ALGEBRAIC FRACTIONS

Exam Questions

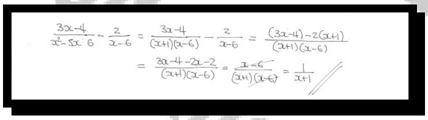
Question 1 (**)

Express

$$\frac{3x-4}{x^2-5x-6}-\frac{2}{x-6},$$

as a single fraction in its simplest form.

$\boxed{}$, $\boxed{\dfrac{1}{x+1}}$



$$\frac{3x-4}{x^2-5x-6}-\frac{2}{x-6}=\frac{3x-4}{(x+1)(x-6)}-\frac{2}{x-6}=\frac{(3x-4)-2(x+1)}{(x+1)(x-6)}$$

$$=\frac{3x-4-2x-2}{(x+1)(x-6)}=\frac{\cancel{x-6}}{(x+1)\cancel{(x-6)}}=\frac{1}{x+1}$$

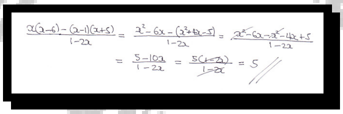
Question 2 (**)

Show that

$$\frac{x(x-6)-(x-1)(x+5)}{1-2x}=k,$$

where k is an integer to be found.

$\boxed{k=5}$



$$\frac{x(x-6)-(x-1)(x+5)}{1-2x}=\frac{x^2-6x-(x^2+4x-5)}{1-2x}=\frac{\cancel{x^2}-6x-\cancel{x^2}-4x+5}{1-2x}$$

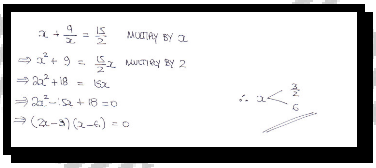
$$=\frac{5-10x}{1-2x}=\frac{5\cancel{(1-2x)}}{\cancel{1-2x}}=5$$

Question 3 (**)

Solve the equation

$$x+\frac{9}{x}=\frac{15}{2},\ x\neq 0.$$

$$\boxed{x=\frac{3}{2},6}$$



$x+\frac{9}{x}=\frac{15}{2}$ MULTIPLY BY x

$\Rightarrow x^2+9=\frac{15}{2}x$ MULTIPLY BY 2

$\Rightarrow 2x^2+18=15x$

$\Rightarrow 2x^2-15x+18=0$

$\Rightarrow (2x-3)(x-6)=0$

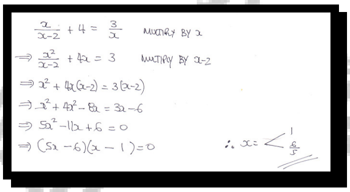
$\therefore x \begin{cases} \frac{3}{2} \\ 6 \end{cases}$

Question 4 (+)**

Solve the equation

$$\frac{x}{x-2}+4=\frac{3}{x},\ x\neq 0.$$

$$\boxed{x=1,\frac{6}{5}}$$



$\frac{x}{x-2}+4=\frac{3}{x}$ MULTIPLY BY x

$\Rightarrow \frac{x^2}{x-2}+4x=3$ MULTIPLY BY $x-2$

$\Rightarrow x^2+4x(x-2)=3(x-2)$

$\Rightarrow x^2+4x^2-8x=3x-6$

$\Rightarrow 5x^2-11x+6=0$

$\Rightarrow (5x-6)(x-1)=0$

$\therefore x= \begin{cases} 1 \\ \frac{6}{5} \end{cases}$

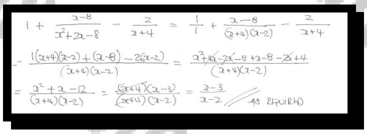
Question 5 (+)**

Show clearly that

$$1+\frac{x-8}{x^2+2x-8}-\frac{2}{x+4}\equiv\frac{x-p}{x-q},$$

stating the value of each of the integer constants, p and q.

$\boxed{\quad}$, $\boxed{p=3}$, $\boxed{q=2}$



$$1+\frac{x-8}{x^2+2x-8}-\frac{2}{x+4}=\frac{1}{1}+\frac{x-8}{(x+4)(x-2)}-\frac{2}{x+4}$$

$$=\frac{1(x+4)(x-2)+(x-8)-2(x-2)}{(x+4)(x-2)}=\frac{x^2+\cancel{4x}-\cancel{2x}-8+x-8-\cancel{2x}+4}{(x+4)(x-2)}$$

$$=\frac{x^2+x-12}{(x+4)(x-2)}=\frac{\cancel{(x+4)}(x-3)}{\cancel{(x+4)}(x-2)}=\frac{x-3}{x-2}\ //$$

AS REQUIRED

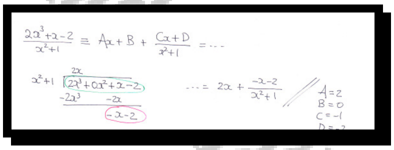
Question 6 (+)**

Given that

$$\frac{2x^3+x-2}{x^2+1}\equiv Ax+B+\frac{Cx+D}{x^2+1},$$

use polynomial division, or another appropriate method, to find the value of each of the constants A, B, C and D.

$\boxed{\quad}$, $\boxed{(A,B,C,D)=(2,0,-1,-2)}$



$$\frac{2x^3+x-2}{x^2+1}\equiv Ax+B+\frac{Cx+D}{x^2+1}=\dots$$

$$\begin{array}{r|l} & 2x \\ \hline x^2+1 & 2x^3+0x^2+x-2 \\ & -2x^3 \qquad -2x \\ \hline & \qquad\quad -x-2 \end{array}$$

$$\dots=2x+\frac{-x-2}{x^2+1}\ //$$

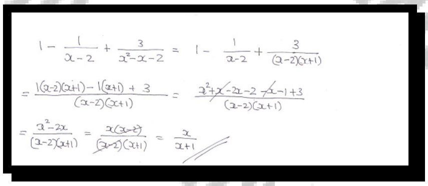
A = 2
B = 0
C = -1
D = -2

Question 7 **(**+)**

$$1-\frac{1}{x-2}+\frac{3}{x^2-x-2},\ x\neq 2,\ x\neq 1.$$

Write the above algebraic expression as a single simplified fraction

$$\boxed{},\ \boxed{\frac{x}{x+1}}$$



$$1-\frac{1}{x-2}+\frac{3}{x^2-x-2}=1-\frac{1}{x-2}+\frac{3}{(x-2)(x+1)}$$

$$=\frac{1(x-2)(x+1)-1(x+1)+3}{(x-2)(x+1)}=\frac{x^2+\cancel{x}-2x-2\cancel{-x}-1+3}{(x-2)(x+1)}$$

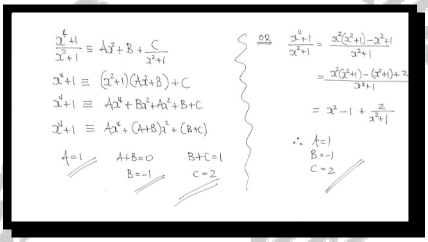
$$=\frac{x^2-2x}{(x-2)(x+1)}=\frac{x\cancel{(x-2)}}{\cancel{(x-2)}(x+1)}=\frac{x}{x+1}$$

Question 8 **(**+)**

$$\frac{x^4+1}{x^2+1}\equiv Ax^2+B+\frac{C}{x^2+1}.$$

Find the value of each of the constants A, B and C.

$$\boxed{A=1},\ \boxed{B=-1},\ \boxed{C=2}$$



$$\frac{x^4+1}{x^2+1}\equiv Ax^2+B+\frac{C}{x^2+1}$$

$$x^4+1\equiv(x^2+1)(Ax^2+B)+C$$

$$x^4+1\equiv Ax^4+Bx^2+Ax^2+B+C$$

$$x^4+1\equiv Ax^4+(A+B)x^2+(B+C)$$

$A=1$ $A+B=0$ $B+C=1$

$B=-1$ $C=2$

OR

$$\frac{x^4+1}{x^2+1}=\frac{x^2(x^2+1)-x^2+1}{x^2+1}$$

$$=\frac{x^2(x^2+1)-(x^2+1)+2}{x^2+1}$$

$$=x^2-1+\frac{2}{x^2+1}$$

$\therefore$ $A=1$

$B=-1$

$C=2$

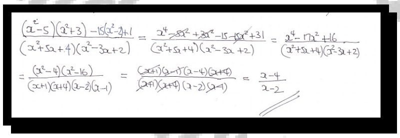
Question 9 **(***)**

Show that

$$\frac{\left(x^2-5\right)\left(x^2+3\right)-15\left(x^2-2\right)+1}{\left(x^2+5x+4\right)\left(x^2-3x+2\right)} \equiv \frac{x+A}{x+B},$$

stating the value of each of the constants A and B.

$\boxed{A=-4}$, $\boxed{B=-2}$



$$\frac{(x^2-5)(x^2+3)-15(x^2-2)+1}{(x^2+5x+4)(x^2-3x+2)} = \frac{x^4\cancel{-5x^2}+\cancel{3x^2}-15\cancel{-15x^2}+31}{(x^2+5x+4)(x^2-3x+2)} = \frac{x^4-17x^2+16}{(x^2+5x+4)(x^2-3x+2)}$$

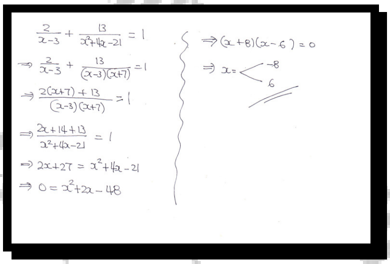
$$= \frac{(x^2-4)(x^2-16)}{(x+1)(x+4)(x-2)(x-1)} = \frac{\cancel{(x+1)}\cancel{(x-1)}(x-4)\cancel{(x+4)}}{\cancel{(x+1)}\cancel{(x+4)}(x-2)\cancel{(x-1)}} = \frac{x-4}{x-2}$$

Question 10 **(***)**

Solve the equation

$$\frac{2}{x-3}+\frac{13}{x^2+4x-21}=1,\ x\neq 3,\ x\neq 7.$$

$\boxed{x=-8,6}$



$$\frac{2}{x-3}+\frac{13}{x^2+4x-21}=1$$

$$\Rightarrow \frac{2}{x-3}+\frac{13}{(x-3)(x+7)}=1$$

$$\Rightarrow \frac{2(x+7)+13}{(x-3)(x+7)}=1$$

$$\Rightarrow \frac{2x+14+13}{x^2+4x-21}=1$$

$$\Rightarrow 2x+27=x^2+4x-21$$

$$\Rightarrow 0=x^2+2x-48$$

$$\Rightarrow (x+8)(x-6)=0$$

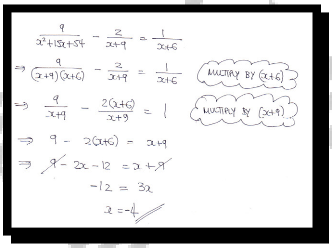
$$\Rightarrow x=\begin{cases}-8\\6\end{cases}$$

Question 11 **(***)**

Solve the equation

$$\frac{9}{x^2+15x+54}-\frac{2}{x+9}=\frac{1}{x+6},\ x\neq -6,\ x\neq -9.$$

$\boxed{x=-4}$



$$\frac{9}{x^2+15x+54}-\frac{2}{x+9}=\frac{1}{x+6}$$

$$\Rightarrow \frac{9}{(x+9)(x+6)}-\frac{2}{x+9}=\frac{1}{x+6}$$

MULTIPLY BY $(x+6)$

$$\Rightarrow \frac{9}{x+9}-\frac{2(x+6)}{x+9}=1$$

MULTIPLY BY $(x+9)$

$$\Rightarrow 9-2(x+6)=x+9$$

$$\Rightarrow \cancel{9}-2x-12=x+\cancel{9}$$

$$-12=3x$$

$$x=-4$$

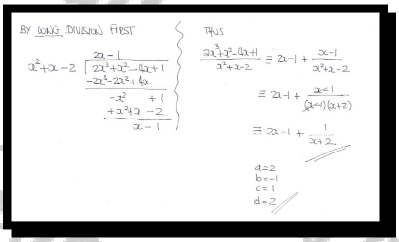
Question 12 **(***)**

Given that

$$\frac{2x^3+x^2-4x+1}{x^2+x-2}\equiv Ax+B+\frac{C}{x+D},$$

use polynomial division, or another appropriate method, to find the value of each of the constants A, B, C and D.

$\boxed{\quad}$, $\boxed{(A,B,C,D)=(2,-1,1,2)}$



BY LONG DIVISION FIRST

$$\begin{array}{r|l} & 2x-1 \\ x^2+x-2 & 2x^3+x^2-4x+1 \\ & -2x^3-2x^2+4x \\ & \quad -x^2 \quad\quad +1 \\ & \quad +x^2+x-2 \\ & \quad\quad\quad x-1 \end{array}$$

THUS

$$\frac{2x^3+x^2-4x+1}{x^2+x-2}\equiv 2x-1+\frac{x-1}{x^2+x-2}$$

$$\equiv 2x-1+\frac{\cancel{x-1}}{\cancel{(x-1)}(x+2)}$$

$$\equiv 2x-1+\frac{1}{x+2}$$

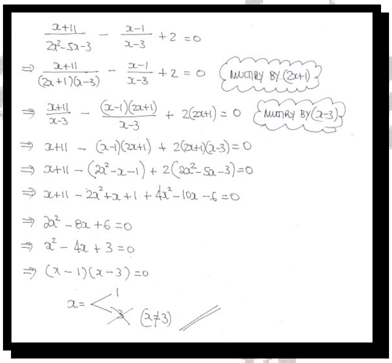
a=2
b=-1
c=1
d=2

Question 13 **(***)**

Solve the equation

$$\frac{x+11}{2x^2-5x-3}-\frac{x-1}{x-3}+2=0\,,\ x\neq-\frac{1}{2}\,,\ x\neq 3\,.$$

$\boxed{x=1}$



$$\frac{x+11}{2x^2-5x-3}-\frac{x-1}{x-3}+2=0$$

$$\Rightarrow \frac{x+11}{(2x+1)(x-3)}-\frac{x-1}{x-3}+2=0$$ MULTIPLY BY $(2x+1)$

$$\Rightarrow \frac{x+11}{x-3}-\frac{(x-1)(2x+1)}{x-3}+2(2x+1)=0$$ MULTIPLY BY $(x-3)$

$$\Rightarrow x+11-(x-1)(2x+1)+2(2x+1)(x-3)=0$$

$$\Rightarrow x+11-(2x^2-x-1)+2(2x^2-5x-3)=0$$

$$\Rightarrow x+11-2x^2+x+1+4x^2-10x-6=0$$

$$\Rightarrow 2x^2-8x+6=0$$

$$\Rightarrow x^2-4x+3=0$$

$$\Rightarrow (x-1)(x-3)=0$$

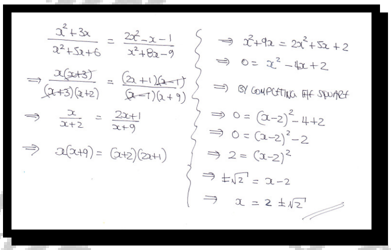
$$x=\begin{cases}1\\ \cancel{3}\ (x\neq3)\end{cases}$$

Question 14 **(***)**

Find, in exact surd form, the roots of the equation

$$\frac{x^2+3x}{x^2+5x+6}=\frac{2x^2-x-1}{x^2+8x-9}\,,\ x\neq-3,\ x\neq1\,.$$

$\boxed{x=2\pm\sqrt{2}}$



$$\frac{x^2+3x}{x^2+5x+6}=\frac{2x^2-x-1}{x^2+8x-9}$$

$$\Rightarrow \frac{x\cancel{(x+3)}}{\cancel{(x+3)}(x+2)}=\frac{(2x+1)\cancel{(x-1)}}{\cancel{(x-1)}(x+9)}$$

$$\Rightarrow \frac{x}{x+2}=\frac{2x+1}{x+9}$$

$$\Rightarrow x(x+9)=(x+2)(2x+1)$$

$$\Rightarrow x^2+9x=2x^2+5x+2$$

$$\Rightarrow 0=x^2-4x+2$$

$\Rightarrow$ BY COMPLETING THE SQUARE

$$\Rightarrow 0=(x-2)^2-4+2$$

$$\Rightarrow 0=(x-2)^2-2$$

$$\Rightarrow 2=(x-2)^2$$

$$\Rightarrow \pm\sqrt{2}=x-2$$

$$\Rightarrow x=2\pm\sqrt{2}$$

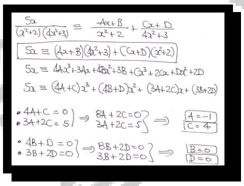
Question 15 **(***)**

$$\frac{5x}{\left(x^2+2\right)\left(4x^2+3\right)} \equiv \frac{Ax+B}{\left(x^2+2\right)}+\frac{Cx+D}{\left(4x^2+3\right)},$$

where A, B, C and D are constants.

Determine the value of each of the constants A, B, C and D.

$\boxed{A=-1}$, $\boxed{B=0}$, $\boxed{C=4}$, $\boxed{D=0}$



$$\frac{5x}{(x^2+2)(4x^2+3)} \equiv \frac{Ax+B}{x^2+2}+\frac{Cx+D}{4x^2+3}$$

$$\boxed{5x \equiv (Ax+B)(4x^2+3)+(Cx+D)(x^2+2)}$$

$$5x \equiv 4Ax^3+3Ax+4Bx^2+3B+Cx^3+2Cx+Dx^2+2D$$

$$5x \equiv (4A+C)x^3+(4B+D)x^2+(3A+2C)x+(3B+2D)$$

$$\left.\begin{array}{l}\bullet\ 4A+C=0\\ \bullet\ 3A+2C=5\end{array}\right\} \Rightarrow \left.\begin{array}{l}8A+2C=0\\ 3A+2C=5\end{array}\right\} \Rightarrow \begin{array}{l}\boxed{A=-1}\\ \boxed{C=4}\end{array}$$

$$\left.\begin{array}{l}\bullet\ 4B+D=0\\ \bullet\ 3B+2D=0\end{array}\right\} \Rightarrow \left.\begin{array}{l}8B+2D=0\\ 3B+2D=0\end{array}\right\} \Rightarrow \begin{array}{l}\boxed{B=0}\\ \boxed{D=0}\end{array}$$

Question 16 **(***)**

Show clearly that the expression

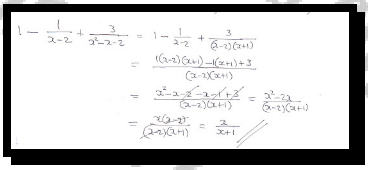
$$1-\frac{1}{x-2}+\frac{3}{x^2-x-2},$$

can be written in the form

$$\frac{x+a}{x+b},$$

where a and b are integer constants to be found.

$\boxed{\quad}$, $\boxed{a=0}$, $\boxed{b=1}$



$$1-\frac{1}{x-2}+\frac{3}{x^2-x-2} = 1-\frac{1}{x-2}+\frac{3}{(x-2)(x+1)}$$

$$= \frac{1(x-2)(x+1)-1(x+1)+3}{(x-2)(x+1)}$$

$$= \frac{x^2-x-\cancel{2}-x-\cancel{1}+\cancel{3}}{(x-2)(x+1)} = \frac{x^2-2x}{(x-2)(x+1)}$$

$$= \frac{x\cancel{(x-2)}}{\cancel{(x-2)}(x+1)} = \frac{x}{x+1} \quad //$$

Question 17 **(***)**

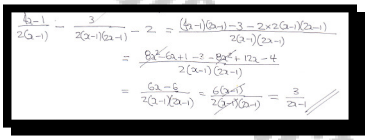
$$f(x) \equiv \frac{4x-1}{2(x-1)} - \frac{3}{2(x-1)(2x-1)} - 2,\ x \in \mathbb{R},\ x \neq 1,\ x \neq \frac{1}{2}.$$

Show clearly that

$$f(x) \equiv \frac{k}{2x-1},\ x \in \mathbb{R},\ x \neq \frac{1}{2},$$

where k is an integer to be found.

$\boxed{}$, $\boxed{k=3}$



$$\frac{4x-1}{2(x-1)} - \frac{3}{2(x-1)(2x-1)} - 2 = \frac{(4x-1)(2x-1) - 3 - 2\times2(x-1)(2x-1)}{2(x-1)(2x-1)}$$

$$= \frac{\cancel{8x^2} - 6x + 1 - 3 - \cancel{8x^2} + 12x - 4}{2(x-1)(2x-1)}$$

$$= \frac{6x-6}{2(x-1)(2x-1)} = \frac{6\cancel{(x-1)}}{2\cancel{(x-1)}(2x-1)} = \frac{3}{2x-1}$$

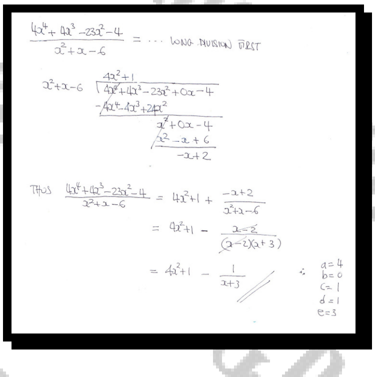
Question 18 **(***)**

Given that

$$\frac{4x^4 + 4x^3 - 23x^2 - 4}{x^2 + x - 6} \equiv Ax^2 + Bx + C - \frac{C}{x+E},$$

find the value of each of the constants A, B, C, D and E.

$\boxed{}$, $\boxed{(A,B,C,D,E) = (4,0,1,1,3)}$



$$\frac{4x^4 + 4x^3 - 23x^2 - 4}{x^2 + x - 6} = \dots$$ LONG DIVISION FIRST

$$\begin{array}{r l}
 & 4x^2 + 1 \\
x^2 + x - 6 & \overline{)\, 4x^4 + 4x^3 - 23x^2 + 0x - 4} \\
 & -4x^4 - 4x^3 + 24x^2 \\
\hline
 & x^2 + 0x - 4 \\
 & x^2 - x + 6 \\
\hline
 & -x + 2
\end{array}$$

THUS $$\frac{4x^4 + 4x^3 - 23x^2 - 4}{x^2 + x - 6} = 4x^2 + 1 + \frac{-x+2}{x^2+x-6}$$

$$= 4x^2 + 1 - \frac{\cancel{x-2}}{\cancel{(x-2)}(x+3)}$$

$$= 4x^2 + 1 - \frac{1}{x+3}$$

$\therefore$ a = 4
b = 0
c = 1
d = 1
e = 3

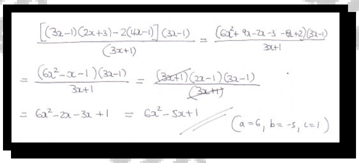
Question 19 **(***)**

Show clearly that

$$\frac{\left[(3x-1)(2x+3)-2(4x-1)\right](3x-1)}{3x+1} \equiv ax^2+bx+c\,,$$

where a, b and c are integers to be found.

$\boxed{}$, $\boxed{a=6}$, $\boxed{b=-5}$, $\boxed{c=1}$



$$\frac{\left[(3x-1)(2x+3)-2(4x-1)\right](3x-1)}{(3x+1)} = \frac{(6x^2+9x-2x-3-8x+2)(3x-1)}{3x+1}$$

$$= \frac{(6x^2-x-1)(3x-1)}{3x+1} = \frac{\cancel{(3x+1)}(2x-1)(3x-1)}{\cancel{(3x+1)}}$$

$$= 6x^2-2x-3x+1 = 6x^2-5x+1$$

$(a=6,\ b=-5,\ c=1)$

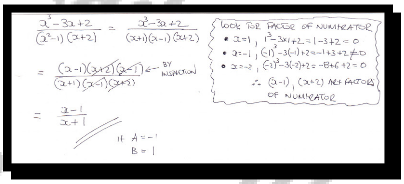
Question 20 **(***+)**

Show clearly that

$$\frac{x^3-3x+2}{(x^2-1)(x+2)} \equiv \frac{x+A}{x+B}\,,$$

stating the value of each of the constants A and B.

$\boxed{A=-1}$, $\boxed{B=1}$



$$\frac{x^3-3x+2}{(x^2-1)(x+2)} = \frac{x^3-3x+2}{(x+1)(x-1)(x+2)}$$

$$= \frac{(x-1)\cancel{(x+2)}\cancel{(x-1)}}{(x+1)\cancel{(x-1)}\cancel{(x+2)}} \leftarrow \text{BY INSPECTION}$$

$$= \frac{x-1}{x+1}$$

if $A=-1$
$B=1$

LOOK FOR FACTOR OF NUMERATOR
- $x=1$, $1^3-3\times1+2=1-3+2=0$
- $x=-1$, $(-1)^3-3(-1)+2=-1+3+2\neq0$
- $x=-2$, $(-2)^3-3(-2)+2=-8+6+2=0$

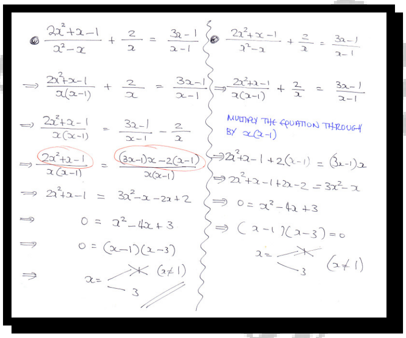
$\therefore$ $(x-1)$, $(x+2)$ ARE FACTORS OF NUMERATOR

Question 21 (***+)

Find the solution of the equation

$$\frac{2x^2+x-1}{x^2-x}+\frac{2}{x}=\frac{3x-1}{x-1}.$$

$\boxed{x=3, x\neq 1}$



$\frac{2x^2+x-1}{x^2-x}+\frac{2}{x}=\frac{3x-1}{x-1}$

$\Rightarrow \frac{2x^2+x-1}{x(x-1)}+\frac{2}{x}=\frac{3x-1}{x-1}$

$\Rightarrow \frac{2x^2+x-1}{x(x-1)}=\frac{3x-1}{x-1}-\frac{2}{x}$

$\Rightarrow \frac{2x^2+x-1}{x(x-1)}=\frac{(3x-1)x-2(x-1)}{x(x-1)}$

$\Rightarrow 2x^2+x-1=3x^2-x-2x+2$

$\Rightarrow 0=x^2-4x+3$

$\Rightarrow 0=(x-1)(x-3)$

$\Rightarrow x = \begin{cases} \times & (x\neq 1) \\ 3 \end{cases}$

$\frac{2x^2+x-1}{x^2-x}+\frac{2}{x}=\frac{3x-1}{x-1}$

$\Rightarrow \frac{2x^2+x-1}{x(x-1)}+\frac{2}{x}=\frac{3x-1}{x-1}$

MULTIPLY THE EQUATION THROUGH BY $x(x-1)$

$\Rightarrow 2x^2+x-1+2(x-1)=(3x-1)x$

$\Rightarrow 2x^2+x-1+2x-2=3x^2-x$

$\Rightarrow 0=x^2-4x+3$

$\Rightarrow (x-1)(x-3)=0$

$x = \begin{cases} \times \\ 3 \end{cases} \quad (x\neq 1)$

Question 22 (***+)

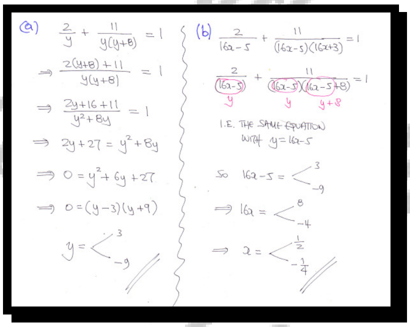
a) Find the solutions of the equation

$$\frac{2}{y}+\frac{11}{y(y+8)}=1,\ y\neq -8,\ y\neq 0.$$

b) Hence, or otherwise, solve the equation

$$\frac{2}{16x-5}+\frac{11}{(16x-5)(16x+3)}=1,\ x\neq -\frac{3}{16},\ x\neq \frac{5}{16}.$$

$\boxed{x=-9,3}$, $\boxed{x=-\frac{1}{4},\frac{1}{2}}$



(a) $\frac{2}{y}+\frac{11}{y(y+8)}=1$

$\Rightarrow \frac{2(y+8)+11}{y(y+8)}=1$

$\Rightarrow \frac{2y+16+11}{y^2+8y}=1$

$\Rightarrow 2y+27=y^2+8y$

$\Rightarrow 0=y^2+6y+27$

$\Rightarrow 0=(y-3)(y+9)$

$y=\begin{cases}3\\-9\end{cases}$

(b) $\frac{2}{16x-5}+\frac{11}{(16x-5)(16x+3)}=1$

$\frac{2}{\underbrace{(16x-5)}_{y}}+\frac{11}{\underbrace{(16x-5)}_{y}\underbrace{(16x-5+8)}_{y+8}}=1$

I.E. THE SAME EQUATION WITH $y=16x-5$

So $16x-5=\begin{cases}3\\-9\end{cases}$

$\Rightarrow 16x=\begin{cases}8\\-4\end{cases}$

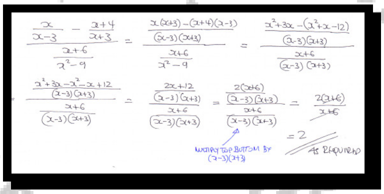
$\Rightarrow x=\begin{cases}\frac{1}{2}\\-\frac{1}{4}\end{cases}$

Question 23 (***+)

Show clearly that

$$\frac{\frac{x}{x-3}-\frac{x+4}{x+3}}{\frac{x+6}{x^2-9}}\equiv 2.$$

$\boxed{\text{proof}}$



$\frac{\frac{x}{x-3}-\frac{x+4}{x+3}}{\frac{x+6}{x^2-9}}=\frac{\frac{x(x+3)-(x+4)(x-3)}{(x-3)(x+3)}}{\frac{x+6}{x^2-9}}=\frac{\frac{x^2+3x-(x^2+x-12)}{(x-3)(x+3)}}{\frac{x+6}{(x-3)(x+3)}}$

$=\frac{\frac{x^2+3x-x^2-x+12}{(x-3)(x+3)}}{\frac{x+6}{(x-3)(x+3)}}=\frac{\frac{2x+12}{(x-3)(x+3)}}{\frac{x+6}{(x-3)(x+3)}}=\frac{\frac{2(x+6)}{(x-3)(x+3)}}{\frac{x+6}{(x-3)(x+3)}}=\frac{2(x+6)}{\cancel{x+6}}$

MULTIPLY TOP BOTTOM BY $(x-3)(x+3)$

$=2$

AS REQUIRED

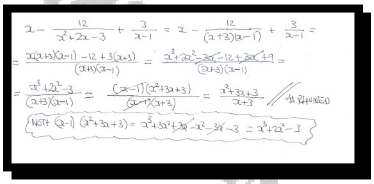
Question 24 (*+)**

$$f(x) \equiv x - \frac{12}{x^2+2x-3} + \frac{3}{x-1}, \quad x \in \mathbb{R}, \; x \neq -3, \; x \neq 1 \; .$$

Show clearly that

$$f(x) \equiv \frac{x^2+3x+3}{x+3}, \quad x \in \mathbb{R}, \; x \neq -3, \; x \neq 1 \; .$$

$\boxed{}$, $\boxed{\text{proof}}$



$$x - \frac{12}{x^2+2x-3} + \frac{3}{x-1} = x - \frac{12}{(x+3)(x-1)} + \frac{3}{x-1} =$$

$$= \frac{x(x+3)(x-1) - 12 + 3(x+3)}{(x+3)(x-1)} = \frac{x^3+2x^2\cancel{-3x}-12\cancel{+3x}+9}{(x+3)(x-1)} =$$

$$= \frac{x^3+2x^2-3}{(x+3)(x-1)} = \frac{\cancel{(x-1)}(x^2+3x+3)}{\cancel{(x-1)}(x+3)} = \frac{x^2+3x+3}{x+3} \;//\; \text{AS REQUIRED}$$

NOTE $(x-1)(x^2+3x+3) = x^3+3x^2\cancel{+3x}-x^2\cancel{-3x}-3 = x^3+2x^2-3$

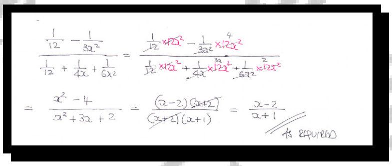
Question 25 (*+)**

Show clearly that

$$\frac{\dfrac{1}{12} - \dfrac{1}{3x^2}}{\dfrac{1}{12} + \dfrac{1}{4x} + \dfrac{1}{6x^2}} \equiv \frac{x+a}{x+b} \, .$$

where a and b are integers to be found

$\boxed{}$, $\boxed{\dfrac{x-2}{x+1}}$



$$\frac{\frac{1}{12} - \frac{1}{3x^2}}{\frac{1}{12} + \frac{1}{4x} + \frac{1}{6x^2}} = \frac{\frac{1}{\cancel{12}} \times \cancel{12}x^2 - \frac{1}{\cancel{3x^2}} \times \overset{4}{\cancel{12x^2}}}{\frac{1}{\cancel{12}} \times \cancel{12}x^2 + \frac{1}{\cancel{4x}} \times \overset{3x}{\cancel{12x^2}} + \frac{1}{\cancel{6x^2}} \times \overset{2}{\cancel{12x^2}}}$$

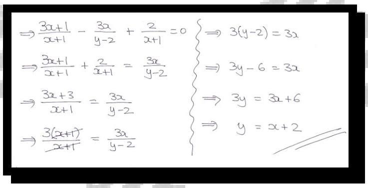
$$= \frac{x^2-4}{x^2+3x+2} = \frac{(x-2)\cancel{(x+2)}}{\cancel{(x+2)}(x+1)} = \frac{x-2}{x+1} \;//\; \text{AS REQUIRED}$$

Question 26 **(****)**

$$\frac{3x+1}{x+1}-\frac{3x}{y-2}+\frac{2}{x+1}=0,\ x\neq -1,\ y\neq 2.$$

Show that the above expression represents a straight line and give its equation.

$\boxed{y=x+2}$



$\Rightarrow \frac{3x+1}{x+1}-\frac{3x}{y-2}+\frac{2}{x+1}=0$

$\Rightarrow \frac{3x+1}{x+1}+\frac{2}{x+1}=\frac{3x}{y-2}$

$\Rightarrow \frac{3x+3}{x+1}=\frac{3x}{y-2}$

$\Rightarrow \frac{3\cancel{(x+1)}}{\cancel{x+1}}=\frac{3x}{y-2}$

$\Rightarrow 3(y-2)=3x$

$\Rightarrow 3y-6=3x$

$\Rightarrow 3y=3x+6$

$\Rightarrow y=x+2$

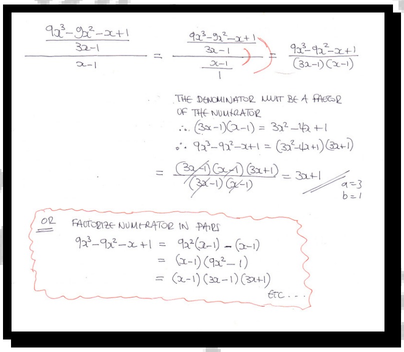
Question 27 **(****)**

Show clearly that

$$\frac{\frac{9x^3-9x^2-x+1}{3x-1}}{x-1}\equiv ax+b,$$

where a and b are integers to be found.

$\boxed{a=1}$, $\boxed{b=1}$



$\frac{\frac{9x^3-9x^2-x+1}{3x-1}}{x-1}=\frac{\frac{9x^3-9x^2-x+1}{3x-1}}{\frac{x-1}{1}}=\frac{9x^3-9x^2-x+1}{(3x-1)(x-1)}$

THE DENOMINATOR MUST BE A FACTOR OF THE NUMERATOR

$\therefore (3x-1)(x-1)=3x^2-4x+1$

$\therefore 9x^3-9x^2-x+1=(3x^2-4x+1)(3x+1)$

$=\frac{\cancel{(3x-1)}\cancel{(x-1)}(3x+1)}{\cancel{(3x-1)}\cancel{(x-1)}}=3x+1$

$a=3$
$b=1$

OR FACTORIZE NUMERATOR IN PAIRS

$9x^3-9x^2-x+1=9x^2(x-1)-(x-1)$

$=(x-1)(9x^2-1)$

$=(x-1)(3x-1)(3x+1)$

ETC...

Question 28 **(****)**

$$f(x) \equiv \frac{6x^2 - 21x + 17}{(x-3)(x-1)^2}, \quad x \in \mathbb{R}, \quad x \neq 3, \quad x \neq 1.$$

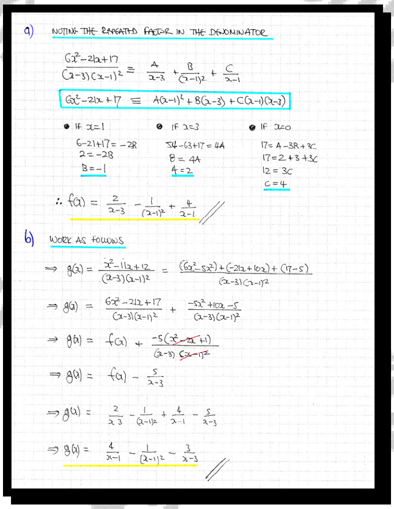
a) Express $f(x)$ into partial fractions.

$$g(x) \equiv \frac{(x-12)(x+1)}{(x-3)(x-1)^2}, \quad x \in \mathbb{R}, \quad x \neq 3, \quad x \neq 1.$$

b) Use the result of part **(a)** to express $g(x)$ into partial fractions.

*No credit will be given in this part by repeating the method used in part **(a)***

$\boxed{\text{SYN-K}}$, $\boxed{f(x) \equiv \frac{4}{x-1} + \frac{2}{x-3} - \frac{1}{(x-1)^2}}$ $\boxed{f(x) \equiv \frac{4}{x-1} - \frac{3}{x-3} - \frac{1}{(x-1)^2}}$



a) NOTING THE REPEATED FACTOR IN THE DENOMINATOR

$$\frac{6x^2-21x+17}{(x-3)(x-1)^2} \equiv \frac{A}{x-3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$$

$$\boxed{6x^2-21x+17 \equiv A(x-1)^2 + B(x-3) + C(x-1)(x-3)}$$

- IF $x=1$
  $6-21+17 = -2B$
  $2 = -2B$
  $B = -1$
- IF $x=3$
  $54-63+17 = 4A$
  $8 = 4A$
  $A = 2$
- IF $x=0$
  $17 = A - 3B + 3C$
  $17 = 2 + 3 + 3C$
  $12 = 3C$
  $C = 4$

$$\therefore f(x) = \frac{2}{x-3} - \frac{1}{(x-1)^2} + \frac{4}{x-1}$$

b) WORK AS FOLLOWS

$$\Rightarrow g(x) = \frac{x^2-11x+12}{(x-3)(x-1)^2} = \frac{(6x^2-5x^2)+(-21x+10x)+(17-5)}{(x-3)(x-1)^2}$$

$$\Rightarrow g(x) = \frac{6x^2-21x+17}{(x-3)(x-1)^2} + \frac{-5x^2+10x-5}{(x-3)(x-1)^2}$$

$$\Rightarrow g(x) = f(x) + \frac{-5(\cancel{x^2-2x+1})}{(x-3)\cancel{(x-1)^2}}$$

$$\Rightarrow g(x) = f(x) - \frac{5}{x-3}$$

$$\Rightarrow g(x) = \frac{2}{x-3} - \frac{1}{(x-1)^2} + \frac{4}{x-1} - \frac{5}{x-3}$$

$$\Rightarrow g(x) = \frac{4}{x-1} - \frac{1}{(x-1)^2} - \frac{3}{x-3}$$

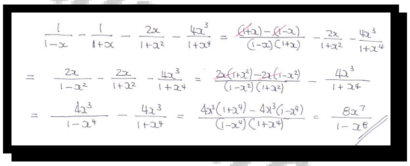
Question 29 (****+)

Simplify the following expression

$$\frac{1}{1-x}-\frac{1}{1+x}-\frac{2x}{1+x^2}-\frac{4x^3}{1+x^4},$$

giving the final answer as a single algebraic fraction in its simplest form.

$$\boxed{\frac{8x^7}{1-x^8}}$$



$$\frac{1}{1-x}-\frac{1}{1+x}-\frac{2x}{1+x^2}-\frac{4x^3}{1+x^4}=\frac{\cancel{1}+x)-(\cancel{1}-x)}{(1-x)(1+x)}-\frac{2x}{1+x^2}-\frac{4x^3}{1+x^4}$$

$$=\frac{2x}{1-x^2}-\frac{2x}{1+x^2}-\frac{4x^3}{1+x^4}=\frac{\cancel{2x}(1+x^2)-\cancel{2x}(1-x^2)}{(1-x^2)(1+x^2)}-\frac{4x^3}{1+x^4}$$

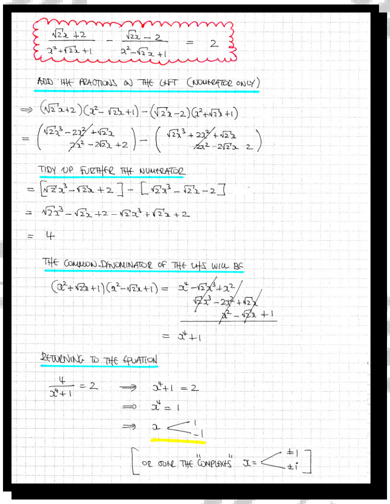
$$=\frac{4x^3}{1-x^4}-\frac{4x^3}{1+x^4}=\frac{4x^3(1+x^4)-4x^3(1-x^4)}{(1-x^4)(1+x^4)}=\frac{8x^7}{1-x^8}$$

Question 30 (****+)

Solve the following equation.

$$\frac{2+\sqrt{2}\,x}{x^2+\sqrt{2}\,x+1}+\frac{2-\sqrt{2}\,x}{x^2-\sqrt{2}\,x+1}=2,\quad x\in\mathbb{R}.$$

$\boxed{}$, $\boxed{x=\pm 1}$



$$\frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1}-\frac{\sqrt{2}x-2}{x^2-\sqrt{2}x+1}=2$$

ADD THE FRACTIONS ON THE LEFT (NUMERATOR ONLY)

$$\Rightarrow (\sqrt{2}x+2)(x^2-\sqrt{2}x+1)-(\sqrt{2}x-2)(x^2+\sqrt{2}x+1)$$

$$=\left(\begin{array}{l}\sqrt{2}x^3-\cancel{2x^2}+\sqrt{2}x\\ \quad\cancel{2x^2}-2\sqrt{2}x+2\end{array}\right)-\left(\begin{array}{l}\sqrt{2}x^3+\cancel{2x^2}+\sqrt{2}x\\ \quad-\cancel{2x^2}-2\sqrt{2}x-2\end{array}\right)$$

TIDY UP FURTHER THE NUMERATOR

$$=\left[\sqrt{2}x^3-\sqrt{2}x+2\right]-\left[\sqrt{2}x^3-\sqrt{2}x-2\right]$$

$$=\sqrt{2}x^3-\sqrt{2}x+2-\sqrt{2}x^3+\sqrt{2}x+2$$

$$=4$$

THE COMMON DENOMINATOR OF THE LHS WILL BE

$$(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)=\begin{array}{l}x^4-\cancel{\sqrt{2}x^3}+\cancel{x^2}\\ \quad\cancel{\sqrt{2}x^3}-2x^2+\cancel{\sqrt{2}x}\\ \qquad\cancel{x^2}-\cancel{\sqrt{2}x}+1\end{array}$$

$$=x^4+1$$

RETURNING TO THE EQUATION

$$\frac{4}{x^4+1}=2\implies x^4+1=2$$

$$\implies x^4=1$$

$$\implies x\ \begin{cases}1\\-1\end{cases}$$

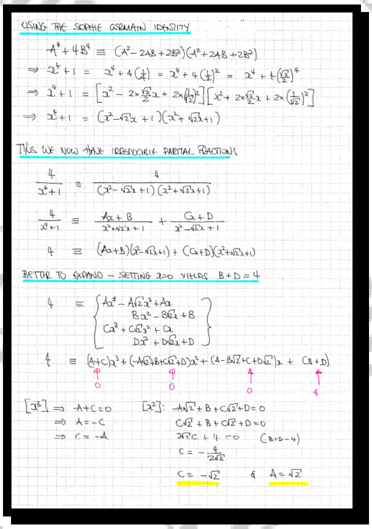
$$\left[\text{OR OVER THE "COMPLEXES"}\quad x=\begin{cases}\pm 1\\ \pm i\end{cases}\right]$$

Question 31 (*****)

$$f(x) \equiv \frac{4}{x^4+1}, \quad x \in \mathbb{R}.$$

Express $f(x)$ into partial fractions over the set of real numbers.

$\boxed{}$, $\boxed{f(x) \equiv \frac{2+\sqrt{2}\,x}{x^2+\sqrt{2}\,x+1} + \frac{2-\sqrt{2}\,x}{x^2-\sqrt{2}\,x+1} = 2}$



USING THE SOPHIE GERMAIN IDENTITY

$A^4 + 4B^4 \equiv (A^2 - 2AB + 2B^2)(A^2 + 2AB + 2B^2)$

$\Rightarrow x^4 + 1 = x^4 + 4\left(\frac{1}{4}\right) = x^4 + 4\left(\frac{1}{2}\right)^2 = x^4 + 4\left(\frac{\sqrt{2}}{2}\right)^4$

$\Rightarrow x^4 + 1 = \left[x^2 - 2\times\frac{\sqrt{2}}{2}x + 2\times\left(\frac{1}{\sqrt{2}}\right)^2\right]\left[x^2 + 2\times\frac{\sqrt{2}}{2}x + 2\times\left(\frac{1}{\sqrt{2}}\right)^2\right]$

$\Rightarrow x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$

THUS WE NOW HAVE IRREDUCIBLE PARTIAL FRACTIONS

$\frac{4}{x^4+1} \equiv \frac{4}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)}$

$\frac{4}{x^4+1} \equiv \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$

$4 \equiv (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1)$

BETTER TO EXPAND – SETTING $x=0$ YIELDS $B+D=4$

$$4 \equiv \left\{\begin{array}{l} Ax^3 - A\sqrt{2}x^2 + Ax \\ \qquad Bx^2 - B\sqrt{2}x + B \\ Cx^3 + C\sqrt{2}x^2 + Cx \\ \qquad Dx^2 + D\sqrt{2}x + D \end{array}\right\}$$

$$4 \equiv \underset{\substack{\uparrow \\ 0}}{(A+C)}x^3 + \underset{\substack{\uparrow \\ 0}}{(-A\sqrt{2}+B+C\sqrt{2}+D)}x^2 + \underset{\substack{\uparrow \\ 0}}{(A-B\sqrt{2}+C+D\sqrt{2})}x + \underset{\substack{\uparrow \\ 4}}{(B+D)}$$

$[x^3] \Rightarrow A+C=0$

$\Rightarrow A=-C$

$\Rightarrow C=-A$

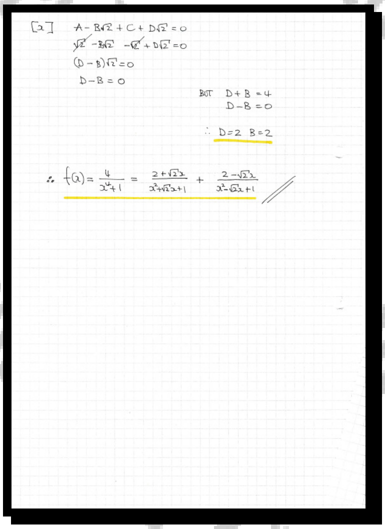
$[x^2]$: $-A\sqrt{2}+B+C\sqrt{2}+D=0$

$C\sqrt{2}+B+C\sqrt{2}+D=0$

$2\sqrt{2}C+4=0 \quad (B+D=4)$

$C = -\frac{4}{2\sqrt{2}}$

$C = -\sqrt{2}$ & $A = \sqrt{2}$



$[x]$ $A - B\sqrt{2} + C + D\sqrt{2} = 0$

$\cancel{\sqrt{2}} - B\sqrt{2} - \cancel{\sqrt{2}} + D\sqrt{2} = 0$

$(D-B)\sqrt{2}=0$

$D-B=0$

BUT $D+B=4$

$D-B=0$

$\therefore D=2 \quad B=2$

$\therefore f(x) = \frac{4}{x^4+1} = \frac{2+\sqrt{2}x}{x^2+\sqrt{2}x+1} + \frac{2-\sqrt{2}x}{x^2-\sqrt{2}x+1}$ //

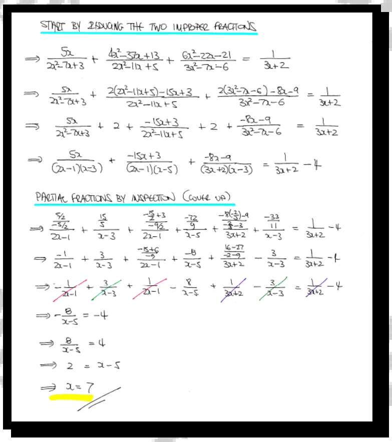
Question 32 (*****)

Solve the following rational equation, over the set of real numbers.

$$\frac{5x}{2x^2-7x+3}+\frac{4x^2-37x+13}{2x^2-11x+5}+\frac{6x^2-22x-21}{3x^2-7x-6}=\frac{1}{3x+2}.$$

You may ignore non finite solutions.

$\boxed{}$, $\boxed{x=7}$



START BY REDUCING THE TWO IMPROPER FRACTIONS

$$\Longrightarrow \frac{5x}{2x^2-7x+3}+\frac{4x^2-37x+13}{2x^2-11x+5}+\frac{6x^2-22x-21}{3x^2-7x-6}=\frac{1}{3x+2}$$

$$\Longrightarrow \frac{5x}{2x^2-7x+3}+\frac{2(2x^2-11x+5)-15x+3}{2x^2-11x+5}+\frac{2(3x^2-7x-6)-8x-9}{3x^2-7x-6}=\frac{1}{3x+2}$$

$$\Longrightarrow \frac{5x}{2x^2-7x+3}+2+\frac{-15x+3}{2x^2-11x+5}+2+\frac{-8x-9}{3x^2-7x-6}=\frac{1}{3x+2}$$

$$\Longrightarrow \frac{5x}{(2x-1)(x-3)}+\frac{-15x+3}{(2x-1)(x-5)}+\frac{-8x-9}{(3x+2)(x-3)}=\frac{1}{3x+2}-4$$

PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$\Longrightarrow \frac{\frac{5/2}{-5/2}}{2x-1}+\frac{\frac{15}{5}}{x-3}+\frac{\frac{-\frac{15}{2}+3}{-9/2}}{2x-1}+\frac{\frac{-72}{9}}{x-5}+\frac{\frac{-8(-\frac{2}{3})-9}{-\frac{2}{3}-3}}{3x+2}+\frac{\frac{-33}{11}}{x-3}=\frac{1}{3x+2}-4$$

$$\Longrightarrow \frac{-1}{2x-1}+\frac{3}{x-3}+\frac{\frac{-15+6}{-9}}{2x-1}+\frac{-8}{x-5}+\frac{\frac{16-27}{-2-9}}{3x+2}-\frac{3}{x-3}=\frac{1}{3x+2}-4$$

$$\Longrightarrow -\cancel{\frac{1}{2x-1}}+\cancel{\frac{3}{x-3}}+\cancel{\frac{1}{2x-1}}-\frac{8}{x-5}+\cancel{\frac{1}{3x+2}}-\cancel{\frac{3}{x-3}}=\cancel{\frac{1}{3x+2}}-4$$

$$\Longrightarrow -\frac{8}{x-5}=-4$$

$$\Longrightarrow \frac{8}{x-5}=4$$

$$\Longrightarrow 2=x-5$$

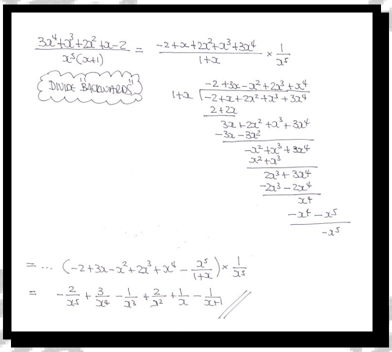
$$\Longrightarrow x=7$$

Question 33 **(*****)**

$$f(x) \equiv \frac{3x^4 + x^3 + 2x^2 + x - 2}{(x+1)x^5}, \ x \in \mathbb{R}, \ x \neq 0, x \neq -1$$

Express $f(x)$ into partial fractions in their simplest form.

$$\boxed{}, \ \boxed{f(x) \equiv -\frac{2}{x^5} + \frac{3}{x^4} - \frac{1}{x^3} + \frac{2}{x^2} + \frac{1}{x} - \frac{1}{x+1}}$$



$$\frac{3x^4+x^3+2x^2+x-2}{x^5(x+1)} = \frac{-2+x+2x^2+x^3+3x^4}{1+x} \times \frac{1}{x^5}$$

"DIVIDE BACKWARDS"

$$\begin{array}{r} -2+3x-x^2+2x^3+x^4 \\ 1+x \ \overline{)\ -2+x+2x^2+x^3+3x^4} \\ \underline{2+2x} \\ 3x+2x^2+x^3+3x^4 \\ \underline{-3x-3x^2} \\ -x^2+x^3+3x^4 \\ \underline{x^2+x^3} \\ 2x^3+3x^4 \\ \underline{-2x^3-2x^4} \\ x^4 \\ \underline{-x^4-x^5} \\ -x^5 \end{array}$$

$$= \dots \left(-2+3x-x^2+2x^3+x^4-\frac{x^5}{1+x}\right) \times \frac{1}{x^5}$$

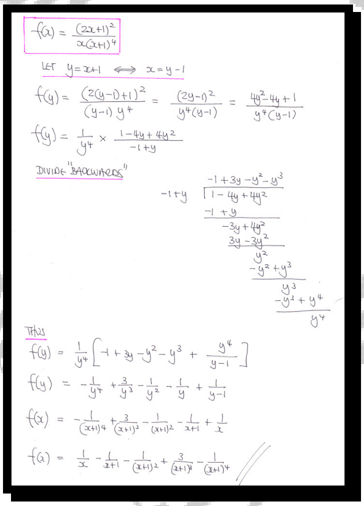
$$= -\frac{2}{x^5}+\frac{3}{x^4}-\frac{1}{x^3}+\frac{2}{x^2}+\frac{1}{x}-\frac{1}{x+1} \ //$$

Question 34 (*****)

$$f(x) \equiv \frac{(2x+1)^2}{x(x+1)^4}, \; x \in \mathbb{R}, \; x \neq 0, x \neq -1$$

Express $f(x)$ into partial fractions in their simplest form.

☐ , $\boxed{f(x) \equiv \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} + \frac{3}{(x+1)^3} - \frac{1}{(x+1)^4}}$



$\boxed{f(x) = \frac{(2x+1)^2}{x(x+1)^4}}$

LET $y = x+1 \iff x = y-1$

$f(y) = \frac{(2(y-1)+1)^2}{(y-1)y^4} = \frac{(2y-1)^2}{y^4(y-1)} = \frac{4y^2-4y+1}{y^4(y-1)}$

$f(y) = \frac{1}{y^4} \times \frac{1-4y+4y^2}{-1+y}$

DIVIDE "BACKWARDS"

$$\begin{array}{r l}
 & -1+3y-y^2-y^3 \\
-1+y & \overline{)\,1-4y+4y^2} \\
 & \underline{-1+y} \\
 & \quad -3y+4y^2 \\
 & \quad \underline{3y-3y^2} \\
 & \qquad y^2 \\
 & \qquad \underline{-y^2+y^3} \\
 & \qquad\quad y^3 \\
 & \qquad\quad \underline{-y^3+y^4} \\
 & \qquad\qquad y^4
\end{array}$$

THUS

$f(y) = \frac{1}{y^4}\left[-1+3y-y^2-y^3+\frac{y^4}{y-1}\right]$

$f(y) = -\frac{1}{y^4}+\frac{3}{y^3}-\frac{1}{y^2}-\frac{1}{y}+\frac{1}{y-1}$

$f(x) = -\frac{1}{(x+1)^4}+\frac{3}{(x+1)^3}-\frac{1}{(x+1)^2}-\frac{1}{x+1}+\frac{1}{x}$

$f(x) = \frac{1}{x}-\frac{1}{x+1}-\frac{1}{(x+1)^2}+\frac{3}{(x+1)^3}-\frac{1}{(x+1)^4}$ //