \arctan x=C+x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\ldots$$

let $x=0 \Rightarrow C=0$

$$\Rightarrow \arctan x=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\ldots=\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{2n-1}x^{2n-1}\right]$$

$$\Rightarrow \arctan 1=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow \frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow \pi=4\sum_{n=0}^{\infty}\frac{(-1)^{n+2}}{2n+1}=4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

$$\Rightarrow 6\pi=24\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

$$\Rightarrow 6\pi=24\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\right]$$

$$\Rightarrow 6\pi=24+24\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}$$

$$\Rightarrow 24\sum_{n=1}\frac{(-1)^n}{2n+1}=6\pi-24$$

FINALLY COLLECTING ALL THE RESULTS

$$\frac{1}{1}-\frac{1}{1+4}+\frac{1}{1+4+9}-\frac{1}{1+4+9+16}+\frac{1}{1+4+9+16+25}-\ldots$$

$$=6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}+6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}-24\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n+1}$$

$$=6\ln 2+(6-6\ln 2)+(6\pi-24)$$

$$=6\pi-18=6(\pi-3)$$

Question 108 (*****)

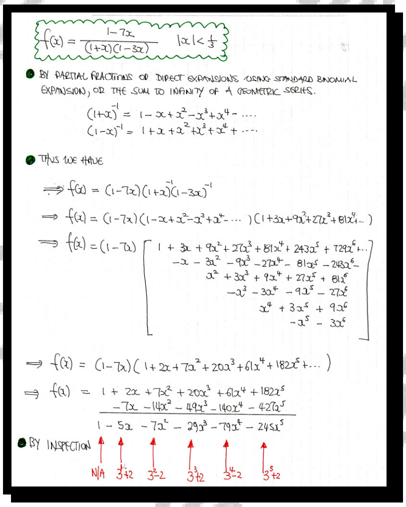
$$f(x) \equiv \frac{1-7x}{(1+x)(1-3x)}, \quad -\tfrac{1}{3} < x < \tfrac{1}{3}.$$

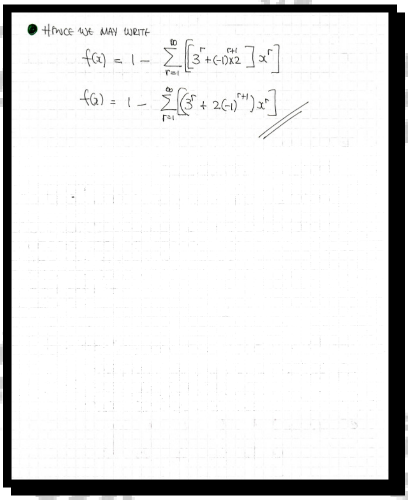
Show that $f(x)$ can be written in the form

$$f(x) = 1 - \sum_{r=1}^{\infty} \left[x^r g(r) \right],$$

where $g(r)$ is a simplified function to be found.

$\boxed{}$, $\boxed{g(r) = 3^r + 2\times(-1)^{r+1}}$





Question 109 (*****)

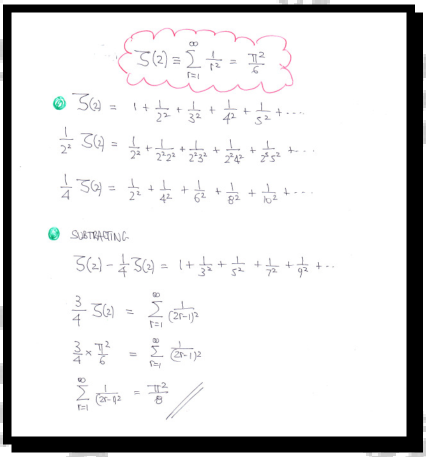
It is given that

$$\zeta(2) = \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}.$$

By using this fact alone find the exact value of

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}.$$

$$\boxed{\frac{\pi^2}{8}}$$

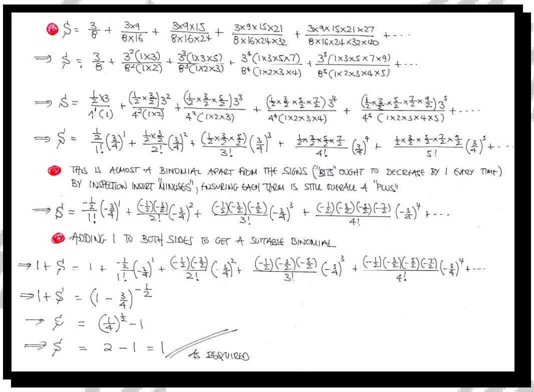


Question 110 (*****)

$$S = \frac{3}{8} + \frac{3\times 9}{8\times 16} + \frac{3\times 9\times 15}{8\times 16\times 24} + \frac{3\times 9\times 15\times 21}{8\times 16\times 24\times 32} + \frac{3\times 9\times 15\times 21\times 27}{8\times 16\times 24\times 32\times 40} \ldots$$

By considering a suitable binomial expansion, show that $S = 1$.

SP-Z , proof


$$S = \frac{3}{8} + \frac{3\times9}{8\times16} + \frac{3\times9\times15}{8\times16\times24} + \frac{3\times9\times15\times21}{8\times16\times24\times32} + \frac{3\times9\times15\times21\times27}{8\times16\times24\times32\times40} + \ldots$$

$$\Rightarrow S = \frac{3}{8} + \frac{3^2(1\times3)}{8^2(1\times2)} + \frac{3^3(1\times3\times5)}{8^3(1\times2\times3)} + \frac{3^4(1\times3\times5\times7)}{8^4(1\times2\times3\times4)} + \frac{3^5(1\times3\times5\times7\times9)}{8^5(1\times2\times3\times4\times5)} + \ldots$$

$$\Rightarrow S = \frac{\frac{1}{2}\times3}{4^1(1)} + \frac{(\frac{1}{2}\times\frac{3}{2})3^2}{4^2(1\times2)} + \frac{(\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2})3^3}{4^3(1\times2\times3)} + \frac{(\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2})3^4}{4^4(1\times2\times3\times4)} + \frac{(\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2}\times\frac{9}{2})3^5}{4^5(1\times2\times3\times4\times5)} + \ldots$$

$$\Rightarrow S = \frac{\frac{1}{2}}{1!}\left(\frac{3}{4}\right)^1 + \frac{\frac{1}{2}\times\frac{3}{2}}{2!}\left(\frac{3}{4}\right)^2 + \frac{(\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2})}{3!}\left(\frac{3}{4}\right)^3 + \frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2}}{4!}\left(\frac{3}{4}\right)^4 + \frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}\times\frac{7}{2}\times\frac{9}{2}}{5!}\left(\frac{3}{4}\right)^5 + \ldots$$

- THIS IS ALMOST A BINOMIAL APART FROM THE SIGNS ("BITS" OUGHT TO DECREASE BY 1 EVERY TIME) BY INSPECTION INSERT "MINUSES", ENSURING EACH TERM IS STILL OVERALL A "PLUS"

$$\Rightarrow S = \frac{-\frac{1}{2}}{1!}\left(-\frac{3}{4}\right)^1 + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{3}{4}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\frac{3}{4}\right)^3 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{4!}\left(-\frac{3}{4}\right)^4 + \ldots$$

- ADDING 1 TO BOTH SIDES TO GET A SUITABLE BINOMIAL

$$\Rightarrow 1 + S = 1 + \frac{-\frac{1}{2}}{1!}\left(-\frac{3}{4}\right)^1 + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{3}{4}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\frac{3}{4}\right)^3 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{4!}\left(-\frac{3}{4}\right)^4 + \ldots$$

$$\Rightarrow 1 + S = \left(1 - \frac{3}{4}\right)^{-\frac{1}{2}}$$

$$\Rightarrow S = \left(\frac{1}{4}\right)^{-\frac{1}{2}} - 1$$

$$\Rightarrow S = 2 - 1 = 1$$

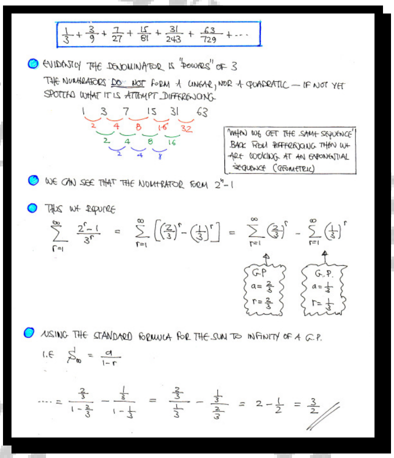
AS REQUIRED

Question 111 (*****)

Sum the following series of infinite terms.

$$\frac{1}{3} + \frac{3}{9} + \frac{7}{27} + \frac{15}{81} + \frac{31}{243} + \frac{63}{729} + \ldots$$

$\boxed{}$, $\boxed{\frac{3}{2}}$


$$\frac{1}{3} + \frac{3}{9} + \frac{7}{27} + \frac{15}{81} + \frac{31}{243} + \frac{63}{729} + \ldots$$

- EVIDENTLY THE DENOMINATOR IS "POWERS" OF 3

THE NUMERATORS DO NOT FORM A LINEAR, NOR A QUADRATIC — IF NOT YET SPOTTED WHAT IT IS ATTEMPT DIFFERENCING

1 3 7 15 31 63

2 4 8 16 32

2 4 8 16

2 4 8

"WHEN WE GET THE SAME SEQUENCE BACK FROM DIFFERENCING THEN WE ARE LOOKING AT AN EXPONENTIAL SEQUENCE (GEOMETRIC)"

- WE CAN SEE THAT THE NUMERATOR FORM $2^n - 1$
- THUS WE REQUIRE

$$\sum_{r=1}^{\infty} \frac{2^r - 1}{3^r} = \sum_{r=1}^{\infty}\left[\left(\frac{2}{3}\right)^r - \left(\frac{1}{3}\right)^r\right] = \sum_{r=1}^{\infty}\left(\frac{2}{3}\right)^r - \sum_{r=1}^{\infty}\left(\frac{1}{3}\right)^r$$

G.P. $a = \frac{2}{3}$, $r = \frac{2}{3}$ G.P. $a = \frac{1}{3}$, $r = \frac{1}{3}$

- USING THE STANDARD FORMULA FOR THE SUM TO INFINITY OF A G.P.

I.E. $S_\infty = \frac{a}{1-r}$

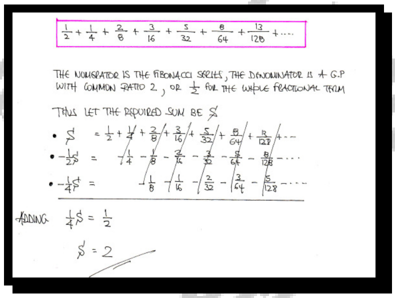
$$\ldots = \frac{\frac{2}{3}}{1-\frac{2}{3}} - \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} - \frac{\frac{1}{3}}{\frac{2}{3}} = 2 - \frac{1}{2} = \frac{3}{2}$$

Question 112 (*****)

Sum the following series of infinite terms.

$$\frac{1}{2}+\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{5}{32}+\frac{8}{64}+\frac{13}{128}+\dots$$

[SYN], [2]


$\frac{1}{2}+\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{5}{32}+\frac{8}{64}+\frac{13}{128}+\dots$

THE NUMERATOR IS THE FIBONACCI SERIES, THE DENOMINATOR IS A G.P WITH COMMON RATIO 2, OR $\frac{1}{2}$ FOR THE WHOLE FRACTIONAL TERM

THUS LET THE REQUIRED SUM BE S

- $S = \frac{1}{2}+\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{5}{32}+\frac{8}{64}+\frac{13}{128}+\dots$
- $-\frac{1}{2}S = -\frac{1}{4}-\frac{1}{8}-\frac{2}{16}-\frac{3}{32}-\frac{5}{64}-\frac{8}{128}-\dots$
- $-\frac{1}{4}S = -\frac{1}{8}-\frac{1}{16}-\frac{2}{32}-\frac{3}{64}-\frac{5}{128}-\dots$

ADDING $\frac{1}{4}S=\frac{1}{2}$

$S=2$

Question 113 (*****)

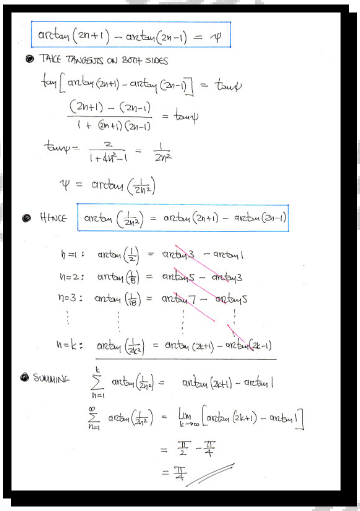
By considering the simplification of

$$\arctan(2n+1)-\arctan(2n-1),$$

determine the exact value of

$$\sum_{n=1}^{\infty}\left[\arctan\left(\frac{1}{2n^2}\right)\right].$$

$\boxed{}$, $\boxed{\frac{\pi}{4}}$


$\arctan(2n+1)-\arctan(2n-1)=\psi$

TAKE TANGENTS ON BOTH SIDES

$\tan[\arctan(2n+1)-\arctan(2n-1)]=\tan\psi$

$\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}=\tan\psi$

$\tan\psi=\frac{2}{1+4n^2-1}=\frac{1}{2n^2}$

$\psi=\arctan\left(\frac{1}{2n^2}\right)$

HENCE $\arctan\left(\frac{1}{2n^2}\right)=\arctan(2n+1)-\arctan(2n-1)$

$n=1: \arctan\left(\frac{1}{2}\right)=\arctan 3-\arctan 1$

$n=2: \arctan\left(\frac{1}{8}\right)=\arctan 5-\arctan 3$

$n=3: \arctan\left(\frac{1}{18}\right)=\arctan 7-\arctan 5$

$n=k: \arctan\left(\frac{1}{2k^2}\right)=\arctan(2k+1)-\arctan(2k-1)$

SUMMING $\sum_{n=1}^{k}\arctan\left(\frac{1}{2n^2}\right)=\arctan(2k+1)-\arctan 1$

$\sum_{n=1}^{\infty}\arctan\left(\frac{1}{2n^2}\right)=\lim_{k\to\infty}\left[\arctan(2k+1)-\arctan 1\right]$

$=\frac{\pi}{2}-\frac{\pi}{4}$

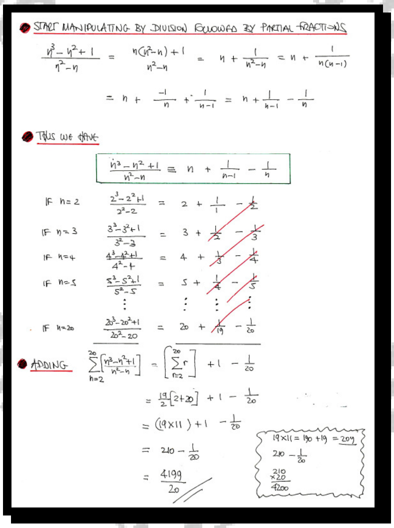
$=\frac{\pi}{4}$

Question 114 (*****)

Determine the exact value of the following sum.

$$\sum_{n=2}^{20}\left[\frac{n^3-n^2+1}{n^2-n}\right].$$

$\boxed{}$, $\boxed{\dfrac{4199}{20}}$



START MANIPULATING BY DIVISION FOLLOWED BY PARTIAL FRACTIONS

$$\frac{n^3-n^2+1}{n^2-n} = \frac{n(n^2-n)+1}{n^2-n} = n + \frac{1}{n^2-n} = n + \frac{1}{n(n-1)}$$

$$= n + \frac{-1}{n} + \frac{1}{n-1} = n + \frac{1}{n-1} - \frac{1}{n}$$

THUS WE HAVE

$$\boxed{\frac{n^3-n^2+1}{n^2-n} \equiv n + \frac{1}{n-1} - \frac{1}{n}}$$

IF $n=2$ $\quad \frac{2^3-2^2+1}{2^2-2} = 2 + \frac{1}{1} - \frac{1}{2}$

IF $n=3$ $\quad \frac{3^3-3^2+1}{3^2-3} = 3 + \frac{1}{2} - \frac{1}{3}$

IF $n=4$ $\quad \frac{4^3-4^2+1}{4^2-4} = 4 + \frac{1}{3} - \frac{1}{4}$

IF $n=5$ $\quad \frac{5^3-5^2+1}{5^2-5} = 5 + \frac{1}{4} - \frac{1}{5}$

$\vdots$

IF $n=20$ $\quad \frac{20^3-20^2+1}{20^2-20} = 20 + \frac{1}{19} - \frac{1}{20}$

ADDING

$$\sum_{n=2}^{20}\left[\frac{n^3-n^2+1}{n^2-n}\right] = \left[\sum_{r=2}^{20} r\right] + 1 - \frac{1}{20}$$

$$= \frac{19}{2}[2+20] + 1 - \frac{1}{20}$$

$$= (19\times 11) + 1 - \frac{1}{20}$$

$$= 210 - \frac{1}{20}$$

$$= \frac{4199}{20}$$

$19\times 11 = 190 + 19 = 209$

$210 - \frac{1}{20}$

$210 \times 20 = 4200$

Question 115 (*****)

$$\sum_{r=1}^{\infty}\left[\frac{1}{r^2}\right]=L.$$

It is given that the above infinite series converges to a limit L.

Find, in terms of L where appropriate, the limit of each of the following infinite series.

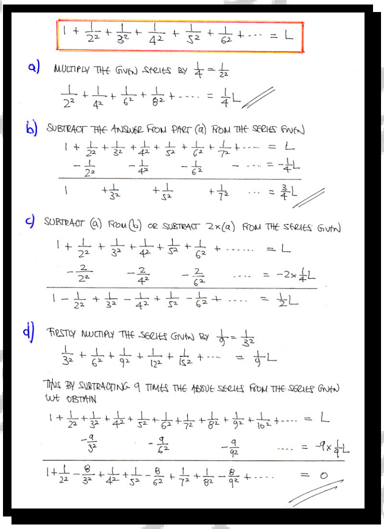
a) $\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\frac{1}{10^2}+\dots$

b) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\dots$

c) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\dots$

d) $\frac{1}{1^2}+\frac{1}{2^2}-\frac{8}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}-\frac{8}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}-\frac{8}{9^2}+\dots$

$\boxed{}$, $\boxed{\frac{1}{4}L}$, $\boxed{\frac{3}{4}L}$, $\boxed{\frac{1}{2}L}$, $\boxed{0}$



$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\dots=L$

a) MULTIPLY THE GIVEN SERIES BY $\frac{1}{4}=\frac{1}{2^2}$

$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\dots=\frac{1}{4}L$

b) SUBTRACT THE ANSWER FROM PART (a) FROM THE SERIES GIVEN

$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\dots=L$

$-\frac{1}{2^2}\quad-\frac{1}{4^2}\quad-\frac{1}{6^2}\quad-\dots=-\frac{1}{4}L$

$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}\dots=\frac{3}{4}L$

c) SUBTRACT (a) FROM (b) OR SUBTRACT $2\times$(a) FROM THE SERIES GIVEN

$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\dots=L$

$-\frac{2}{2^2}\quad-\frac{2}{4^2}\quad-\frac{2}{6^2}\quad\dots=-2\times\frac{1}{4}L$

$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\dots=\frac{1}{2}L$

d) FIRSTLY MULTIPLY THE SERIES GIVEN BY $\frac{1}{9}=\frac{1}{3^2}$

$\frac{1}{3^2}+\frac{1}{6^2}+\frac{1}{9^2}+\frac{1}{12^2}+\frac{1}{15^2}+\dots=\frac{1}{9}L$

THUS BY SUBTRACTING 9 TIMES THE ABOVE SERIES FROM THE SERIES GIVEN WE OBTAIN

$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9^2}+\frac{1}{10^2}+\dots=L$

$-\frac{9}{3^2}\quad-\frac{9}{6^2}\quad-\frac{9}{9^2}\quad\dots=-9\times\frac{1}{9}L$

$1+\frac{1}{2^2}-\frac{8}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}-\frac{8}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}-\frac{8}{9^2}+\dots=0$

Question 116 (*****)

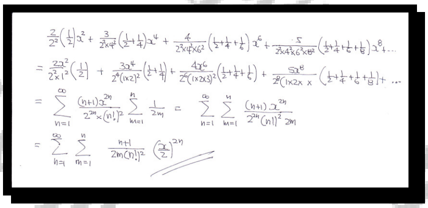
Consider the infinite series

$$\frac{2}{2^2}\left(\frac{1}{2}\right)x^2+\frac{3}{2^2\times 4^2}\left(\frac{1}{2}+\frac{1}{4}\right)x^4+\frac{4}{2^2\times 4^2\times 6^2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)x^6+\frac{5}{2^2\times 4^2\times 6^2\times 8^2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}\right)x^8+\dots$$

Write the above series in Sigma notation, in its simplest form.

[*You are not required to investigate its convergence or to sum it.*]

$$\boxed{\sum_{n=1}^{\infty}\left[\frac{n+1}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\sum_{m=1}^{n}\frac{1}{2m}\right] \text{ or } \sum_{n=1}^{\infty}\sum_{m=1}^{n}\left[\frac{n+1}{2m(n!)^2}\left(\frac{x}{2}\right)^{2n}\right]}$$

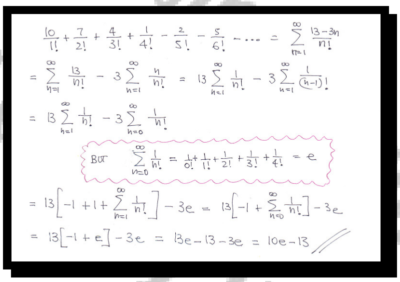


Question 117 (*****)

Determine the sum to infinity of the following series

$$\frac{10}{1!}+\frac{7}{2!}+\frac{4}{3!}+\frac{1}{4!}-\frac{2}{5!}-\frac{5}{6!}+\dots$$

$$\boxed{10\mathrm{e}-13}$$



Question 118 (*****)

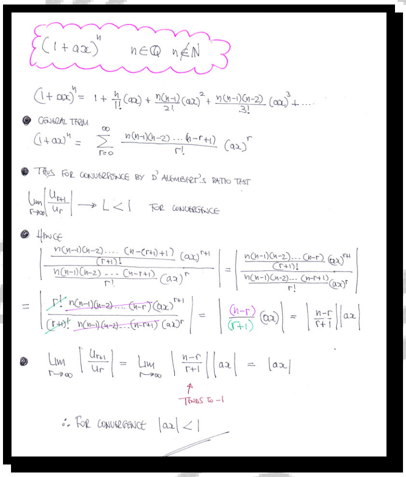
Consider the binomial infinite series expansion

$$(1+ax)^n,$$

where $a \in \mathbb{R}$, $n \in \mathbb{Q}$, $n \notin \mathbb{N}$.

Show that the series converges if $|ax| < 1$.

proof



Question 119 (*****)

The n^{th} term of a series is given recursively by

$$u_{n+1} = \frac{n}{2n+1} u_n, \quad n \in \mathbb{N}, \quad u_1 = 2.$$

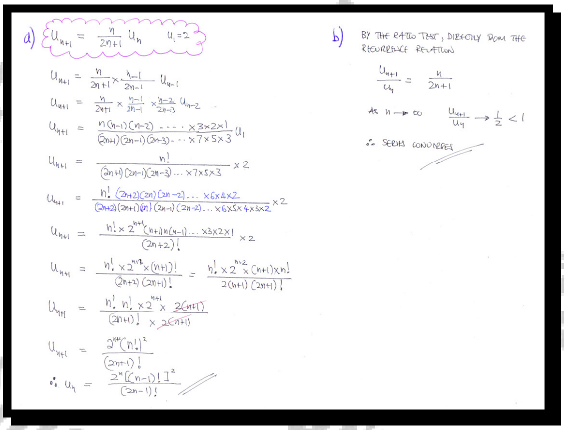
a) Show, by direct manipulation, that

$$u_n = \frac{2^n \times \left[(n-1)!\right]^2}{(2n-1)!}.$$

[*you may not use proof by induction in this part*]

b) Determine whether $\sum_{n}^{\infty} u_n$ converges or diverges.

converges

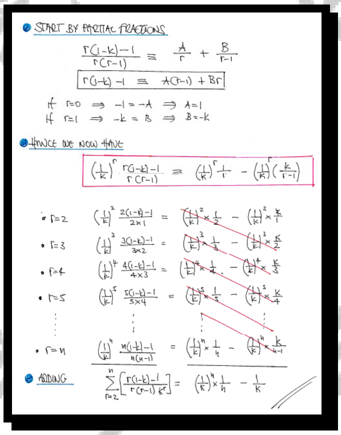


Question 120 (*****)

Determine, in terms of k and n, a simplified expression

$$\sum_{r=2}^{n}\left[\frac{r(1-k)-1}{r(r-1)k^{r}}\right].$$

$\boxed{}$, $\boxed{\frac{1}{n}\left(\frac{1}{k}\right)^{n}-\frac{1}{k}}$



START BY PARTIAL FRACTIONS

$$\frac{r(1-k)-1}{r(r-1)} \equiv \frac{A}{r}+\frac{B}{r-1}$$

$$\boxed{r(1-k)-1 \equiv A(r-1)+Br}$$

If $r=0 \Rightarrow -1=-A \Rightarrow A=1$

If $r=1 \Rightarrow -k=B \Rightarrow B=-k$

HENCE WE NOW HAVE

$$\boxed{\left(\frac{1}{k}\right)^{r}\frac{r(1-k)-1}{r(r-1)} \equiv \left(\frac{1}{k}\right)^{r}\frac{1}{r}-\left(\frac{1}{k}\right)^{r}\left(\frac{k}{r-1}\right)}$$

- $r=2$: $\left(\frac{1}{k}\right)^{2}\frac{2(1-k)-1}{2\times1} = \left(\frac{1}{k}\right)^{2}\times\frac{1}{2}-\left(\frac{1}{k}\right)^{2}\times\frac{k}{1}$
- $r=3$: $\left(\frac{1}{k}\right)^{3}\frac{3(1-k)-1}{3\times2} = \left(\frac{1}{k}\right)^{3}\times\frac{1}{3}-\left(\frac{1}{k}\right)^{3}\times\frac{k}{2}$
- $r=4$: $\left(\frac{1}{k}\right)^{4}\frac{4(1-k)-1}{4\times3} = \left(\frac{1}{k}\right)^{4}\times\frac{1}{4}-\left(\frac{1}{k}\right)^{4}\times\frac{k}{3}$
- $r=5$: $\left(\frac{1}{k}\right)^{5}\frac{5(1-k)-1}{5\times4} = \left(\frac{1}{k}\right)^{5}\times\frac{1}{5}-\left(\frac{1}{k}\right)^{5}\times\frac{k}{4}$

$\vdots$

- $r=n$: $\left(\frac{1}{k}\right)^{n}\frac{n(1-k)-1}{n(n-1)} = \left(\frac{1}{k}\right)^{n}\times\frac{1}{n}-\left(\frac{1}{k}\right)^{n}\times\frac{k}{n-1}$

ADDING

$$\sum_{r=2}^{n}\left[\frac{r(1-k)-1}{r(r-1)k^{r}}\right]=\left(\frac{1}{k}\right)^{n}\times\frac{1}{n}-\frac{1}{k}$$

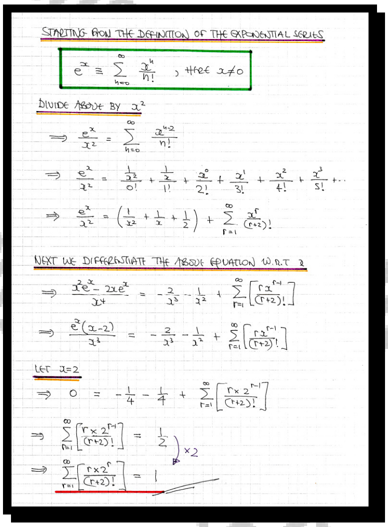
Question 121 (*****)

Use an appropriate method to sum the following series

$$\sum_{r=1}^{\infty} \frac{r \times 2^r}{(r+2)!}.$$

You may assume the series converges.

$\boxed{}$, $\boxed{1}$



STARTING FROM THE DEFINITION OF THE EXPONENTIAL SERIES

$$e^x \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ HERE } x \neq 0$$

DIVIDE ABOVE BY x^2

$$\Longrightarrow \frac{e^x}{x^2} = \sum_{n=0}^{\infty} \frac{x^{n-2}}{n!}$$

$$\Longrightarrow \frac{e^x}{x^2} = \frac{\frac{1}{x^2}}{0!} + \frac{\frac{1}{x}}{1!} + \frac{x^0}{2!} + \frac{x^1}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \dots$$

$$\Longrightarrow \frac{e^x}{x^2} = \left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{2}\right) + \sum_{r=1}^{\infty} \frac{x^r}{(r+2)!}$$

NEXT WE DIFFERENTIATE THE ABOVE EQUATION W.R.T x

$$\Longrightarrow \frac{x^2e^x - 2xe^x}{x^4} = -\frac{2}{x^3} - \frac{1}{x^2} + \sum_{r=1}^{\infty}\left[\frac{rx^{r-1}}{(r+2)!}\right]$$

$$\Longrightarrow \frac{e^x(x-2)}{x^3} = -\frac{2}{x^3} - \frac{1}{x^2} + \sum_{r=1}^{\infty}\left[\frac{rx^{r-1}}{(r+2)!}\right]$$

LET $x=2$

$$\Longrightarrow 0 = -\frac{1}{4} - \frac{1}{4} + \sum_{r=1}^{\infty}\left[\frac{r \times 2^{r-1}}{(r+2)!}\right]$$

$$\Longrightarrow \sum_{r=1}^{\infty}\left[\frac{r \times 2^{r-1}}{(r+2)!}\right] = \frac{1}{2} \quad \times 2$$

$$\Longrightarrow \sum_{r=1}^{\infty}\left[\frac{r \times 2^{r}}{(r+2)!}\right] = 1$$

Question 122 (*****)

The n^{th} term of a series is given recursively by

$$u_n = \frac{2n}{2n+1}u_{n-1}, \quad n \in \mathbb{N}, \quad u_0 = 1 .$$

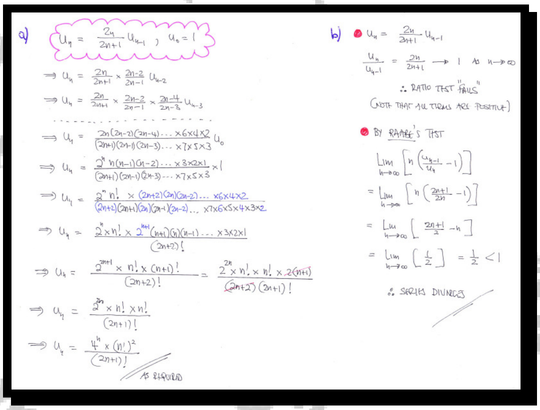
a) Show, by direct manipulation, that

$$u_n = \frac{4^n \times (n!)^2}{(2n+1)!} .$$

[*you may not use proof by induction in this part*]

b) Determine whether $\displaystyle\sum_{n}^{\infty} u_n$ converges or diverges.

diverges



Question 123 (*****)

The following convergent series S is given below

$$S = \frac{\sin\theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \ldots$$

By considering the sum to infinity of a suitable series involving the complex exponential function, show that

$$S = e^{-\cos\theta}\sin(\sin\theta).$$

[ ] , [proof]

DEFINE SERIES, C & S, BASED ON COMPLEX NUMBERS

$$C = \frac{\cos\theta}{1!} - \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} - \frac{\cos 4\theta}{4!} + \ldots$$

$$S = \frac{\sin\theta}{1!} - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \frac{\sin 4\theta}{4!} + \ldots$$

COMBINE TO FORM A COMPLEX EXPONENTIAL SERIES

$$C + iS = \frac{1}{1!}(\cos\theta + i\sin\theta) - \frac{1}{2!}(\cos 2\theta + i\sin 2\theta) + \frac{1}{3!}(\cos 3\theta + i\sin 3\theta) - \ldots$$

$$C + iS = \frac{1}{1!}e^{i\theta} - \frac{1}{2!}e^{2i\theta} + \frac{1}{3!}e^{3i\theta} - \frac{1}{4!}e^{4i\theta} + \ldots$$

NOW CONSIDER SOME SIMPLE STANDARD EXPANSIONS

$$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \ldots$$

$$z - \frac{z^2}{2!} + \frac{z^3}{3!} - \frac{z^4}{4!} = 1 - e^{-z}$$

HENCE WE NOW HAVE

$$C + iS = (e^{i\theta}) - \frac{(e^{i\theta})^2}{2!} + \frac{(e^{i\theta})^3}{3!} - \frac{(e^{i\theta})^4}{4!} + \ldots$$

$$C + iS = 1 - e^{-e^{i\theta}}$$

$$C + iS = 1 - e^{-(\cos\theta + i\sin\theta)}$$

$$C + iS = 1 - e^{-\cos\theta} \times e^{-i\sin\theta}$$

$$C + iS = 1 - e^{-\cos\theta}\left[\cos(\sin\theta) - i\sin(\sin\theta)\right]$$

$$C + iS = \left[1 - e^{-\cos\theta}\cos(\sin\theta)\right] + i\left[e^{-\cos\theta}\sin(\sin\theta)\right]$$

SELECTING IMAGINARY PART WE OBTAIN

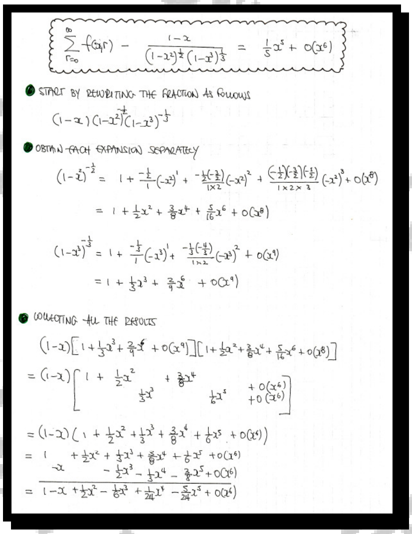
$$\sum_{r=1}^{\infty} \frac{(-1)^r \cos(r\theta)}{r!} = e^{-\cos\theta}\sin(\sin\theta)$$ //

Question 124 (*****)

$$g(x) \equiv \sum_{r=0}^{\infty} f(x,r) \;-\; \frac{1-x}{\sqrt{1-x^2}\,\sqrt[3]{1-x^3}}, \quad -1 < x < 1.$$

Given that the first term of the series expansion of $g(x)$ is $\frac{1}{5}x^5$, determine in exact simplified form a simplified expression of $f(x,r)$.

$$\boxed{}, \quad \boxed{f(x,r) = \frac{(-x)^r}{r!}}$$



$$\sum_{r=0}^{\infty} f(x,r) - \frac{1-x}{(1-x^2)^{\frac{1}{2}}(1-x^3)^{\frac{1}{3}}} = \frac{1}{5}x^5 + O(x^6)$$

- START BY REWRITING THE FRACTION AS FOLLOWS

$$(1-x)(1-x^2)^{-\frac{1}{2}}(1-x^3)^{-\frac{1}{3}}$$

- OBTAIN EACH EXPANSION SEPARATELY

$$(1-x^2)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}(-x^2)^1 + \frac{-\frac{1}{2}(-\frac{3}{2})}{1\times2}(-x^2)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1\times2\times3}(-x^2)^3 + O(x^8)$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + O(x^8)$$

$$(1-x^3)^{-\frac{1}{3}} = 1 + \frac{-\frac{1}{3}}{1}(-x^3)^1 + \frac{-\frac{1}{3}(-\frac{4}{3})}{1\times2}(-x^3)^2 + O(x^9)$$

$$= 1 + \frac{1}{3}x^3 + \frac{2}{9}x^6 + O(x^9)$$

- COLLECTING ALL THE RESULTS

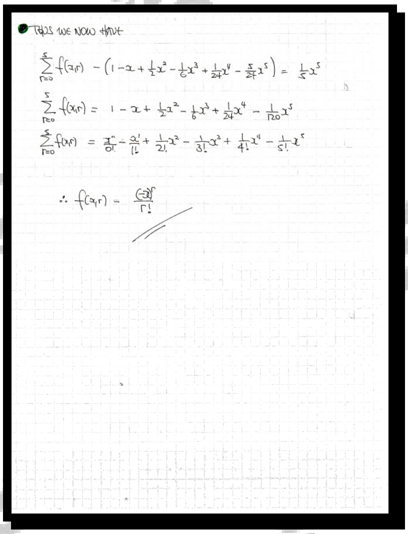
$$(1-x)\left[1 + \frac{1}{3}x^3 + \frac{2}{9}x^6 + O(x^9)\right]\left[1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + O(x^8)\right]$$

$$= (1-x)\left[1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{3}{8}x^4 + \frac{1}{6}x^5 + O(x^6)\right]$$

$$= (1-x)\left(1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{3}{8}x^4 + \frac{1}{6}x^5 + O(x^6)\right)$$

$$= 1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{3}{8}x^4 + \frac{1}{6}x^5 + O(x^6) - x - \frac{1}{2}x^3 - \frac{1}{3}x^4 - \frac{3}{8}x^5 + O(x^6)$$

$$= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{5}{24}x^5 + O(x^6)$$



- THUS WE NOW HAVE

$$\sum_{r=0}^{5} f(x,r) - \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{5}{24}x^5\right) = \frac{1}{5}x^5$$

$$\sum_{r=0}^{5} f(x,r) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5$$

$$\sum_{r=0}^{5} f(x,r) = \frac{x^0}{0!} - \frac{x^1}{1!} + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5$$

$$\therefore f(x,r) = \frac{(-x)^r}{r!}$$

Question 125 (*****)

Determine the value of the following infinite convergent sum.

$$\sum_{r=2}^{\infty}\left[\frac{4r-1}{r(r-1)}\left(-\frac{1}{3}\right)^{r}\right].$$

SP , $\boxed{\frac{1}{3}}$

START BY PARTIAL FRACTIONS (BY INSPECTION)

$$\frac{4r-1}{r(r-1)} \equiv \frac{1}{r} + \frac{3}{r-1}$$

HENCE WE NOW HAVE

$$\boxed{\frac{4r-1}{r(r-1)}\left(-\frac{1}{3}\right)^{r} \equiv \frac{1}{r}\left(-\frac{1}{3}\right)^{r} + \frac{3}{r-1}\left(-\frac{1}{3}\right)^{r}}$$

- $r=2$: $\frac{7}{2\times1}\left(-\frac{1}{3}\right)^{2} = \frac{1}{2}\left(-\frac{1}{3}\right)^{2} + \frac{3}{1}\left(-\frac{1}{3}\right)^{2} = \frac{1}{2}\left(-\frac{1}{3}\right)^{2} + \frac{3}{1}\left(-\frac{1}{3}\right)^{2}$
- $r=3$: $\frac{11}{3\times2}\left(-\frac{1}{3}\right)^{3} = \frac{1}{3}\left(-\frac{1}{3}\right)^{3} + \frac{3}{2}\left(-\frac{1}{3}\right)^{3} = \frac{1}{3}\left(-\frac{1}{3}\right)^{3} - \frac{1}{2}\left(-\frac{1}{3}\right)^{2}$
- $r=4$: $\frac{15}{4\times3}\left(-\frac{1}{3}\right)^{4} = \frac{1}{4}\left(-\frac{1}{3}\right)^{4} + \frac{3}{3}\left(-\frac{1}{3}\right)^{4} = \frac{1}{4}\left(-\frac{1}{3}\right)^{4} - \frac{1}{3}\left(-\frac{1}{3}\right)^{3}$
- $r=5$: $\frac{19}{5\times4}\left(-\frac{1}{3}\right)^{5} = \frac{1}{5}\left(-\frac{1}{3}\right)^{5} + \frac{3}{4}\left(-\frac{1}{3}\right)^{5} = \frac{1}{5}\left(-\frac{1}{3}\right)^{5} - \frac{1}{4}\left(-\frac{1}{3}\right)^{4}$

$\vdots$

$r=n$: $\frac{4n-1}{n(n-1)}\left(-\frac{1}{3}\right)^{n} = \frac{1}{n}\left(-\frac{1}{3}\right)^{n} - \frac{1}{n-1}\left(-\frac{1}{3}\right)^{n-1}$

ADDING

$$\sum_{r=2}^{n}\left[\frac{4r-1}{r(r-1)}\left(-\frac{1}{3}\right)^{r}\right] = \frac{1}{n}\left(-\frac{1}{3}\right)^{n} + \frac{1}{3}$$

$$\Rightarrow \sum_{r=2}^{\infty}\left[\frac{4r-1}{r(r-1)}\left(-\frac{1}{3}\right)^{r}\right] = \lim_{n\to\infty}\left[\frac{1}{n}\left(-\frac{1}{3}\right)^{n} + \frac{1}{3}\right]$$

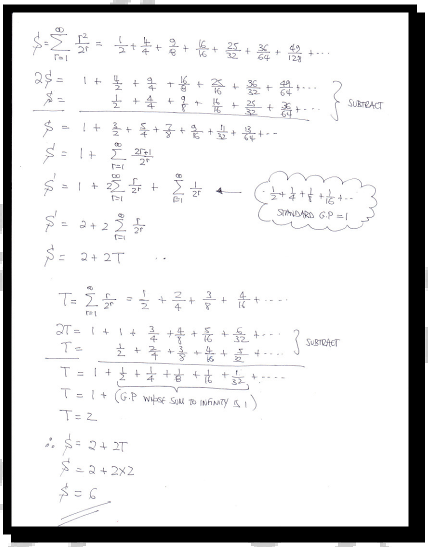
$$\Rightarrow \sum_{r=2}^{\infty}\left[\frac{4r-1}{r(r-1)}\left(-\frac{1}{3}\right)^{r}\right] = \frac{1}{3}$$

Question 126 **(*****)**

Show clearly that

$$\sum_{r=1}^{\infty} \frac{r^2}{2^r} = 6 .$$

proof



$S = \sum_{r=1}^{\infty} \frac{r^2}{2^r} = \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32} + \frac{36}{64} + \frac{49}{128} + \dots$

$2S = 1 + \frac{4}{2} + \frac{9}{4} + \frac{16}{8} + \frac{25}{16} + \frac{36}{32} + \frac{49}{64} + \dots$

$S = \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32} + \frac{36}{64} + \dots$ } SUBTRACT

$S = 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \frac{9}{16} + \frac{11}{32} + \frac{13}{64} + \dots$

$S = 1 + \sum_{r=1}^{\infty} \frac{2r+1}{2^r}$

$S = 1 + 2\sum_{r=1}^{\infty} \frac{r}{2^r} + \sum_{r=1}^{\infty} \frac{1}{2^r}$ ← $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ STANDARD G.P $= 1$

$S = 2 + 2\sum_{r=1}^{\infty} \frac{r}{2^r}$

$S = 2 + 2T$

$T = \sum_{r=1}^{\infty} \frac{r}{2^r} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \dots$

$2T = 1 + 1 + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \dots$

$T = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \dots$ } SUBTRACT

$T = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$

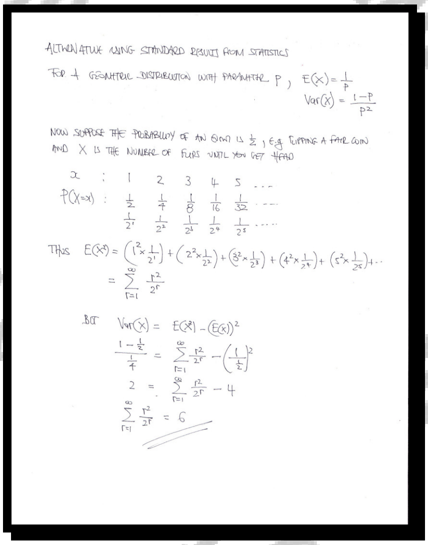
$T = 1 +$ (G.P WHOSE SUM TO INFINITY IS 1)

$T = 2$

$\therefore S = 2 + 2T$

$S = 2 + 2 \times 2$

$S = 6$



ALTERNATIVE USING STANDARD RESULTS FROM STATISTICS

FOR A GEOMETRIC DISTRIBUTION WITH PARAMETER p, $E(X) = \frac{1}{p}$, $\text{Var}(X) = \frac{1-p}{p^2}$

NOW SUPPOSE THE PROBABILITY OF AN EVENT IS $\frac{1}{2}$, e.g. FLIPPING A FAIR COIN AND X IS THE NUMBER OF FLIPS UNTIL YOU GET HEAD

x	1	2	3	4	5	...
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	...
	$\frac{1}{2^1}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$	...

THUS $E(X^2) = \left(1^2 \times \frac{1}{2^1}\right) + \left(2^2 \times \frac{1}{2^2}\right) + \left(3^2 \times \frac{1}{2^3}\right) + \left(4^2 \times \frac{1}{2^4}\right) + \left(5^2 \times \frac{1}{2^5}\right) + \dots$

$= \sum_{r=1}^{\infty} \frac{r^2}{2^r}$

BUT $\text{Var}(X) = E(X^2) - (E(X))^2$

$\frac{1 - \frac{1}{2}}{\frac{1}{4}} = \sum_{r=1}^{\infty} \frac{r^2}{2^r} - \left(\frac{1}{\frac{1}{2}}\right)^2$

$2 = \sum_{r=1}^{\infty} \frac{r^2}{2^r} - 4$

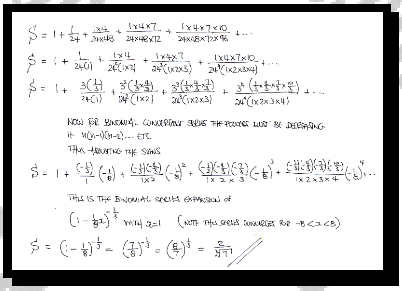
$\sum_{r=1}^{\infty} \frac{r^2}{2^r} = 6$

Question 127 **(*****)**

Show clearly that

$$1 + \frac{1}{24} + \frac{1 \cdot 4}{24 \cdot 48} + \frac{1 \cdot 4 \cdot 7}{24 \cdot 48 \cdot 72} + \frac{1 \cdot 4 \cdot 7 \cdot 10}{24 \cdot 48 \cdot 72 \cdot 96} - \dots = \frac{2}{\sqrt[3]{7}} .$$

, proof



$S = 1 + \frac{1}{24} + \frac{1 \times 4}{24 \times 48} + \frac{1 \times 4 \times 7}{24 \times 48 \times 72} + \frac{1 \times 4 \times 7 \times 10}{24 \times 48 \times 72 \times 96} + \dots$

$S = 1 + \frac{1}{24(1)} + \frac{1 \times 4}{24^2(1 \times 2)} + \frac{1 \times 4 \times 7}{24^3(1 \times 2 \times 3)} + \frac{1 \times 4 \times 7 \times 10}{24^4(1 \times 2 \times 3 \times 4)} + \dots$

$S = 1 + \frac{3\left(\frac{1}{3}\right)}{24(1)} + \frac{3^2\left(\frac{1}{3} \times \frac{4}{3}\right)}{24^2(1 \times 2)} + \frac{3^3\left(\frac{1}{3} \times \frac{4}{3} \times \frac{7}{3}\right)}{24^3(1 \times 2 \times 3)} + \frac{3^4\left(\frac{1}{3} \times \frac{4}{3} \times \frac{7}{3} \times \frac{10}{3}\right)}{24^4(1 \times 2 \times 3 \times 4)} + \dots$

NOW FOR BINOMIAL CONVERGENT SERIES THE POWERS MUST BE DECREASING IE $n(n-1)(n-2)\dots$ ETC

THUS ADJUSTING THE SIGNS

$S = 1 + \frac{\left(-\frac{1}{3}\right)}{1}\left(-\frac{1}{8}\right) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{1 \times 2}\left(-\frac{1}{8}\right)^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{1 \times 2 \times 3}\left(-\frac{1}{8}\right)^3 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\left(-\frac{10}{3}\right)}{1 \times 2 \times 3 \times 4}\left(-\frac{1}{8}\right)^4 + \dots$

THIS IS THE BINOMIAL SERIES EXPANSION OF

$\left(1 - \frac{1}{8}x\right)^{-\frac{1}{3}}$ WITH $x = 1$ (NOTE THIS SERIES CONVERGES FOR $-8 < x < 8$)

$S = \left(1 - \frac{1}{8}\right)^{-\frac{1}{3}} = \left(\frac{7}{8}\right)^{-\frac{1}{3}} = \left(\frac{8}{7}\right)^{\frac{1}{3}} = \frac{2}{\sqrt[3]{7}}$

Question 128 (*****)

$$S_n = \sum_{r=1}^{n} \left(r^2 \times 2^r \right)$$

Use the standard techniques for the summation of a geometric series, to show that

$$S_n = \left(n^2 - 2n + 3 \right) \times 2^{n+1} - 6 .$$

[*You may not use proof by induction in this question.*]

proof

$$\sum_{r=1}^{n} r^2 2^r = S_n = 1^2 \times 2^1 + 2^2 \times 2^2 + 3^2 \times 2^3 + 4^2 \times 2^4 + 5^2 \times 2^5 + \dots + n^2 \times 2^n$$

$$-2S_n = \quad - 1^2 \times 2^2 - 2^2 \times 2^3 - 3^2 \times 2^4 - 4^2 \times 2^5 - \dots - (n-1)^2 \times 2^n - n^2 \times 2^{n+1}$$

ADD

$$-S_n = 1 \times 2^1 + 3 \times 2^2 + 5 \times 2^3 + 7 \times 2^4 + 9 \times 2^5 + \dots + (2n-1) \times 2^n - n^2 \times 2^{n+1}$$

HENCE

$$+2S_n = \quad - 1 \times 2^2 - 3 \times 2^3 - 5 \times 2^4 - 7 \times 2^5 - \dots - (2n-3) \times 2^n - (2n-1) \times 2^{n+1} + n^2 \times 2^{n+2}$$

ADD AGAIN

$$S_n = 1 \times 2 + 2 \times 2^2 + 2 \times 2^3 + 2 \times 2^4 + 2 \times 2^5 + \dots + 2 \times 2^n + (-n^2 - 2n + 1) \times 2^{n+1} + 2n^2 \times 2^{n+1}$$

$$S_n = 2 + 2\left[2^2 + 2^3 + 2^4 + 2^5 + \dots + 2^n \right] + (n^2 - 2n + 1) \times 2^{n+1}$$

GP WITH $a = 4$, $r = 2$, $n = n-1$

$$S_n = 2 + 2\left(\frac{4(2^{n-1} - 1)}{2 - 1} \right) + (n^2 - 2n + 1) \times 2^{n+1}$$

$$S_n = 2 + 8(2^{n-1} - 1) + (n^2 - 2n + 1) \times 2^{n+1}$$

$$S_n = 2 + 2 \times 2^{n+1} - 8 + (n^2 - 2n + 1) \times 2^{n+1}$$

$$S_n = (n^2 - 2n + 3) \times 2^{n+1} - 6$$

AS REQUIRED

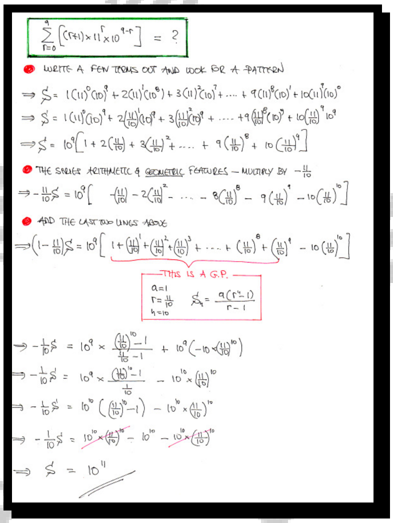
Question 129 (*****)

By showing a detailed method, sum the following series.

$$\sum_{r=0}^{9}\left[(r+1)\times 11^{r}\times 10^{9-r}\right].$$

You may leave the answer in index form.

$$\boxed{},\quad \boxed{\sum_{r=0}^{9}\left[(r+1)\times 11^{r}\times 10^{9-r}\right]=10^{11}}$$



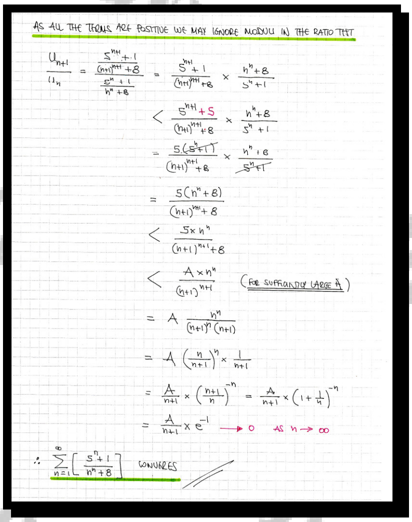
Question 130 (*****)

Use the ratio test to show that the following series converges

$$\sum_{n=1}^{\infty}\left[\frac{5^n+1}{n^n+8}\right].$$

You may assume without proof that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}=\frac{1}{e}$.

$\boxed{}$, $\boxed{\text{proof}}$



AS ALL THE TERMS ARE POSITIVE WE MAY IGNORE MODULI IN THE RATIO TEST

$$\frac{u_{n+1}}{u_n}=\frac{\frac{5^{n+1}+1}{(n+1)^{n+1}+8}}{\frac{5^n+1}{n^n+8}}=\frac{5^{n+1}+1}{(n+1)^{n+1}+8}\times\frac{n^n+8}{5^n+1}$$

$$<\frac{5^{n+1}+5}{(n+1)^{n+1}+8}\times\frac{n^n+8}{5^n+1}$$

$$=\frac{5(5^n+1)}{(n+1)^{n+1}+8}\times\frac{n^n+8}{5^n+1}$$

$$=\frac{5(n^n+8)}{(n+1)^{n+1}+8}$$

$$<\frac{5\times n^n}{(n+1)^{n+1}+8}$$

$$<\frac{A\times n^n}{(n+1)^{n+1}}\quad\text{(FOR SUFFICIENTLY LARGE } A\text{)}$$

$$=A\,\frac{n^n}{(n+1)^n(n+1)}$$

$$=A\left(\frac{n}{n+1}\right)^n\times\frac{1}{n+1}$$

$$=\frac{A}{n+1}\times\left(\frac{n+1}{n}\right)^{-n}=\frac{A}{n+1}\times\left(1+\frac{1}{n}\right)^{-n}$$

$$=\frac{A}{n+1}\times e^{-1}\longrightarrow 0\quad\text{AS } n\to\infty$$

$\therefore \sum_{n=1}^{\infty}\left[\frac{5^n+1}{n^n+8}\right]$ CONVERGES

Question 131 (*****)

$$f(x)=\frac{1}{\sqrt{1-x}}, \quad -1<x<1.$$

a) By manipulating the general term of binomial expansion of $f(x)$ show that

$$f(x)=\sum_{r=0}^{\infty}\binom{2r}{r}\left(\tfrac{1}{4}x\right)^{r}.$$

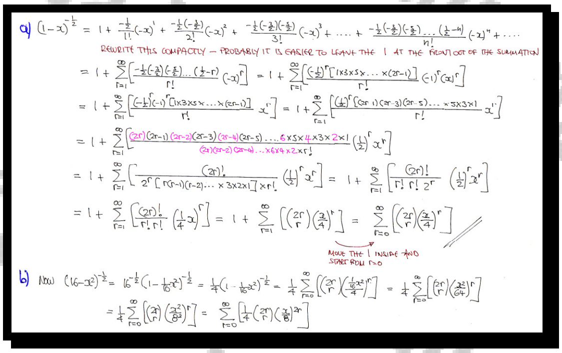
b) Find a similar expression for $\dfrac{1}{\sqrt{16-x^2}}$ and show further that

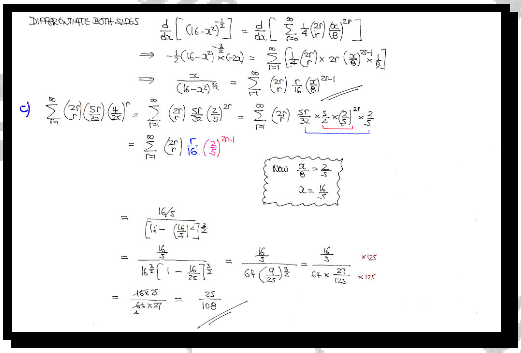
$$\frac{x}{\left(16-x^2\right)^{\frac{3}{2}}}=\sum_{r=1}^{\infty}\binom{2r}{r}\left(\tfrac{1}{16}r\right)\left(\tfrac{1}{8}x\right)^{2r-1}.$$

c) Determine the exact value of

$$\sum_{r=1}^{\infty}\binom{2r}{r}\left(\tfrac{5}{32}r\right)\left(\tfrac{4}{25}\right)^{r}.$$

$\boxed{}$, $\boxed{\dfrac{25}{108}}$





Question 132 (*****)

Determine, in terms of n, a simplified expression

$$\sum_{r=1}^{n}\left[\frac{r^2+9r+19}{(r+5)!}\right],$$

and hence, or otherwise, deduce the value of

$$\sum_{r=1}^{\infty}\left[\frac{r^2+7r+11}{(r+4)!}\right].$$

$$\boxed{\sum_{r=1}^{n}\left[\frac{r^2+9r+19}{(r+5)!}\right]=\frac{1}{6}-\frac{n+5}{(n+5)!}}, \quad \boxed{\sum_{r=1}^{\infty}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\frac{5}{24}}$$

- START WITH PARTIAL FRACTIONS — NOTE THAT THE NUMERATOR IS A QUADRATIC IN r, SO WE HAVE TO TRY "2 FACTORIALS APART"

I.E $\dfrac{r^2+9r+19}{(r+5)!} \equiv \dfrac{A}{(r+5)!}+\dfrac{B}{(r+3)!}$

$\Rightarrow r^2+9r+19 \equiv A+B(r+5)(r+4)$

$\Rightarrow r^2+9r+19 \equiv Br^2+9Br+(20B+A)$

$\therefore B=1$ & $A=-1$

- HENCE BY THE METHOD OF DIFFERENCES

$$\frac{r^2+9r+19}{(r+5)!}=\frac{1}{(r+3)!}-\frac{1}{(r+5)!}$$

$r=1$: $\dfrac{1+9+19}{6!}=\dfrac{1}{4!}-\dfrac{1}{6!}$

$r=2$: $\dfrac{4+18+19}{7!}=\dfrac{1}{5!}-\dfrac{1}{7!}$

$r=3$: $\dfrac{9+27+19}{8!}=\dfrac{1}{6!}-\dfrac{1}{8!}$

$r=4$: $\dfrac{16+36+19}{9!}=\dfrac{1}{7!}-\dfrac{1}{9!}$

$\vdots$

$r=n-1$: $\dfrac{(n-1)^2+9(n-1)+19}{(n+4)!}=\dfrac{1}{(n+2)!}-\dfrac{1}{(n+4)!}$

$r=n$: $\dfrac{n^2+9n+19}{(n+5)!}=\dfrac{1}{(n+3)!}-\dfrac{1}{(n+5)!}$

- ADDING

$$\sum_{r=1}^{n}\left[\frac{r^2+9r+19}{(r+5)!}\right]=\frac{1}{4!}+\frac{1}{5!}-\left[\frac{1}{(n+4)!}+\frac{1}{(n+5)!}\right]$$

$$=\frac{5}{5!}+\frac{1}{5!}-\left[\frac{n+5}{(n+5)!}+\frac{1}{(n+5)!}\right]$$

$$\therefore \sum_{r=1}^{n}\left[\frac{r^2+9r+19}{(r+5)!}\right]=\frac{6}{5!}-\frac{n+6}{(n+5)!}=\frac{1}{6}-\frac{n+6}{(n+5)!}$$

- NOW PROCEED AS FOLLOWS

$$\Rightarrow \sum_{r=1}^{n}\left[\frac{r^2+9r+19}{(r+5)!}\right]=\frac{6}{5!}-\frac{n+6}{(n+5)!}$$

(BOTH ARE THE SAME)

$$\Rightarrow \sum_{r=2}^{n}\left[\frac{(r-1)^2+9(r-1)+19}{(r-1+5)!}\right]=\frac{6}{5!}-\frac{n+6}{(n+5)!}$$

$$\Rightarrow \sum_{r=2}^{n}\left[\frac{r^2-2r+1+9r-9+19}{(r+4)!}\right]=\frac{6}{5!}-\frac{n+6}{(n+5)!}$$

$$\Rightarrow \sum_{r=2}^{n}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\frac{6}{5!}-\frac{n+6}{(n+5)!}$$

$$\Rightarrow \sum_{r=1}^{n}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\underbrace{\frac{1+7+11}{5!}}_{r=1}+\frac{6}{5!}-\frac{n+6}{(n+5)!}$$

$$\Rightarrow \sum_{r=1}^{n}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\frac{25}{5!}-\frac{n+6}{(n+5)!}$$

$$\Rightarrow \sum_{r=1}^{n}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\frac{5}{4!}-\frac{n+6}{(n+5)!}$$

$$\Rightarrow \sum_{r=1}^{\infty}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\lim_{n\to\infty}\left[\frac{5}{24}-\frac{n+6}{(n+5)!}\right]$$

$$\Rightarrow \sum_{r=1}^{\infty}\left[\frac{r^2+7r+11}{(r+4)!}\right]=\frac{5}{24}$$

Question 133 (*****)

The n^{th} term of a series is given recursively by

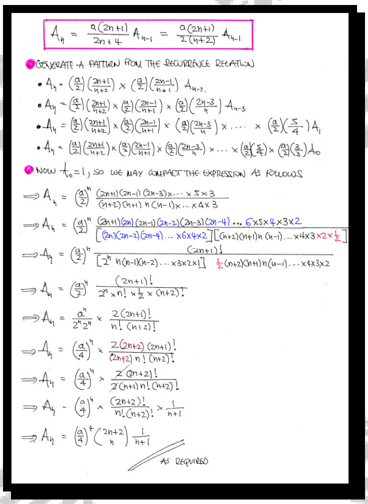
$$A_n = \frac{a(2n+1)}{2n+4} A_{n-1}, \; n \in \mathbb{N}, \; n \geq 1,$$

where a is a positive constant.

Given further that $A_0 = 1$, show that

$$A_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n} \frac{1}{n+1}.$$

SP-V, proof



$$A_n = \frac{a(2n+1)}{2n+4} A_{n-1} = \frac{a(2n+1)}{2(n+2)} A_{n-1}$$

GENERATE A PATTERN FROM THE RECURRENCE RELATION

- $A_n = \left(\frac{a}{2}\right)\left(\frac{2n+1}{n+2}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-1}{n+1}\right) A_{n-2}$
- $A_n = \left(\frac{a}{2}\right)\left(\frac{2n+1}{n+2}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-1}{n+1}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-3}{n}\right) A_{n-3}$
- $A_n = \left(\frac{a}{2}\right)\left(\frac{2n+1}{n+2}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-1}{n+1}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-3}{n}\right) \times \dots \times \left(\frac{a}{2}\right)\left(\frac{5}{4}\right) A_1$
- $A_n = \left(\frac{a}{2}\right)\left(\frac{2n+1}{n+2}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-1}{n+1}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-3}{n}\right) \times \dots \times \left(\frac{a}{2}\right)\left(\frac{5}{4}\right) \times \left(\frac{a}{2}\right)\left(\frac{3}{3}\right) A_0$

NOW $A_0 = 1$, SO WE MAY COMPACT THE EXPRESSION AS FOLLOWS

$$\Rightarrow A_n = \left(\frac{a}{2}\right)^n \frac{(2n+1)(2n-1)(2n-3) \times \dots \times 5 \times 3}{(n+2)(n+1)n(n-1) \times \dots \times 4 \times 3}$$

$$\Rightarrow A_n = \left(\frac{a}{2}\right)^n \frac{(2n+1)(2n)(2n-1)(2n-2)(2n-3)(2n-4)\dots 6 \times 5 \times 4 \times 3 \times 2}{[(2n)(2n-2)(2n-4)\dots \times 6 \times 4 \times 2][(n+2)(n+1)n(n-1)\dots \times 4 \times 3 \times 2 \times \frac{1}{2}]}$$

$$\Rightarrow A_n = \left(\frac{a}{2}\right)^n \frac{(2n+1)!}{[2^n n(n-1)(n-2)\dots \times 3 \times 2 \times 1] \; \frac{1}{2}(n+2)(n+1)n(n-1)\dots \times 4 \times 3 \times 2}$$

$$\Rightarrow A_n = \left(\frac{a}{2}\right)^n \frac{(2n+1)!}{2^n \times n! \times \frac{1}{2} \times (n+2)!}$$

$$\Rightarrow A_n = \frac{a^n}{2^n 2^n} \times \frac{2(2n+1)!}{n!(n+2)!}$$

$$\Rightarrow A_n = \left(\frac{a}{4}\right)^n \times \frac{2(2n+2)(2n+1)!}{(2n+2)n!(n+2)!}$$

$$\Rightarrow A_n = \left(\frac{a}{4}\right)^n \times \frac{2(2n+2)!}{2(n+1)n!(n+2)!}$$

$$\Rightarrow A_n = \left(\frac{a}{4}\right)^n \times \frac{(2n+2)!}{n!(n+2)!} \times \frac{1}{n+1}$$

$$\Rightarrow A_n = \left(\frac{a}{4}\right)^4 \binom{2n+2}{n} \frac{1}{n+1}$$

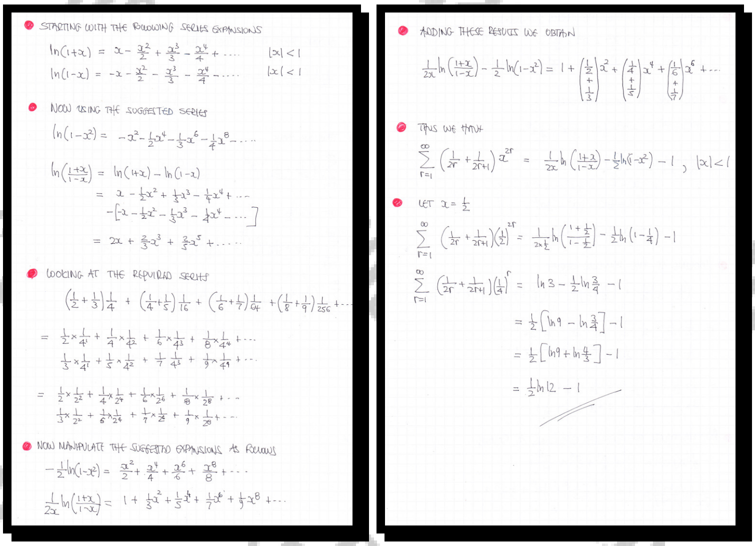
AS REQUIRED

Question 134 (*****)

By considering the series expansions of $\ln\left(1-x^2\right)$ and $\ln\left(\frac{1+x}{1-x}\right)$, or otherwise, find the exact value of the following series.

$$\sum_{r=1}^{\infty}\left[\left(\frac{1}{2r}+\frac{1}{2r+1}\right)\left(\frac{1}{4}\right)^r\right].$$

$\boxed{}$, $\boxed{-1+\frac{1}{2}\ln 12}$

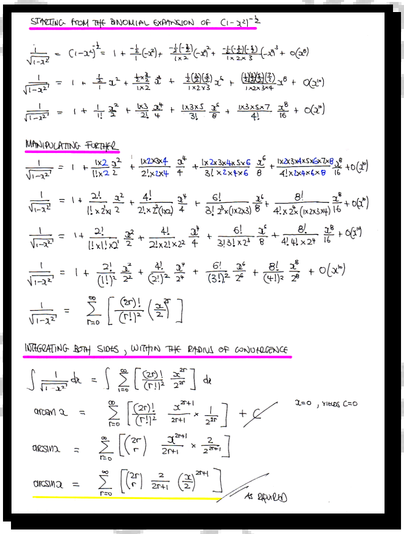


Question 135 (*****)

By considering a suitable binomial expansion, show that

$$\arcsin x = \sum_{r=0}^{\infty}\left[\binom{2r}{r}\frac{2}{2r+1}\left(\frac{x}{2}\right)^{2r+1}\right].$$

X J , proof



STARTING FROM THE BINOMIAL EXPANSION OF $(1-x^2)^{-\frac{1}{2}}$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}(-x^2) + \frac{-\frac{1}{2}(-\frac{3}{2})}{1\times 2}(-x^2)^2 + \frac{-\frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})}{1\times 2\times 3}(-x^2)^3 + O(x^8)$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{\frac{1}{2}}{1}x^2 + \frac{\frac{1}{2}\times\frac{3}{2}}{1\times 2}x^4 + \frac{\frac{1}{2}(\frac{3}{2})(\frac{5}{2})}{1\times 2\times 3}x^6 + \frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})(\frac{7}{2})}{1\times 2\times 3\times 4}x^8 + O(x^{10})$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{1!}\frac{x^2}{2} + \frac{1\times 3}{2!}\frac{x^4}{4} + \frac{1\times 3\times 5}{3!}\frac{x^6}{8} + \frac{1\times 3\times 5\times 7}{4!}\frac{x^8}{16} + O(x^{10})$$

MANIPULATING FURTHER

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1\times 2}{1!\times 2}\frac{x^2}{2} + \frac{1\times 2\times 3\times 4}{2!\times 2\times 4}\frac{x^4}{4} + \frac{1\times 2\times 3\times 4\times 5\times 6}{3!\times 2\times 4\times 6}\frac{x^6}{8} + \frac{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8}{4!\times 2\times 4\times 6\times 8}\frac{x^8}{16} + O(x^{10})$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{2!}{1!\times 2^1\times 1}\frac{x^2}{2} + \frac{4!}{2!\times 2^2(1\times 2)}\frac{x^4}{4} + \frac{6!}{3!\,2^3\times(1\times 2\times 3)}\frac{x^6}{8} + \frac{8!}{4!\times 2^4\times(1\times 2\times 3\times 4)}\frac{x^8}{16} + O(x^{10})$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{2!}{1!\times 1!\times 2^1}\frac{x^2}{2} + \frac{4!}{2!\times 2!\times 2^2}\frac{x^4}{4} + \frac{6!}{3!\,3!\times 2^3}\frac{x^6}{8} + \frac{8!}{4!\,4!\times 2^4}\frac{x^8}{16} + O(x^{10})$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{2!}{(1!)^2}\frac{x^2}{2^2} + \frac{4!}{(2!)^2}\frac{x^4}{2^4} + \frac{6!}{(3!)^2}\frac{x^6}{2^6} + \frac{8!}{(4!)^2}\frac{x^8}{2^8} + O(x^{10})$$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{r=0}^{\infty}\left[\frac{(2r)!}{(r!)^2}\left(\frac{x}{2}\right)^{2r}\right]$$

INTEGRATING BOTH SIDES, WITHIN THE RADIUS OF CONVERGENCE

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \sum_{r=0}^{\infty}\left[\frac{(2r)!}{(r!)^2}\frac{x^{2r}}{2^{2r}}\right]dx$$

$$\arcsin x = \sum_{r=0}^{\infty}\left[\frac{(2r)!}{(r!)^2}\frac{x^{2r+1}}{2r+1}\times\frac{1}{2^{2r}}\right] + C$$

$x=0$, YIELDS $C=0$

$$\arcsin x = \sum_{r=0}^{\infty}\left[\binom{2r}{r}\frac{x^{2r+1}}{2r+1}\times\frac{2}{2^{2r+1}}\right]$$

$$\arcsin x = \sum_{r=0}^{\infty}\left[\binom{2r}{r}\frac{2}{2r+1}\left(\frac{x}{2}\right)^{2r+1}\right]$$

AS REQUIRED

Question 136 (*****)

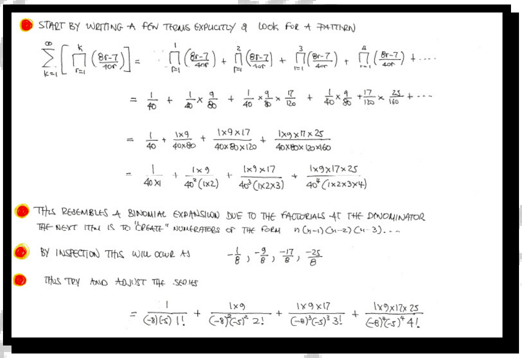
The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Find the sum to infinity of the following expression

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right].$$

SP-K , $\boxed{\sqrt[8]{\frac{5}{4}}-1}$



START BY WRITING A FEW TERMS EXPLICITLY & LOOK FOR A PATTERN

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right] = \prod_{r=1}^{1}\left(\frac{8r-7}{40r}\right) + \prod_{r=1}^{2}\left(\frac{8r-7}{40r}\right) + \prod_{r=1}^{3}\left(\frac{8r-7}{40r}\right) + \prod_{r=1}^{4}\left(\frac{8r-7}{40r}\right) + \ldots$$

$$= \frac{1}{40} + \frac{1}{40}\times\frac{9}{80} + \frac{1}{40}\times\frac{9}{80}\times\frac{17}{120} + \frac{1}{40}\times\frac{9}{80}\times\frac{17}{120}\times\frac{25}{160} + \ldots$$

$$= \frac{1}{40} + \frac{1\times9}{40\times80} + \frac{1\times9\times17}{40\times80\times120} + \frac{1\times9\times17\times25}{40\times80\times120\times160}$$

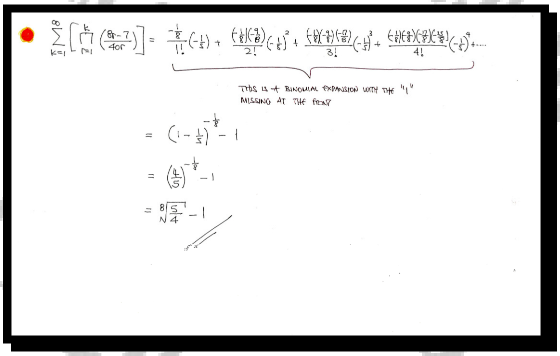
$$= \frac{1}{40\times1} + \frac{1\times9}{40^2(1\times2)} + \frac{1\times9\times17}{40^3(1\times2\times3)} + \frac{1\times9\times17\times25}{40^4(1\times2\times3\times4)}$$

THIS RESEMBLES A BINOMIAL EXPANSION DUE TO THE FACTORIALS AT THE DENOMINATOR. THE NEXT ITEM IS TO "CREATE" NUMERATORS OF THE FORM $n(n-1)(n-2)(n-3)\ldots$

BY INSPECTION THIS WILL OCCUR AS $-\frac{1}{8}, -\frac{9}{8}, -\frac{17}{8}, -\frac{25}{8}$

THUS TRY AND ADJUST THE SERIES

$$= \frac{1}{(-8)(-5)\,1!} + \frac{1\times9}{(-8)^2(-5)^2\,2!} + \frac{1\times9\times17}{(-8)^3(-5)^3\,3!} + \frac{1\times9\times17\times25}{(-8)^4(-5)^4\,4!}$$



$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right] = \frac{-\frac{1}{8}}{1!}\left(-\frac{1}{5}\right) + \frac{\left(-\frac{1}{8}\right)\left(-\frac{9}{8}\right)}{2!}\left(-\frac{1}{8}\right)^2 + \frac{\left(-\frac{1}{8}\right)\left(-\frac{9}{8}\right)\left(-\frac{17}{8}\right)}{3!}\left(-\frac{1}{5}\right)^3 + \frac{\left(-\frac{1}{8}\right)\left(-\frac{9}{8}\right)\left(-\frac{17}{8}\right)\left(-\frac{25}{8}\right)}{4!}\left(-\frac{1}{5}\right)^4 + \ldots$$

THIS IS A BINOMIAL EXPANSION WITH THE "1" MISSING AT THE FRONT

$$= \left(1-\frac{1}{5}\right)^{-\frac{1}{8}} - 1$$

$$= \left(\frac{4}{5}\right)^{-\frac{1}{8}} - 1$$

$$= \sqrt[8]{\frac{5}{4}} - 1$$

Question 137 (*****)

A sequence u_1, u_2, u_3, u_5, u_6, … is generated by the recurrence relation

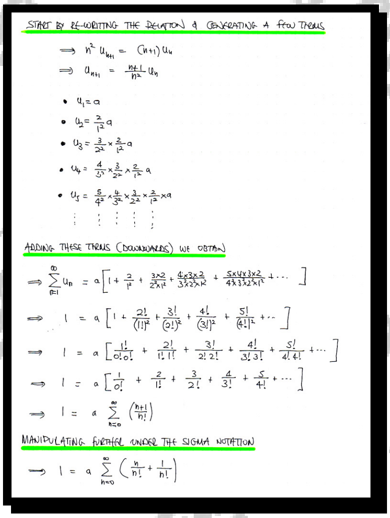
$$n^2 u_{n+1} = (n+1)u_n, \quad n = 1, 2, 3, 4, \ldots$$

It is further given that

$$\sum_{n=1}^{\infty} u_n = 1.$$

Find in exact form the value of u_1.

I.X-C, $u_1 = \dfrac{1}{2\mathrm{e}}$



START BY RE-WRITING THE RELATION & GENERATING A FEW TERMS

$\Rightarrow n^2 u_{n+1} = (n+1)u_n$

$\Rightarrow u_{n+1} = \frac{n+1}{n^2}u_n$

- $u_1 = a$
- $u_2 = \frac{2}{1^2}a$
- $u_3 = \frac{3}{2^2}\times\frac{2}{1^2}a$
- $u_4 = \frac{4}{3^2}\times\frac{3}{2^2}\times\frac{2}{1^2}a$
- $u_5 = \frac{5}{4^2}\times\frac{4}{3^2}\times\frac{3}{2^2}\times\frac{2}{1^2}\times a$

ADDING THESE TERMS (DOWNWARDS) WE OBTAIN

$\Rightarrow \sum_{n=1}^{\infty} u_n = a\left[1 + \frac{2}{1^2} + \frac{3\times2}{2^2\times1^2} + \frac{4\times3\times2}{3^2\times2^2\times1^2} + \frac{5\times4\times3\times2}{4^2\times3^2\times2^2\times1^2} + \ldots\right]$

$\Rightarrow 1 = a\left[1 + \frac{2!}{(1!)^2} + \frac{3!}{(2!)^2} + \frac{4!}{(3!)^2} + \frac{5!}{(4!)^2} + \ldots\right]$

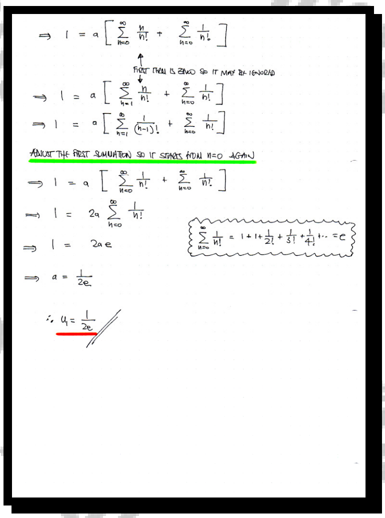
$\Rightarrow 1 = a\left[\frac{1!}{0!0!} + \frac{2!}{1!1!} + \frac{3!}{2!2!} + \frac{4!}{3!3!} + \frac{5!}{4!4!} + \ldots\right]$

$\Rightarrow 1 = a\left[\frac{1}{0!} + \frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + \frac{5}{4!} + \ldots\right]$

$\Rightarrow 1 = a\sum_{n=0}^{\infty}\left(\frac{n+1}{n!}\right)$

MANIPULATING FURTHER UNDER THE SIGMA NOTATION

$\Rightarrow 1 = a\sum_{n=0}^{\infty}\left(\frac{n}{n!} + \frac{1}{n!}\right)$



$\Rightarrow 1 = a\left[\sum_{n=0}^{\infty}\frac{n}{n!} + \sum_{n=0}^{\infty}\frac{1}{n!}\right]$

FIRST TERM IS ZERO SO IT MAY BE IGNORED

$\Rightarrow 1 = a\left[\sum_{n=1}^{\infty}\frac{n}{n!} + \sum_{n=0}^{\infty}\frac{1}{n!}\right]$

$\Rightarrow 1 = a\left[\sum_{n=1}^{\infty}\frac{1}{(n-1)!} + \sum_{n=0}^{\infty}\frac{1}{n!}\right]$

ADJUST THE FIRST SUMMATION SO IT STARTS FROM $n=0$ AGAIN

$\Rightarrow 1 = a\left[\sum_{n=0}^{\infty}\frac{1}{n!} + \sum_{n=0}^{\infty}\frac{1}{n!}\right]$

$\Rightarrow 1 = 2a\sum_{n=0}^{\infty}\frac{1}{n!}$

$\sum_{n=0}^{\infty}\frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots = e$

$\Rightarrow 1 = 2ae$

$\Rightarrow a = \frac{1}{2e}$

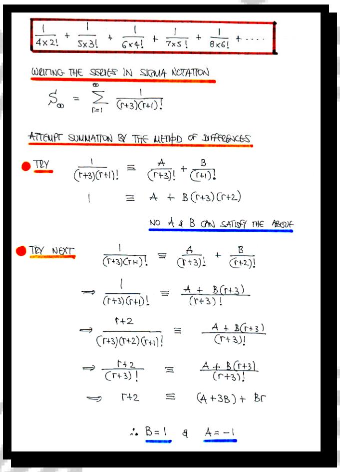
$\therefore u_1 = \frac{1}{2e}$

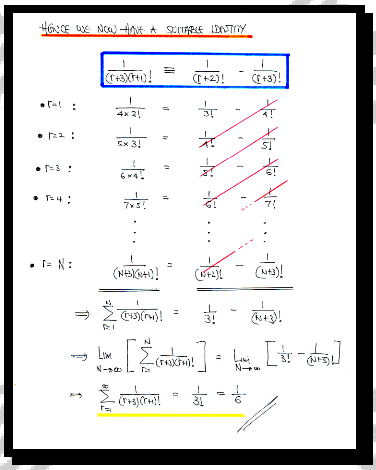
Question 138 (*****)

Find the sum to infinity of the following convergent series.

$$\frac{1}{4\times 2!}+\frac{1}{5\times 3!}+\frac{1}{6\times 4!}+\frac{1}{7\times 5!}+\frac{1}{8\times 6!}+\ldots$$

P2- , $\boxed{\frac{1}{6}}$





Question 139 (*****)

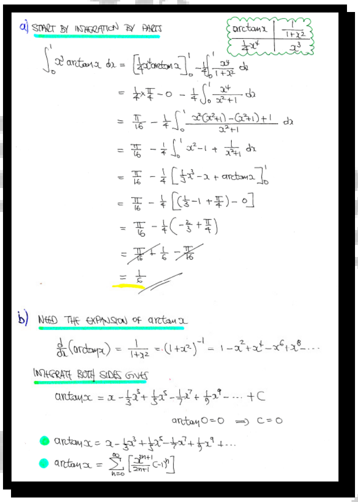
a) Use an appropriate integration method to evaluate the following integral.

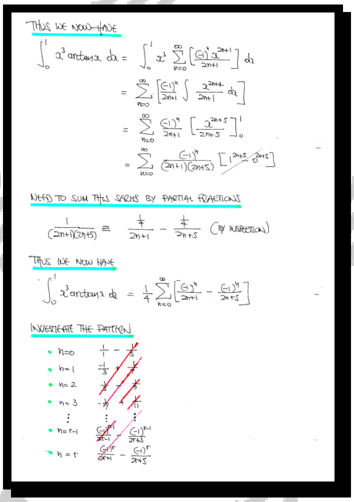
$$\int_0^1 x^3 \arctan x \; dx \, .$$

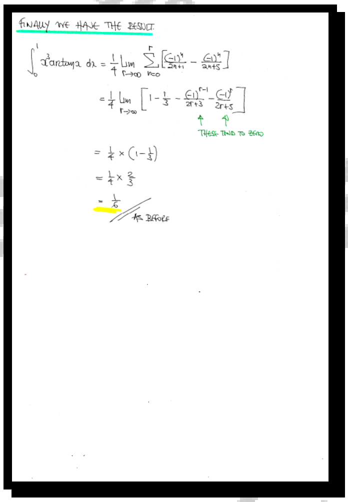
b) Obtain an infinite series expansion for $\arctan x$ and use this series expansion to verify the answer obtained for the above integral in part **(a)**.

[*you may assume that integration and summation commute*]

$\boxed{}\,,\ \boxed{\frac{1}{6}}$





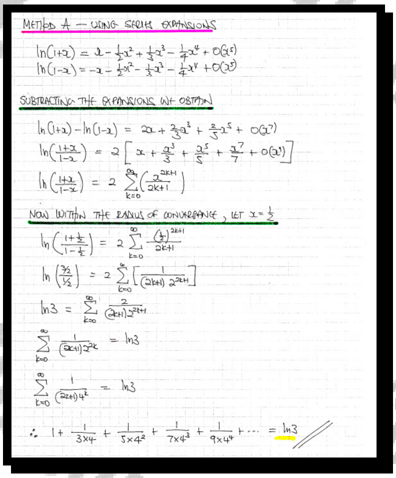


Question 140 (*****)

Find the sum to infinity of the following series.

$$1+\frac{1}{3\times 4}+\frac{1}{5\times 4^2}+\frac{1}{7\times 4^3}+\frac{1}{9\times 4^4}+\ldots$$

$\boxed{\ln 3}$



METHOD A – USING SERIES EXPANSIONS

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$

$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + O(x^5)$

SUBTRACTING THE EXPANSIONS WE OBTAIN

$\ln(1+x) - \ln(1-x) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + O(x^7)$

$\ln\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + O(x^9)\right]$

$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{k=0}^{\infty}\left(\frac{x^{2k+1}}{2k+1}\right)$

NOW WITHIN THE RADIUS OF CONVERGENCE, LET $x=\frac{1}{2}$

$\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = 2\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{2k+1}}{2k+1}$

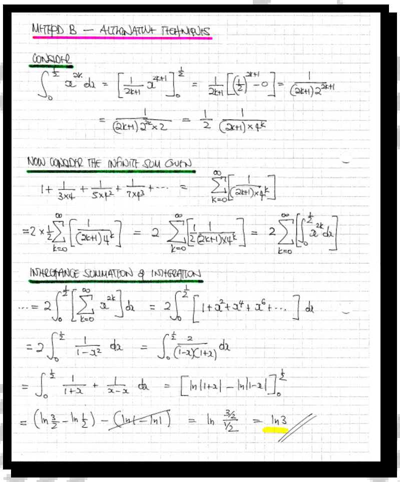
$\ln\left(\frac{3/2}{1/2}\right) = 2\sum_{k=0}^{\infty}\left[\frac{1}{(2k+1)\,2^{2k+1}}\right]$

$\ln 3 = \sum_{k=0}^{\infty}\frac{2}{(2k+1)2^{2k+1}}$

$\sum_{k=0}^{\infty}\frac{1}{(2k+1)2^{2k}} = \ln 3$

$\sum_{k=0}^{\infty}\frac{1}{(2k+1)4^{k}} = \ln 3$

$\therefore\ 1 + \frac{1}{3\times 4} + \frac{1}{5\times 4^2} + \frac{1}{7\times 4^3} + \frac{1}{9\times 4^4} + \cdots = \ln 3$



METHOD B – ALTERNATIVE TECHNIQUES

CONSIDER

$\int_0^{\frac{1}{2}} x^{2k}\,dx = \left[\frac{1}{2k+1}x^{2k+1}\right]_0^{\frac{1}{2}} = \frac{1}{2k+1}\left[\left(\frac{1}{2}\right)^{2k+1} - 0\right] = \frac{1}{(2k+1)2^{2k+1}}$

$= \frac{1}{(2k+1)2^{2k}\times 2} = \frac{1}{2}\,\frac{1}{(2k+1)\times 4^k}$

NOW CONSIDER THE INFINITE SUM GIVEN

$1 + \frac{1}{3\times 4} + \frac{1}{5\times 4^2} + \frac{1}{7\times 4^3} + \cdots = \sum_{k=0}^{\infty}\left[\frac{1}{(2k+1)\times 4^k}\right]$

$= 2\times\frac{1}{2}\sum_{k=0}^{\infty}\left[\frac{1}{(2k+1)4^k}\right] = 2\sum_{k=0}^{\infty}\left[\frac{1}{2}\,\frac{1}{(2k+1)\times 4^k}\right] = 2\sum_{k=0}^{\infty}\left[\int_0^{\frac{1}{2}} x^{2k}\,dx\right]$

INTERCHANGE SUMMATION & INTEGRATION

$\ldots = 2\int_0^{\frac{1}{2}}\left[\sum_{k=0}^{\infty} x^{2k}\right]dx = 2\int_0^{\frac{1}{2}}\left[1 + x^2 + x^4 + x^6 + \ldots\right]dx$

$= 2\int_0^{\frac{1}{2}}\frac{1}{1-x^2}\,dx = \int_0^{\frac{1}{2}}\frac{2}{(1-x)(1+x)}\,dx$

$= \int_0^{\frac{1}{2}}\frac{1}{1+x} + \frac{1}{x-x}\,dx = \Big[\ln|1+x| - \ln|1-x|\Big]_0^{\frac{1}{2}}$

$= \left(\ln\frac{3}{2} - \ln\frac{1}{2}\right) - (\ln 1 - \ln 1) = \ln\frac{3/2}{1/2} = \ln 3$

Question 141 (*****)

By showing a detailed method, sum the following series.

$$\sum_{n=0}^{\infty}\left[\frac{\cos^2\left(\frac{1}{6}n\pi\right)}{2^n}\right].$$

$\boxed{}$, $\boxed{\frac{3}{2}}$

START BY TRIGONOMETRIC IDENTITIES

$$\sum_{n=0}^{\infty}\frac{\cos^2(\frac{1}{6}n\pi)}{2^n} = \sum_{n=0}^{\infty}\frac{\frac{1}{2}+\frac{1}{2}\cos(\frac{1}{3}n\pi)}{2^n} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1+\cos(\frac{1}{3}n\pi)}{2^n}$$

SPLIT INTO A GEOMETRIC PROGRESSION AND ANOTHER SERIES

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2^n}\right) + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\cos(\frac{1}{3}n\pi)}{2^n}$$

$$= \frac{1}{2}\left[1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\right] + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\cos(\frac{1}{3}n\pi)}{2^n}$$

" $S_{\infty} = \frac{a}{1-r}$ " ↓ USING COMPLEX NUMBERS

$$= \left[\frac{1}{2}\times\frac{1}{1-\frac{1}{2}}\right] + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\text{Re}\left[e^{\frac{1}{3}n\pi i}\right]}{2^n}$$

$$= \left[\frac{1}{2}\times\frac{1}{\frac{1}{2}}\right] + \frac{1}{2}\text{Re}\left[\sum_{n=0}^{\infty}\left(\frac{e^{\frac{1}{3}n\pi i}}{2^n}\right)\right]$$

$$= 1 + \frac{1}{2}\text{Re}\left[\sum_{n=0}^{\infty}\left(\frac{e^{\frac{1}{3}\pi i}}{2}\right)^n\right]$$

NOTE THAT THE SERIES CONVERGES SINCE $\left|\frac{e^{\frac{1}{3}\pi i}}{2}\right| = \frac{|e^{\frac{1}{3}\pi i}|}{2} = \frac{1}{2}$

$$= 1 + \frac{1}{2}\text{Re}\left[1+\frac{1}{2}e^{\frac{1}{3}\pi i}+\frac{1}{4}e^{\frac{2}{3}\pi i}+\frac{1}{8}e^{\pi i}+\frac{1}{16}e^{\frac{4}{3}\pi i}+\dots\right]$$

AGAIN USING THE SUM TO INFINITY OF A G.P. " $S_{\infty} = \frac{a}{1-r}$ "

$$= 1 + \frac{1}{2}\text{R}\left[\frac{1}{1-\frac{1}{2}e^{\frac{1}{3}\pi i}}\right]$$

$$= 1 + \text{Re}\,\frac{1}{2-e^{\frac{1}{3}\pi i}}\Bigg]$$

MANIPULATE THE EXPRESSION TO EXTRACT THE REAL PART

$$= 1 + \text{Re}\left[\frac{2-e^{-\frac{1}{3}\pi i}}{(2-e^{\frac{1}{3}\pi i})(2-e^{-\frac{1}{3}\pi i})}\right]$$

$$= 1 + \text{Re}\left[\frac{2-e^{-\frac{1}{3}\pi i}}{4-2e^{\frac{1}{3}\pi i}-2e^{-\frac{1}{3}\pi i}+1}\right]$$

$$= 1 + \text{Re}\left[\frac{2-e^{-\frac{1}{3}\pi i}}{5-4(\frac{1}{2}e^{\frac{1}{3}\pi i}+\frac{1}{2}e^{-\frac{1}{3}\pi i})}\right]$$

$$= 1 + \text{Re}\left[\frac{2-(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3})}{5-4\cosh(\frac{1}{3}\pi i)}\right]$$

$$= 1 + \text{Re}\left[\frac{2-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}{5-4\cos\frac{\pi}{3}}\right] \qquad \cosh(iz)=\cos z$$

$$= 1 + \text{Re}\left[\frac{\frac{3}{2}+i\frac{\sqrt{3}}{2}}{3}\right]$$

$$= 1 + \frac{3/2}{3}$$

$$= 1 + \frac{1}{2}$$

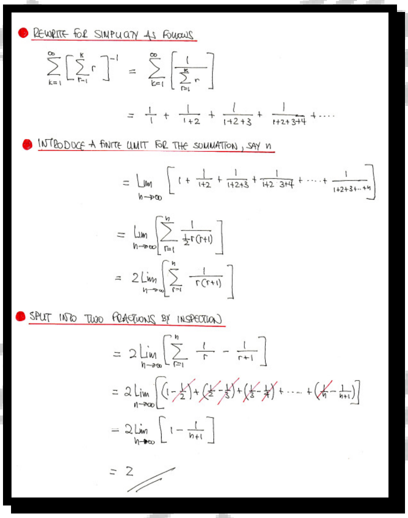
$$\therefore \sum_{n=0}^{\infty}\left(\frac{\cos^2(\frac{1}{6}n\pi)}{2^n}\right) = \frac{3}{2}$$

Question 142 (*****)

Evaluate the following expression

$$\sum_{k=1}^{\infty}\left[\sum_{r=1}^{k} r\right]^{-1}.$$

SP , 2



REWRITE FOR SIMPLICITY AS FOLLOWS

$$\sum_{k=1}^{\infty}\left[\sum_{r=1}^{k} r\right]^{-1} = \sum_{k=1}^{\infty}\left[\frac{1}{\sum_{r=1}^{k} r}\right]$$

$$= \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots$$

INTRODUCE A FINITE LIMIT FOR THE SUMMATION, SAY n

$$= \lim_{n\to\infty}\left[1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots + \frac{1}{1+2+3+\dots+n}\right]$$

$$= \lim_{n\to\infty}\left[\sum_{r=1}^{n} \frac{1}{\frac{1}{2}r(r+1)}\right]$$

$$= 2\lim_{n\to\infty}\left[\sum_{r=1}^{n} \frac{1}{r(r+1)}\right]$$

SPLIT INTO TWO FRACTIONS BY INSPECTION

$$= 2\lim_{n\to\infty}\left[\sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+1}\right]$$

$$= 2\lim_{n\to\infty}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$= 2\lim_{n\to\infty}\left[1-\frac{1}{n+1}\right]$$

$$= 2$$

Question 143 (*****)

The first three terms of a series S are

$$S = 7 + 9x + 8x^2 + \ldots$$

The n^{th} term of S is given by

$$A\left(\tfrac{3}{4}x\right)^n + B\left(\tfrac{1}{3}x\right)^n,$$

where A and B are non zero constants.

Given that the sum to infinity of S is 19, determine the value of x.

SP-F, $x = \frac{12}{19}$

$S = 7 + 9x + 8x^2 + \ldots$ $\quad \left[A\left(\frac{3}{4}\right)^n + B\left(\frac{1}{3}\right)^n\right]x^n$

- If $n=0$: $(A+B)x^0 = 7$
- If $n=1$: $\left(\frac{3}{4}A + \frac{1}{3}B\right)x^1 = 9$

$A + B = 7$ ($\times 9$), $\frac{3}{4}A + \frac{1}{3}B = 9$ ($\times 12$)

$9A + 9B = 63$
$9A + 4B = 108$
$5B = -45$
$B = -9$
$A = 16$

- NOW THE SERIES IS

$S = \sum_{n=0}^{\infty}\left[16\left(\frac{3}{4}\right)^n - 9\left(\frac{1}{3}\right)^n\right]x^n$

$S = 16\sum_{n=0}^{\infty}\left[\left(\frac{3}{4}\right)^n x^n\right] - 9\sum_{n=0}^{\infty}\left[\left(\frac{1}{3}\right)^n x^n\right]$

$S = 16\sum_{n=0}^{\infty}\left(\frac{3}{4}x\right)^n - 9\sum_{n=0}^{\infty}\left(\frac{1}{3}x\right)^n$

$S = 16\left[1 + \frac{3}{4}x + \frac{9}{16}x^2 + \frac{27}{64}x^3 + \ldots\right] - 9\left[1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3 + \ldots\right]$

$S = 16 \times \left(\frac{1}{1 - \frac{3}{4}x}\right) - 9\left(\frac{1}{1 - \frac{1}{3}x}\right)$

NOW THE SUM TO INFINITY IS 19

$\Rightarrow \frac{16}{1 - \frac{3}{4}x} - \frac{9}{1 - \frac{1}{3}x} = 19$

$\Rightarrow \frac{64}{4 - 3x} - \frac{27}{3 - x} = 19$

$\Rightarrow 64(3-x) - 27(4-3x) = 19(3-x)(4-3x)$

$\Rightarrow 192 - 64x - 108 + 81x = 19(3x-4)(x-3)$

$\Rightarrow 84 + 17x = 19(3x^2 - 13x + 12)$

$\Rightarrow 84 + 17x = 57x^2 - 19 \times 13x + 19 \times 12$

$\Rightarrow 0 = 57x^2 - 247x - 17x + 228 - 84$

$\Rightarrow 0 = 57x^2 - 264x + 144$

$\Rightarrow 0 = 19x^2 - 88x + 48$

$\Rightarrow 0 = (19x - 12)(x - 4)$

$\Rightarrow x = 4$ or $\frac{12}{19}$

BUT IN ORDER TO CONVERGE $\left|\frac{3}{4}x\right| < 1$ & $\left|\frac{1}{3}x\right| < 1$

$|x| < \frac{4}{3}$ & $|x| < 3$

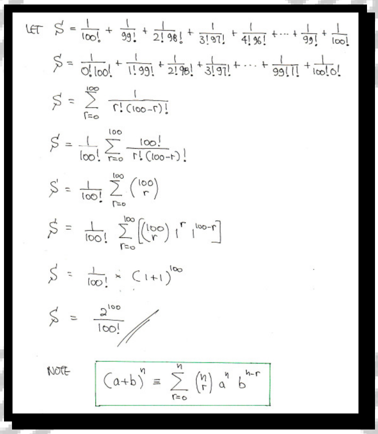
$\therefore x = \frac{12}{19}$ (AS 4 IS GREATER THAN $\frac{4}{3}$ OR EVEN 3)

Question 144 (*****)

Find a simplified expression for the following sum

$$\frac{1}{100!}+\frac{1}{99!}+\frac{1}{2!98!}+\frac{1}{3!97!}+\frac{1}{4!96!}+\ldots+\frac{1}{2!98!}+\frac{1}{99!}+\frac{1}{100!}$$

SP-Y , $\boxed{\frac{2^{100}}{100!}}$

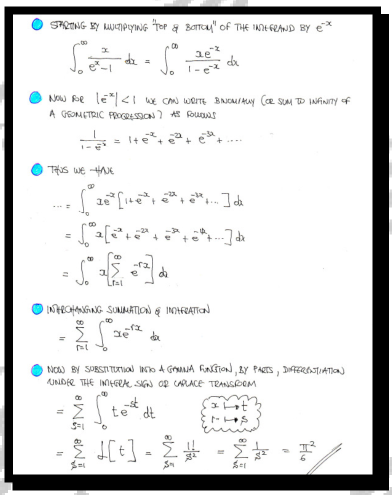


Question 145 (*****)

Show by detailed workings that

$$\int_0^{\infty} \frac{x}{e^x-1}\,dx = \frac{\pi^2}{6}$$

proof



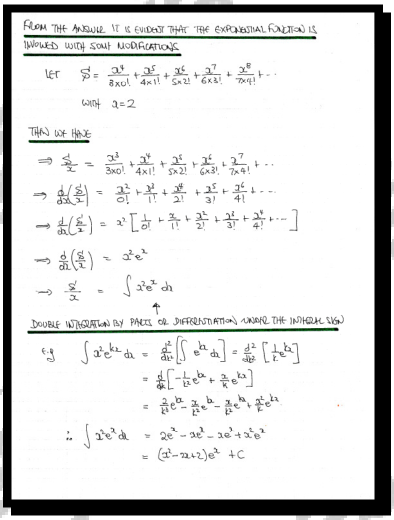
Question 146 (*****)

Consider the following convergent infinite series.

$$\sum_{r=0}^{\infty} \frac{2^{r+4}}{(r+3)\,r!}$$

Use appropriate techniques to show that the sum to infinity of the above series is $4\left(e^2-1\right)$

[ ], [proof]



FROM THE ANSWER IT IS EVIDENT THAT THE EXPONENTIAL FUNCTION IS INVOLVED WITH SOME MODIFICATIONS

LET $S = \frac{x^4}{3\times 0!} + \frac{x^5}{4\times 1!} + \frac{x^6}{5\times 2!} + \frac{x^7}{6\times 3!} + \frac{x^8}{7\times 4!} + \dots$ WITH $x=2$

THEN WE HAVE

$\Rightarrow \frac{S}{x} = \frac{x^3}{3\times 0!} + \frac{x^4}{4\times 1!} + \frac{x^5}{5\times 2!} + \frac{x^6}{6\times 3!} + \frac{x^7}{7\times 4!} + \dots$

$\Rightarrow \frac{d}{dx}\left(\frac{S}{x}\right) = \frac{x^2}{0!} + \frac{x^3}{1!} + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \dots$

$\Rightarrow \frac{d}{dx}\left(\frac{S}{x}\right) = x^2\left[\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right]$

$\Rightarrow \frac{d}{dx}\left(\frac{S}{x}\right) = x^2 e^x$

$\Rightarrow \frac{S}{x} = \int x^2 e^x \, dx$

DOUBLE INTEGRATION BY PARTS OR DIFFERENTIATION UNDER THE INTEGRAL SIGN

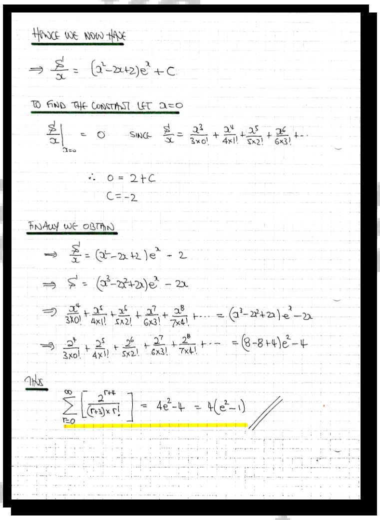
e.g. $\int x^2 e^{kx}\,dx = \frac{d^2}{dk^2}\left[\int e^{kx}\,dx\right] = \frac{d^2}{dk^2}\left[\frac{1}{k}e^{kx}\right]$

$= \frac{d}{dk}\left[-\frac{1}{k^2}e^{kx} + \frac{x}{k}e^{kx}\right]$

$= \frac{2}{k^3}e^{kx} - \frac{x}{k^2}e^{kx} - \frac{x}{k^2}e^{kx} + \frac{x^2}{k}e^{kx}$

$\therefore \int x^2 e^x\,dx = 2e^x - xe^x - xe^x + x^2e^x$

$= (x^2-2x+2)e^x + C$



HENCE WE NOW HAVE

$\Rightarrow \frac{S}{x} = (x^2-2x+2)e^x + C$

TO FIND THE CONSTANT LET $x=0$

$\left.\frac{S}{x}\right|_{x=0} = 0$ SINCE $\frac{S}{x} = \frac{x^3}{3\times 0!} + \frac{x^4}{4\times 1!} + \frac{x^5}{5\times 2!} + \frac{x^6}{6\times 3!} + \dots$

$\therefore 0 = 2 + C$

$C = -2$

FINALLY WE OBTAIN

$\Rightarrow \frac{S}{x} = (x^2-2x+2)e^x - 2$

$\Rightarrow S = (x^3-2x^2+2x)e^x - 2x$

$\Rightarrow \frac{x^4}{3\times 0!} + \frac{x^5}{4\times 1!} + \frac{x^6}{5\times 2!} + \frac{x^7}{6\times 3!} + \frac{x^8}{7\times 4!} + \dots = (x^3-2x^2+2x)e^x - 2x$

$\Rightarrow \frac{2^4}{3\times 0!} + \frac{2^5}{4\times 1!} + \frac{2^6}{5\times 2!} + \frac{2^7}{6\times 3!} + \frac{2^8}{7\times 4!} + \dots = (8-8+4)e^2 - 4$

THUS

$\sum_{r=0}^{\infty}\left[\frac{2^{r+4}}{(r+3)\times r!}\right] = 4e^2 - 4 = 4(e^2-1)$

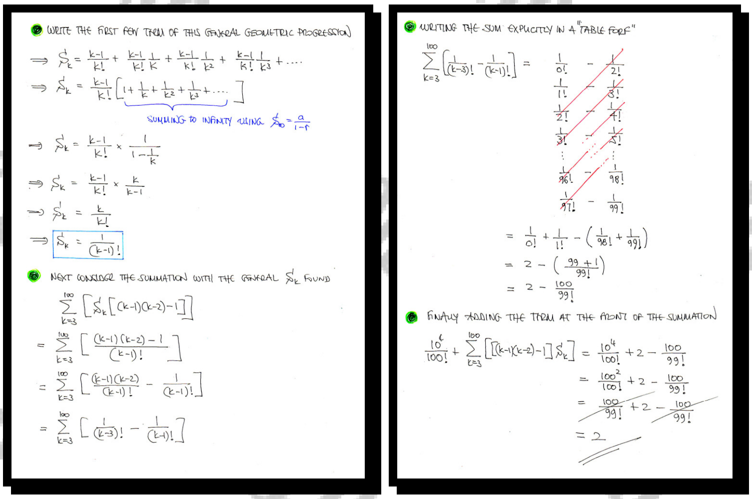
Question 147 (*****)

A family of infinite geometric series S_k, has first term $\frac{k-1}{k!}$ and common ratio $\frac{1}{k}$, where $k = 3,4,5,6,\ldots,99,100$.

Find the value of

$$\frac{10^4}{100!} + \sum_{k=3}^{100}\left[\left[(k-1)(k-2)-1\right]S_k\right].$$

$\boxed{}$, $\boxed{2}$



Question 148 (*****)

A discrete random variable X is geometrically distributed with parameter p.

Show that …

a) … $\mathrm{E}(X) = \dfrac{1}{p}$.

b) … $\mathrm{E}(X) = \dfrac{1-p}{p^2}$.

[ ], [proof]

a) LET $X \sim \text{Geo}(p)$ $\quad 0 < p < 1$

x	1	2	3	4	5	....
$P(X=x)$	p	$(1-p)p$	$(1-p)^2p$	$(1-p)^3p$	$(1-p)^4p$	....

$$E(X) = p + 2p(1-p) + 3p(1-p)^2 + 4p(1-p)^3 + 5p(1-p)^4 + \dots$$
$$E(X) = p\left[1 + 2(1-p) + 3(1-p)^2 + 4p(1-p)^3 + 5p(1-p)^4 + \dots\right.$$

MULTIPLY THE ABOVE LINE BY $-(1-p)$

$$-(1-p)E(X) = p\left[\quad -(1-p) - 2(1-p)^2 - 3(1-p)^3 - 4(1-p)^4 - \dots\right]$$
$$E(X) = p\left[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + 5(1-p)^4 + \dots\right]$$

ADDING THE TWO LINES ABOVE WE OBTAIN

$$\left[-(1-p)+1\right]E(X) = p\left[1 + (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 + \dots\right]$$
$$\cancel{p}\,E(X) = \cancel{p}\left[1 + (1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 + \dots\right]$$
$$E(X) = 1 + (1-p) + (1-p) + (1-p)^3 + (1-p)^4 + \dots$$

THIS IS A GEOMETRIC PROGRESSION WITH $a = 1$, $r = 1-p$, $S_\infty = \frac{a}{1-r}$

$$E(X) = \frac{1}{1-(1-p)} \quad \leftarrow a, \ \leftarrow 1-r$$
$$E(X) = \frac{1}{p} \quad \text{// AS REQUIRED}$$

ALTERNATIVE VARIATION

HAVING FORMED AN EXPRESSION FOR EXPECTATION AS BEFORE

$$E(X) = p\left[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \dots\right]$$

LET $q = 1-p$

$$\Rightarrow E(X) = (1-q)\left[1 + 2q + 3q^2 + 4q^3 + \dots\right]$$
$$\Rightarrow E(X) = (1-q)\frac{d}{dq}\left[q + q^2 + q^3 + q^4 + \dots\right]$$

CONVERGENT G.P WITH $a = q$, $r = q^2$, $S_\infty = \frac{a}{1-r}$

$$\Rightarrow E(X) = (1-q)\frac{d}{dq}\left[\frac{q}{1-q}\right]$$
$$\Rightarrow E(X) = (1-q) \times \frac{(1-q)\times 1 - q(-1)}{(1-q)^2}$$
$$\Rightarrow E(X) = (1-q) \times \frac{1}{(1-q)^2}$$
$$\Rightarrow E(X) = \frac{1}{1-q}$$
$$\Rightarrow E(X) = \frac{1}{p} \quad //$$

b) • NEXT THE VARIANCE

$$E(X^2) = \left[1^2 \times p\right] + \left[2^2 \times p(1-p)\right] + \left[3^2 \times p(1-p)^2\right] + \left[4^2 \times p \times (1-p)^3\right] + \dots$$
$$E(X^2) = p\left[1 + 4(1-p) + 9(1-p)^2 + 16(1-p)^3 + 25(1-p)^4 + \dots\right]$$

• MULTIPLY THE EXPRESSION BY $-(1-p)$

$$-(1-p)E(X^2) = p\left[-(1-p) - 4(1-p)^2 - 9(1-p)^3 - 16(1-p)^4 - 25(1-p)^5 - \dots\right]$$
$$E(X^2) = p\left[1 + 4(1-p) + 9(1-p)^2 + 16(1-p)^3 + 25(1-p)^4 + 36(1-p)^5 + \dots\right]$$

• ADDING THE TWO LINES ABOVE

$$\left[1-(1-p)\right]E(X^2) = p\left[1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + 9(1-p)^4 + \dots\right]$$
$$\cancel{p}\,E(X^2) = \cancel{p}\left[1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + 9(1-p)^4 + \dots\right]$$
$$E(X^2) = 1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + 9(1-p)^4 + \dots$$

• MULTIPLY THE ABOVE LINE AGAIN BY $-(1-p)$

$$-(1-p)E(X^2) = \quad -(1-p) - 3(1-p)^2 - 5(1-p)^3 - 7(1-p)^4 - 9(1-p)^5 \dots$$
$$E(X^2) = 1 + 3(1-p) + 5(1-p)^2 + 7(1-p)^3 + 9(1-p)^4 + 11(1-p)^5 \dots$$

• ADDING AGAIN THE TWO LINES ABOVE

$$\left[1-(1-p)\right]E(X^2) = 1 + 2(1-p) + 2(1-p)^2 + 2(1-p)^3 + 2(1-p)^4 + \dots$$
$$p\,E(X^2) = 1 + 2\left[(1-p) + (1-p)^2 + (1-p)^3 + (1-p)^4 + \dots\right]$$

THIS IS A CONVERGENT G.P: $a = 1-p$, $r = 1-p$, $S_\infty = \frac{a}{1-r}$

$$p\,E(X^2) = 1 + 2 \times \frac{1-p}{1-(1-p)}$$

• HENCE, FURTHER TIDYING UP

$$p\,E(X^2) = 1 + \frac{2(1-p)}{p}$$
$$p^2E(X^2) = p + 2 - 2p$$
$$p^2E(X^2) = 2 - p$$
$$E(X^2) = \frac{2-p}{p^2}$$

• FINALLY THE VARIANCE

$$\text{Var}(X) = E(X^2) - \left[E(X)\right]^2$$
$$= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2$$
$$= \frac{2-p}{p^2} - \frac{1}{p^2}$$
$$= \frac{1-p}{p^2} \quad //$$

Question 149 (*****)

Find the sum to infinity of the following convergent series

$$1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\frac{5^3}{5!}+\frac{6^3}{6!}+\dots$$

$\boxed{5e}$

$$1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\frac{5^3}{5!}+\dots=?$$

- $S = 1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\frac{5^3}{5!}+\dots = \sum_{n=1}^{\infty}\frac{n^3}{n!}$ CANCEL DOWN n

$$= \sum_{n=1}^{\infty}\frac{n^2}{(n-1)!} = \sum_{n=0}^{\infty}\frac{(n+1)^2}{n!}$$

- NOW EXPAND THE NUMERATOR – SPLIT THE FRACTION – CANCEL DOWN AND READJUST THE DUMMY VARIABLE

$$\Rightarrow S = \sum_{n=0}^{\infty}\frac{n^2+2n+1}{n!} = \left[\sum_{n=0}^{\infty}\frac{n^2}{n!}\right]+\left[2\sum_{n=0}^{\infty}\frac{n}{n!}\right]+\left[\sum_{n=0}^{\infty}\frac{1}{n!}\right]$$

FIRST TERM ZERO; $\sum_{n=0}^{\infty}\frac{1}{n!} \to e$

$$\Rightarrow S = \left[\sum_{n=1}^{\infty}\frac{n^2}{n!}\right]+\left[2\sum_{n=1}^{\infty}\frac{n}{n!}\right]+e$$

$$\Rightarrow S = \left[\sum_{n=1}^{\infty}\frac{n}{(n-1)!}\right]+2\left[\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\right]+e$$

$$\Rightarrow S = \left[\sum_{n=0}^{\infty}\frac{n+1}{n!}\right]+2\left[\sum_{n=0}^{\infty}\frac{1}{n!}\right]+e$$

($\sum_{n=0}^{\infty}\frac{1}{n!} \to e$)

- REPEAT THE PROCESS WITH THE FIRST SUM

$$\Rightarrow S = \left[\sum_{n=0}^{\infty}\frac{n}{n!}\right]+\left[\sum_{n=0}^{\infty}\frac{1}{n!}\right]+2e+e$$

FIRST TERM ZERO; $\sum_{n=0}^{\infty}\frac{1}{n!} \to e$

$$\Rightarrow S = \left[\sum_{n=1}^{\infty}\frac{n}{n!}\right]+4e$$

$$\Rightarrow S = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}+4e$$

$$\Rightarrow S = \sum_{n=0}^{\infty}\frac{1}{n!}+4e$$

$$\Rightarrow S = e+4e$$

$$\therefore\ 1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\frac{5^3}{5!}+\dots = 5e$$

Question 150 (*****)

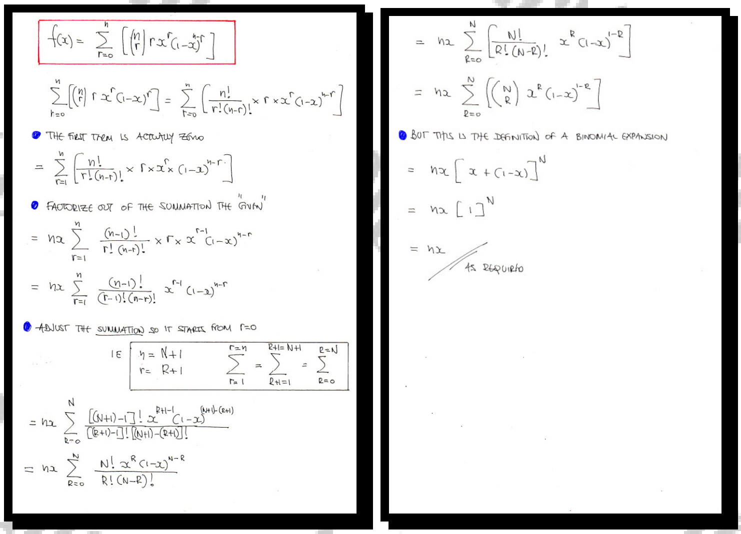
The function f is defined as

$$f(n,x) \equiv \sum_{r=0}^{n} \binom{n}{r} r\, x^{r} (1-x)^{n-r},$$

where $n \in \mathbb{N}$, $x \in \mathbb{R}$, $0 < x < 1$.

Show that $f(n,x) \equiv nx$.

[ ], [proof]



Question 151 (*****)

The binomial probability distribution $X \sim \mathrm{B}(n, p)$ satisfies

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r},$$

where $r = 0, 1, 2, 3, \ldots, n$ and $0 < p < 1$.

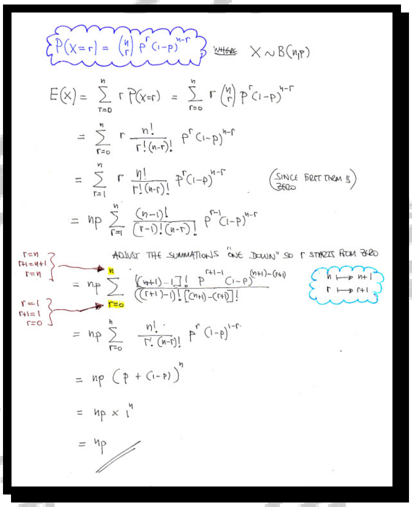
The expectation of X is defined as

$$\mathrm{E}(X) \equiv \sum_{r=0}^{n} \left[r\,\mathrm{P}(X = r) \right]$$

Show that

$$\mathrm{E}(X) = np.$$

[ ] , [proof]


$P(X=r) = \binom{n}{r} p^r (1-p)^{n-r}$ where $X \sim B(n,p)$

$E(X) = \sum_{r=0}^{n} r P(X=r) = \sum_{r=0}^{n} r \binom{n}{r} p^r (1-p)^{n-r}$

$= \sum_{r=0}^{n} r \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}$

$= \sum_{r=1}^{n} r \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}$ (SINCE FIRST TERM IS ZERO)

$= np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!(n-r)!} p^{r-1} (1-p)^{n-r}$

ADJUST THE SUMMATIONS "ONE DOWN" SO r STARTS FROM ZERO

$= np \sum_{r=0}^{n} \frac{[(n+1)-1]!}{[(r+1)-1]![(n+1)-(r+1)]!} p^{r+1-1} (1-p)^{(n+1)-(r+1)}$ ($n \mapsto n+1$, $r \mapsto r+1$)

$= np \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}$

$= np (p + (1-p))^n$

$= np \times 1^n$

$= np$

Question 152 (*****)

The function f is defined in terms of the real constants, a, b and c, by

$$f(x) = \left(a + bx + cx^2\right)(1-x)^{-3}, \quad x \in \mathbb{R}, \quad |x| < 1.$$

a) Show that

$$f(x) = a + (3a+b)x + \frac{1}{2}\sum_{n=2}^{\infty}\Big[\big[a(n+1)(n+2) + bn(n+1) + cn(n-1)\big]x^n\Big].$$

b) Use the expression of part **(a)** to deduce the value of

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

$\boxed{}$, $\boxed{6}$

a) • STARTING WITH THE BINOMIAL EXPANSION OF $(1-x)^{-3}$

$$\Rightarrow (1-x)^{-3} = 1 + \frac{-3}{1}(-x)^1 + \frac{(-3)(-4)}{1\times2}(-x)^2 + \frac{(-3)(-4)(-5)}{1\times2\times3}(-x)^3 + \frac{(-3)(-4)(-5)(-6)}{1\times2\times3\times4}(-x)^4 + \dots$$

$$\Rightarrow (1-x)^{-3} = 1 + \frac{3}{1}x + \frac{3\times4}{1\times2}x^2 + \frac{3\times4\times5}{1\times2\times3}x^3 + \frac{3\times4\times5\times6}{1\times2\times3\times4}x^4 + \dots$$

$$\dots + \frac{3\times4\times5\times6\times\dots(n+2)}{1\times2\times3\times4\times\dots n}x^n$$

• THUS THE COEFFICIENT OF x^n IS

$$\frac{1}{2}\left(\frac{2\times3\times5\times\dots\times(n+2)}{n!}\right) = \frac{1}{2}\frac{(n+2)!}{n!} = \frac{1}{2}\frac{(n+2)(n+1)\times n!}{n!}$$

$$(1-x)^{-3} = \sum_{n=0}^{\infty}\tfrac{1}{2}(n+1)(n+2)x^n$$

• THUS LOOKING NOW AT $f(x)$

$$\Rightarrow f(x) = (a+bx+cx^2)(1-x)^{-3} = (a+bx+cx^2)\sum_{n=0}^{\infty}\tfrac{1}{2}(n+1)(n+2)x^n$$

$$\Rightarrow f(x) = \tfrac{1}{2}a\sum_{n=0}^{\infty}(n+1)(n+2)x^n + \tfrac{1}{2}b\sum_{n=0}^{\infty}(n+1)(n+2)x^{n+1} + \tfrac{1}{2}c\sum_{n=0}^{\infty}(n+1)(n+2)x^{n+2}$$

$$\Rightarrow f(x) = \tfrac{1}{2}a\times2\times x^0 + \tfrac{1}{2}b\times1\times2\times x^1$$

$$+ \tfrac{1}{2}a\times2\times3\times x^1$$

$$+ \tfrac{1}{2}a\sum_{n=2}^{\infty}(n+1)(n+2)x^n + \tfrac{1}{2}b\sum_{n=1}^{\infty}(n+1)(n+2)x^{n+1} + \tfrac{1}{2}c\sum_{n=0}^{\infty}(n+1)(n+2)x^{n+2}$$

• $f(x) = (a+bx+cx^2)(1-x)^{-3}$

$$= a + (b+3a)x + \tfrac{1}{2}a\sum_{n=2}^{\infty}(n+1)(n+2)x^n$$

$$+ \tfrac{1}{2}b\sum_{n=1}^{\infty}(n+1)(n+2)x^{n+1}$$

$$+ \tfrac{1}{2}c\sum_{n=0}^{\infty}(n+1)(n+2)x^{n+2}$$

• "ADJUST" THE SUMMATIONS SO THEY ALL START FROM $n=2$

$$\Rightarrow f(x) = a + (3a+b)x + \tfrac{1}{2}a\sum_{n=2}^{\infty}(n+1)(n+2)x^n$$

$$+ \tfrac{1}{2}b\sum_{n=2}^{\infty}n(n+1)x^n$$

$$+ \tfrac{1}{2}c\sum_{n=2}^{\infty}(n-1)(n)x^n$$

$$\Rightarrow f(x) = a + (3a+b)x + \frac{1}{2}\sum_{n=2}^{\infty}\big[a(n+1)(n+2) + bn(n+1) + cn(n-1)\big]x^n$$

b) • NOW LOOKING AT THE COEFFICIENT OF x^n (MULTIPLIED THE $\frac{1}{2}$ INTO THE SUM)

$$\begin{pmatrix}\tfrac{1}{2}an^2 + \tfrac{3}{2}an + a\\ \tfrac{1}{2}bn^2 + \tfrac{1}{2}bn\\ \tfrac{1}{2}cn^2 - \tfrac{1}{2}cn\end{pmatrix}x^n$$

LOOK FOR THIS TO REDUCE TO n^2x^n

$\therefore \boxed{a=0}$ $\quad \tfrac{1}{2}b - \tfrac{1}{2}c = 0$, $\boxed{b=c}$ $\quad \tfrac{1}{2}b + \tfrac{1}{2}c = 1$, $\tfrac{1}{2}b + \tfrac{1}{2}b = 1$, $\boxed{b=c=1}$

$$\Rightarrow f(x) = (a+bx+cx^2)(1-x)^{-3} = a + (3a+b)x + \frac{1}{2}\sum_{n=2}^{\infty}\big[a(n+1)(n+2) + bn(n+1) + cn(n-1)\big]x^n$$

• LET $a=0$, $b=1$, $c=1$

$$\Rightarrow f(x) = (x+x^2)(1-x)^{-3} = x + \sum_{n=2}^{\infty}n^2x$$

$$\Rightarrow f(\tfrac{1}{2}) = \left(\tfrac{1}{2}+\tfrac{1}{4}\right)\left(1-\tfrac{1}{2}\right)^{-3} = \tfrac{1}{2} + \sum_{n=2}^{\infty}n^2\left(\tfrac{1}{2}\right)^n$$

$$\Rightarrow \tfrac{3}{4}\times\left(\tfrac{1}{2}\right)^{-3} = \tfrac{1}{2} + \sum_{n=2}^{\infty}\frac{n^2}{2^n}$$

$$\Rightarrow \tfrac{3}{4}\times8 = \tfrac{1}{2} + \sum_{n=2}^{\infty}\frac{n^2}{2^n}$$

$$\Rightarrow 6 = \sum_{n=1}^{\infty}\frac{n^2}{2^n} \quad \left(\text{AS } \frac{1^2}{2^1} = \frac{1}{2}\right)$$

$$\therefore \sum_{n=1}^{\infty}\frac{n^2}{2^n} = 6$$

Question 153 (*****)

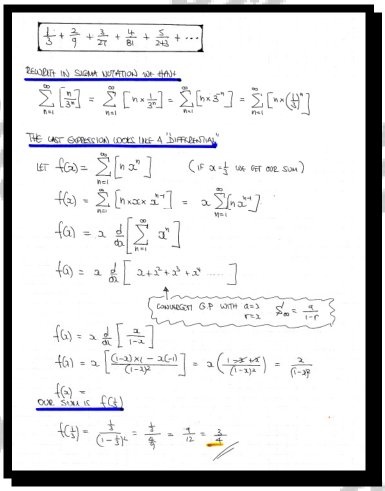
The function f is defined by

$$f(x) \equiv \sum_{n=1}^{\infty}\left[nx^{n}\right], \quad x \in \mathbb{R}, \quad |x|<1.$$

Use the above function to find the sum to infinity of the following series.

$$\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\frac{5}{243}+\ldots$$

$\boxed{}$, $\boxed{\frac{3}{4}}$



$\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\frac{5}{243}+\ldots$

REWRITE IN SIGMA NOTATION WE HAVE

$$\sum_{n=1}^{\infty}\left[\frac{n}{3^n}\right]=\sum_{n=1}^{\infty}\left[n\times\frac{1}{3^n}\right]=\sum_{n=1}^{\infty}\left[n\times 3^{-n}\right]=\sum_{n=1}^{\infty}\left[n\times\left(\frac{1}{3}\right)^n\right]$$

THE LAST EXPRESSION LOOKS LIKE A "DIFFERENTIAL"

LET $f(x)=\sum_{n=1}^{\infty}\left[n x^n\right]$ (IF $x=\frac{1}{3}$ WE GET OUR SUM)

$$f(x)=\sum_{n=1}^{\infty}\left[n\times x\times x^{n-1}\right]=x\sum_{n=1}^{\infty}\left[n x^{n-1}\right]$$

$$f(x)=x\frac{d}{dx}\left[\sum_{n=1}^{\infty}x^n\right]$$

$$f(x)=x\frac{d}{dx}\left[x+x^2+x^3+x^4\ldots\right]$$

CONVERGENT G.P WITH $a=x$, $r=x$, $S_\infty=\frac{a}{1-r}$

$$f(x)=x\frac{d}{dx}\left[\frac{x}{1-x}\right]$$

$$f(x)=x\left[\frac{(1-x)\times 1-x(-1)}{(1-x)^2}\right]=x\left(\frac{1-x+x}{(1-x)^2}\right)=\frac{x}{(1-x)^2}$$

OUR SUM IS $f\left(\frac{1}{3}\right)$

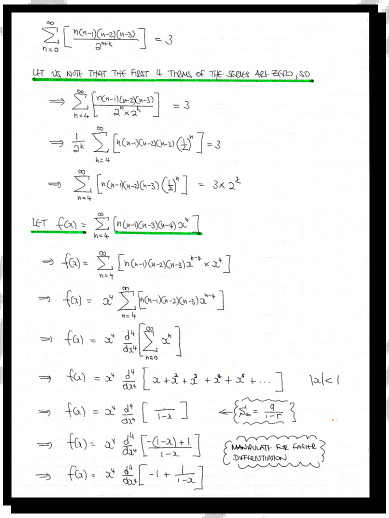
$$f\left(\tfrac{1}{3}\right)=\frac{\frac{1}{3}}{\left(1-\frac{1}{3}\right)^2}=\frac{\frac{1}{3}}{\frac{4}{9}}=\frac{9}{12}=\frac{3}{4}$$

Question 154 (*****)

Find the value of $x \in \mathbb{R}$ in the following equation

$$\sum_{n=0}^{\infty}\left[\frac{n(n-1)(n-2)(n-3)}{2^{n+k}}\right] = 3.$$

$\boxed{}$, $\boxed{k=4}$



$$\sum_{n=0}^{\infty}\left[\frac{n(n-1)(n-2)(n-3)}{2^{n+k}}\right] = 3$$

LET US NOTE THAT THE FIRST 4 TERMS OF THE SERIES ARE ZERO, SO

$$\Rightarrow \sum_{n=4}^{\infty}\left[\frac{n(n-1)(n-2)(n-3)}{2^n \times 2^k}\right] = 3$$

$$\Rightarrow \frac{1}{2^k}\sum_{n=4}^{\infty}\left[n(n-1)(n-2)(n-3)\left(\tfrac{1}{2}\right)^n\right] = 3$$

$$\Rightarrow \sum_{n=4}^{\infty}\left[n(n-1)(n-2)(n-3)\left(\tfrac{1}{2}\right)^n\right] = 3\times 2^k$$

LET $f(x) = \sum_{n=4}^{\infty}\left[n(n-1)(n-3)(n-4)x^n\right]$

$$\Rightarrow f(x) = \sum_{n=4}^{\infty}\left[n(n-1)(n-2)(n-3)x^{n-4}\times x^4\right]$$

$$\Rightarrow f(x) = x^4\sum_{n=4}^{\infty}\left[n(n-1)(n-2)(n-3)x^{n-4}\right]$$

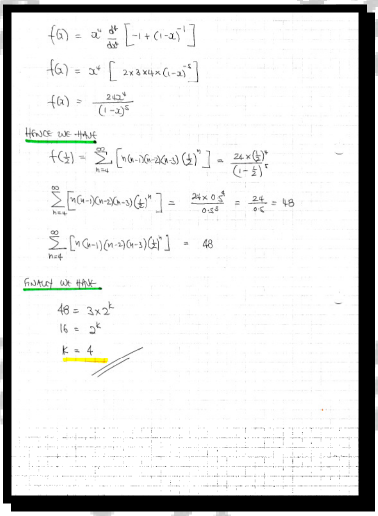
$$\Rightarrow f(x) = x^4\frac{d^4}{dx^4}\left[\sum_{n=0}^{\infty}x^n\right]$$

$$\Rightarrow f(x) = x^4\frac{d^4}{dx^4}\left[x + x^2 + x^3 + x^4 + x^5 + \dots\right] \quad |x|<1$$

$$\Rightarrow f(x) = x^4\frac{d^4}{dx^4}\left[\frac{1}{1-x}\right] \quad \leftarrow \left\{S_\infty = \frac{a}{1-r}\right\}$$

$$\Rightarrow f(x) = x^4\frac{d^4}{dx^4}\left[\frac{-(1-x)+1}{1-x}\right] \quad \left\{\text{MANIPULATE FOR EASIER DIFFERENTIATION}\right\}$$

$$\Rightarrow f(x) = x^4\frac{d^4}{dx^4}\left[-1+\frac{1}{1-x}\right]$$



$$f(x) = x^4\frac{d^4}{dx^4}\left[-1+(1-x)^{-1}\right]$$

$$f(x) = x^4\left[2\times3\times4\times(1-x)^{-5}\right]$$

$$f(x) = \frac{24x^4}{(1-x)^5}$$

HENCE WE HAVE

$$f\left(\tfrac{1}{2}\right) = \sum_{n=4}^{\infty}\left[n(n-1)(n-2)(n-3)\left(\tfrac{1}{2}\right)^n\right] = \frac{24\times\left(\frac{1}{2}\right)^4}{\left(1-\frac{1}{2}\right)^5}$$

$$\sum_{n=4}^{\infty}\left[n(n-1)(n-2)(n-3)\left(\tfrac{1}{2}\right)^n\right] = \frac{24\times0.5^4}{0.5^5} = \frac{24}{0.5} = 48$$

$$\sum_{n=4}^{\infty}\left[n(n-1)(n-2)(n-3)\left(\tfrac{1}{2}\right)^n\right] = 48$$

FINALLY WE HAVE

$$48 = 3\times2^k$$

$$16 = 2^k$$

$$k = 4$$

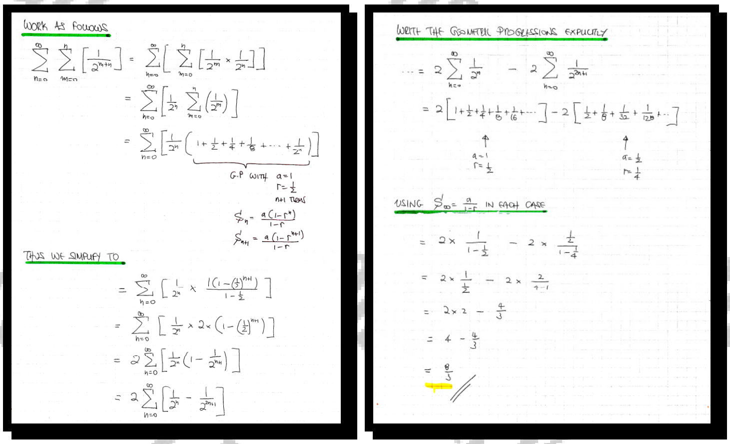
Question 155 (*****)

Evaluate the following expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\frac{1}{2^{m+n}}\right].$$

Detailed workings must be shown.

$\boxed{}$, $\boxed{\frac{8}{3}}$



WORK AS FOLLOWS

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\frac{1}{2^{m+n}}\right] = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\left[\frac{1}{2^m}\times\frac{1}{2^n}\right]\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\sum_{m=0}^{n}\left(\frac{1}{2^m}\right)\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^n}\right)\right]$$

G.P WITH $a=1$, $r=\frac{1}{2}$, $n+1$ TERMS

$S_n = \frac{a(1-r^n)}{1-r}$

$S_{n+1} = \frac{a(1-r^{n+1})}{1-r}$

THUS WE SIMPLIFY TO

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\times\frac{1\left(1-\left(\frac{1}{2}\right)^{n+1}\right)}{1-\frac{1}{2}}\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\times 2\times\left(1-\left(\frac{1}{2}\right)^{n+1}\right)\right]$$

$$= 2\sum_{n=0}^{\infty}\left[\frac{1}{2^n}\left(1-\frac{1}{2^{n+1}}\right)\right]$$

$$= 2\sum_{n=0}^{\infty}\left[\frac{1}{2^n}-\frac{1}{2^{2n+1}}\right]$$

WRITE THE GEOMETRIC PROGRESSIONS EXPLICITLY

$$\dots = 2\sum_{n=0}^{\infty}\frac{1}{2^n} - 2\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}$$

$$= 2\left[1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots\right] - 2\left[\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}+\dots\right]$$

$a=1$, $r=\frac{1}{2}$ $\qquad$ $a=\frac{1}{2}$, $r=\frac{1}{4}$

USING $S_\infty = \frac{a}{1-r}$ IN EACH CASE

$$= 2\times\frac{1}{1-\frac{1}{2}} - 2\times\frac{\frac{1}{2}}{1-\frac{1}{4}}$$

$$= 2\times\frac{1}{\frac{1}{2}} - 2\times\frac{2}{4-1}$$

$$= 2\times 2 - \frac{4}{3}$$

$$= 4 - \frac{4}{3}$$

$$= \frac{8}{3}$$

Question 156 (*****)

It is given that for $x \in \mathbb{R}$, $-\frac{1}{k} < x < \frac{1}{k}$, $k > 0$,

$$f(x,k) \equiv \frac{k+1}{(1-x)(1+kx)}.$$

Given further that

$$f(x,k) \equiv \sum_{r=0}^{\infty}\left[a_r x^r\right],$$

where a_r are functions of k, show that

$$\sum_{r=0}^{\infty}\left[{a_r}^2 x^r\right] = \frac{(1-kx)(1+k)^2}{(1-x)(1+kx)(1-k^2x)}.$$

You may assume that $\sum_{r=0}^{\infty}\left[{a_r}^2 x^r\right]$ converges.

[ ], proof

$$f(x,k) = \frac{k+1}{(1-x)(1+kx)} \qquad -\frac{1}{k} < x < \frac{1}{k}, \quad k>0$$

- REWRITE & EXPAND BINOMIALLY

$$\Rightarrow f(x,k) = (k+1)(1-x)^{-1}(1+kx)^{-1} = (1+k)\left[1+x+x^2+x^3+x^4+O(x^5)\right]\left[1-kx+k^2x^2-k^3x^3+k^4x^4+O(x^5)\right]$$

$$(1-x)^{-1} = 1+x+x^2+x^3+x^4+\dots$$
$$(1+x)^{-1} = 1-x+x^2-x^3+x^4+\dots$$

- TIDYING UP THE EXPRESSION

$$\Rightarrow f(x,k) = (1+k)\begin{bmatrix} 1 - kx + k^2x^2 - k^3x^3 + k^4x^4 - \dots \\ x - kx^2 + k^2x^3 - k^3x^4 + \dots \\ x^2 - kx^3 + k^2x^4 - \dots \\ x^3 - kx^4 + \dots \\ x^4 - \dots \end{bmatrix}$$

$$\Rightarrow f(x,k) = (1+k)\left[1 + (1-k)x + (1-k+k^2)x^2 + (1-k+k^2-k^3)x^3 + (1-k+k^2-k^3+k^4)x^4 + \dots\right]$$

- USING THE IDENTITY $a^n - b^n \equiv (a^{n-1} - a^{n-2}b^1 + a^{n-3}b^2 - a^{n-4}b^3 + \dots\ b^n)(a-b)$

$$\Rightarrow f(x,k) = \overset{a_0}{(1+k)} + \overset{a_1}{(1-k^2)}x + \overset{a_2}{(1+k^3)}x^2 + \overset{a_3}{(1-k^4)}x^3 + \overset{a_4}{(1-k^5)}x^4 + O(x^5)$$

$$\Rightarrow f(x,k) = \sum_{r=0}^{\infty}\left[\left[1+k^{r+1}(-1)^r\right]x^r\right]$$

$$\Rightarrow f(x,k) = \sum_{r=0}^{\infty}\left[\left[1+k(-k)^r\right]x^r\right]$$

- NEXT CONSIDER THE REQUIRED SERIES

$$\Rightarrow g(x,k) = \sum_{r=0}^{\infty}\left[\left[1+k(-k)^r\right]^2 x^r\right] = \sum_{r=0}^{\infty}\left[\left[1+2k(-k)^r+k^2(-k)^{2r}\right]x^r\right]$$

$$\Rightarrow g(x,k) = \sum_{r=0}^{\infty}\left[x^r\right] + 2k\sum_{r=0}^{\infty}\left[(-k)^r x^r\right] + k^2\sum_{r=0}^{\infty}\left[(k^2)^r x^r\right]$$

$$\Rightarrow g(x,k) = \sum_{r=0}^{\infty}(x^r) + 2k\sum_{r=0}^{\infty}(-kx)^r + k^2\sum(k^2x)^r$$

- NEXT RECALLING THE STANDARD EXPANSIONS WE USED EARLIER

$$(1-x)^{-1} = 1+x+x^2+x^3+\dots = \sum_{r=0}^{\infty}x^r$$
$$(1+x)^{-1} = 1-x+x^2-x^3+\dots = \sum_{r=0}^{\infty}(-x)^r$$

$$\Rightarrow g(x,k) = \frac{1}{1-x} + \frac{2k}{1+kx} + \frac{k^2}{1-k^2x}$$

$$\Rightarrow g(x,k) = \frac{(1+kx)(1-k^2x) + 2k(1-x)(1-k^2x) + k^2(1-x)(1+kx)}{(1-x)(1+kx)(1-k^2x)}$$

$$\Rightarrow g(x,k) = \frac{\left\{\begin{array}{l} 1 + kx - k^2x - k^3x^2 \\ 2k - 2k^3x + 2k^3x^2 \\ \quad -2kx \\ k^2 + k^3x \\ \quad -k^2x - k^3x^2 \end{array}\right\}}{(1-x)(1+kx)(1-k^2x)}$$

$$\Rightarrow g(x,k) = \frac{(k^2+2k+1) - (k^3+2k^2+k)x}{(1-x)(1+kx)(1-k^2x)}$$

$$\Rightarrow g(x,k) = \frac{(k^2+2k+1) - (k^2+2k+1)kx}{(1-x)(1+kx)(1-k^2x)}$$

$$\Rightarrow g(x,k) = \frac{(k^2+2k+1)(1-kx)}{(1-x)(1+kx)(1-k^2x)}$$

$$\Rightarrow g(x,k) = \frac{(1+k)^2(1-kx)}{(1-x)(1+kx)(1-k^2x)}$$

Question 157 (*****)

It is given that

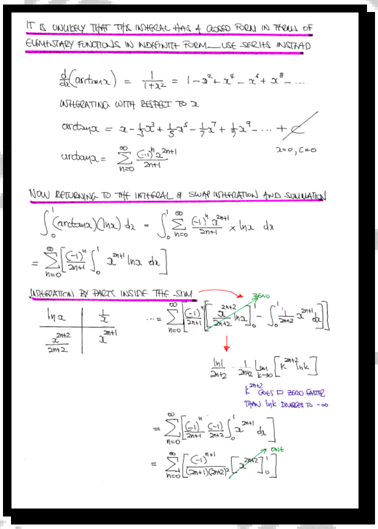
- $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots=\frac{1}{4}\pi$
- $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\quad\ldots=\frac{1}{12}\pi^2$
- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots=\ln 2$

Assuming the following integral converges find its exact value.

$$\int_0^1 (\ln x)(\arctan x)\ dx\ .$$

[*you may assume that integration and summation commute*]

$\boxed{}$, $\boxed{\frac{1}{48}\left[\pi^2-12\pi+24\ln 2\right]}$



IT IS UNLIKELY THAT THE INTEGRAL HAS A CLOSED FORM IN TERMS OF ELEMENTARY FUNCTIONS IN INDEFINITE FORM — USE SERIES INSTEAD

$$\frac{d}{dx}(\arctan x)=\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8-\ldots$$

INTEGRATING WITH RESPECT TO x

$$\arctan x = x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9-\ldots+C \qquad x=0,\ C=0$$

$$\arctan x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

NOW RETURNING TO THE INTEGRAL & SWAP INTEGRATION AND SUMMATION

$$\int_0^1(\arctan x)(\ln x)\,dx=\int_0^1\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\times\ln x\,dx$$

$$=\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2n+1}\int_0^1 x^{2n+1}\ln x\,dx\right]$$

INTEGRATION BY PARTS INSIDE THE SUM

$\ln x$	$\frac{1}{x}$
$\frac{x^{2n+2}}{2n+2}$	x^{2n+1}

$$\ldots=\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2n+1}\left[\left[\frac{x^{2n+2}}{2n+2}\ln x\right]_0^1-\int_0^1\frac{1}{2n+2}x^{2n+1}\,dx\right]\right]$$

ZERO

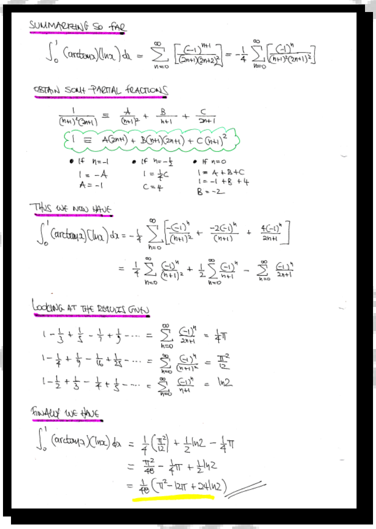
$$\frac{\ln 1}{2n+2}-\frac{1}{2n+2}\lim_{k\to 0}\left[k^{2n+2}\ln k\right]$$

k^{2n+2} GOES TO ZERO FASTER THAN $\ln k$ DIVERGES TO $-\infty$

$$=\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2n+1}\frac{(-1)}{2n+2}\int_0^1 x^{2n+1}\,dx\right]$$

$$=\sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{(2n+1)(2n+2)^2}\left[x^{2n+2}\right]_0^1\right]$$

ONE



SUMMARIZING SO FAR

$$\int_0^1(\arctan x)(\ln x)\,dx=\sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{(2n+1)(2n+2)^2}\right]=-\frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(n+1)^2(2n+1)}\right]$$

OBTAIN SOME PARTIAL FRACTIONS

$$\frac{1}{(n+1)^2(2n+1)}\equiv\frac{A}{(n+1)^2}+\frac{B}{n+1}+\frac{C}{2n+1}$$

$$1\equiv A(2n+1)+B(n+1)(2n+1)+C(n+1)^2$$

- IF $n=-1$: $1=-A$, $A=-1$
- IF $n=-\frac{1}{2}$: $1=\frac{1}{4}C$, $C=4$
- IF $n=0$: $1=A+B+C$, $1=-1+B+4$, $B=-2$

THUS WE NOW HAVE

$$\int_0^1(\arctan x)(\ln x)\,dx=-\frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{-(-1)^n}{(n+1)^2}+\frac{-2(-1)^n}{(n+1)}+\frac{4(-1)^n}{2n+1}\right]$$

$$=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}-\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

LOOKING AT THE RESULTS GIVEN

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\frac{1}{4}\pi$$

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}=\frac{\pi^2}{12}$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}=\ln 2$$

FINALLY WE HAVE

$$\int_0^1(\arctan x)(\ln x)\,dx=\frac{1}{4}\left(\frac{\pi^2}{12}\right)+\frac{1}{2}\ln 2-\frac{1}{4}\pi$$

$$=\frac{\pi^2}{48}-\frac{1}{4}\pi+\frac{1}{2}\ln 2$$

$$=\frac{1}{48}\left(\pi^2-12\pi+24\ln 2\right)$$

Question 158 (*****)

Given that p and q are positive, shown that the natural logarithm of their arithmetic mean exceeds the arithmetic mean of their natural logarithms by

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right].$$

You may find the series expansion of $\text{artanh}\left(x^2\right)$ useful in this question.

[ ] , [proof]

- STARTING FROM THE SERIES EXPANSION OF $\text{artanh}\,x$ IN LOG FORM

$$\Rightarrow \text{artanh}\,x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$$

$$\Rightarrow \text{artanh}\,x = \frac{1}{2}\left[\begin{array}{l} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \dots \\ -\left(x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} - \dots\right)\end{array}\right]$$

$$\Rightarrow \text{artanh}\,x = \frac{1}{2}\left[2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \dots\right]$$

$$\Rightarrow \text{artanh}\,x = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \dots$$

$$\Rightarrow \text{artanh}(x^2) = x^2 + \frac{1}{3}x^6 + \frac{1}{5}x^{10} + \frac{1}{7}x^{14} + \dots$$

$$\therefore \boxed{\text{artanh}(x^2) = \sum_{r=1}^{\infty}\left[\frac{x^{4r-2}}{2r-1}\right] = \frac{1}{2}\ln\left(\frac{1+x^2}{1-x^2}\right)}$$

- NEXT LET $x = \frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}$ IN THE ARGUMENT OF THE LOG ABOVE

$$\Rightarrow \frac{1+x^2}{1-x^2} = \frac{1 + \left[\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right]^2}{1 - \left[\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right]^2}$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY

$$\frac{1+x^2}{1-x^2} = \frac{(\sqrt{p}+\sqrt{q})^2 + (\sqrt{p}-\sqrt{q})^2}{(\sqrt{p}+\sqrt{q})^2 - (\sqrt{p}-\sqrt{q})^2}$$

$$\frac{1+x^2}{1-x^2} = \frac{p + 2\sqrt{pq} + q + p - 2\sqrt{pq} + q}{p + 2\sqrt{pq} + q - (p - 2\sqrt{pq} + q)}$$

$$\frac{1+x^2}{1-x^2} = \frac{2p+2q}{4\sqrt{pq}} = \frac{p+q}{2\sqrt{pq}}$$

- PUTTING ALL THE RESULTS TOGETHER

$$\sum_{r=1}^{\infty}\left[\frac{x^{4r-2}}{2r-1}\right] = \frac{1}{2}\ln\left[\frac{1+x^2}{1-x^2}\right]$$

$$\sum_{r=1}^{\infty}\left[\frac{1}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}-\sqrt{q}}\right)^{4r-2}\right] = \frac{1}{2}\ln\left(\frac{p+q}{2\sqrt{pq}}\right)$$

$$2\sum_{r=1}^{\infty}\left[\frac{1}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right] = \ln\left[\frac{p+q}{2\sqrt{pq}}\right]$$

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right] = \ln\left(\frac{p+q}{2}\right) - \ln\sqrt{pq}$$

$$\sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right] = \ln\left(\frac{p+q}{2}\right) - \frac{1}{2}\ln(pq)$$

THUS WE FINALLY HAVE THE DESIRED RESULT

$$\ln\left(\frac{p+q}{2}\right) - \frac{\ln p + \ln q}{2} = \sum_{r=1}^{\infty}\left[\frac{2}{2r-1}\left(\frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}+\sqrt{q}}\right)^{4r-2}\right]$$

Question 159 (*****)

Show that

$$1+\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^9}{9!}+\frac{x^{12}}{12!}+\frac{x^{15}}{15!}+\ldots=\frac{1}{3}\left[e^x+2e^{\frac{1}{2}x}\cos\left(\tfrac{1}{2}\sqrt{3}x\right)\right].$$

You may find useful in this question the fact that if $z=e^{i\frac{2}{3}\pi}$ then $1+z+z^2=0$.

[ ], [proof]