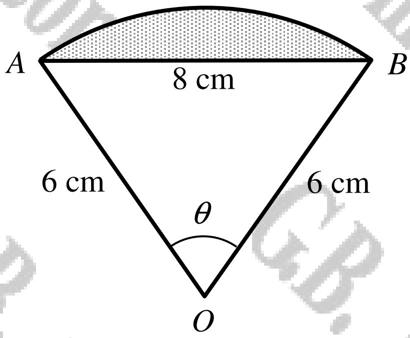
GEOMETRIC MENSURATION

Question 1 **(**)**



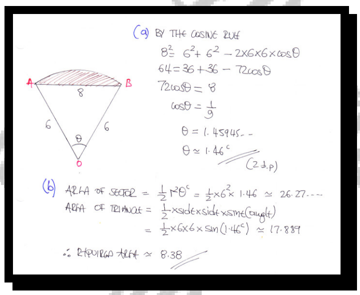
The figure above shows a circular sector OAB, subtending an angle of θ radians at its centre O.

The radius of the sector is 6 cm and the length of the **chord** AB is 8 cm.

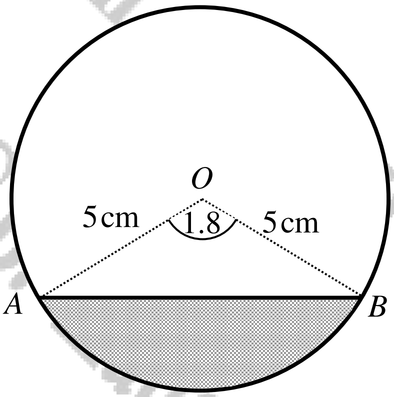
a) Find the size of the angle θ in radians, correct to two decimal places.

b) Determine the area of the circular **segment**, shown shaded in the figure.

$\boxed{}$, $\boxed{\theta \approx 1.46^{c}}$, $\boxed{\text{area} \approx 8.38 \text{ to } 8.39}$



Question 2 (**)



The figure above shows a circle with centre at O and radius 5 cm.

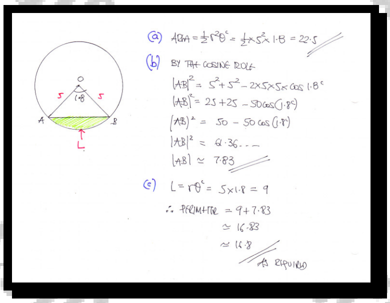
The points A and B lie on the circle so that the angle AOB is 1.8 radians.

a) Find the area of the sector OAB.

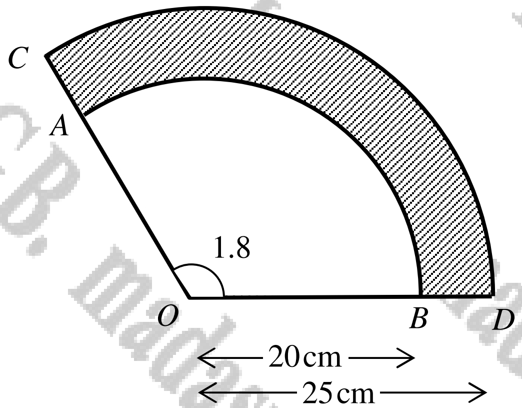
b) Determine the length of the chord AB.

c) Hence show that the perimeter of the minor segment, shown shaded in the figure, is approximately 16.8 cm.

$\boxed{\text{area} = 22.5}$, $\boxed{|AB| \approx 7.83}$



Question 3 (**)



The figure above shows two concentric circular sectors OAB and OCD, where O is their common centre. Both sectors subtend an angle of 1.8 radians at O.

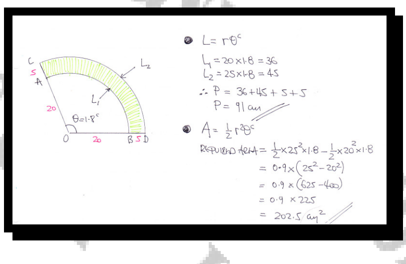
The point A lies on OC and similarly the point B lies on OD.

It is further given that $|OA| = |OB| = 20\text{cm}$ and $|OC| = |OD| = 25\text{cm}$.

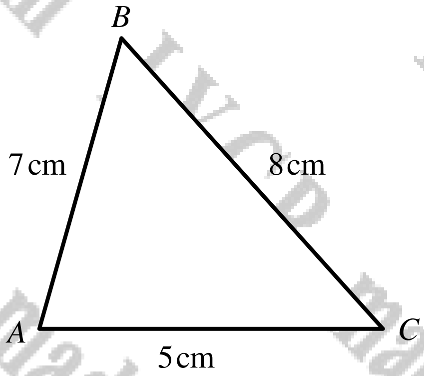
The finite region $ACDB$ is shown shaded in the above figure.

Determine the perimeter and the area of $ACDB$.

$\boxed{P = 91 \text{ cm}}$, $\boxed{A = 202.5 \text{ cm}^2}$



Question 4 (**)

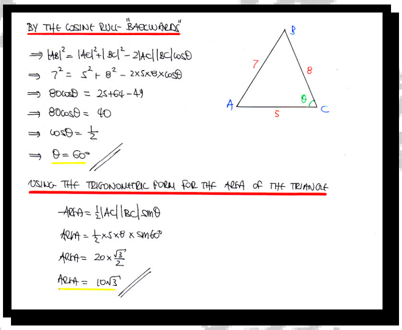


The figure above shows a triangle ABC where the following information is given.

$$|AB| = 7 \text{ cm}, \quad |BC| = 8 \text{ cm} \text{ and } \quad |AC| = 5 \text{ cm}.$$

Find the size of the angle $\measuredangle ACB$ in degrees, and hence determine as an exact surd the area of the triangle ABC.

$\boxed{\quad}$, $\boxed{\measuredangle ACB = 60°}$, $\boxed{\text{Area} = 10\sqrt{3} \text{ cm}^2}$


BY THE COSINE RULE "BACKWARDS"

$\Rightarrow |AB|^2 = |AC|^2 + |BC|^2 - 2|AC||BC|\cos\theta$

$\Rightarrow 7^2 = 5^2 + 8^2 - 2 \times 5 \times 8 \times \cos\theta$

$\Rightarrow 80\cos\theta = 25 + 64 - 49$

$\Rightarrow 80\cos\theta = 40$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = 60°$

USING THE TRIGONOMETRIC FORM FOR THE AREA OF THE TRIANGLE

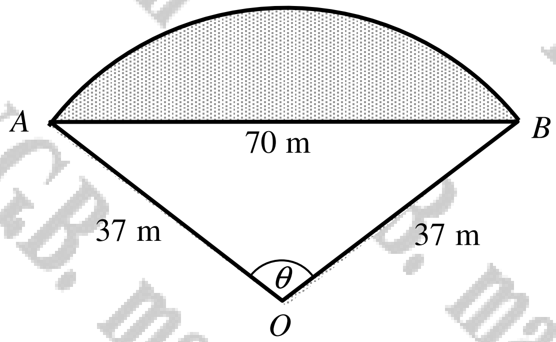
$\text{AREA} = \frac{1}{2}|AC||BC|\sin\theta$

$\text{AREA} = \frac{1}{2} \times 5 \times 8 \times \sin 60°$

$\text{AREA} = 20 \times \frac{\sqrt{3}}{2}$

$\text{AREA} = 10\sqrt{3}$

Question 5 (**)



The figure above shows a circular sector OAB, subtending an angle of θ radians at its centre O.

The radius of the sector is 37 m and the length of the **chord** AB is 70 m.

a) Show that θ is approximately 2.481 radians.

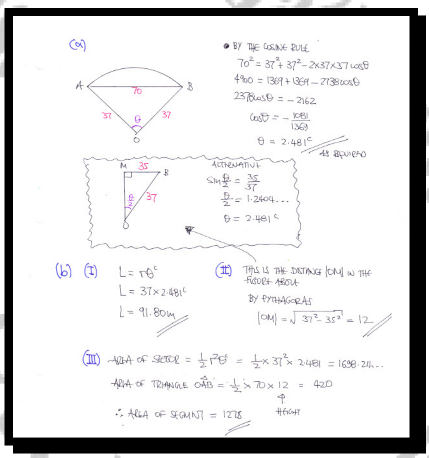
b) Calculate to an appropriate degree of accuracy…

i. … the length of the **arc** AB.

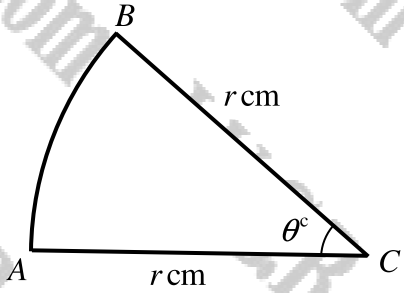
ii. … the shortest distance from O to the **chord** AB.

iii. … the area of the circular segment, shown shaded in the figure.

[ ] , [91.8 m] , [12 m] , [area $\approx 1278 \text{ m}^2$]



Question 6 (**)

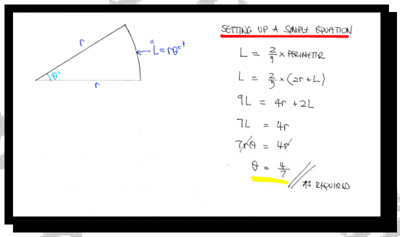


The figure above shows a circular sector ABC of radius r cm subtending an angle θ radians at C.

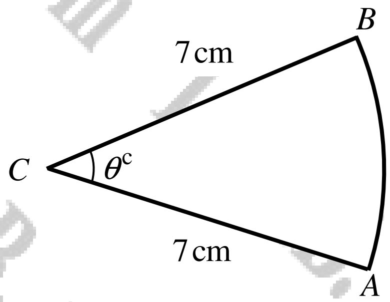
The length of the arc AB is $\frac{2}{9}$ of the perimeter of the sector.

Show that $\theta = \frac{4}{7}$ radians.

[ ] , [proof]



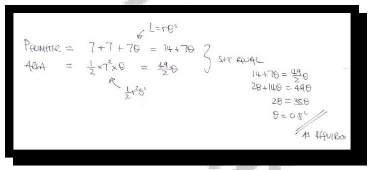
Question 7 (**)



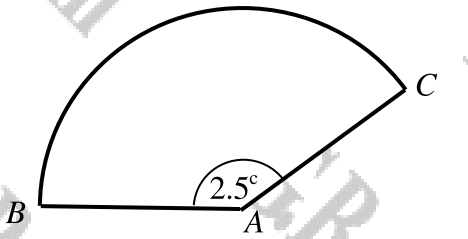
The figure above shows a circular sector ABC of radius 7 cm subtending an angle θ radians at C.

Given the perimeter of the sector is **numerically equal** to the area of the sector show that θ is 0.8 radians.

proof



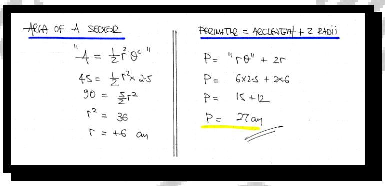
Question 8 (**)



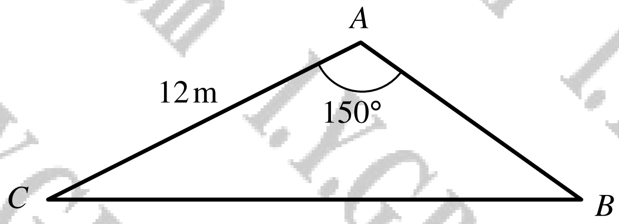
The figure above shows a circular sector ABC subtending an angle of 2.5 radians at the point A.

Given that the area of the sector is 45 cm^2, find its perimeter.

$\boxed{}$, $\boxed{P = 27 \text{ cm}}$



Question 9 (**)



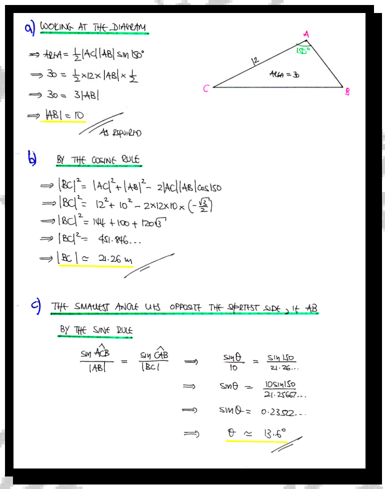
The triangle ABC is such so that AC is 12 m and the angle CAB is $150°$.

a) Given that the area of the triangle ABC is 30 m^2, show that the length of AB is 10 m.

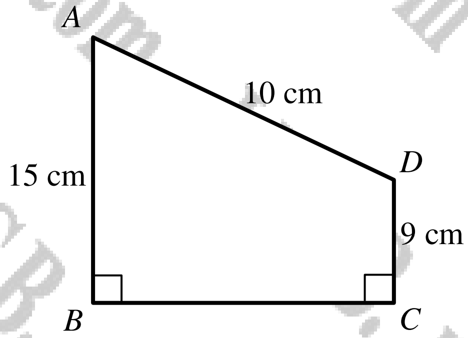
b) Find the length of BC, giving the answer in m, correct to 2 decimal places.

c) Calculate the smallest angle of the triangle ABC, giving the answer in degrees, correct to one decimal place.

[ ] , [21.26 m] , [13.6°]



Question 10 (**)



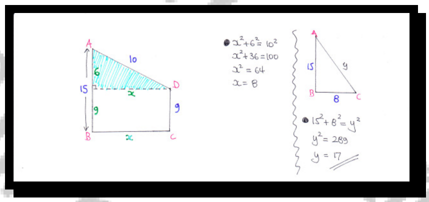
The figure above shows a right angled trapezium $ABCD$.

It is given that

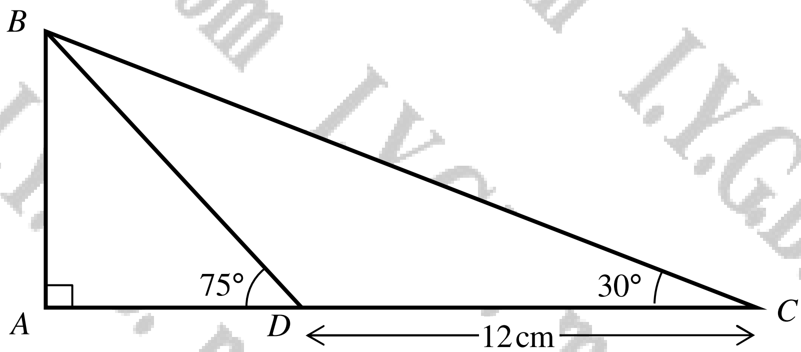
$$|AB| = 15 \text{ cm}, \ |DC| = 9 \text{ cm}, \ |AD| = 10 \text{ cm} \quad \text{and} \quad \measuredangle ABC = \measuredangle BCD = 90°.$$

Determine the length of the straight line AC.

$$\boxed{|AC| = 17 \text{ cm}}$$



Question 11 (**+)



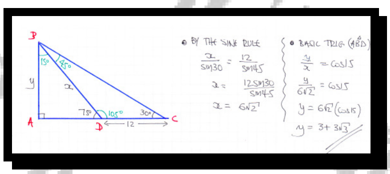
The figure above shows a right angled triangle ABC, where the angle BCA is $30°$.

The point D lies on AC so that the angle BDA is $75°$.

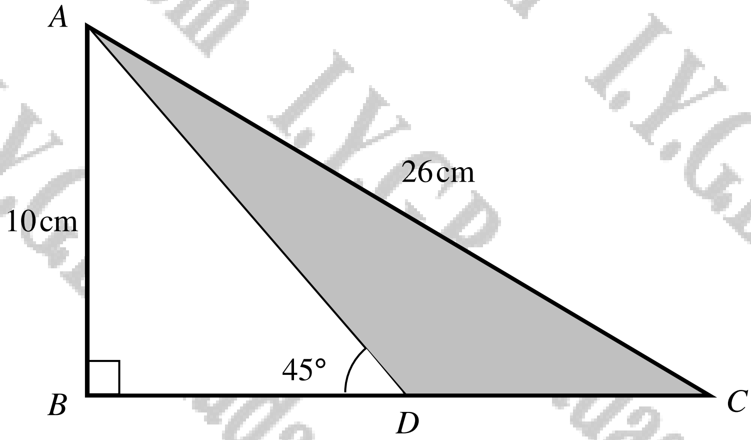
The length of DC is 12 cm.

Calculate the length of AB.

$$\boxed{|AB| = 3 + 3\sqrt{3} \approx 8.20}$$



Question 12 (**+)

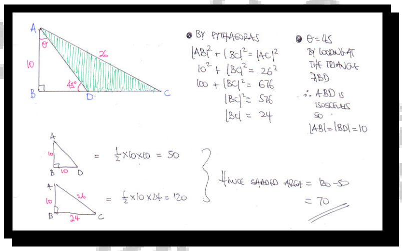


The figure above shows the right angled triangle ABC where AB is 10 cm, AC is 26 cm and the angle ABC is $90°$.

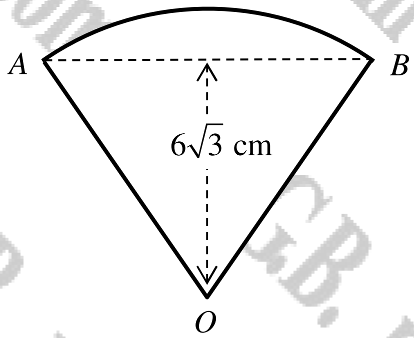
The point D lies on BC so that the angle ADB is $45°$.

Find the area of the triangle ACD, shown shaded in the figure above.

70



Question 13 (**+)



The figure above shows a badge in the shape of a circular sector OAB, centred at O.

The triangle OAB is equilateral and its perpendicular height is $6\sqrt{3}$ cm.

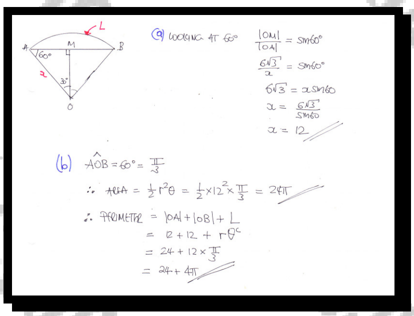
a) Find the length of OA.

b) Determine in terms of π …

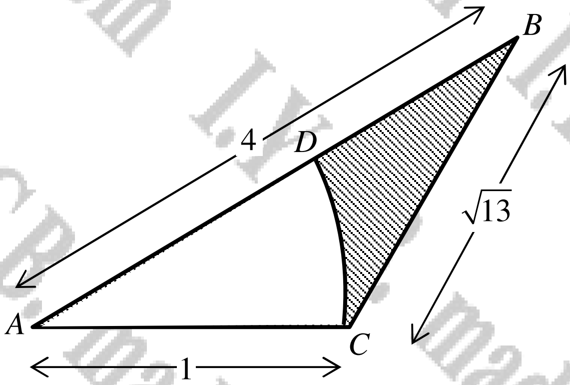
i. … the area of the badge.

ii. … the perimeter of the badge.

$\boxed{|OA| = 12}$, $\boxed{\text{area} = 24\pi}$, $\boxed{\text{perimeter} = 24 + 4\pi}$



Question 14 (**+)



The figure above shows the triangle ABC, where $|AB| = 4$, $|AC| = 1$ and $|BC| = \sqrt{13}$.

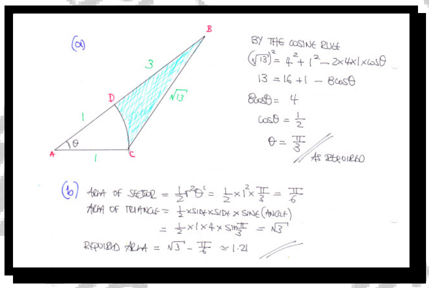
a) Show that $\measuredangle BAC = \frac{\pi}{3}$.

A circular sector ACD, where D lies on AB, is drawn inside the triangle ABC.

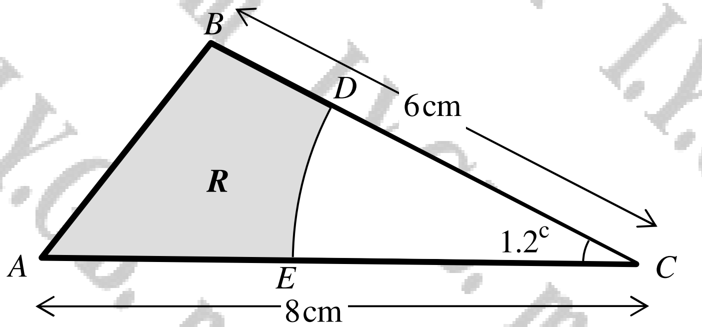
The centre of the sector is at A and its radius is 1.

b) Determine the area of the shaded region BCD.

[ ] , [area ≈ 1.21]



Question 15 (**+)



The figure above shows a triangle ABC where the lengths of AC and BC are 8 cm and 6 cm, respectively. The angle BCA is 1.2 radians.

a) Find the length of AB.

b) Determine the area of the triangle ABC.

A circular arc with centre at C and radius 4 cm is drawn inside the triangle.

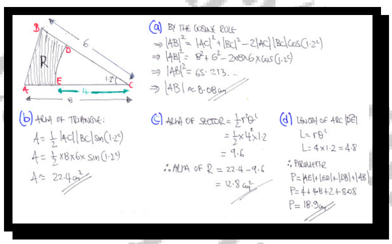
The arc intersects the triangle at the points D and E.

The shaded region R is bounded by the straight lines EA, AB, BD and the arc ED.

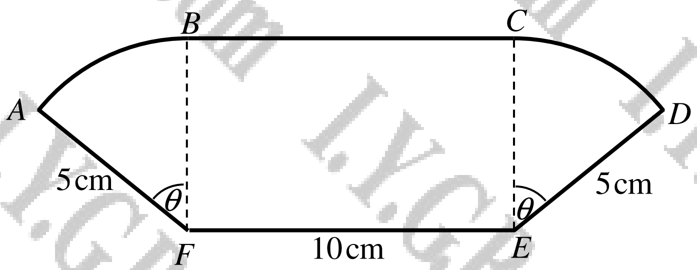
c) Calculate the area of R.

d) Calculate the perimeter of R.

$|AB| \approx 8.08$, $\text{area}_{ABC} \approx 22.4$, $\text{area}_R \approx 12.8$, $\text{perimeter}_R \approx 18.9$



Question 16 (**+)



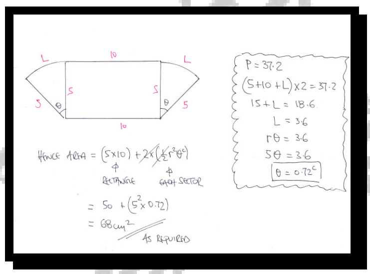
The figure above shows a rectangle $FBCE$ with two identical circular sectors attached to its sides FB and EC.

Each of these circular sectors has radius 5 cm and subtends an angle of θ radians at its respective centre, F and E.

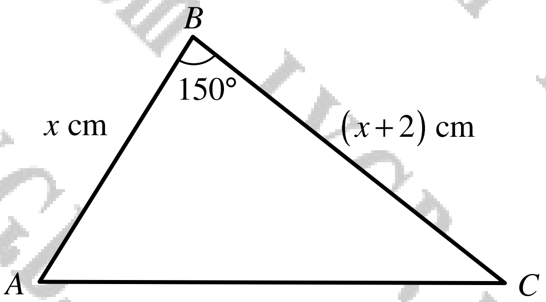
The length of FE is 10 cm.

Given that the perimeter of the **entire** shape $ABCDEF$ is 37.2 cm, show clearly that its area is 68 cm^2.

proof



Question 17 (+)**



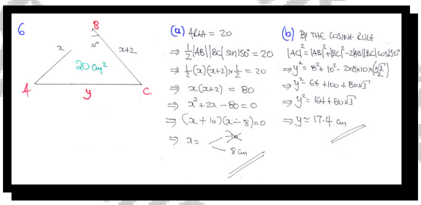
The figure above shows a triangle ABC whose area is 20 cm^2.

The lengths of AB and BC are x cm and $(x+2)$ cm respectively, and the size of the angle ABC is $150°$.

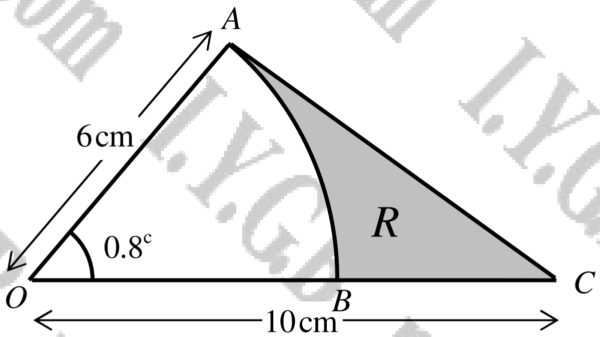
a) Find the value of x.

b) Determine the length of AC.

$\boxed{x=8}$, $\boxed{|AC| \approx 17.4}$



Question 18 (***)



The figure above shows a triangle OAC where $|OA| = 6$ cm, $|OC| = 10$ cm.

The angle AOC is 0.8 radians.

a) Calculate the area of the triangle OAC.

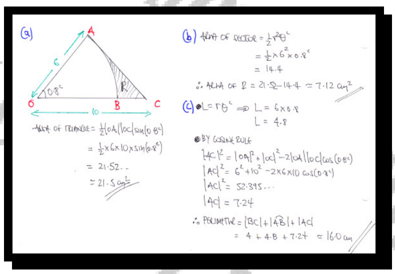
An arc centred at O with radius 6 cm is drawn inside the triangle, meeting OC at B.

The shaded region R is bounded by AC, OC and the arc AB.

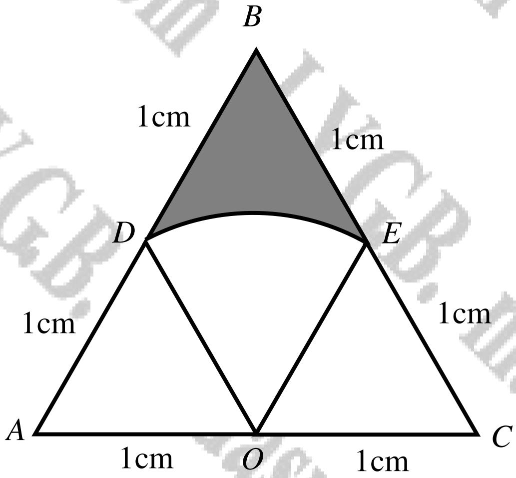
b) Find the area of R.

c) Determine the perimeter of R.

[ ], [area of triangle ≈ 21.52 cm^2], [area of $R \approx 7.12$ cm^2], [perimeter of $R \approx 16.0$ cm]



Question 19 (***)



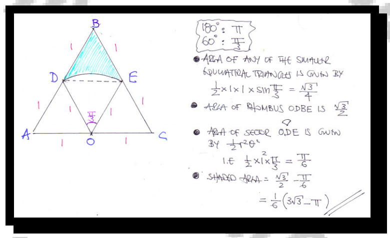
The figure above shows an equilateral triangle ABC of side length 2 cm.

The points O, D and E are the midpoints of AC, AB and BC, respectively.

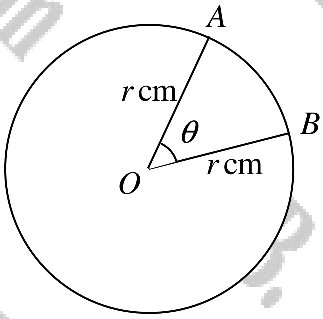
A circular arc, centred at O, having OD and OE as radii is drawn.

Determine the exact area of the shaded region.

$$\boxed{\frac{1}{6}\left(3\sqrt{3}-\pi\right)}$$



Question 20 (***)



The figure above shows a circle with centre at O and radius r cm.

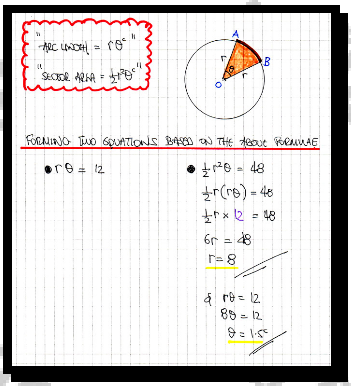
The **minor** sector AOB subtends an angle of θ radians at O.

The area of the **minor sector** AOB is 48 cm^2.

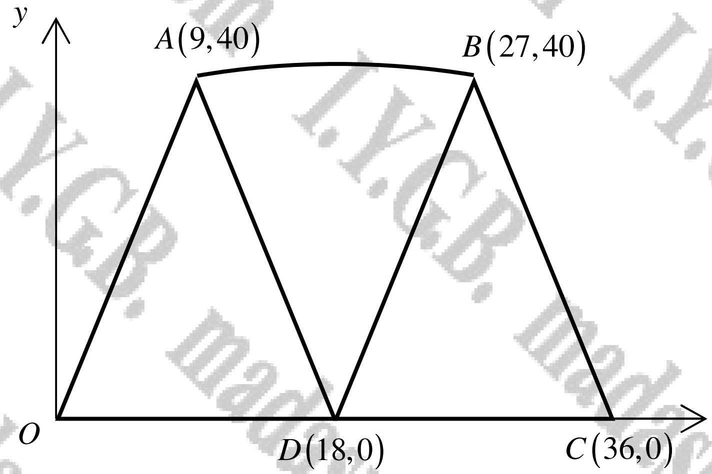
The length of the **minor arc** AB is 12 cm.

Determine the value of r and the value of θ.

$r = 8, \quad \theta = 1.5$



Question 21 (***)



The figure above shows the cross section of a river dam modelled in a system of coordinate axes where all units are in metres.

The cross section of the dam consists of a circular sector ADB and two isosceles triangles OAD and DBC.

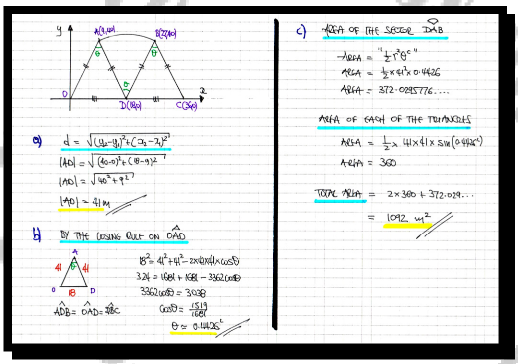
The coordinates of the points A, B, C and D are $(9,40)$, $(27,40)$, $(36,0)$ and $(18,0)$, respectively.

a) Find the length of AD.

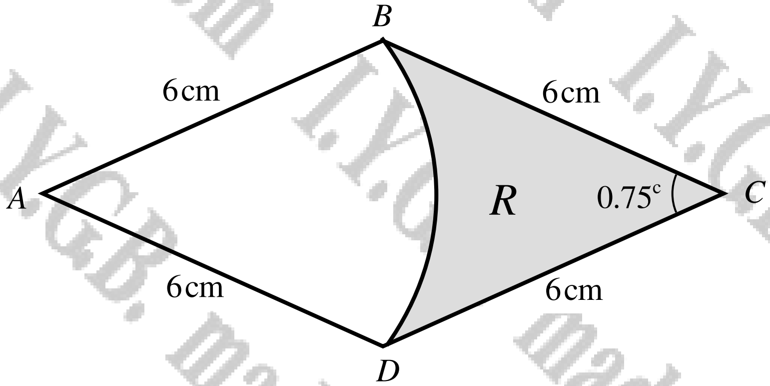
b) Show that the angle ADB is approximately 0.4426 radians.

c) Hence determine, to the nearest m^2, the cross sectional area of the dam.

$\boxed{|AD| = 41}$, $\boxed{\text{area} \approx 1092}$



Question 22 (***)



The figure above shows a rhombus $ABCD$ with side length 6 cm .

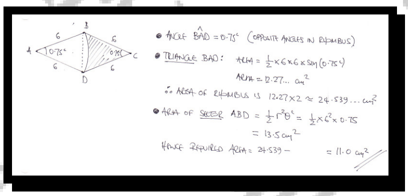
The angle BCD is 0.75 radians.

A circular arc BD is drawn inside the rhombus with centre at A and radius 6 cm .

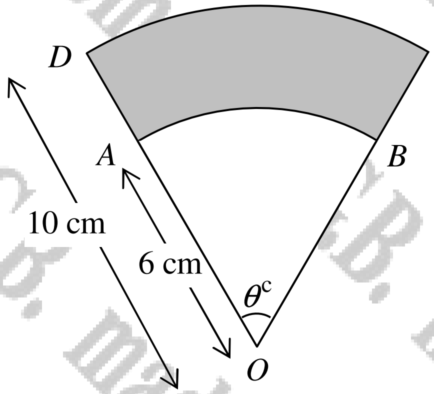
The arc BD divides the rhombus into two regions, the smaller of the two regions shown shaded in the figure, is denoted by R .

Find, to three significant figures, the area of R .

area $\approx 11.0 \text{ cm}^2$



Question 23 (***)

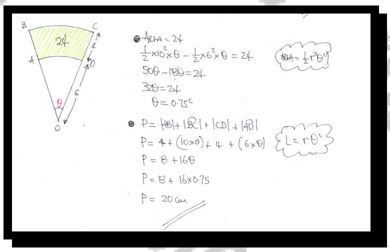


The figure above shows two circular arcs AB and DC, which are parts of circular sectors whose centre is at O. Both sectors subtend an angle θ radians at O.

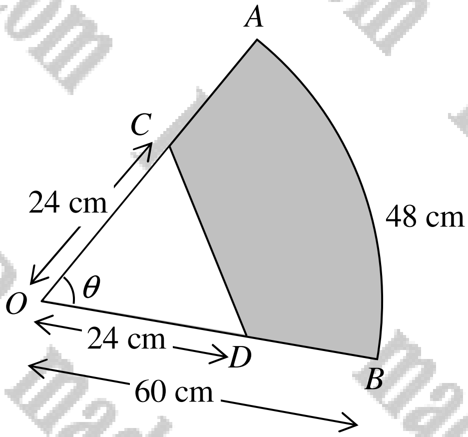
OAD is a straight line segment with $|OA| = 6\text{cm}$ and $|OB| = 10\text{cm}$.

Given that the area of the shaded region $ABCD$ is 24 cm^2, calculate the perimeter of $ABCD$.

perimeter = 20 cm



Question 24 (***)



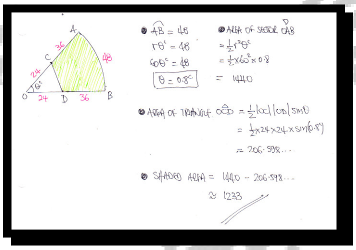
The figure above shows a circular sector OAB whose centre is at O.

The radius of the sector is 60 cm.

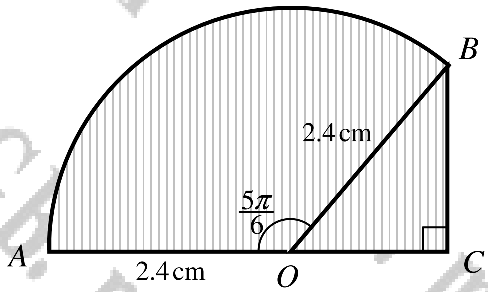
The points C and D lie on OA and OB respectively, so that $|OC| = |OD| = 24$ cm.

Given that the length of the arc AB is 48 cm, find the area of the shaded region $ABDC$, correct to the nearest cm^2.

[ ], area $= 1233$



Question 25 **(***)**



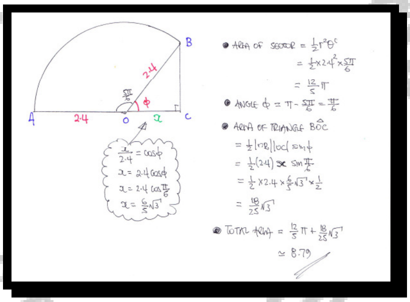
The figure above shows a composite shape.

The composite shape consists of a circular sector AOB centred at O, where it subtends an angle of $\frac{5\pi}{6}$ radians.

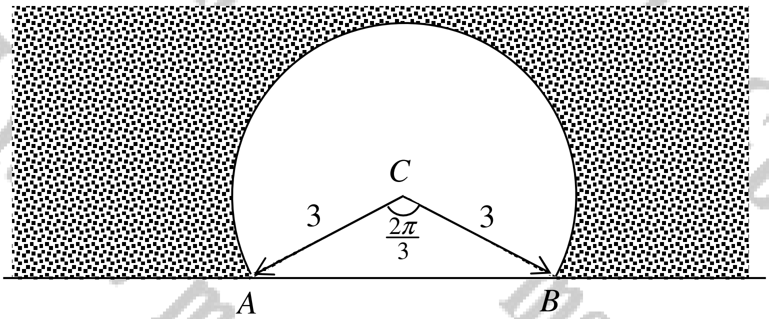
The straight sides of the sector have length of 2.4 cm. The triangle OBC is right angled at C and is attached to the sector so that AOC is a straight line.

Find, to two decimal places, the area of the composite shape.

$\boxed{}$, $\boxed{\text{area} \approx 8.79 \text{ cm}^2}$



Question 26 (***)



The figure above shows the cross section of a railway tunnel, modelled as the **major segment** of a circle, centre at C and radius of 3 m.

The angle ACB is $\frac{2\pi}{3}$ radians.

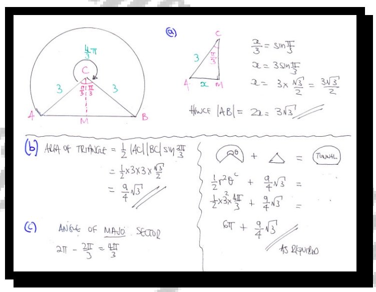
a) Find the exact length of AB.

b) Determine the area of the triangle ACB.

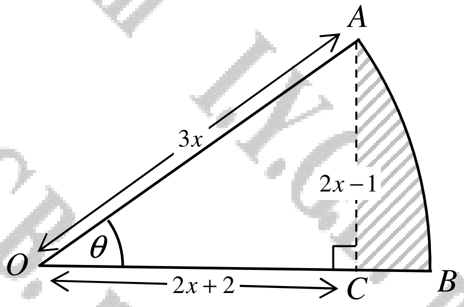
c) Show that the cross sectional area of the tunnel is

$$6\pi + \frac{9}{4}\sqrt{3}.$$

$\boxed{}$, $\boxed{|AB| = 3\sqrt{3}}$, $\boxed{\text{area} = \frac{9\sqrt{3}}{4}}$



Question 27 (***+)



The figure above shows a circular sector OAB of radius $3x$ cm, subtending an angle θ radians at O.

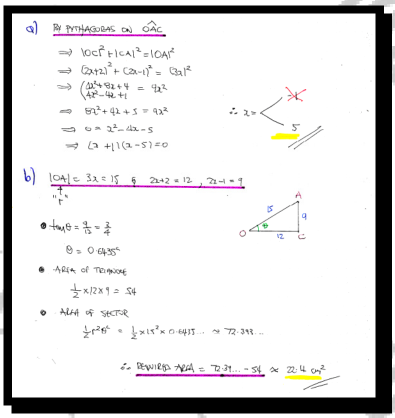
The line AC is perpendicular to OB and has length $(2x-1)$ cm.

The length of OC is $(2x+2)$ cm.

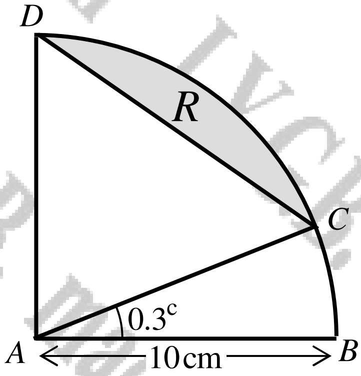
a) Show that $x=5$.

b) Find the area of the shaded region ACB.

[ ], [area ≈ 18.4]



Question 28 (***+)



The figure above shows a quarter circle ABD of radius 10 cm, whose centre is at A.

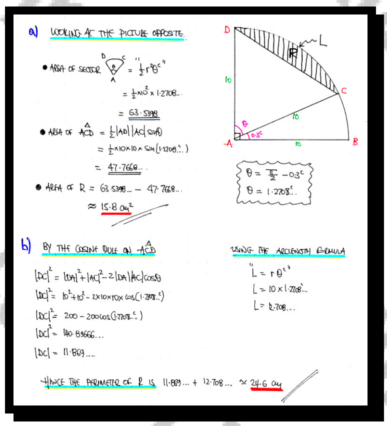
The point C lies on the arc BD so that the angle CAB is 0.3 radians.

The segment bounded by the semicircle and the chord CD is denoted by R.

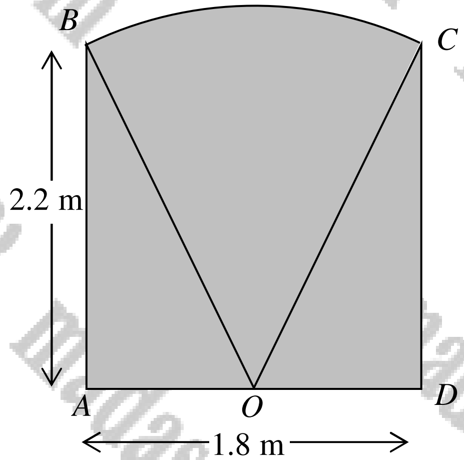
a) Determine the area of R.

b) Find the perimeter of R.

[ ] , area $\approx 15.8 \text{ cm}^2$, perimeter ≈ 24.6 cm



Question 29 (***+)



The figure above shows a design of a door.

The door design consists of two congruent right angled triangles ABO and DCO where $\measuredangle BAO = \measuredangle CDO = 90°$, and a circular sector BOC centred at O, where O is the midpoint of AD.

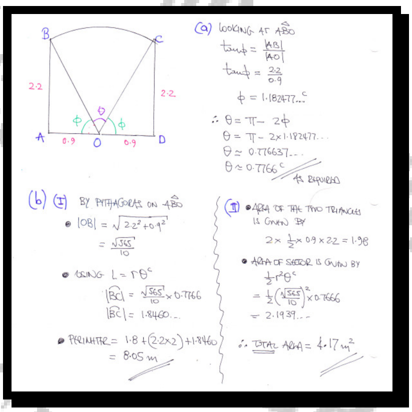
a) Show that the angle BOC is approximately 0.7766 radians.

b) Hence determine …

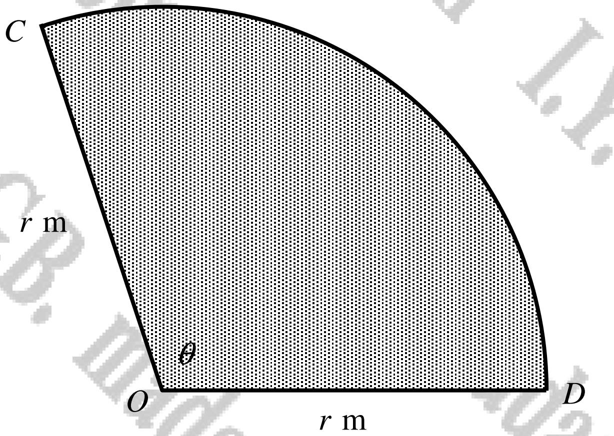
i. … the perimeter of the door design.

ii. … the area of the door design.

$\boxed{}$, $\boxed{P \approx 8.05 \text{ m}}$, $\boxed{A \approx 4.17 \text{ m}^2}$



Question 30 (***+)

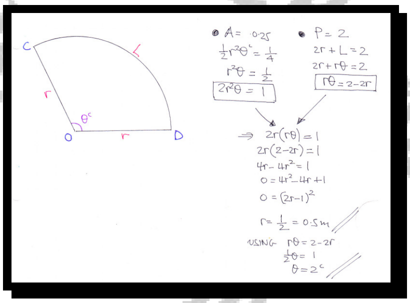


A circular sector OCD, subtending an angle θ radians at its centre O, has a radius of r m.

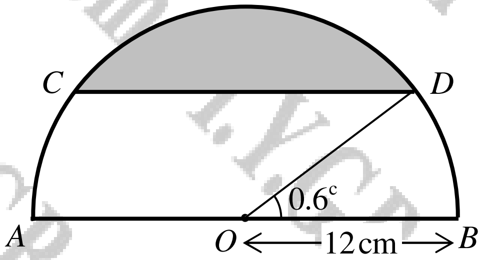
The sector has an area of 0.25 m^2 and a perimeter of 2 m.

Determine the values of r and θ.

$\boxed{r = 0.5 \text{ m}}$, $\boxed{\theta = 2^{\text{c}}}$



Question 31 **(***+)**



The figure above shows a semi circle with centre at O and radius 12 cm.

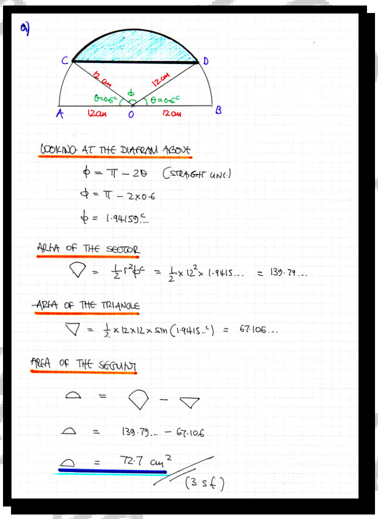
The diameter of the semicircle is AOB, the chord CD is parallel to AOB.

It is further given that the angle DOB is 0.6^c.

a) Find the area of the shaded segment.

b) Find the perimeter of the shaded segment.

[ ], area $\approx 72.7 \text{ cm}^2$, perimeter ≈ 43.1 cm



a)

LOOKING AT THE DIAGRAM ABOVE

$\phi = \pi - 2\theta$ (STRAIGHT LINE)

$\phi = \pi - 2 \times 0.6$

$\phi = 1.94159...^c$

AREA OF THE SECTOR

$\text{sector} = \frac{1}{2}r^2\phi^c = \frac{1}{2} \times 12^2 \times 1.9415... = 139.79...$

AREA OF THE TRIANGLE

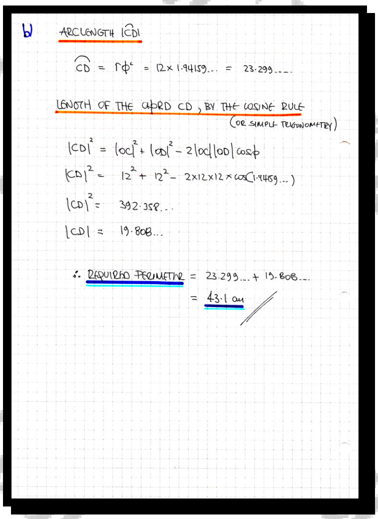
$\text{triangle} = \frac{1}{2} \times 12 \times 12 \times \sin(1.9415...^c) = 67.106...$

AREA OF THE SEGMENT

$\text{segment} = \text{sector} - \text{triangle}$

$\text{segment} = 139.79... - 67.106$

$\text{segment} = 72.7 \text{ cm}^2$ (3 s.f.)



b)

ARCLENGTH $|\widehat{CD}|$

$\widehat{CD} = r\phi^c = 12 \times 1.94159... = 23.299...$

LENGTH OF THE CHORD CD, BY THE COSINE RULE (OR SIMPLE TRIGONOMETRY)

$|CD|^2 = |OC|^2 + |OD|^2 - 2|OC||OD|\cos\phi$

$|CD|^2 = 12^2 + 12^2 - 2 \times 12 \times 12 \times \cos(1.94159...)$

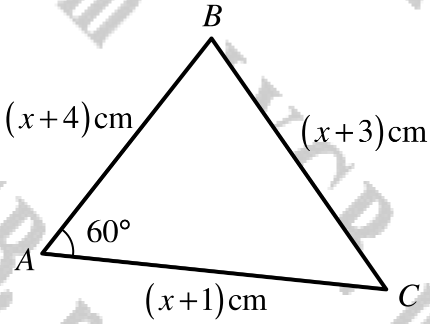
$|CD|^2 = 392.358...$

$|CD| = 19.808...$

$\therefore$ REQUIRED PERIMETER $= 23.299... + 19.808...$

$= 43.1$ cm

Question 32 **(***+)**



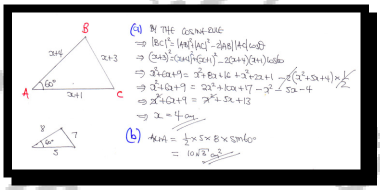
The figure above shows a triangle ABC whose side lengths are given in terms of x.

Given that the angle BAC is $60°$, determine …

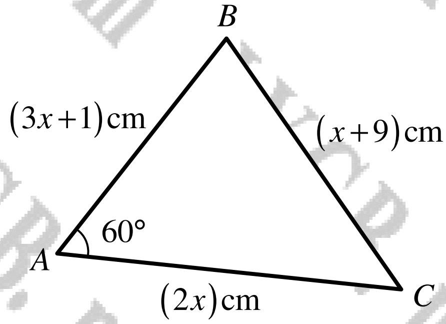
a) … the value of x.

b) … the exact area of the triangle.

$\boxed{x=4}$, $\boxed{\text{area}=10\sqrt{3}}$



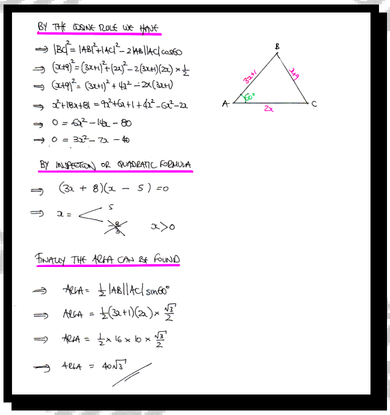
Question 33 (***+)



The figure above shows a triangle ABC whose side lengths are given in terms of x.

Given that the angle BAC is $60°$, determine the exact area of the triangle.

[ ] , $\boxed{\text{area} = 40\sqrt{3}}$



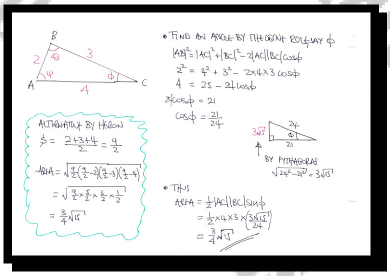
Question 34 (***+)

In the triangle ABC

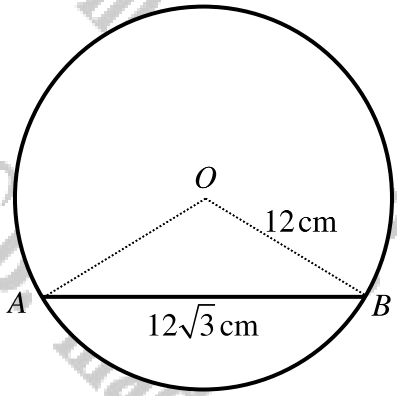
$$|AB| = 2 \text{ cm}, \ |AC| = 4 \text{ cm and } |BC| = 3 \text{ cm}.$$

Find the exact area of the triangle ABC.

$\boxed{}$, $\boxed{\text{area} = \frac{3}{4}\sqrt{15}}$



Question 35 (***+)



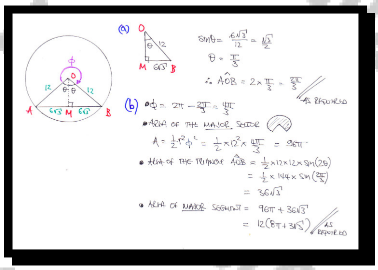
The figure above shows a circle with centre at O and radius 12 cm.

The chord AB has a length of $12\sqrt{3}$ cm.

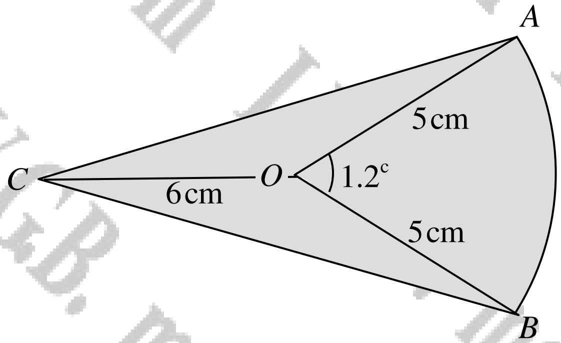
a) Show that the angle AOB is $\dfrac{2\pi}{3}$ radians.

b) Find, in exact form, the area of the **major** segment bounded by the chord AB.

$$\text{area} = 12\left(3\sqrt{3} + 8\pi\right)$$



Question 36 **(***+)**



The figure above shows a template design CAB.

The curve AB is the arc of a circular sector OAB, subtending an angle of 1.2 radians at its centre O. The radius of the sector is 5 cm. The straight lines CA and CB are of equal length. The length of the straight line OC is 6 cm.

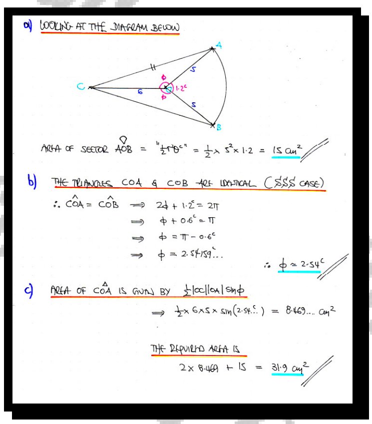
Find, to three significant figures where appropriate, …

a) … the area of the circular sector OAB.

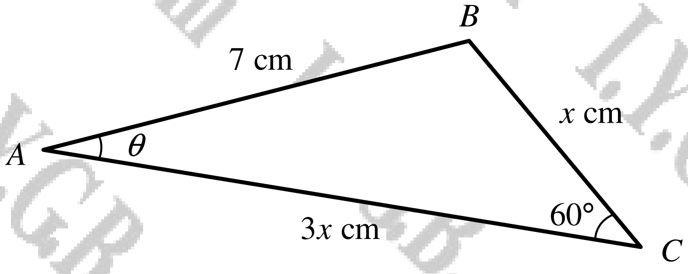
b) … the size of the angle COB, in radians.

c) … the total area of the template design.

$\boxed{}$, $\boxed{\text{area} = 15 \text{ cm}^2}$, $\boxed{\measuredangle COB \approx 2.54^c}$, $\boxed{\text{area} = 31.9 - 32.0 \text{ cm}^2}$



Question 37 **(***+)**



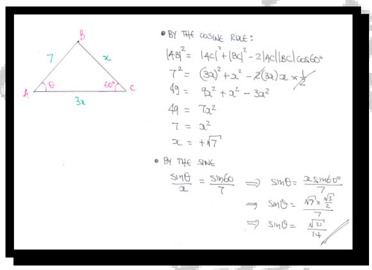
The figure above shows a triangle ABC with side lengths $|AB| = 7$ cm, $|BC| = x$ cm and $|AC| = 3x$ cm.

The sizes of the angles ACB and BAC are $60°$ and $\theta°$, respectively.

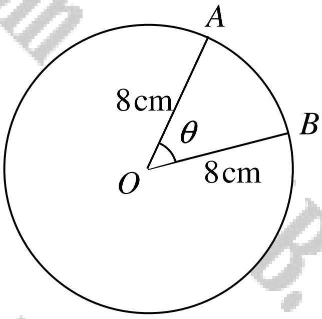
By using the **cosine** rule first and the **sine** rule afterwards, show clearly that

$$\sin\theta = \frac{\sqrt{21}}{14}$$

proof



Question 38 (***+)



The figure above shows a circle with centre at O with radius 8 cm and a **minor** sector AOB, subtending an angle of θ radians at O.

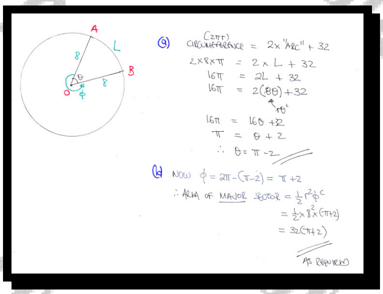
It is further given that the length of the circumference of the circle is twice **plus** 32 cm as large as the minor arc AB.

a) Find the value of θ, in terms of π.

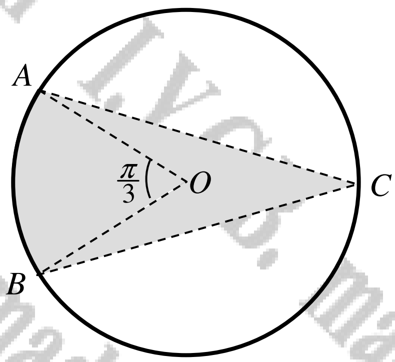
b) Show that the area of the **major** sector AOB is

$$32(\pi+2) \text{ cm}^2.$$

$\boxed{\theta = \pi - 2}$



Question 39 (***+)



The figure above shows the design template of a car logo.

The design consists of a circular ring of radius 6 cm enclosing a region ACB, which is symmetrical about the line OC.

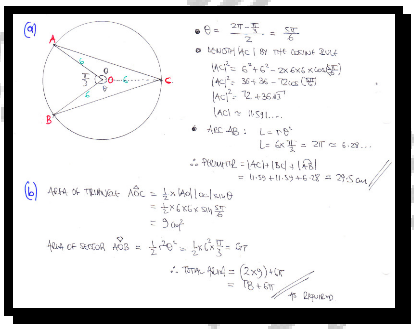
The angle AOB is $\frac{\pi}{3}$.

a) Find, to three significant figures, the perimeter of the shaded region of the logo.

b) Show that the area of the shaded region is

$$(18 + 6\pi)\ \text{cm}^2$$

perimeter ≈ 29.5 cm


(a)

- $\theta = \frac{2\pi - \frac{\pi}{3}}{2} = \frac{5\pi}{6}$
- LENGTH $|AC|$ BY THE COSINE RULE

$|AC|^2 = 6^2 + 6^2 - 2 \times 6 \times 6 \times \cos\left(\frac{5\pi}{6}\right)$

$|AC|^2 = 36 + 36 - 72\cos\left(\frac{5\pi}{6}\right)$

$|AC|^2 = 72 + 36\sqrt{3}$

$|AC| \simeq 11.591\ldots$

- ARC AB: $L = r\theta^c$

$L = 6 \times \frac{\pi}{3} = 2\pi \simeq 6.28\ldots$

$\therefore$ PERIMETER $= |AC| + |BC| + |\widehat{AB}|$

$= 11.59 + 11.59 + 6.28 = 29.5$ cm

(b) AREA OF TRIANGLE $A\hat{O}C = \frac{1}{2} \times |AO||OC| \sin\theta$

$= \frac{1}{2} \times 6 \times 6 \times \sin\frac{5\pi}{6}$

$= 9$ cm²

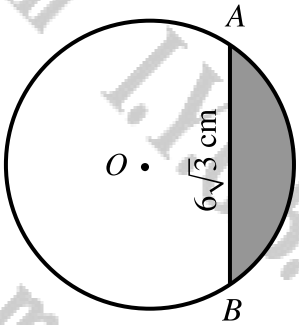
AREA OF SECTOR $A\hat{O}B = \frac{1}{2}r^2\theta^c = \frac{1}{2} \times 6^2 \times \frac{\pi}{3} = 6\pi$

$\therefore$ TOTAL AREA $= (2 \times 9) + 6\pi$

$= 18 + 6\pi$

AS REQUIRED

Question 40 (***+)



The figure above shows a circle with centre at O and radius 6 cm.

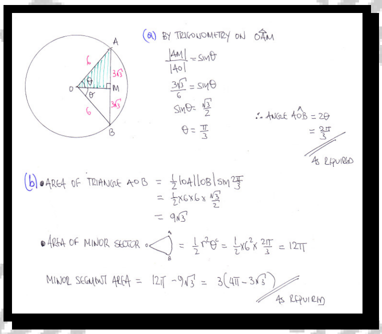
The chord AB has length $6\sqrt{3}$.

a) Show that the angle AOB is $\frac{2\pi}{3}$ radians.

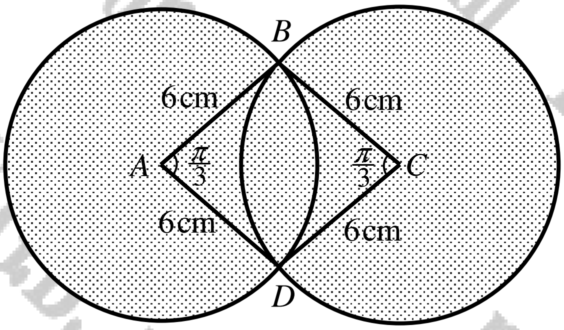
b) Show that the area of the **minor** segment, shown shaded in the figure above, is

$$3\left(4\pi - 3\sqrt{3}\right)\text{cm}^2.$$

proof



Question 41 (***+)



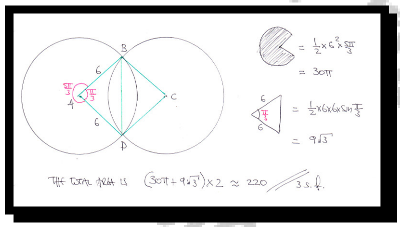
The figure above shows two identical circles with centres at A and C, overlapping each other and meeting at the points B and D.

The radius of each circle is 6 cm. Each of the angles BAD and BCD is $\frac{\pi}{3}$ radians.

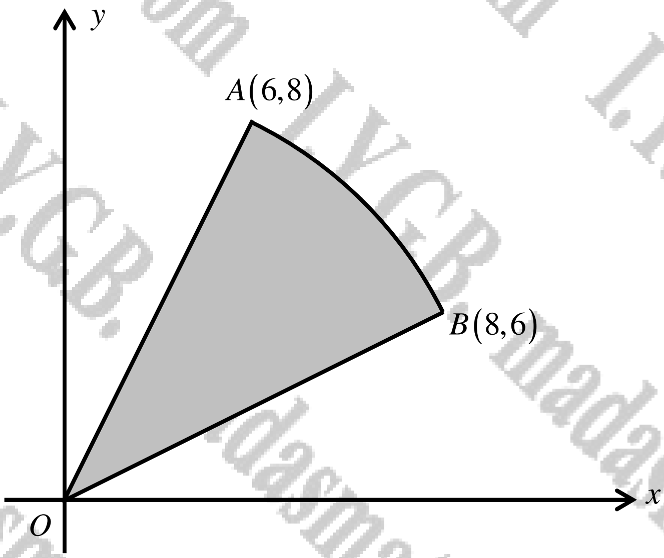
The region, shown shaded in the figure above, enclosed by the two circles, including the overlap, is a car logo design.

Find the area of the logo design.

area ≈ 220 cm^2



Question 42 **(***+)**



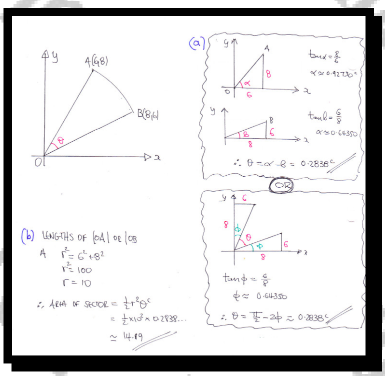
The figure above shows a circular sector OAB with centre at the origin O.

The points A and B have coordinates $(6,8)$ and $(8,6)$, respectively.

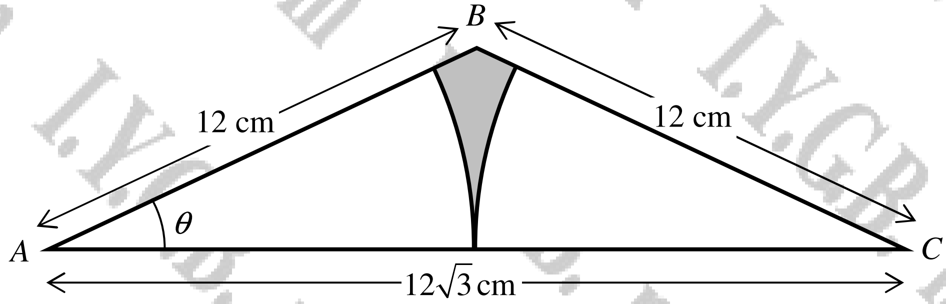
a) Show that the angle AOB is approximately 0.2838 radians.

b) Find, to 2 decimal places, the area of the sector OAB.

[ ] , $\boxed{\text{area} \approx 14.19}$



Question 43 (***+)



An isosceles triangle ABC has $AC = 12\sqrt{3}$ cm and $AB = BC = 12$ cm.

The angle BAC is θ radians.

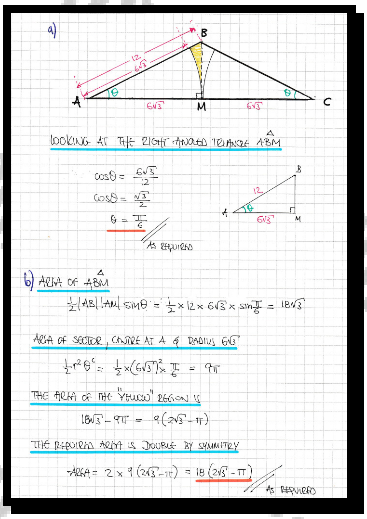
Two identical arcs centred at A and C are drawn inside the triangle. These arcs meet at a point on AC, as shown in the figure above.

a) Show that $\theta = \frac{1}{6}\pi$.

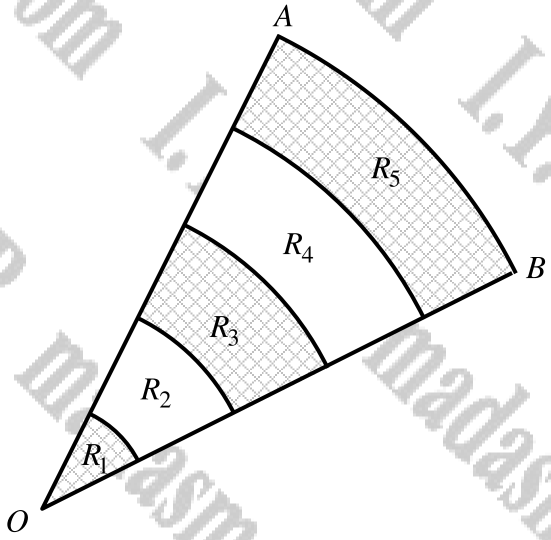
b) Show that the area of the shaded region in the above figure is

$$18\left(2\sqrt{3} - \pi\right)\text{cm}^2.$$

[ ], [proof]



Question 44 **(***+)**



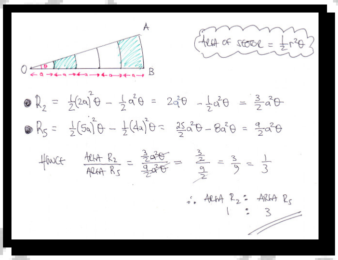
The figure above shows a circular sector OAB.

The sides of the sector are equally divided into five equal parts.

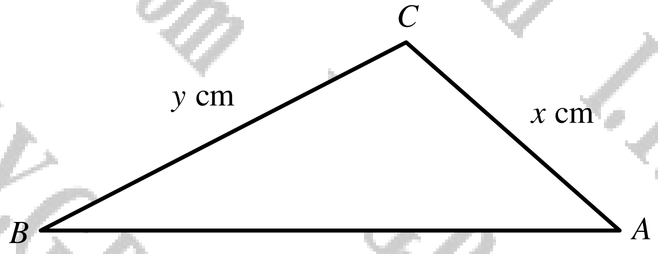
Using these divisions arcs are drawn inside the original sector, creating five distinct regions R_1, R_2, R_3, R_4 and R_5, as shown in the figure.

Show that the areas of the regions R_2 and R_5 are in the ratio $1:3$.

proof



Question 45 (***+)



The figure above shows a triangle ABC.

The lengths of BC and CA are x cm and y cm, respectively.

It is further given that

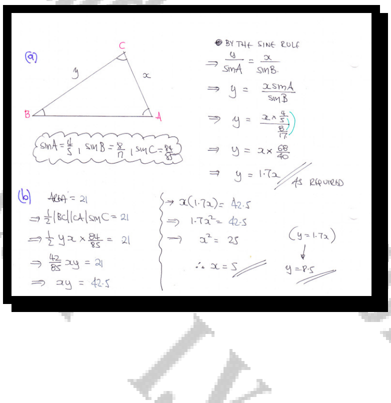
$$\sin A = \frac{4}{5}, \quad \sin B = \frac{8}{17} \quad \text{and} \quad \sin C = \frac{84}{85}.$$

a) Show clearly that $y = 1.7x$.

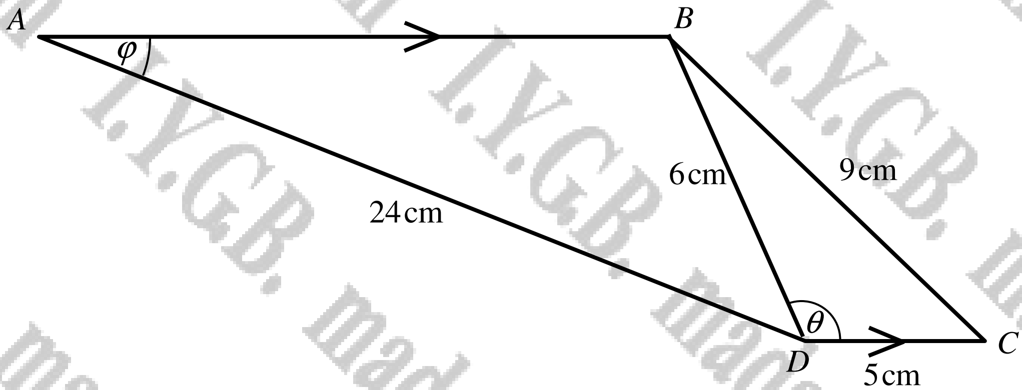
The area of the triangle ABC is 21 cm^2.

b) Find the value of x and the value of y.

$\boxed{}$, $\boxed{x = 5, \quad y = 8.5}$



Question 46 (***+)



The figure above shows a trapezium $ABCD$, where AB is parallel to DC.

The respective lengths of AD, BD, BC and DC are 24 cm, 6 cm, 9 cm and 5 cm.

The angle BDC is θ.

a) Show clearly that

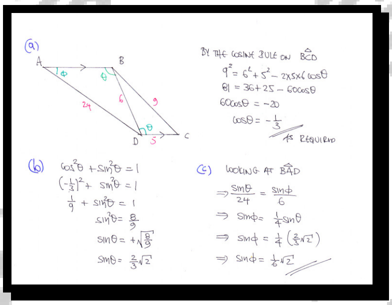
$$\cos\theta = -\frac{1}{3}.$$

b) Hence show further that $\sin\theta = k\sqrt{2}$, where k is a fraction.

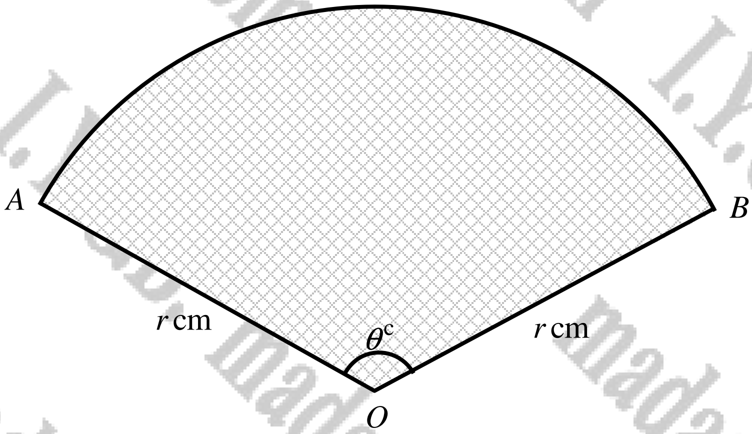
The angle BAD is φ.

c) Find the exact value of $\sin\varphi$.

$\boxed{\sin\theta = \frac{2}{3}\sqrt{2}}$, $\boxed{\sin\varphi = \frac{1}{6}\sqrt{2}}$



Question 47 (***+)



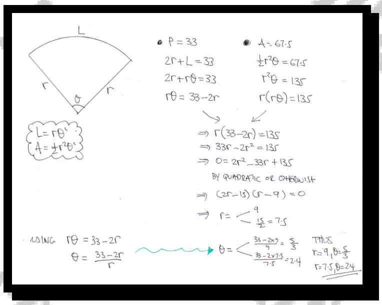
The figure above shows a circular sector OAB, centred at O.

The radius of the sector is r cm and subtends an angle of θ radians at O.

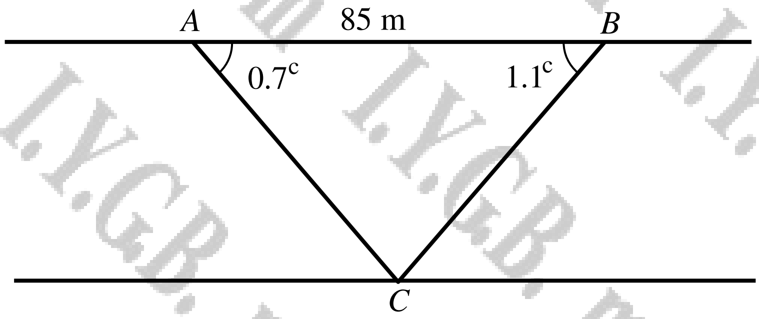
The area of the sector is 67.5 cm^2 and its perimeter is 33 cm.

By forming two suitable equations, or otherwise, determine the two possible pairs of values for r and θ.

$$(r,\theta) = (7.5, 2.4) = \left(9, \frac{5}{3}\right)$$



Question 48 (***+)



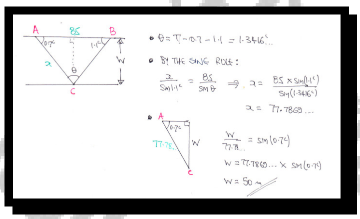
The figure above shows a river of constant width w metres with the points A and B located on one river bank and the point C located on the other river bank.

The distance AB is 85 metres.

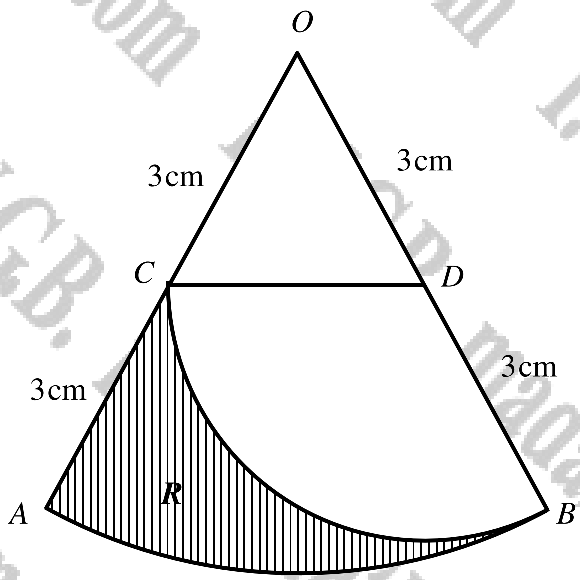
The angles CAB and CBA are 0.7 radians and 1.1 radians, respectively.

Show that w is approximately 50 metres.

[ ], [proof]



Question 49 (***+)



The figure above shows a circular sector OAB, of radius 6 cm, centred at O.

The points C and D are the midpoints of OA and OB, respectively.

The triangle OCD is equilateral.

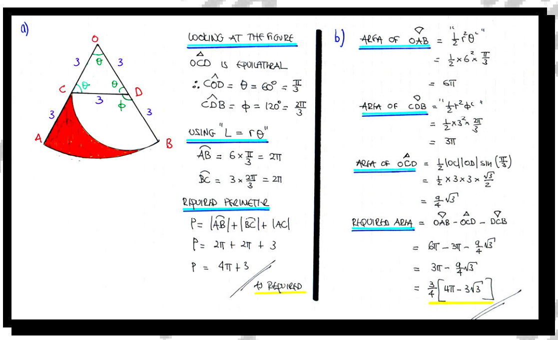
Another circular sector CDB, centred at D and of radius 3 cm, is drawn inside the circular sector OAB.

The finite region R bounded by the circular arcs AB and CB, and the straight line segment AC, is shown shaded in the figure above.

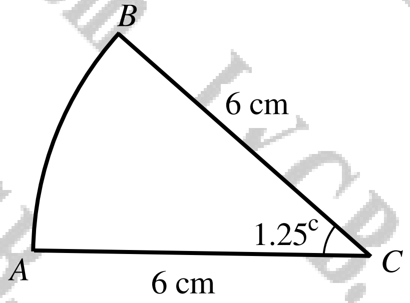
a) Show that the perimeter of R is $(3+4\pi)$ cm.

b) Determine an exact value for the area of R.

[ ] , $3\pi - \frac{9}{4}\sqrt{3}$



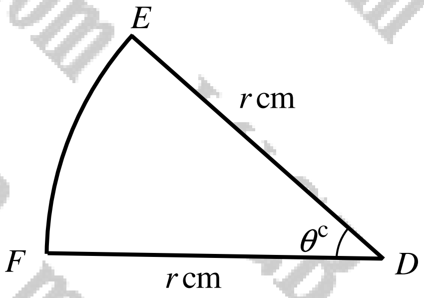
Question 50 (***+)



The figure above shows a circular sector ABC of radius 6 cm subtending an angle 1.25 radians at C.

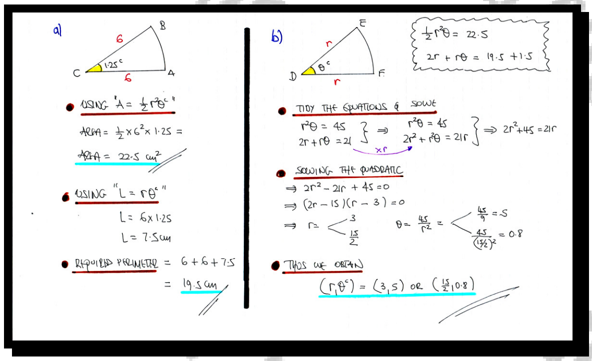
a) Find the perimeter and the area of the sector.

A **different** sector DEF has radius r cm and subtends an angle of θ radians at its centre D.



b) Given that the two sectors have equal area but the perimeter of the sector DEF is 1.5 cm larger than the perimeter of the sector ABC, determine the possible values of r and the corresponding values of θ.

$P = 19.5 \text{ cm}$, $A = 22.5 \text{ cm}^2$, $(r,\theta) = \left(7.5, 0.8^{c}\right)$ or $(r,\theta) = \left(3, 5^{c}\right)$



Question 51 **(***+)**

The triangle ABC has vertices at $A(5,2)$, $B(3,0)$ and $C(-1,6)$.

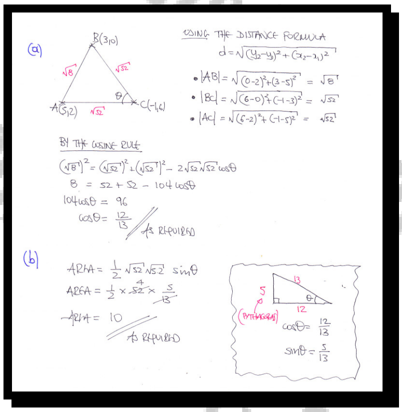
The angle BCA is denoted by θ.

a) Use the cosine rule to show that

$$\cos\theta = \frac{12}{13}.$$

b) Hence, or otherwise, show that the area of the triangle ABC is exactly 10.

proof



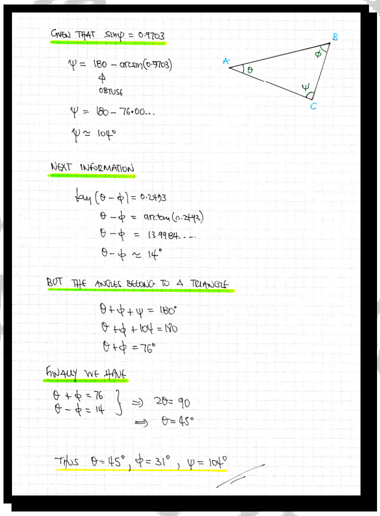
Question 52 (***+)

A triangle has angles θ, φ and ψ, where ψ is an obtuse angle.

It is further given that $\sin\psi = 0.9703$ and $\tan(\theta - \varphi) = 0.2493$.

Calculate, in degrees, the value of each of the angles θ, φ and ψ.

SYN-C, $\boxed{\theta \approx 45°, \ \varphi \approx 31°, \ \psi \approx 104°}$



GIVEN THAT $\sin\psi = 0.9703$

$\psi = 180 - \arcsin(0.9703)$ ← OBTUSE

$\psi = 180 - 76.00...$

$\psi \simeq 104°$

NEXT INFORMATION

$\tan(\theta - \phi) = 0.2493$

$\theta - \phi = \arctan(0.2493)$

$\theta - \phi = 13.9984...$

$\theta - \phi \simeq 14°$

BUT THE ANGLES BELONG TO A TRIANGLE

$\theta + \phi + \psi = 180°$

$\theta + \phi + 104 = 180$

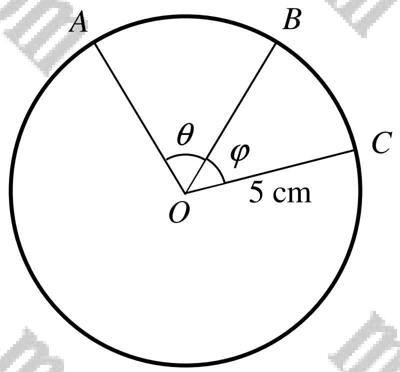
$\theta + \phi = 76°$

FINALLY WE HAVE

$\left.\begin{array}{l}\theta + \phi = 76 \\ \theta - \phi = 14\end{array}\right\} \Rightarrow 2\theta = 90 \Rightarrow \theta = 45°$

THUS $\theta = 45°$, $\phi = 31°$, $\psi = 104°$

Question 53 (***+)



The figure above shows a circle of radius 5 cm, centred at O.

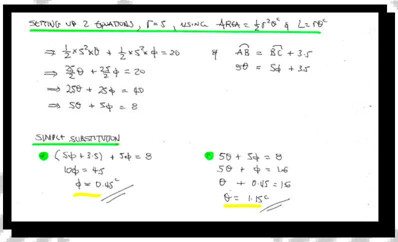
The points A, B and C lie on the circumference of the circle. The angles AOB and BOC are denoted by θ and φ, respectively.

The **sum** of the areas of the sectors AOB and BOC is 20 cm^2.

The length of the arc AB is 3.5 cm **greater** than the length of the arc BC.

Determine the value of θ and the value of φ.

, $\boxed{\theta = 1.15^{\text{c}}}$, $\boxed{\varphi = 0.45^{\text{c}}}$



SETTING UP 2 EQUATIONS, $r=5$, USING AREA $= \frac{1}{2}r^2\theta^c$ & $L = r\theta^c$

$\Rightarrow \frac{1}{2}\times5^2\times\theta + \frac{1}{2}\times5^2\times\phi = 20$ & $\widehat{AB} = \widehat{BC} + 3.5$

$\Rightarrow \frac{25}{2}\theta + \frac{25}{2}\phi = 20$ $\quad 5\theta = 5\phi + 3.5$

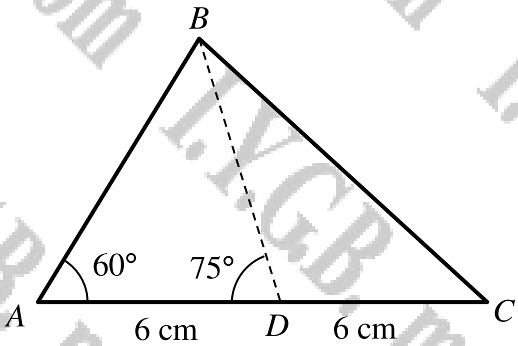
$\Rightarrow 25\theta + 25\phi = 40$

$\Rightarrow 5\theta + 5\phi = 8$

SIMPLE SUBSTITUTION

- $(5\phi + 3.5) + 5\phi = 8$
  $10\phi = 4.5$
  $\phi = 0.45^c$
- $5\theta + 5\phi = 8$
  $5\theta + \phi = 1.6$
  $\theta + 0.45 = 1.6$
  $\theta = 1.15^c$

Question 54 (***+)



The figure above shows a triangle ABC.

The straight line BD is such so that $AD = DC = 6$ cm.

The angles BAD and BDA are $60°$ and $75°$, respectively.

Find in appropriate degree of accuracy …

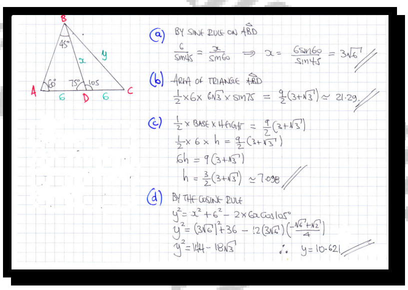
a) … the length of BD.

b) … the area of the triangle of ABD.

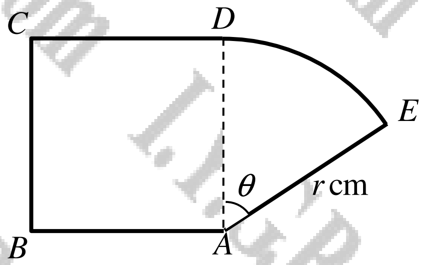
c) … the shortest distance from the vertex B to the straight line AC.

d) … the length of BC.

$\boxed{3\sqrt{6} \approx 7.348}$, $\boxed{\frac{9}{2}\left(3+\sqrt{3}\right) \approx 21.29}$, $\boxed{\frac{3}{2}\left(3+\sqrt{3}\right) \approx 7.098}$, $\boxed{\approx 10.6}$



Question 55 (***+)



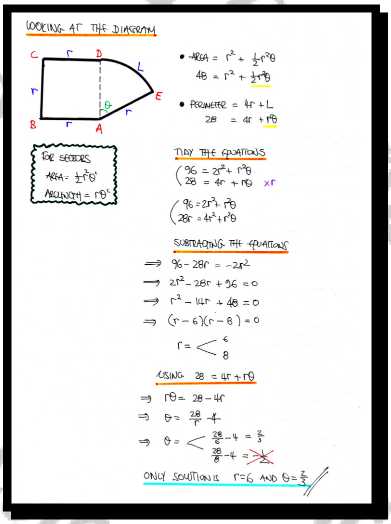
A minor sector ADE with radius r cm, subtends an angle of θ radians at A.

The sector is attached to a square $ABCD$, forming a composite shape S, as shown in figure above.

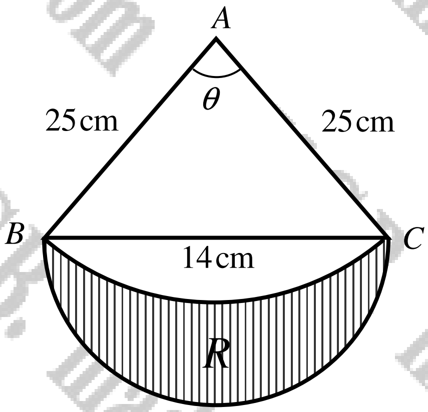
The area and the perimeter of S are $48\,\text{cm}^2$ and $28\,\text{cm}$, respectively.

By forming and solving two equations, find the value of r and the value of θ.

$\boxed{}$, $\boxed{r = 6, \quad \theta = \frac{2}{3}}$



Question 56 (***+)



The figure above shows an isosceles triangle ABC attached to a semicircle with BC as its diameter.

It is further given that $|AB| = |AC| = 25$ cm , $|BC| = 14$ cm and the angle BAC is θ radians.

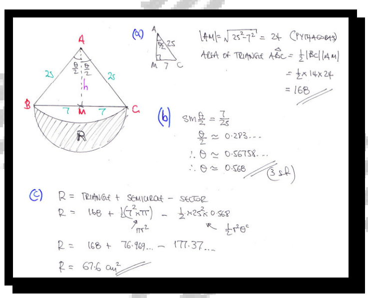
A circular arc BC is drawn inside the semicircle, centred at A with radius 25 cm .

a) Determine the area of the triangle ABC .

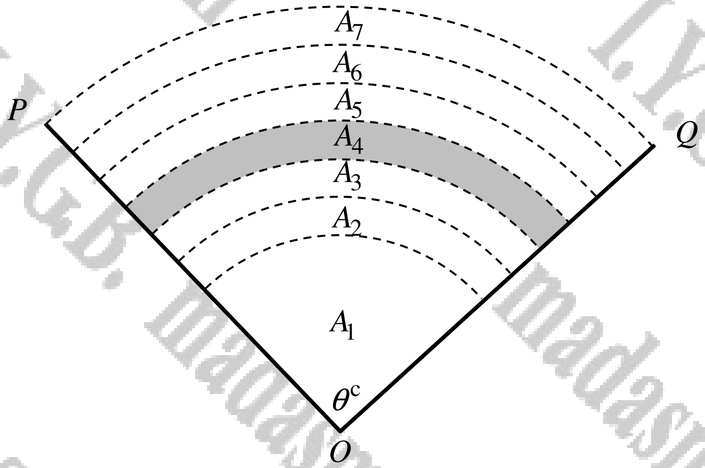
b) Show that $\theta = 0.568$ radians, correct to three significant figures.

c) Find the area of the region R , shown shaded in the figure.

[ ] , [area of triangle $= 168 \text{ cm}^2$] , [area of $R \approx 67.5 - 67.6 \text{ cm}^2$]



Question 57 (****)



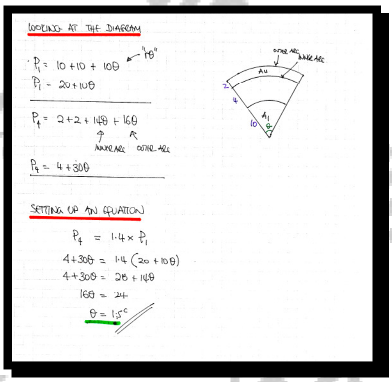
The figure above shows a grid used to help spectators estimate the throwing distances of athletes in a shot put competition. The grid consists of circular sectors each with centre at O and the angle POQ is θ radians.

The radius of the smaller sector is 10 metres and each of the other sectors has a radius 2 metres more than the previous one.

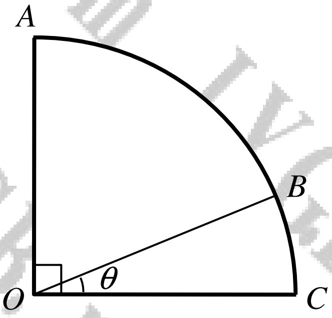
The perimeter of A_4, shown shaded in the figure, is 1.4 times larger than the perimeter of the **sector** A_1.

Determine the value of θ.

$\boxed{}$, $\boxed{\theta = 1.5^c}$



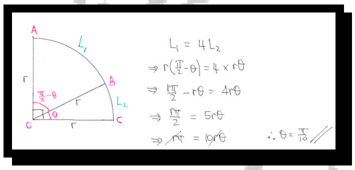
Question 58 (***+)



The figure above shows a quarter circle OAC with centre at O. The point B lies on the curved part of the quarter circle so that the angle BOC is θ radians.

Given that the length of the arc AB is four times as large as the length of the arc BC, show that $\theta = \dfrac{\pi}{10}$.

proof



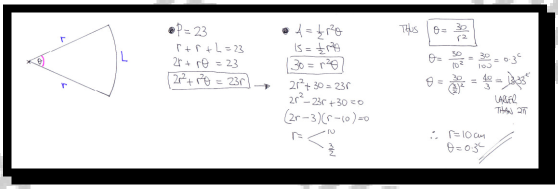
Question 59 (***+)

A circular sector has radius r cm and subtend an angle θ radians at its centre.

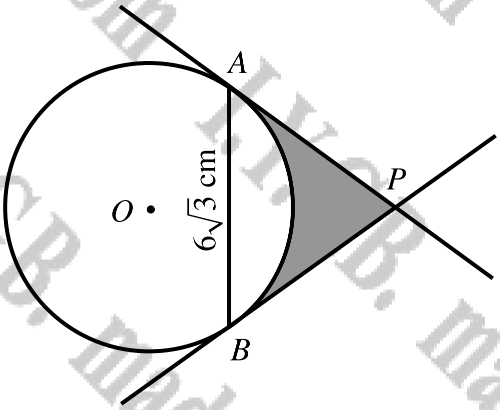
The perimeter of the sector is 23 cm and its area is 15 cm^2.

Find the value of r and the value of θ.

$\boxed{}$, $\boxed{r = 10\text{ cm}}$, $\boxed{\theta = 0.3^c}$



Question 60 (*+)**



The figure above shows a circle with centre at O and radius 6 cm.

The chord AB has length $6\sqrt{3}$ cm.

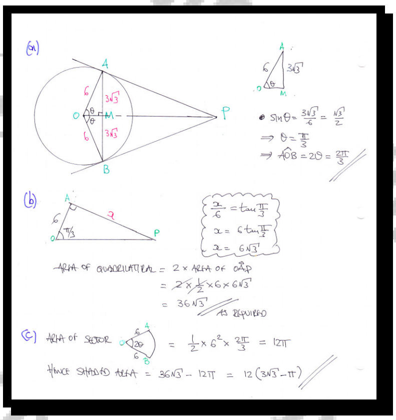
a) Show that the angle AOB is $\frac{2\pi}{3}$ radians.

The tangents to the circle at A and B meet at the point P.

b) Show further that the area of the quadrilateral $OAPB$ is $36\sqrt{3}$ cm^2.

c) Find the area of the shaded region bounded by the tangents and the circle.

area $= 12\left(3\sqrt{3} - \pi\right)$



Question 61 (***+)

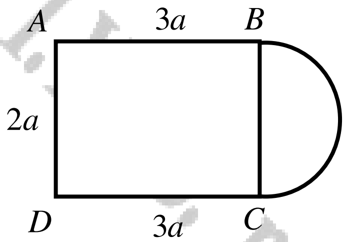


figure 1

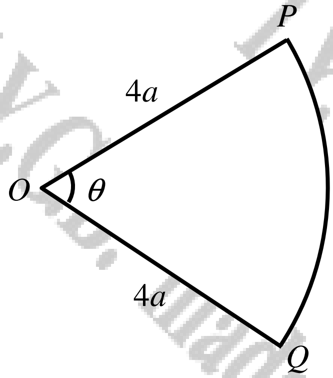


figure 2

Figure 1, shows a rectangle $ABCD$ where $|AB| = |DC| = 3a$ and $|AD| = |BC| = 2a$.

A semicircle with diameter BC is attached to the rectangle. The rectangle and the semicircle are to be considered as a single composite shape X.

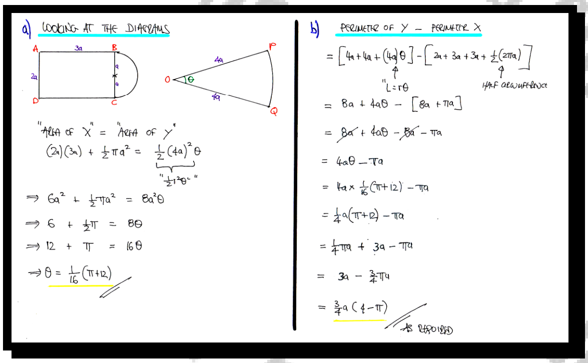
Figure 2, shows a circular sector OPQ where $|OP| = |OQ| = 4a$. The sector has its centre at O, and $\measuredangle POQ = \theta$ radians. The sector is denoted as shape Y.

a) Given that the area of X is equal to the area of Y, express θ in terms of π.

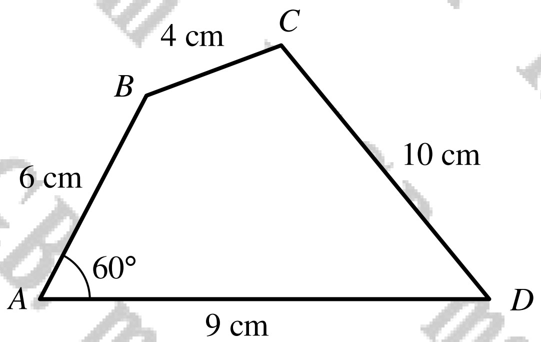
b) Given further that the perimeter Y is greater than the perimeter of X, show that the difference between the perimeter of X and Y is

$$\frac{3}{4}a(4-\pi).$$

$\boxed{}$, $\boxed{\theta = \frac{3}{4} + \frac{\pi}{16}}$



Question 62 (***+)



The figure above shows a quadrilateral $ABCD$, with side lengths AB, BC, CD and DA are 6 cm, 4 cm, 10 cm and 9 cm, respectively.

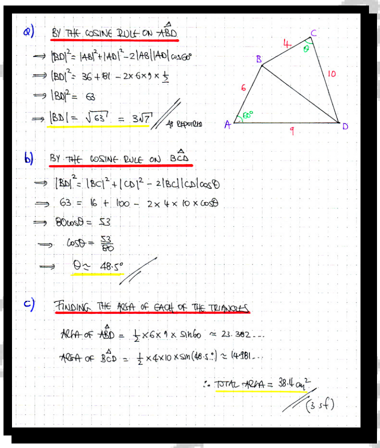
The angle BAD is $60°$.

a) Show that BD is $3\sqrt{7}$ cm.

b) Find, to one decimal place, the size of the angle BCD.

c) Determine, to one decimal place, the area of the quadrilateral $ABCD$.

[ ], [48.5°], [area ≈ 38.4]


a) BY THE COSINE RULE ON $\hat{ABD}$

$\Rightarrow |BD|^2 = |AB|^2 + |AD|^2 - 2|AB||AD|\cos 60°$

$\Rightarrow |BD|^2 = 36 + 81 - 2 \times 6 \times 9 \times \frac{1}{2}$

$\Rightarrow |BD|^2 = 63$

$\Rightarrow |BD| = \sqrt{63} = 3\sqrt{7}$ // AS REQUIRED

b) BY THE COSINE RULE ON $\hat{BCD}$

$\Rightarrow |BD|^2 = |BC|^2 + |CD|^2 - 2|BC||CD|\cos\theta$

$\Rightarrow 63 = 16 + 100 - 2 \times 4 \times 10 \times \cos\theta$

$\Rightarrow 80\cos\theta = 53$

$\Rightarrow \cos\theta = \frac{53}{80}$

$\Rightarrow \theta \simeq 48.5°$ //

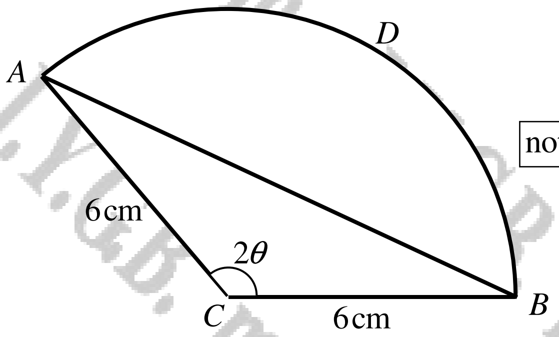
c) FINDING THE AREA OF EACH OF THE TRIANGLES

AREA OF $\hat{ABD} = \frac{1}{2} \times 6 \times 9 \times \sin 60 \simeq 23.382...$

AREA OF $\hat{BCD} = \frac{1}{2} \times 4 \times 10 \times \sin(48.5...°) \simeq 14.981...$

$\therefore$ TOTAL AREA $= 38.4$ cm^2 // (3 s.f.)

Question 63 (***+)



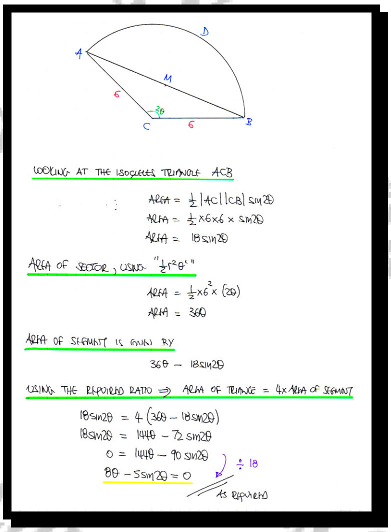
not drawn accurately

The figure above shows a sector $CADB$, of radius 6 cm and angle 2θ radians.

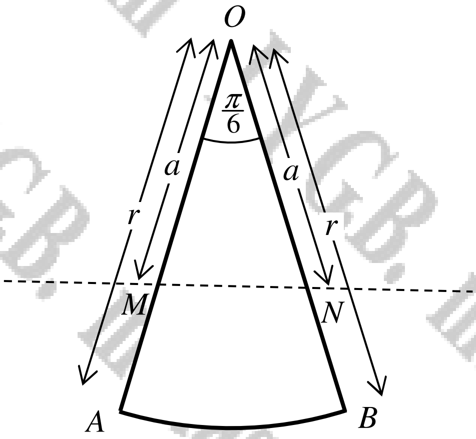
Given that the area of the triangle ABC and the area of segment ABD are in the ratio $4:1$, show that

$$8\theta - 5\sin 2\theta = 0.$$

proof



Question 64 (***+)



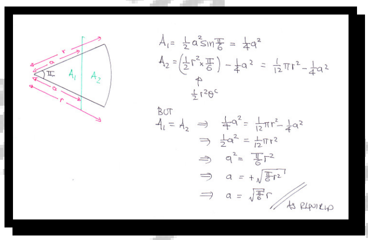
The figure above shows a circular sector OAB. The sector has radius r cm and subtends an angle of $\frac{\pi}{6}$ at O.

The straight line through M and N is such so that $OM = ON = a$ cm.

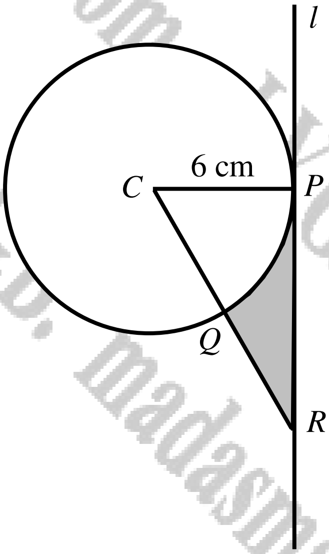
Given that the straight line through M and N divides the sector into two regions of equal area, show that

$$a = \sqrt{\frac{\pi}{6}}\, r.$$

[ ], [proof]



Question 65 (***+)



The figure above shows a circle of radius 6 cm, centre at point C, and the straight line l which is a tangent to the circle at the point P.

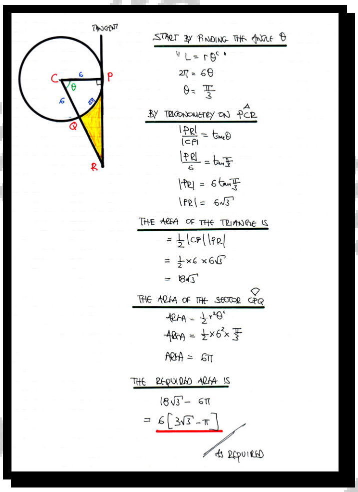
The point R lies on l.

The straight line segment CR meets the circle at the point Q.

Given that the length of the arc QP is 2π cm, show that the area of the finite region bounded by PR, RQ and QP, shown shaded in the figure, is

$$6\left(3\sqrt{3}-\pi\right).$$

[ ], [proof]



START BY FINDING THE ANGLE θ

"$L = r\theta^c$"

$2\pi = 6\theta$

$\theta = \frac{\pi}{3}$

BY TRIGONOMETRY ON $\hat{PCR}$

$\frac{|PR|}{|CP|} = \tan\theta$

$\frac{|PR|}{6} = \tan\frac{\pi}{3}$

$|PR| = 6\tan\frac{\pi}{3}$

$|PR| = 6\sqrt{3}$

THE AREA OF THE TRIANGLE IS

$= \frac{1}{2}|CP||PR|$

$= \frac{1}{2}\times 6 \times 6\sqrt{3}$

$= 18\sqrt{3}$

THE AREA OF THE SECTOR CPQ

AREA $= \frac{1}{2}r^2\theta^c$

AREA $= \frac{1}{2}\times 6^2 \times \frac{\pi}{3}$

AREA $= 6\pi$

THE REQUIRED AREA IS

$18\sqrt{3} - 6\pi$

$= 6\left[3\sqrt{3} - \pi\right]$

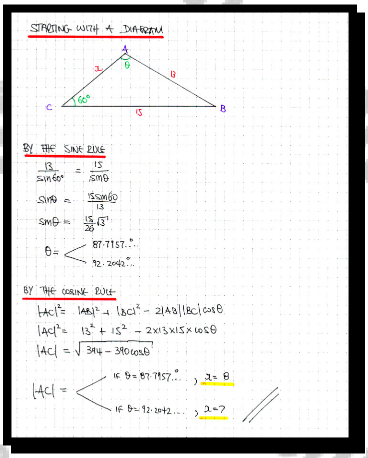
AS REQUIRED

Question 66 (***+)

The triangle ABC has $AB = 13$ cm and $BC = 15$ cm.

Given that $\measuredangle BCA = 60°$, determine the possible values of AC.

$\boxed{}$, $\boxed{|AC| = 7 \text{ cm} \quad \text{or} \quad 8 \text{ cm}}$



Question 67 (***+)

Linda is walking on a long straight horizontal road in a Northern direction.

When Linda reaches a point A on this road, a tree T is observed on a bearing of $30°$.

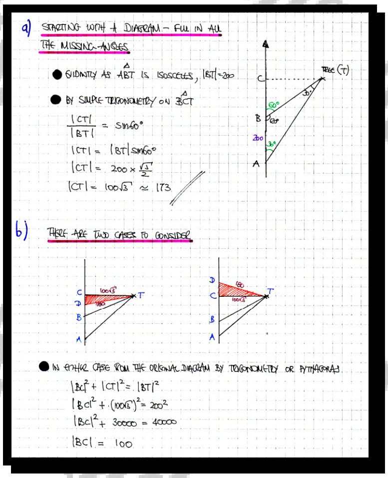
When Linda walks a further distance of 200 m from the point A to the point B on this road, T is now observed on a bearing of $60°$.

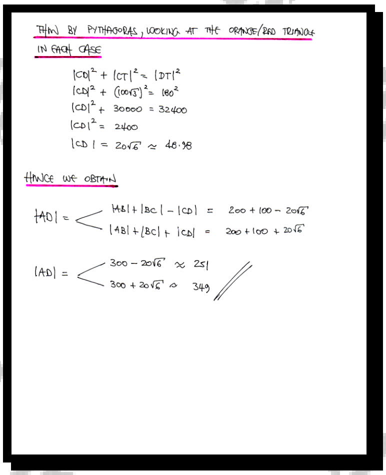
a) Determine the shortest distance of T from the road.

Linda walks further North to some point D, so that the distance DT is 180 m.

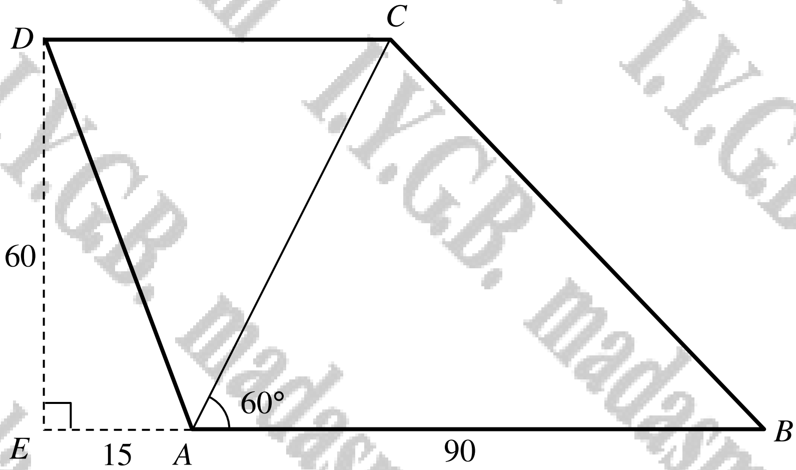
b) Calculate the two possible values for the distance AD.

$\boxed{100\sqrt{3} \approx 173}$, $\boxed{|AD| = 300 \pm 20\sqrt{6}}$





Question 68 (****)



$ABCD$ is a trapezium where AB is parallel to DC.

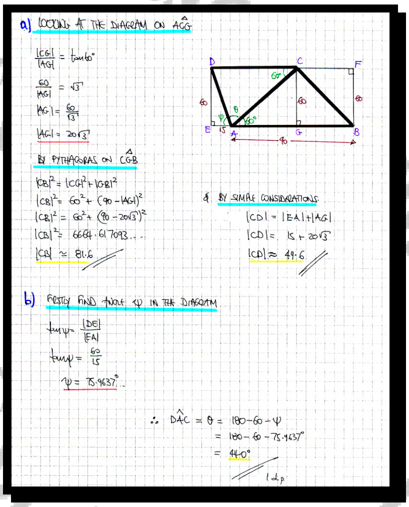
The angle CAB is $60°$ and $|AB| = 90$. The side AB is extended from A to E so that $\measuredangle AED = 90°$, as shown in the figure above.

It is further given that $|EA| = 15$ and $|ED| = 60$.

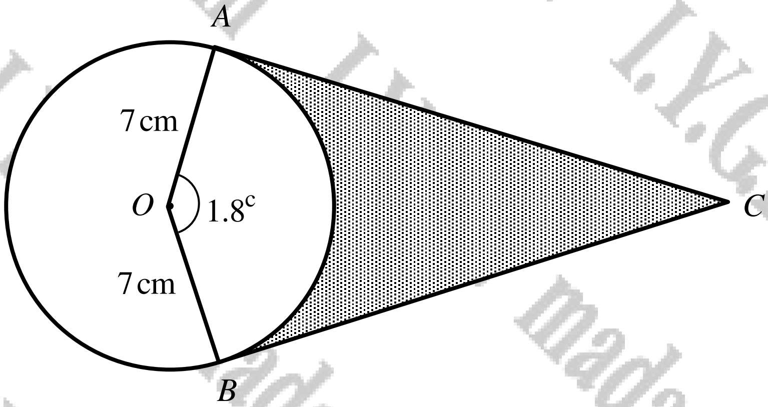
a) Find, correct to 1 decimal place, the value of $|BC|$ and the value of $|CD|$.

b) Calculate, correct to 1 decimal place, the angle DAC.

$\boxed{}$, $\boxed{|BC| \approx 81.6}$, $\boxed{|CD| \approx 49.6}$, $\boxed{\measuredangle DAC \approx 44.0}$



Question 69 (****)



The figure above shows a circle with centre at O and radius 7 cm.

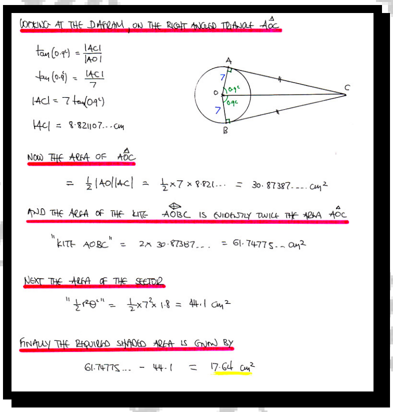
The points A and B lie on the circle so that the angle AOB is 1.8 radians.

The tangents to the circle at the points A and B meet at the point C.

The region shown shaded in the figure above, is enclosed by the two tangents AC and BC, and the circle.

Determine the area of this region.

$\boxed{}$, $\boxed{\text{area} \approx 17.6 \text{ cm}^2}$


LOOKING AT THE DIAGRAM, ON THE RIGHT ANGLED TRIANGLE $\hat{AOC}$

$\tan(0.9^c) = \frac{|AC|}{|AO|}$

$\tan(0.9^c) = \frac{|AC|}{7}$

$|AC| = 7\tan(0.9^c)$

$|AC| = 8.821107\ldots$ cm

NOW THE AREA OF $\hat{AOC}$

$= \frac{1}{2}|AO||AC| = \frac{1}{2} \times 7 \times 8.821\ldots = 30.87387\ldots \text{ cm}^2$

AND THE AREA OF THE KITE AOBC IS EVIDENTLY TWICE THE AREA $\hat{AOC}$

"KITE AOBC" $= 2 \times 30.87387\ldots = 61.74775\ldots \text{ cm}^2$

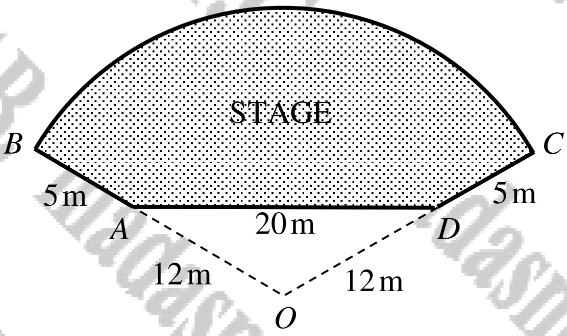
NEXT THE AREA OF THE SECTOR

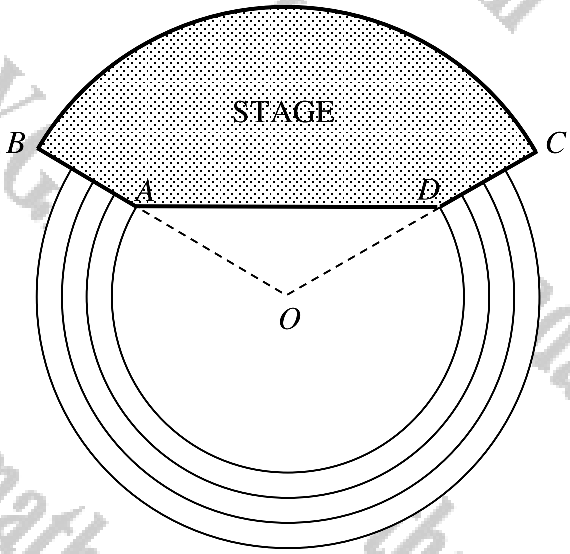
"$\frac{1}{2}r^2\theta^c$" $= \frac{1}{2} \times 7^2 \times 1.8 = 44.1 \text{ cm}^2$

FINALLY THE REQUIRED SHADED AREA IS GIVEN BY

$61.74775\ldots - 44.1 = 17.64 \text{ cm}^2$

Question 70 (****)





The two diagrams above show an orchestral stage $ABCD$ which is part of a circular sector OBC, centred at O and of radius 17 m. The points A and D lie on OB and OC respectively so that $|OA| = |OD| = 12\text{ m}$ and $|AD| = 20\text{ m}$.

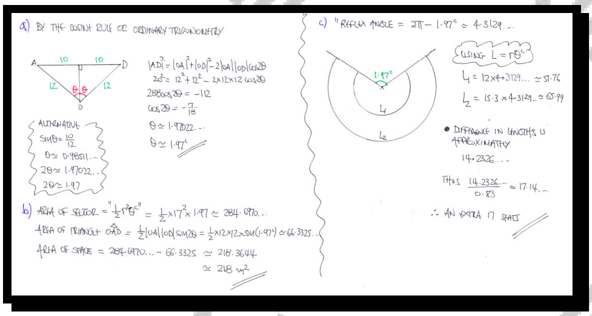
a) Show that $\measuredangle BOC = 1.97$, correct to three significant figures.

b) Calculate the area of the stage.

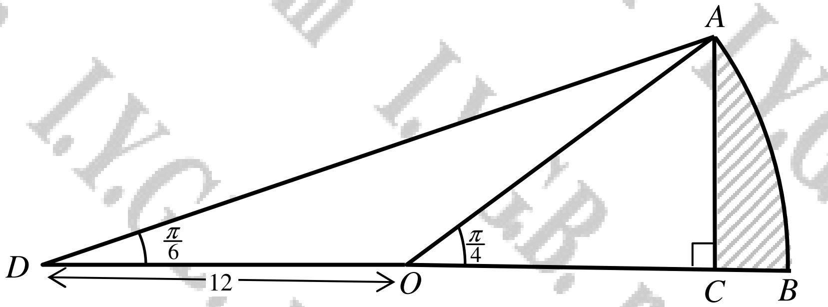
There are 4 rows of seats with their backs arranged in concentric circles, centred at O. The radii of these circles are 12 m, 13.1 m, 14.2 m and 15.3 m.

c) Given further that each seat requires a length of 83 cm along the arc, find approximately how many more seats are in the back row than in the front row.

[ ], [area $\approx 218\text{ m}^2$], [an extra 17 seats]



Question 71 (****)



The figure above shows a triangle OAC with $\measuredangle ACO = \frac{1}{2}\pi$ and $\measuredangle AOC = \frac{1}{4}\pi$.

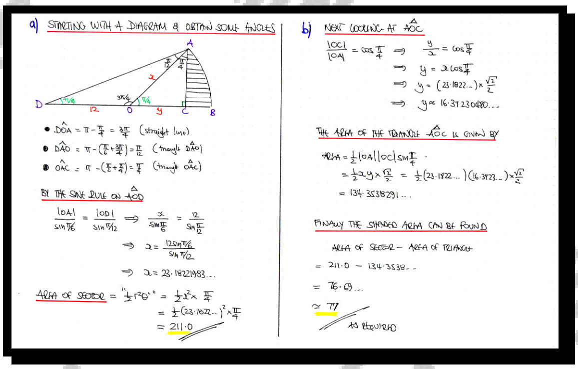
Another triangle AOD is drawn next to the triangle OAC, so that DOC is a straight line, $|DO| = 12$ units and $\measuredangle ADO = \frac{1}{6}\pi$.

Finally a circular sector OAB is drawn, centred at O, with radius OA, so that $DOCB$ is a straight line.

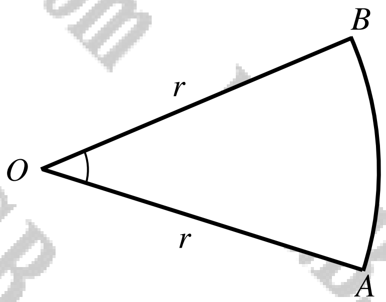
a) Find the area of the sector OAB.

b) Hence show that the area of the shaded region ACB is approximately 77 square units.

[ ], [211.0]



Question 72 (****)



The figure above shows a circular sector OAB of radius r, centred at O, with **perimeter** of 60 units. The area of the sector is denoted by A.

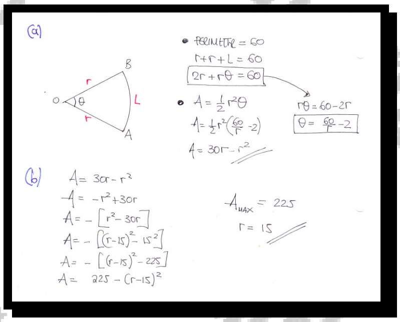
a) Show clearly that

$$A = 30r - r^2 .$$

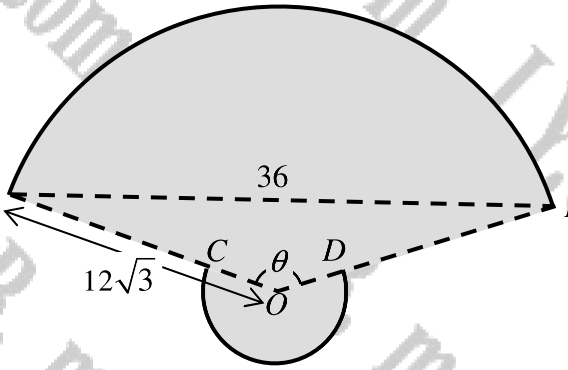
The value of r can vary but the perimeter of the sector is fixed.

b) By completing the square, or otherwise, find the maximum value of A and the value of r which produces this maximum value for A.

$\boxed{A_{\text{max}} = 225}$, $\boxed{r = 15}$



Question 73 (****)



The figure above shows a model of the region used by shot putters in to throw the shot. The throwing region consists of a **minor** circular sector OAB of radius $12\sqrt{3}$ metres subtending an angle θ radians at O. The chord AB is 36 metres.

The shot putter's region COD is a **major** circular sector of radius $3\sqrt{3}$ metres, where C and D lie on OA and OB, respectively.

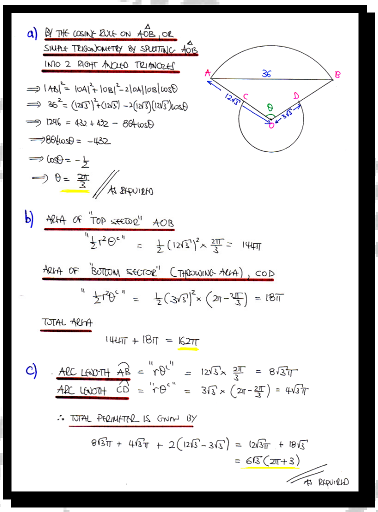
a) Show that $\theta = \frac{2}{3}\pi$.

b) Find, in terms of π, the total area of throwing region and shot putter's region.

c) Show further that the total perimeter of the throwing region and the shot putter's region, shown shaded in the figure above, is

$$6(2\pi+3)\sqrt{3}.$$

$\boxed{}$, $\boxed{162\pi}$



a) BY THE COSINE RULE ON $\hat{AOB}$, OR SIMPLE TRIGONOMETRY BY SPLITTING $\hat{AOB}$ INTO 2 RIGHT ANGLED TRIANGLES

$\Rightarrow |AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta$

$\Rightarrow 36^2 = (12\sqrt{3})^2 + (12\sqrt{3})^2 - 2(12\sqrt{3})(12\sqrt{3})\cos\theta$

$\Rightarrow 1296 = 432 + 432 - 864\cos\theta$

$\Rightarrow 864\cos\theta = -432$

$\Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{2\pi}{3}$ // AS REQUIRED

b) AREA OF "TOP SECTOR" AOB

"$\frac{1}{2}r^2\theta^c$" $= \frac{1}{2}(12\sqrt{3})^2 \times \frac{2\pi}{3} = 144\pi$

AREA OF "BOTTOM SECTOR" (THROWING AREA), COD

"$\frac{1}{2}r^2\theta^c$" $= \frac{1}{2}(3\sqrt{3})^2 \times \left(2\pi - \frac{2\pi}{3}\right) = 18\pi$

TOTAL AREA

$144\pi + 18\pi = 162\pi$

c) ARC LENGTH $\overset{\frown}{AB}$ = "$r\theta^c$" $= 12\sqrt{3} \times \frac{2\pi}{3} = 8\sqrt{3}\pi$

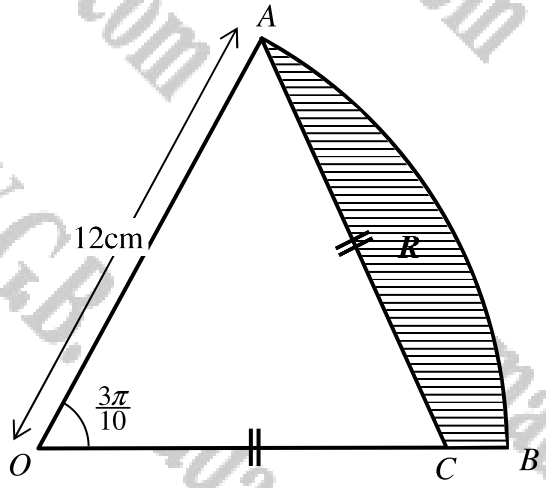
ARC LENGTH $\overset{\frown}{CD}$ = "$r\theta^c$" $= 3\sqrt{3} \times \left(2\pi - \frac{2\pi}{3}\right) = 4\sqrt{3}\pi$

∴ TOTAL PERIMETER IS GIVEN BY

$8\sqrt{3}\pi + 4\sqrt{3}\pi + 2(12\sqrt{3} - 3\sqrt{3}) = 12\sqrt{3}\pi + 18\sqrt{3}$

$= 6\sqrt{3}(2\pi+3)$ // AS REQUIRED

Question 74 (****)



The figure above shows a circular arc OAB of radius 12 cm, subtending an angle of $\frac{3}{10}\pi$ radians at O.

Find to three significant figures …

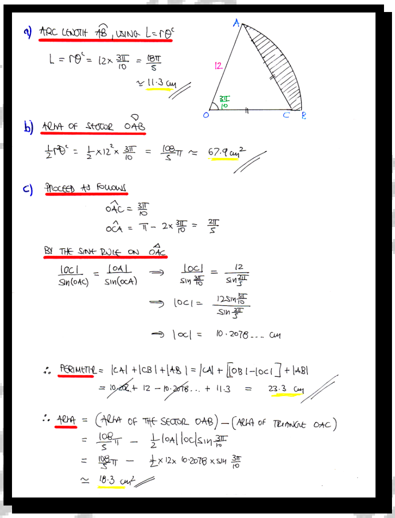
a) … the length of the arc AB.

b) … the area of the sector OAB.

The point C lies on OB so that $OC = AC$. The region R, shown shaded in the figure, is bounded by the arc AB and the straight lines AC and BC.

c) Determine, to three significant figures, the perimeter and area of R.

[ ], [11.3 cm], [67.9 cm^2], [23.3 cm], [18.3–18.4 cm^2]



Question 75 (****)

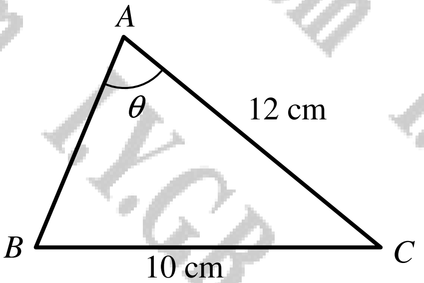


figure 1

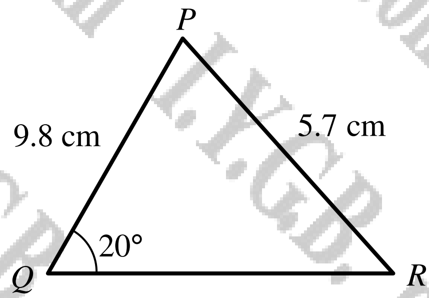


figure 2

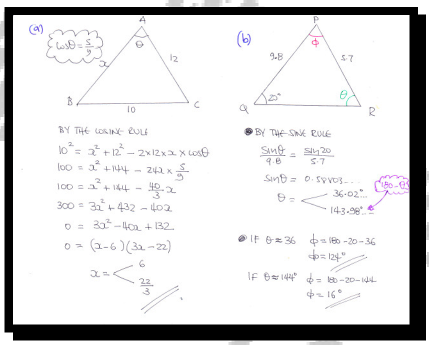
Figure 1 shows the triangle ABC, where $|AC| = 12$ cm, $|BC| = 10$ cm and $\measuredangle BAC = \theta$ so that $\cos\theta = \frac{5}{9}$.

a) Use the cosine rule to form a suitable quadratic, and hence show that one of the **two** possible values for the length of AB is 6 cm and find the other.

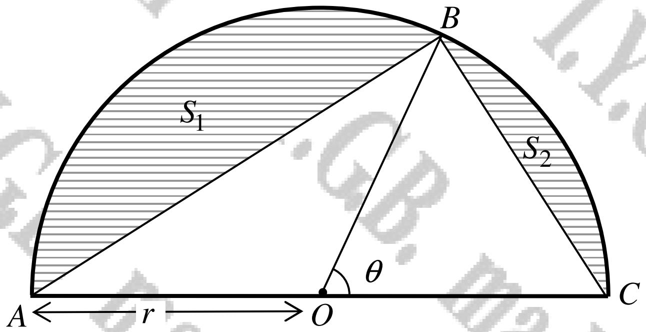
Figure 2 shows a different triangle PQR, where $|PQ| = 9.8$ cm, $|PR| = 5.7$ cm and $\measuredangle PQR = 20°$.

b) Use the sine rule to find, to the nearest degree, the **two** possible values of $\measuredangle QPR$.

$|AB| = \frac{22}{3} \approx 7.33$ cm, $\measuredangle QPR = 16°$ or $124°$



Question 76 **(****)**



The figure above shows a semicircle of radius r cm, where AOC is a diameter with point O the centre of the semicircle.

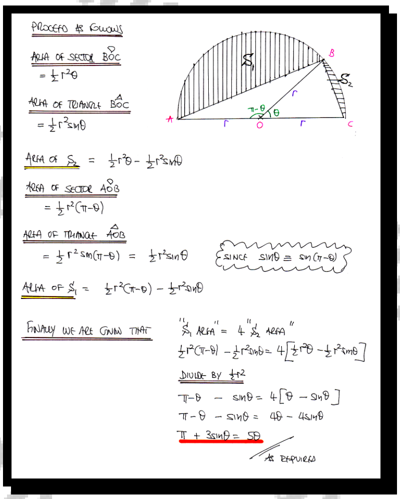
The point B lies on the circular part of the semicircle so that the angle BOC is θ radians.

The chords AB and BC define two segments S_1 and S_2, respectively.

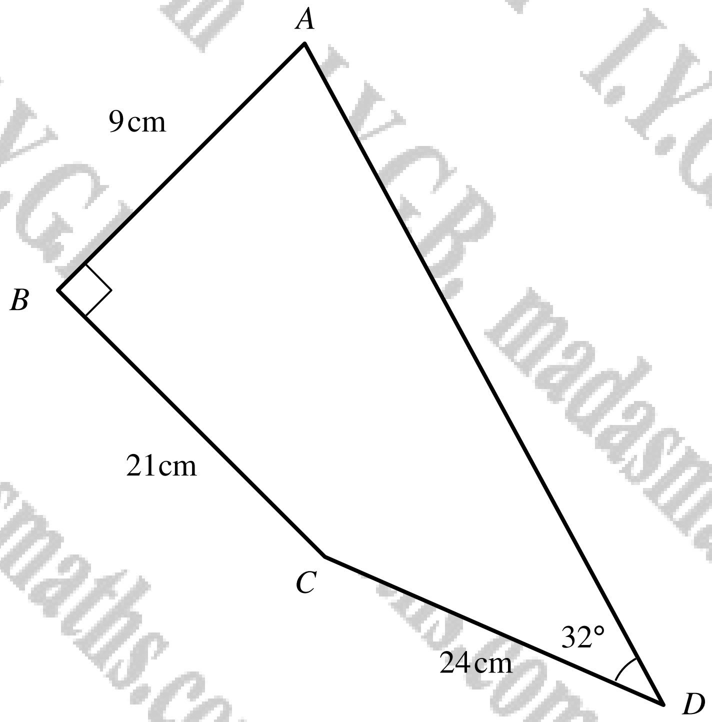
Given that the area of S_1 is four times as large as the area of S_2, show that

$$\pi + 3\sin\theta = 5\theta .$$

[ ], [proof]



Question 77 (****)

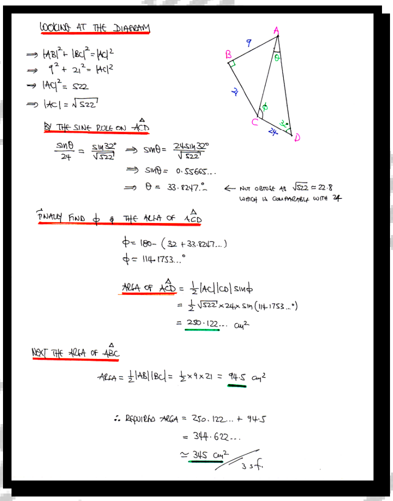


The figure below shows the quadrilateral $ABCD$ where AB is 9 cm, BC is 21 cm and CD is 24 cm.

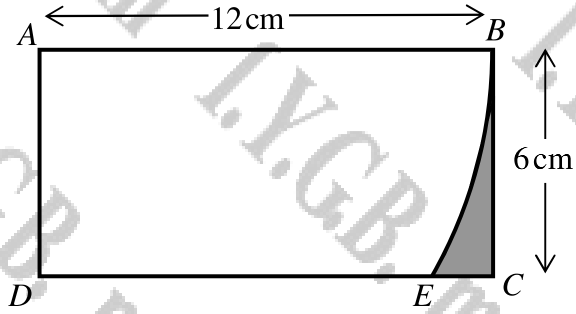
The angle ABC is $90°$ and the angle CDA is $32°$.

Find, to three significant figures, the area of the quadrilateral $ABCD$

[ ], ≈ 345



Question 78 (****)

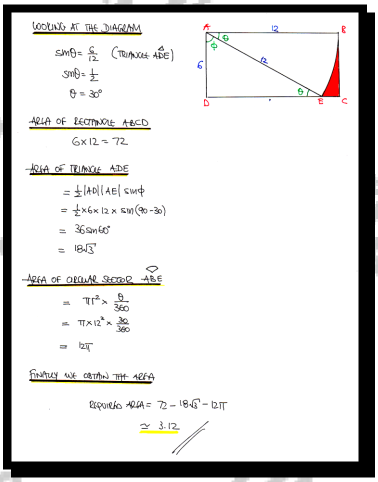


The figure above shows the rectangle $ABCD$ where AB is 12 cm and BC is 6 cm.

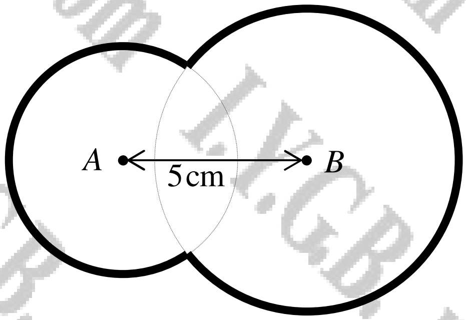
An arc of a circle with centre at A and radius 12 cm is drawn inside the quadrilateral, meeting the side DC at the point E.

Find the area of the shaded region BEC.

[ ], $\boxed{\text{area} = 72 - 12\pi - 18\sqrt{3} \approx 3.12}$



Question 79 (****)



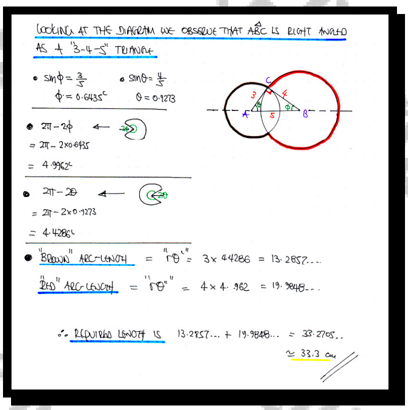
The figure above shows the design for an earring.

The design consists of a part of a circle of radius 3 cm centred at A and another part of a circle of radius 4 cm centred at B.

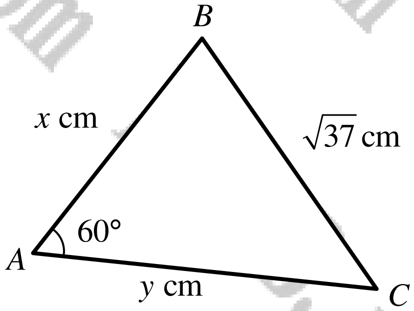
The circles overlap in such a way so that the distance AB is 5 cm.

Find, to three significant figures, the perimeter of the design.

perimeter ≈ 33.3



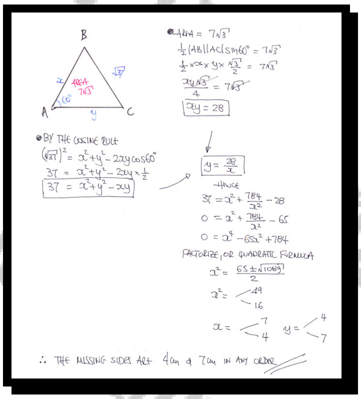
Question 80 (****)



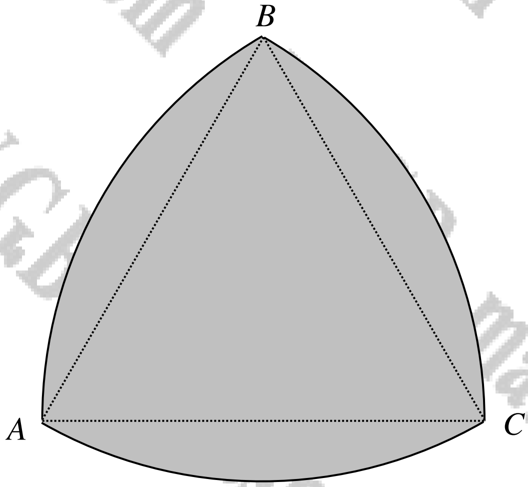
The figure above shows a triangle ABC where AB is x cm, AC is y cm and BC is $\sqrt{37}$ cm. The angle BAC is $60°$.

Given further that the area of the triangle ABC is $7\sqrt{3}$ cm^2, determine by solving two simultaneous equations the value of x and the value of y.

$[x, y] = [4,7] \cup [7,4]$



Question 81 (****)



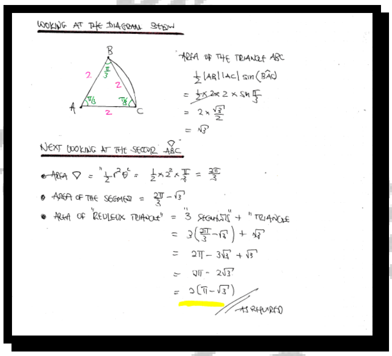
The figure above shows a "curved triangle", known as a Reuleaux triangle, which is constructed as follows.

Starting with an equilateral triangle ABC of side length 2 cm, a circular arc BC is drawn with centre at A. Two more circular arcs AB and AC are drawn with respective centres at C and B.

Show that the area of this Reuleaux triangle is

$$2\left(\pi - \sqrt{3}\right) \text{ cm}^2.$$

[ ], [proof]


LOOKING AT THE DIAGRAM BELOW

AREA OF THE TRIANGLE ABC

$\frac{1}{2}|AB||AC|\sin(\hat{BAC})$

$= \frac{1}{2} \times 2 \times 2 \times \sin\frac{\pi}{3}$

$= 2 \times \frac{\sqrt{3}}{2}$

$= \sqrt{3}$

NEXT LOOKING AT THE SECTOR $\overset{\triangledown}{ABC}$

- AREA $\triangledown$ = "$\frac{1}{2}r^2\theta^c$" = $\frac{1}{2} \times 2^2 \times \frac{\pi}{3} = \frac{2\pi}{3}$
- AREA OF THE SEGMENT = $\frac{2\pi}{3} - \sqrt{3}$
- AREA OF "REULEUX TRIANGLE" = "3 SEGMENTS" + "TRIANGLE"

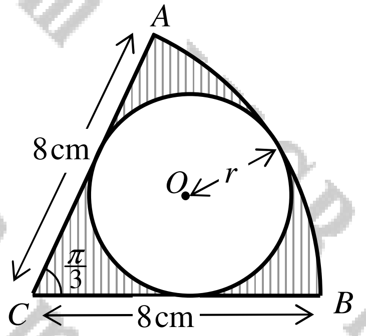
$= 3\left(\frac{2\pi}{3} - \sqrt{3}\right) + \sqrt{3}$

$= 2\pi - 3\sqrt{3} + \sqrt{3}$

$= 2\pi - 2\sqrt{3}$

$= 2\left(\pi - \sqrt{3}\right)$ // AS REQUIRED

Question 82 (****+)

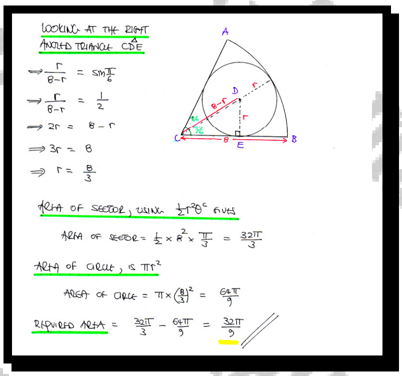


The figure above shows a sector CAB of radius 8 cm, centred at C and subtending an angle of $\frac{\pi}{3}$ radians at C.

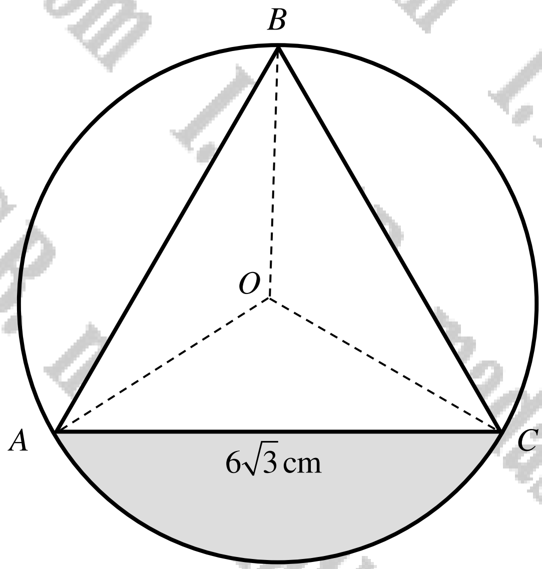
A circle centred at O and of radius r cm is inscribed inside the sector.

Find in terms of π, the area of the shaded region, shown in the figure above.

area $= \frac{32}{9}\pi$



Question 83 (****+)



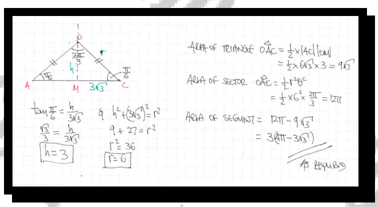
The figure above shows an equilateral triangle ABC circumscribed by a circle of radius 6, with centre at O.

The circular segment, shown shaded region in the figure above, is bounded by the straight line AC.

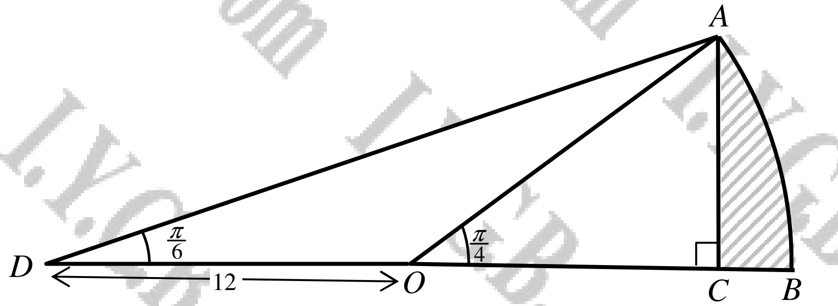
Show that the area of the segment is

$$3\left(4\pi - 3\sqrt{3}\right)\text{cm}^2.$$

proof



Question 84 (****+)



The figure above shows a triangle OAC with $\measuredangle ACO = \frac{1}{2}\pi$ and $\measuredangle AOC = \frac{1}{2}\pi$.

Another triangle AOD is drawn next to the triangle OAC, so that DOC is a straight line, $|DO| = 12$ units and $\measuredangle ADO = \frac{1}{6}\pi$.

Finally a circular sector OAB is drawn, centred at O, with radius OA, so that $DOCB$ is a straight line.

a) Show that the length of OA is

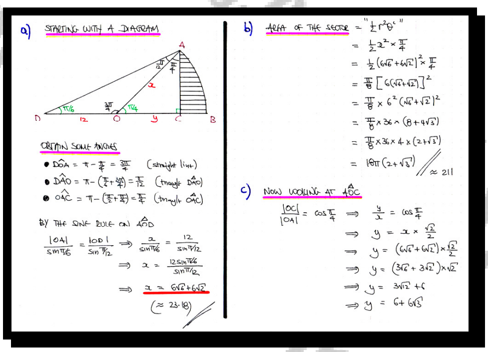
$$6\left(\sqrt{6}+\sqrt{2}\right).$$

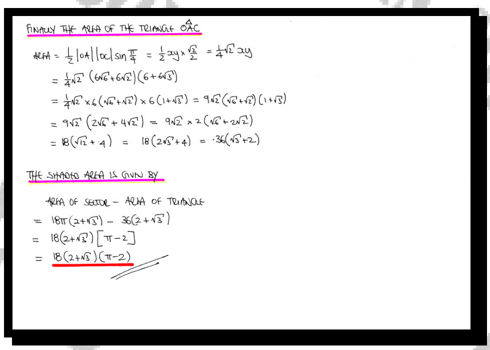
b) Find the exact area of the sector OAB.

c) Hence show that the area of the shaded region ACB is

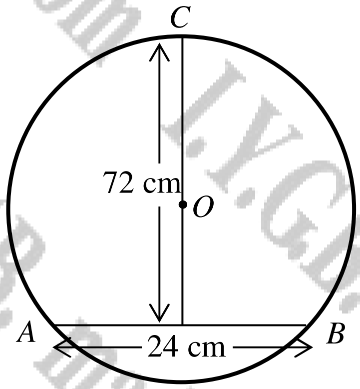
$$18\left(2+\sqrt{3}\right)(\pi-2).$$

$\boxed{}$, $\boxed{18\left(2+\sqrt{3}\right)\pi}$





Question 85 (****+)



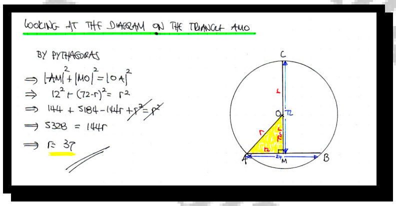
The figure above shows a circle with centre at O and radius r.

The straight line AB is a chord to the circle.

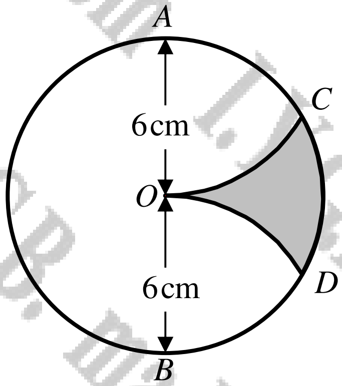
The perpendicular bisector of AB passes through O and meets the circle at the point C, as shown in the figure.

Given that $|AB| = 24 \text{ cm}$ and the length of the perpendicular bisector is 72 cm, determine the value of r.

$\boxed{}$, $\boxed{r = 37}$



Question 86 (****+)



The figure above shows a circle of radius 6 cm, centred at O.

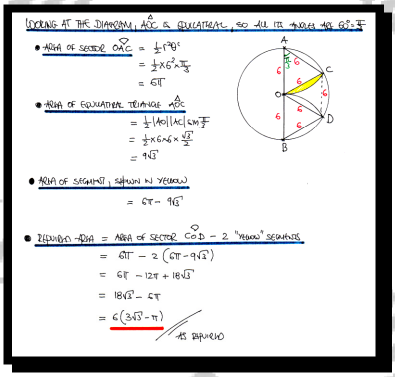
An arc OC with centre at A and radius 6 cm is drawn inside the circle.

A second arc OD is drawn with centre at B and radius 6 cm.

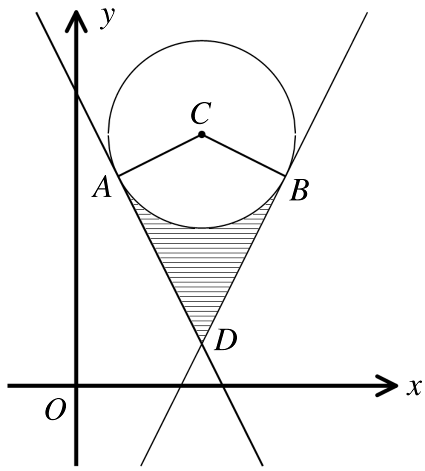
Show clearly that the area of the shaded region OCD is

$$6\left(3\sqrt{3}-\pi\right)\ \text{cm}^2.$$

[ ], proof



Question 87 (****+)



The figure above shows a circle with centre at $C(3,6)$. The points $A(1,5)$ and $B(p,q)$ lie on the circle. The straight lines AD and BD are tangents to the circle.

The kite $CADB$ is symmetrical about the straight line with equation $x = 3$.

a) Calculate the radius of the circle.

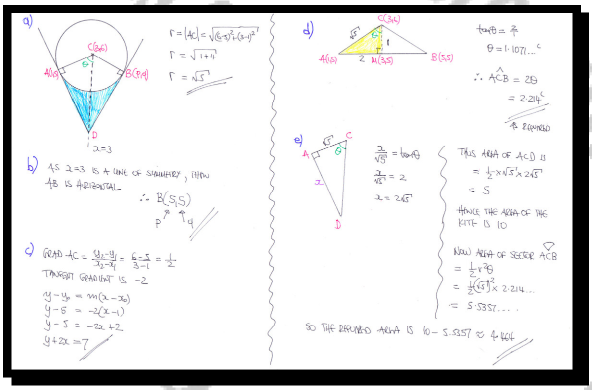
b) State the value of p and the value of q.

c) Find an equation of the tangent to the circle at A.

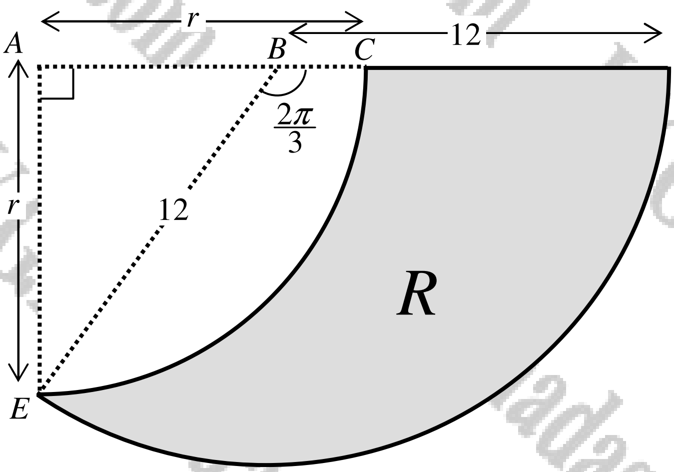
d) Show that the angle ACB is approximately 2.214 radians.

e) Hence determine, to three significant figures, the area of the shaded region bounded by the circle and its tangents at A and B.

$\boxed{}$, $\boxed{r = \sqrt{5}}$, $\boxed{p = q = 5}$, $\boxed{y = 7 - 2x}$, $\boxed{\text{area} \approx 4.46}$



Question 88 (****+)



The figure above is constructed as follows.

- EBD is a circular sector with centre at B and radius 12 units, subtending an angle of $\frac{2\pi}{3}$ radians at B.

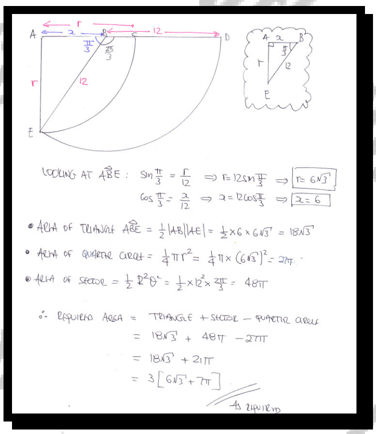
- EAC is a quarter circle with centre at A and radius r units, so that $ABCD$ is a straight line and CAE is a right angle.

The shaded region R is bounded by the arcs ED and EC, and the straight line CD.

Show that the area of R is

$$3\left(7\pi + 6\sqrt{3}\right) \text{ square units.}$$

[ ], [proof]

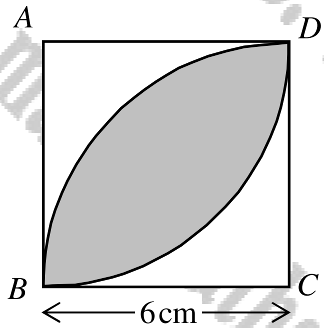


Question 89 (****+)

The figure below shows a square $ABCD$ with side length of 6 cm.

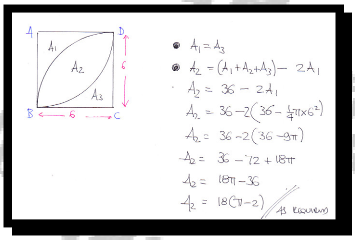
A circular arc BD is drawn inside the square with centre at C and radius of 6 cm.

Another circular arc BD is drawn inside the square with centre at A and radius of 6 cm also, so that the two arcs bound a finite area, shown shaded in the figure above.



Show that area of the shaded region is $18(\pi - 2)$ cm^2.

proof



Question 90 (****+)

The distance between the town of Arundel (A) and the town of Berry (B) is 60 km.

Berry is on bearing of $75°$ from Arundel.

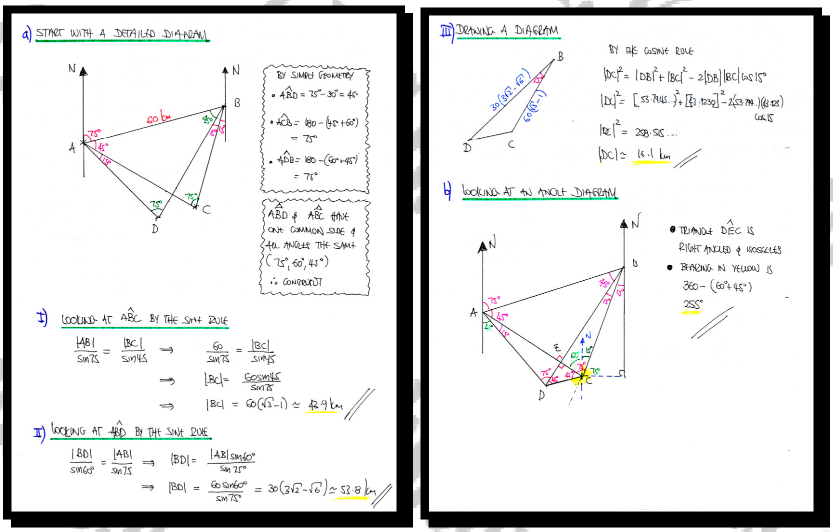
The village of Crake (C) is on a bearing of $120°$ from Arundel and on a bearing of $195°$ from Berry. The village of Dorking (D) is on a bearing of $135°$ from Arundel and on a bearing of $210°$ from Berry.

a) Find, to three significant figures where appropriate, the distance between …

- **i.** … Berry and Crake.
- **ii.** … Berry and Dorking.
- **iii.** … Crake and Dorking.

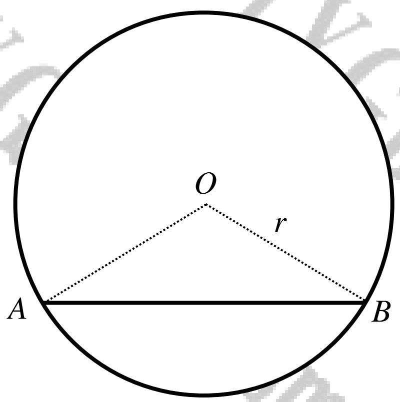
b) State the bearing of Dorking from Crake.

$\boxed{}$, $\boxed{43.9 \text{ km}}$, $\boxed{53.8 \text{ km}}$, $\boxed{16.1 \text{ km}}$, $\boxed{255°}$



Question 91 **(****+)**

The figure below shows a circle with centre at O and radius r. The points A and B lie on the circle so that the angle AOB is θ radians.

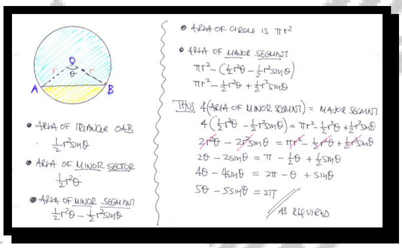


The chord AB divides the circle into a major segment and a minor segment.

Given that the area of the **major segment** is 4 times as large as the area of the **minor segment**, show clearly that

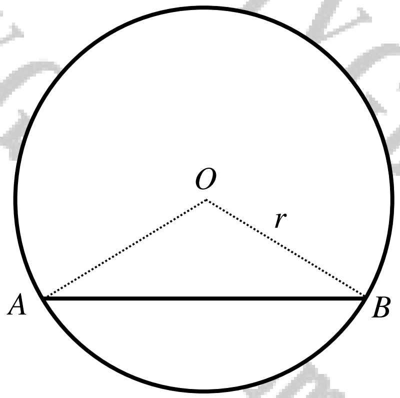
$$5\theta - 5\sin\theta = 2\pi .$$

proof



Question 92 (****+)

The figure below shows a circle with centre at O and radius r. The points A and B lie on the circle so that the angle AOB is 2θ radians.

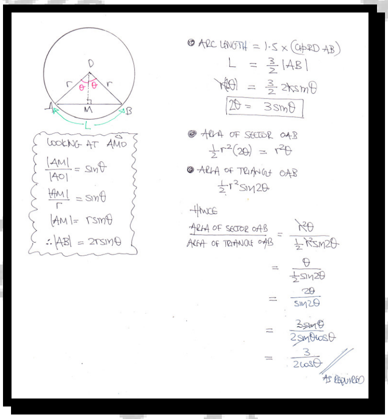


Given that the length of the **arc** AB is 1.5 times as large as the **chord** AB, show clearly that

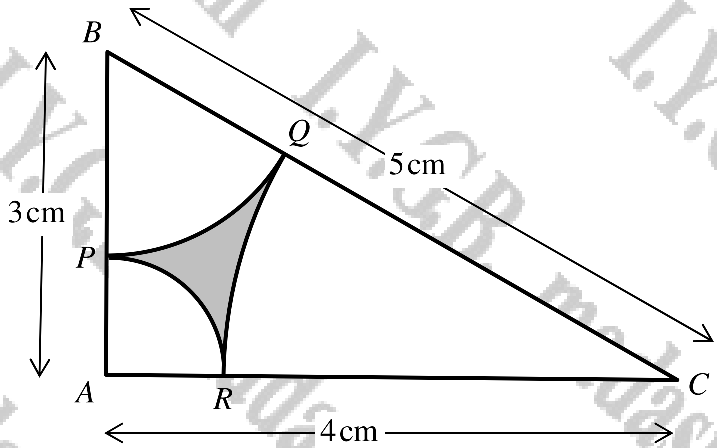
$$\frac{\text{area of the sector } OAB}{\text{area of the triangle } OAB} = \frac{3}{2\cos\theta}.$$

You may use the fact that $\sin 2\theta \equiv 2\sin\theta\cos\theta$.

proof



Question 93 (****+)



The figure above shows a triangle ABC where $\measuredangle BAC = 90°$.

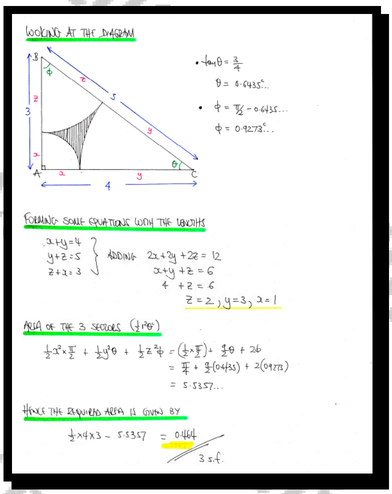
The lengths of AB, AC and BC are 3 cm, 4 cm and 5 cm, respectively.

Three arcs are drawn inside the triangle with centres the three vertices of the triangle.

The arcs so that they touch each other in pairs at the points P, Q and R.

Find the area of the shaded region, correct to three significant figures.

MP2, $\boxed{\text{area} = 0.464 \text{ cm}^2}$



LOOKING AT THE DIAGRAM

- $\tan\theta = \frac{3}{4}$
  $\theta = 0.6435^c\ldots$
- $\phi = \frac{\pi}{2} - 0.6435\ldots$
  $\phi = 0.9273^c\ldots$

FORMING SOME EQUATIONS WITH THE LENGTHS

$x+y=4$, $y+z=5$, $z+x=3$ ADDING $2x+2y+2z=12$
$x+y+z=6$
$4+z=6$
$z=2,\ y=3,\ x=1$

AREA OF THE 3 SECTORS $\left(\frac{1}{2}r^2\theta^c\right)$

$\frac{1}{2}x^2\times\frac{\pi}{2} + \frac{1}{2}y^2\theta + \frac{1}{2}z^2\phi = \left(\frac{1}{2}\times\frac{\pi}{2}\right) + \frac{9}{2}\theta + 2\phi$
$= \frac{\pi}{4} + \frac{9}{2}(0.6435) + 2(0.9273)$
$= 5.5357\ldots$

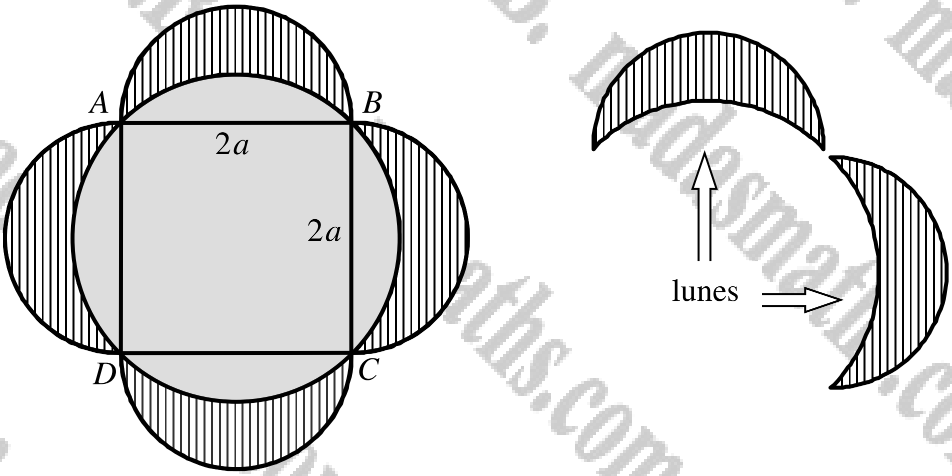
HENCE THE REQUIRED AREA IS GIVEN BY

$\frac{1}{2}\times4\times3 - 5.5357 = 0.464$ (3 s.f.)

Question 94 (****+)

The figure below shows a square $ABCD$ of side length $2a$ cm, circumscribed by a circle.

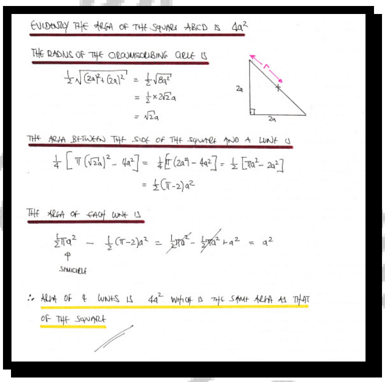
Four semicircles are then drawn outside the square having each of the sides of the square as a diameter.



Each of the four regions bounded by a semicircle and the circumscribing circle is known by the mathematical name of a "lune", i.e. moon shaped.

Show that the area of the four lunes is equal to the area of the square $ABCD$.

[ ], proof



EVIDENTLY THE AREA OF THE SQUARE ABCD IS $4a^2$

THE RADIUS OF THE CIRCUMSCRIBING CIRCLE IS

$$\frac{1}{2}\sqrt{(2a)^2+(2a)^2} = \frac{1}{2}\sqrt{8a^2} = \frac{1}{2}\times 2\sqrt{2}a = \sqrt{2}a$$

THE AREA BETWEEN THE SIDE OF THE SQUARE AND A LUNE IS

$$\frac{1}{4}\left[\pi(\sqrt{2}a)^2 - 4a^2\right] = \frac{1}{4}\left[\pi(2a^2) - 4a^2\right] = \frac{1}{2}\left[\pi a^2 - 2a^2\right] = \frac{1}{2}(\pi-2)a^2$$

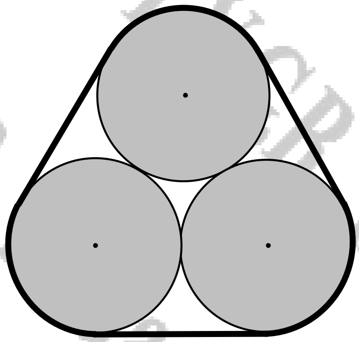
THE AREA OF EACH LUNE IS

$$\underbrace{\frac{1}{2}\pi a^2}_{\text{SEMICIRCLE}} - \frac{1}{2}(\pi-2)a^2 = \frac{1}{2}\pi a^2 - \frac{1}{2}\pi a^2 + a^2 = a^2$$

∴ AREA OF 4 LUNES IS $4a^2$ WHICH IS THE SAME AREA AS THAT OF THE SQUARE

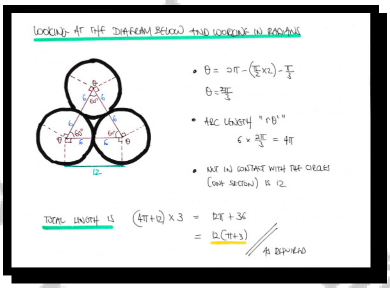
Question 95 (****+)

The figure below shows the plan of three identical circular cylinders of radius 6 cm, held together by an elastic band.



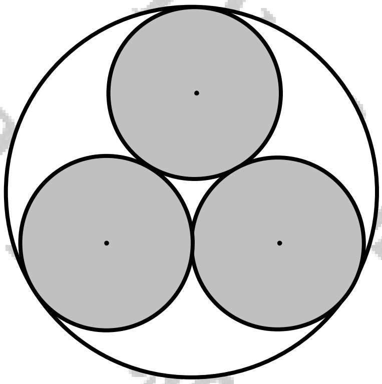
Show that the exact length of the stretched elastic band is $12(\pi+3)$ cm.

[ ], proof



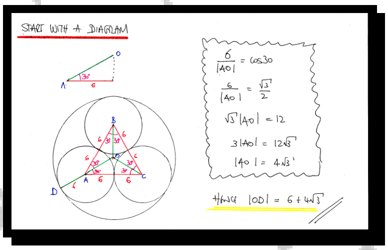
Question 96 (****+)

The figure below shows the plan of three identical circular cylinders of radius 6 cm, that fit snugly inside a larger cylinder.

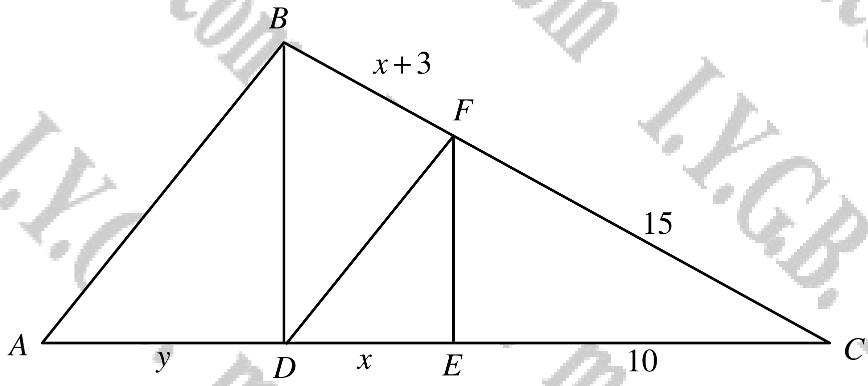


Show that the radius of the larger cylinder is $6+4\sqrt{3}$ cm.

[ ], [proof]



Question 97 (****+)



The figure above shows the triangle ABC.

The point D lies on AC so that the straight line BD meets AC at right angles.

The point E lies on AC and the point F lies on BC, so that the straight line DF is parallel to AB and the straight line EF is parallel to BD.

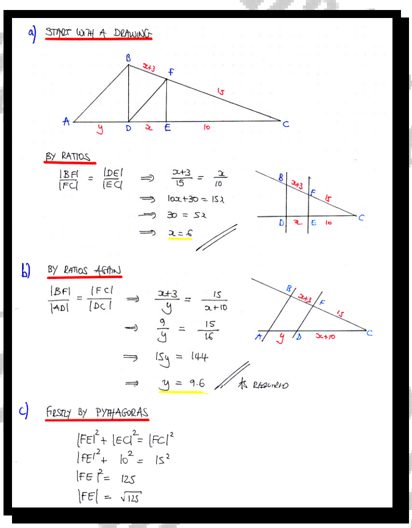
It is further given that the lengths, in cm, of CE, CF, DE, BF and AD are 10, 15, x, $x+3$ and y, respectively.

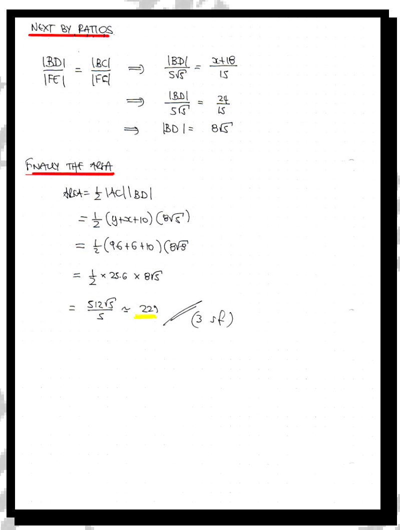
a) Determine the value of x.

b) Show clearly that $y = 9.6$.

c) Find, correct to three significant figures, the area of the triangle ABC.

$\boxed{}$, $\boxed{x=6}$, $\boxed{\text{area}=229}$





Question 98 (****+)

The following information is known about 4 coplanar points.

1. B is north east of A.

2. C is on a bearing of $075°$ from A.

3. B is on a bearing of $285°$ from C.

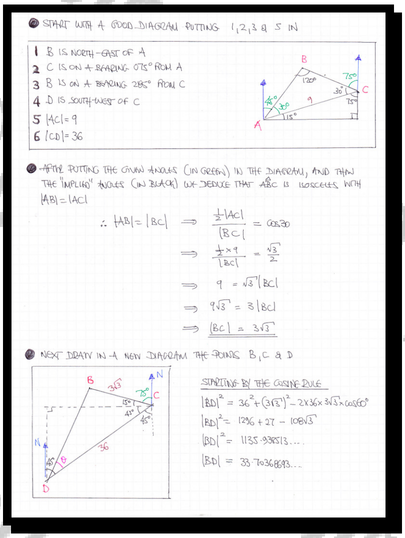
4. D is south west of C.

5. $|AC| = 9$.

6. $|CD| = 36$.

Determine, correct to 2 decimal places, the bearing of B from D.

$\boxed{}$, $\boxed{\approx 37.33°}$


START WITH A GOOD DIAGRAM PUTTING 1, 2, 3 & 5 IN

1 B IS NORTH-EAST OF A
2 C IS ON A BEARING 075° FROM A
3 B IS ON A BEARING 285° FROM C
4 D IS SOUTH-WEST OF C
5 $|AC| = 9$
6 $|CD| = 36$

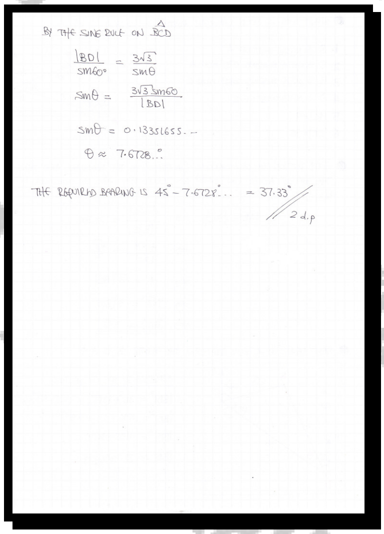
AFTER PUTTING THE GIVEN ANGLES (IN GREEN) IN THE DIAGRAM, AND THEN THE "IMPLIED" ANGLES (IN BLACK) WE DEDUCE THAT $A\hat{B}C$ IS ISOSCELES WITH $|AB| = |AC|$

$$\therefore |AB| = |BC| \implies \frac{\frac{1}{2}|AC|}{|BC|} = \cos 30$$
$$\implies \frac{\frac{1}{2}\times 9}{|BC|} = \frac{\sqrt{3}}{2}$$
$$\implies 9 = \sqrt{3}|BC|$$
$$\implies 9\sqrt{3} = 3|BC|$$
$$\implies |BC| = 3\sqrt{3}$$

NEXT DRAW IN A NEW DIAGRAM THE POINTS B, C & D

STARTING BY THE COSINE RULE

$$|BD|^2 = 36^2 + (3\sqrt{3})^2 - 2\times 36 \times 3\sqrt{3} \times \cos 60°$$
$$|BD|^2 = 1296 + 27 - 108\sqrt{3}$$
$$|BD|^2 = 1135.938513\ldots$$
$$|BD| = 33.70368693\ldots$$


BY THE SINE RULE ON $\hat{BCD}$

$$\frac{|BD|}{\sin 60°} = \frac{3\sqrt{3}}{\sin\theta}$$
$$\sin\theta = \frac{3\sqrt{3}\sin 60}{|BD|}$$
$$\sin\theta = 0.13351655\ldots$$
$$\theta \approx 7.6728\ldots°$$

THE REQUIRED BEARING IS $45° - 7.6728°\ldots = 37.33°$ // 2 d.p

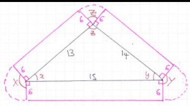
Question 99 (****+)

The island state of Trigland has declared an exclusive economic zone into the sea, which is within 6 miles from every point of its coastline.

The island of Trigland is a rectilinear triangle of sides 13, 14 and 15 miles.

Determine, in exact form, the total economic zone of Trigland, which consists of land and sea.

$\boxed{}$, $\boxed{336+36\pi}$



• FIRSTLY $x+y+z=\pi$

BUT $X=\pi-x$

$Y=\pi-y$

$Z=\pi-z$

THUS $X+Y+Z=(\pi-x)+(\pi-y)+(\pi-z)$

$=3\pi-(x+y+z)$

$=3\pi-\pi$

$=2\pi$

IE A FULL CIRCLE OF RADIUS 6

THUS $\pi\times6^2=36\pi$

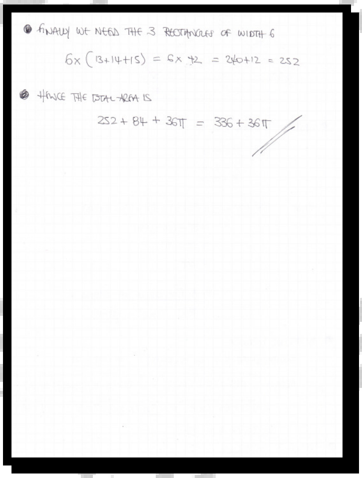
• BY HERONS FORMULA

SEMIPERIMETER $s=\frac{1}{2}(13+14+15)=\frac{1}{2}\times42=21$

AREA $=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{21\times(21-15)(21-14)(21-13)}$

$=\sqrt{21\times6\times7\times8}=\sqrt{3\times7\times3\times2\times7\times8}=\sqrt{9\times49\times16}$

$=3\times7\times4=84$



• FINALLY WE NEED THE 3 RECTANGLES OF WIDTH 6

$6\times(13+14+15)=6\times42=240+12=252$

• HENCE THE TOTAL AREA IS

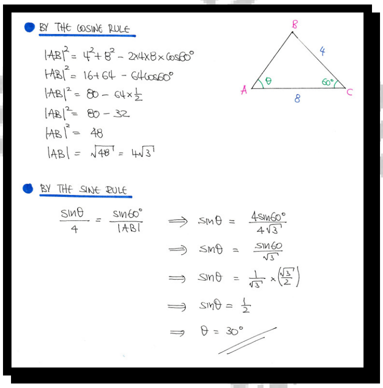
$252+84+36\pi=336+36\pi$

Question 100 **(****+)** **Non Calculator**

A triangle, ABC has $|BC| = 4$ cm, $|AC| = 8$ cm and $\measuredangle ACB = 60°$.

Determine, in degrees, the size of $\measuredangle BAC$.

$\boxed{}$, $\boxed{\measuredangle BAC = 30°}$



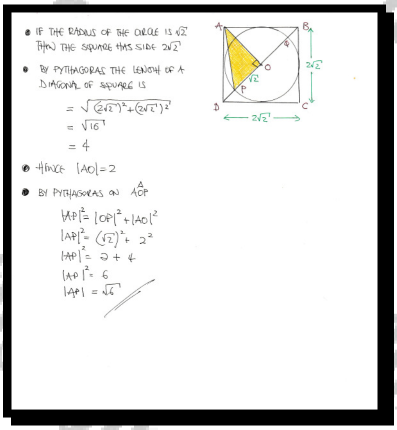
Question 101 (****+)

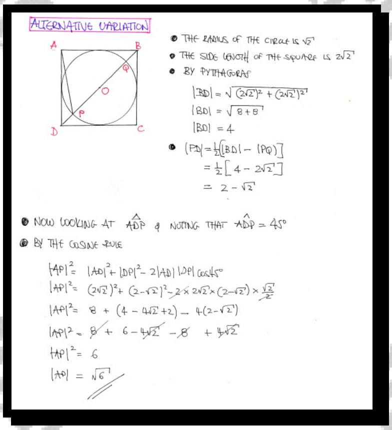
The four sides of a square, $ABCD$, are tangents to a circle of radius $\sqrt{2}$.

The diagonal BD intersects the circle at the points P and Q.

Determine in exact simplified form the length of AP.

$\boxed{}$, $\boxed{|AP| = \sqrt{6}}$





Question 102 (****+)

The triangle ABC is right angled at the vertex B.

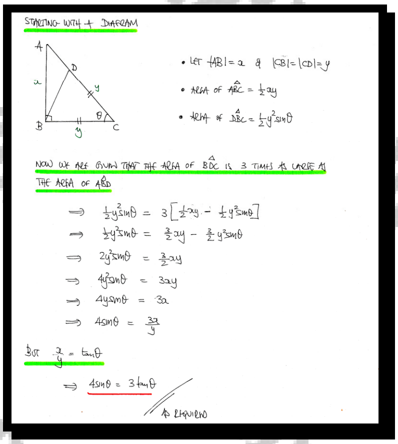
The point D lies on AC so that $|BD| = |BC|$.

Given that the area of the triangle BDC is 3 times as large as the area of the triangle ABD, show that

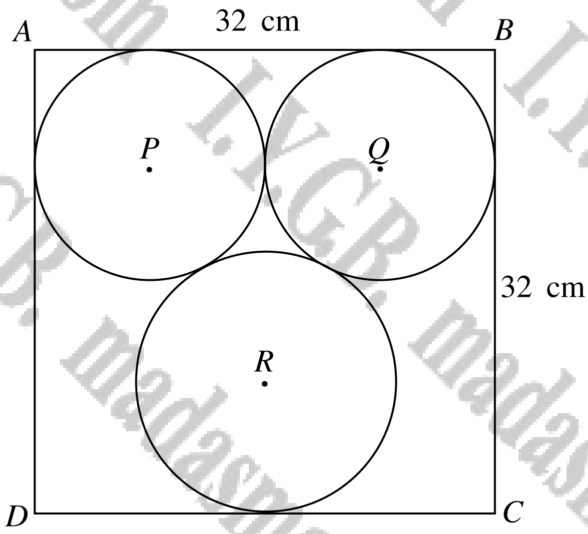
$$4\sin\theta = 3\tan\theta,$$

where θ denotes the angle BCA.

[ ], [proof]



Question 103 (*****)

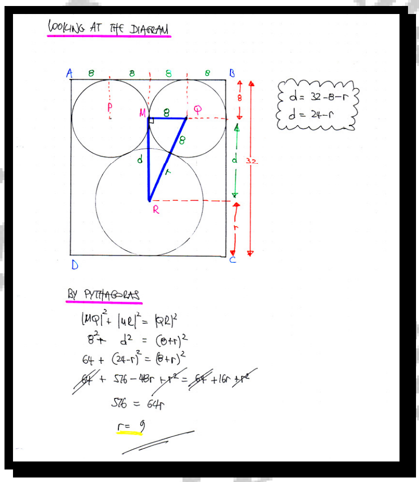


The figure above shows, two identical circles, centred at P and Q, and a third circle, centred at R, are touching each other externally.

The three circles fit snugly inside a square $ABCD$, of side length 32 cm, so that PQ is parallel to AB.

Determine the size of radius of the circle centred at R.

[ ] , $r = 9$ cm

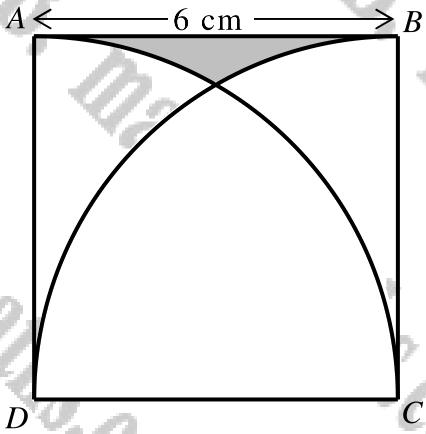


Question 104 **(*****)**

The figure below was constructed as follows.

$ABCD$ is a square with side length 6 cm.

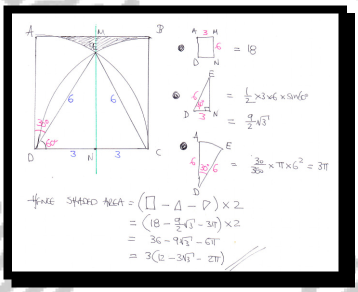
Two quarter circles, with centres at the points C and D, each of radius 6 cm, are drawn inside the square.



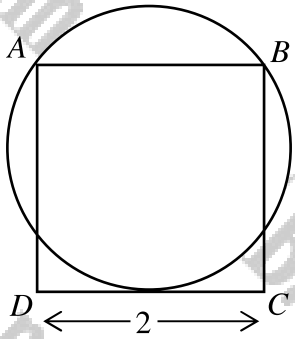
Show that the area of the shaded region is

$$3\left(12-3\sqrt{3}-2\pi\right) \text{ cm}^2.$$

[ ] , [proof]



Question 105 (*****)

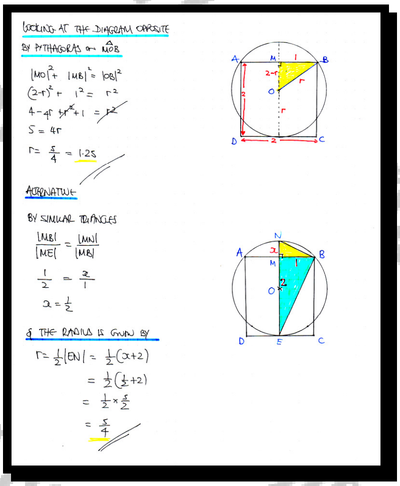


The figure above shows a square $ABCD$ of side length 2 units.

The vertices A and B lie on the circumference of a circle while the side DC is a tangent to the same circle.

Determine the radius of this circle.

$\boxed{}$, $\boxed{r = \frac{5}{4} = 1.25}$



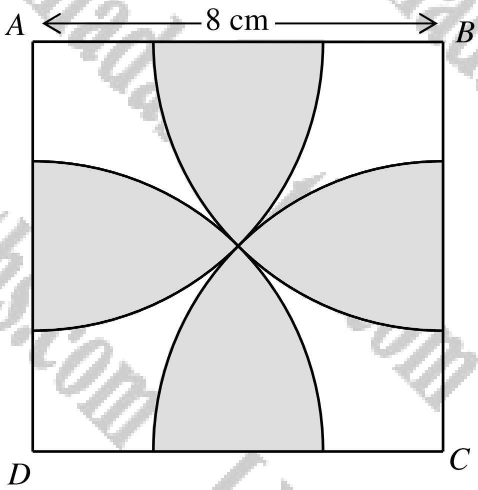
Question 106 (*****)

The figure below was constructed as follows.

$ABCD$ is a square with side length 8 cm.

Four identical quarter circles, whose centres are located at each of the four corners of the square, are drawn inside the square.

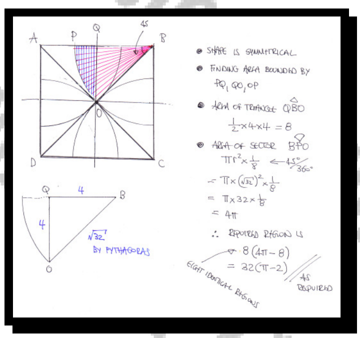
The radii of the quarter circles are such so that the four quarter circles meet at the centre of the square.



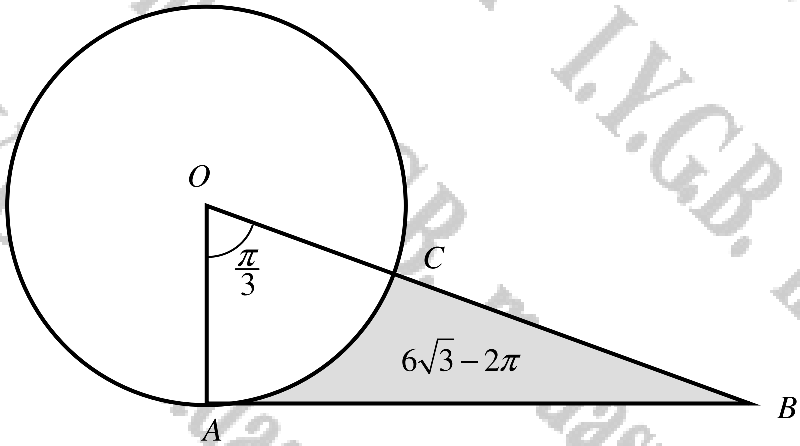
Show that the area of the shaded region is

$$32(\pi - 2) \text{ cm}^2.$$

[ ], proof



Question 107 (*****) (non calculator)

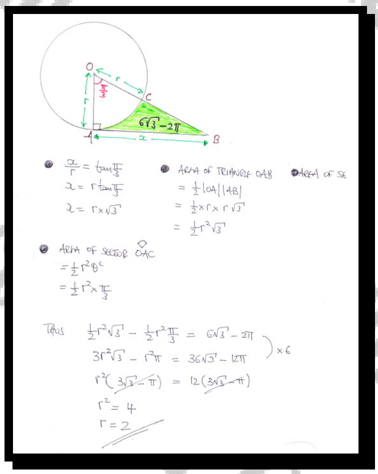


The figure above shows a circle with centre at O.

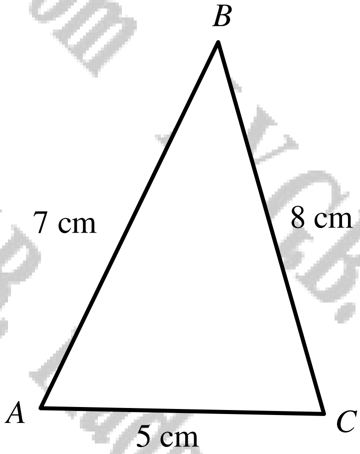
The straight line segment AB is a tangent to the circle at A, so that the angle AOB is $\frac{1}{3}\pi$ radians.

Determine the radius of the circle given further that the area of the shaded region in the figure is $\left(6\sqrt{3}-2\pi\right)$ cm^2.

$\boxed{r=2}$



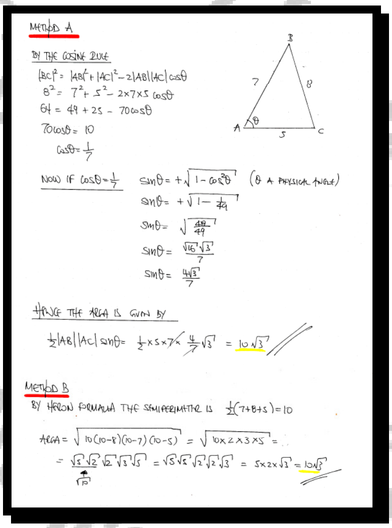
Question 108 (*****) (non calculator)



The figure above shows the triangle ABC where AB is 7 cm, AC is 5 cm and BC is 8 cm.

Show that the exact area of this triangle is $10\sqrt{3}$ cm^2.

[ ], [proof]



METHOD A

BY THE COSINE RULE

$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC|\cos\theta$

$8^2 = 7^2 + 5^2 - 2\times7\times5\cos\theta$

$64 = 49 + 25 - 70\cos\theta$

$70\cos\theta = 10$

$\cos\theta = \frac{1}{7}$

NOW IF $\cos\theta = \frac{1}{7}$

$\sin\theta = +\sqrt{1-\cos^2\theta}$ (θ A PHYSICAL ANGLE)

$\sin\theta = +\sqrt{1-\frac{1}{49}}$

$\sin\theta = \sqrt{\frac{48}{49}}$

$\sin\theta = \frac{\sqrt{16}\sqrt{3}}{7}$

$\sin\theta = \frac{4\sqrt{3}}{7}$

HENCE THE AREA IS GIVEN BY

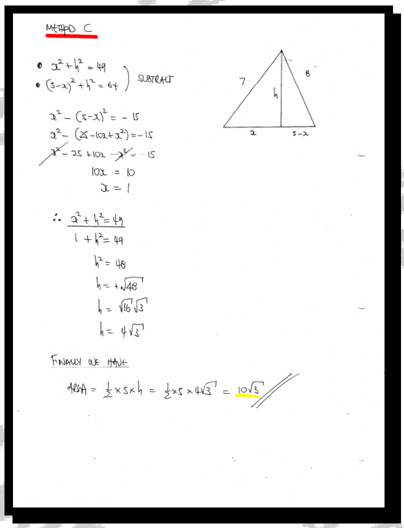
$\frac{1}{2}|AB||AC|\sin\theta = \frac{1}{2}\times5\times7\times\frac{4}{7}\sqrt{3} = 10\sqrt{3}$

METHOD B

BY HERON FORMULA THE SEMIPERIMETER IS $\frac{1}{2}(7+8+5)=10$

AREA $= \sqrt{10(10-8)(10-7)(10-5)} = \sqrt{10\times2\times3\times5} =$

$= \sqrt{5}\sqrt{2}\sqrt{2}\sqrt{3}\sqrt{5} = \sqrt{5}\sqrt{5}\sqrt{2}\sqrt{2}\sqrt{3} = 5\times2\times\sqrt{3} = 10\sqrt{3}$



METHOD C

- $x^2 + h^2 = 49$
- $(5-x)^2 + h^2 = 64$

SUBTRACT

$x^2 - (5-x)^2 = -15$

$x^2 - (25 - 10x + x^2) = -15$

$x^2 - 25 + 10x - x^2 = -15$

$10x = 10$

$x = 1$

$\therefore x^2 + h^2 = 49$

$1 + h^2 = 49$

$h^2 = 48$

$h = +\sqrt{48}$

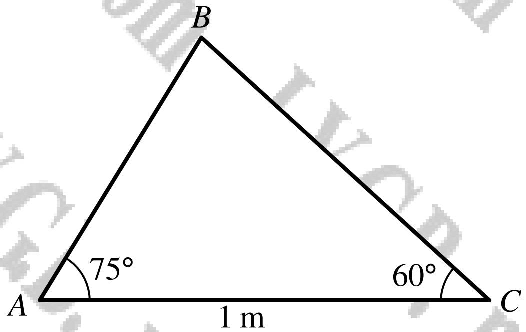
$h = \sqrt{16}\sqrt{3}$

$h = 4\sqrt{3}$

FINALLY WE HAVE

AREA $= \frac{1}{2}\times5\times h = \frac{1}{2}\times5\times4\sqrt{3} = 10\sqrt{3}$

Question 109 (*****)



The figure above shows a triangle ABC.

The length of AC is 1 m.

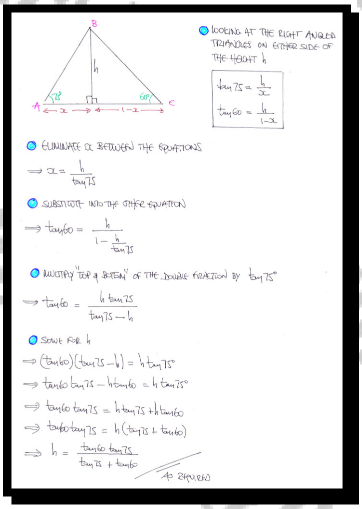
The angles BAC and BCA are $75°$ and $60°$, respectively.

The height of the triangle from the vertex B to the side AC is h cm.

Show that

$$h = \frac{\tan 75° \tan 60°}{\tan 75° + \tan 60°}.$$

[ ], [proof]



LOOKING AT THE RIGHT ANGLED TRIANGLES ON EITHER SIDE OF THE HEIGHT h

$$\tan 75 = \frac{h}{x}$$
$$\tan 60 = \frac{h}{1-x}$$

ELIMINATE x BETWEEN THE EQUATIONS

$$\Rightarrow x = \frac{h}{\tan 75}$$

SUBSTITUTE INTO THE OTHER EQUATION

$$\Rightarrow \tan 60 = \frac{h}{1 - \frac{h}{\tan 75}}$$

MULTIPLY "TOP & BOTTOM" OF THE DOUBLE FRACTION BY $\tan 75°$

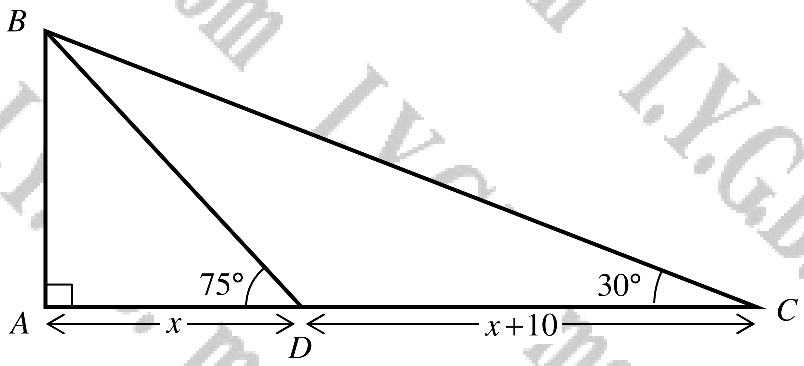
$$\Rightarrow \tan 60 = \frac{h \tan 75}{\tan 75 - h}$$

SOLVE FOR h

$$\Rightarrow (\tan 60)(\tan 75 - h) = h \tan 75°$$
$$\Rightarrow \tan 60 \tan 75 - h \tan 60 = h \tan 75°$$
$$\Rightarrow \tan 60 \tan 75 = h \tan 75 + h \tan 60$$
$$\Rightarrow \tan 60 \tan 75 = h(\tan 75 + \tan 60)$$
$$\Rightarrow h = \frac{\tan 60 \tan 75}{\tan 75 + \tan 60}$$

AS REQUIRED

Question 110 (*****) (non calculator)



The figure above shows a right angled triangle ABC, where the angle BCA is $30°$.

The point D lies on AC so that the angle BDA is $75°$.

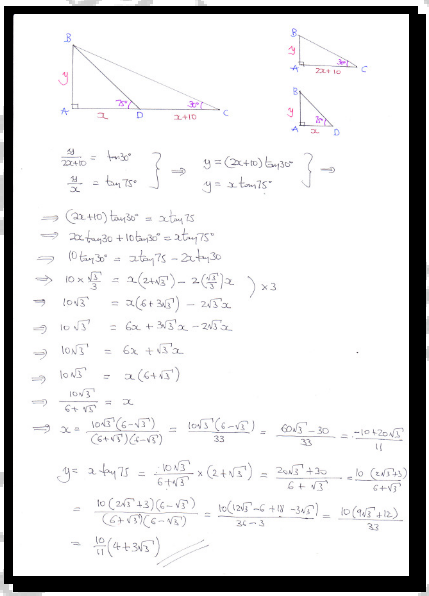
The length of AD is x cm and he length of DC is $x+10$ cm.

Show that the length of AB is

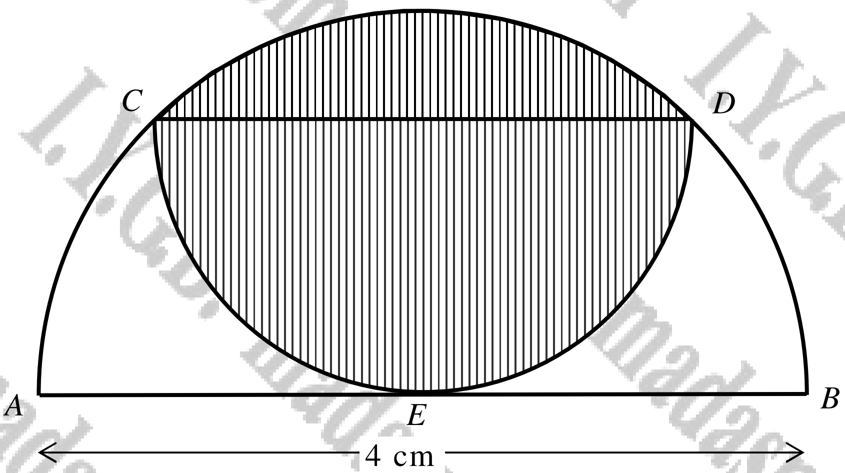
$$\frac{10}{11}\left(4+3\sqrt{3}\right).$$

[*you may assume that* $\tan 75° = 2+\sqrt{3}$]

proof



Question 111 (*****) (non calculator)



The figure above is constructed as follows.

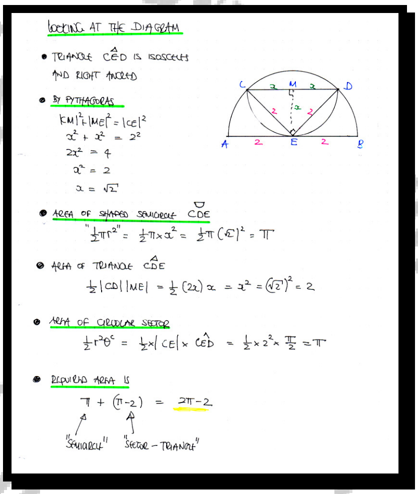
A semicircle with diameter AB of 4 cm is first drawn.

Then another semicircle is drawn, with its diameter CD parallel to AB.

The semicircle with CD as its diameter is circumscribed by the semicircle with AB as its diameter, as shown in the figure.

Show that the area of the shaded region is $(2\pi - 2)$ cm^2.

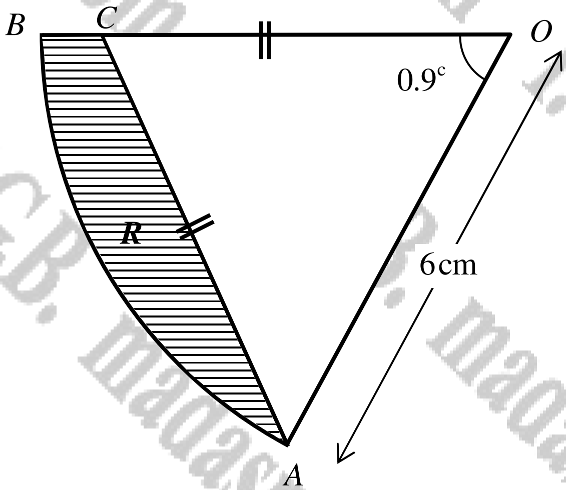
[ ] , [proof]



LOOKING AT THE DIAGRAM

- TRIANGLE $C\hat{E}D$ IS ISOSCELES AND RIGHT ANGLED
- BY PYTHAGORAS
  $|CM|^2 + |ME|^2 = |CE|^2$
  $x^2 + x^2 = 2^2$
  $2x^2 = 4$
  $x^2 = 2$
  $x = \sqrt{2}$
- AREA OF SHADED SEMICIRCLE CDE
  "$\frac{1}{2}\pi r^2$" $= \frac{1}{2}\pi \times x^2 = \frac{1}{2}\pi(\sqrt{2})^2 = \pi$
- AREA OF TRIANGLE CDE
  $\frac{1}{2}|CD||ME| = \frac{1}{2}(2x)x = x^2 = (\sqrt{2})^2 = 2$
- AREA OF CIRCULAR SECTOR
  $\frac{1}{2}r^2\theta^c = \frac{1}{2}\times|CE|\times C\hat{E}D = \frac{1}{2}\times 2^2 \times \frac{\pi}{2} = \pi$
- REQUIRED AREA IS
  $\pi + (\pi - 2) = 2\pi - 2$
  "SEMICIRCLE" "SECTOR − TRIANGLE"

Question 112 (*****) (non calculator)



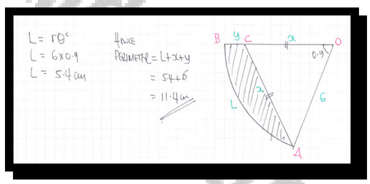
The figure above shows a circular arc OAB of radius 6 cm, subtending an angle of 0.9 radians at O.

The point C lies on OB so that $OC = AC$.

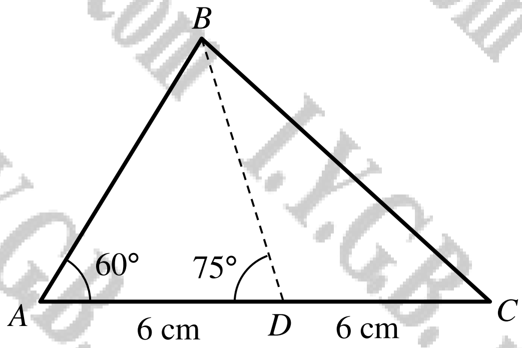
The region R, shown shaded in the figure, is bounded by the arc AB and the straight lines AC and BC.

Determine the perimeter of R.

11.4 cm



Question 113 **(*****)** **(non calculator)**



The figure above shows a triangle ABC.

The line BD is such so that $AD = DC = 6$ cm and the angles BAD and BDA are $60°$ and $75°$, respectively.

Show that

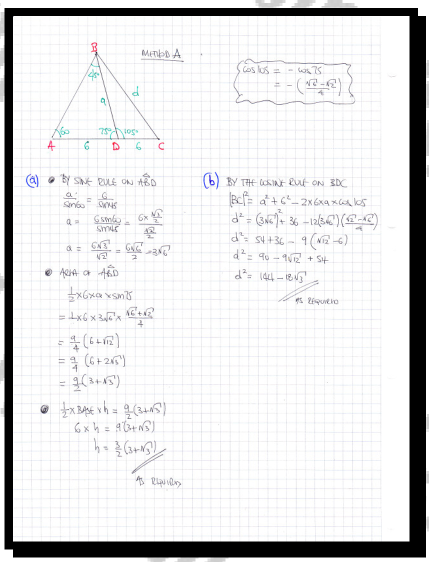
a) The shortest distance from the vertex B to the side AC is

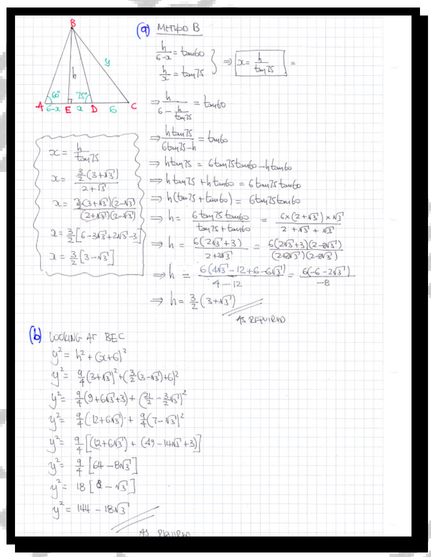
$$\frac{3}{2}\left(3+\sqrt{3}\right).$$

b) The length BC squared is

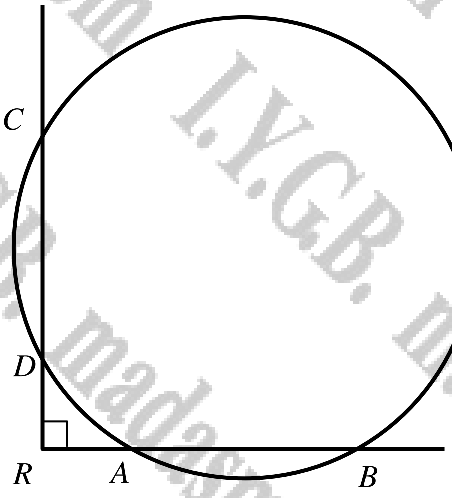
$$144-18\sqrt{3}.$$

proof





Question 114 (*****)



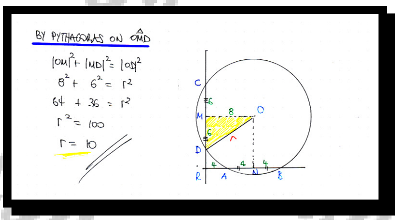
The figure above shows a straight line intersecting a circle at the points A and B so that $|AB| = 8$ units.

Another straight line intersects the same circle at the points C and D so that $|CD| = 12$ units.

The two straight lines intersect each other at right angles at the point R.

Given further that $|AR| = 4$ units, determine the length of the radius of the circle.

$\boxed{}$, $\boxed{r = 10}$

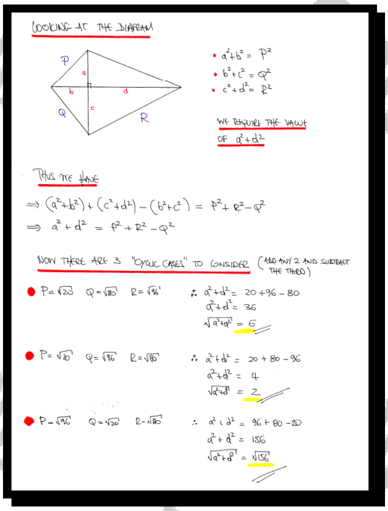


Question 115 (*****)

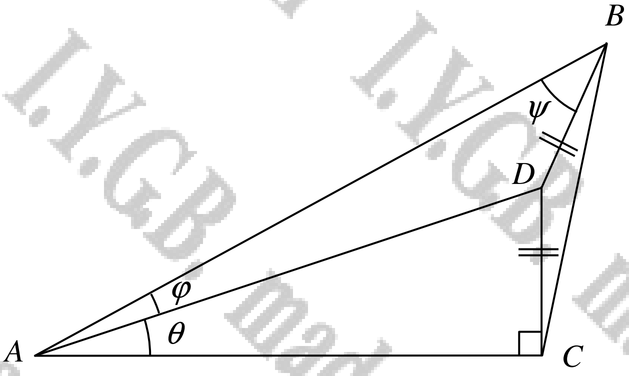
A convex quadrilateral has perpendicular diagonals and three of its sides have lengths of $\sqrt{20}$, $\sqrt{80}$ and $\sqrt{96}$, measured in suitable units.

Determine possible lengths of the fourth side.

$\square$, $\boxed{2 \cup 6 \cup \sqrt{156}}$



Question 116 (*****)



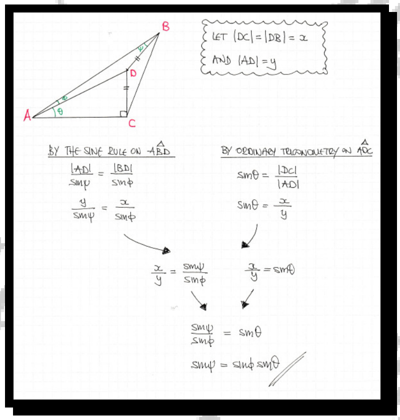
The point D lies inside the triangle ABC, so that $|DB| = |DC|$ and $\measuredangle DCA = \frac{1}{2}\pi$.

Let $\theta = \measuredangle DAC$, $\varphi = \measuredangle BAD$ and $\psi = \measuredangle ABD$.

Show that

$$\sin\psi = \sin\theta \sin\varphi .$$

[ ], [proof]



Question 117 (*****)

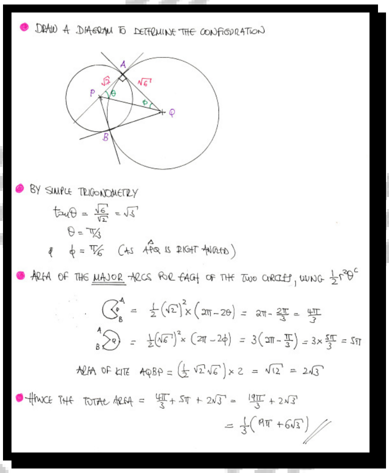
Two coplanar circles, with respective radii $\sqrt{2}$ and $\sqrt{6}$, intersect each other at the points A and B.

The tangent to one of the circles at A, intersects the tangent to the other circle at A at right angles.

Show that the total area enclosed by the two circles is

$$\frac{1}{2}\left(19\pi + 6\sqrt{3}\right).$$

[ ], [proof]



Question 118 (*****)

A hiker on a mountain walk has injured himself.

He rings the rescue station which is located at the point with coordinates $(2,1)$.

He reports that he is lying injured by a river bank where he can see a ruined tower, which his compass indicates that it is located South-West from his position.

It is known to the rescue station that the only river in the area has equation $x=8$ and the ruined tower is located at the point with coordinates $(2,3)$ on the coordinate axes.

The rescuers set off immediately from the Rescue Station and travel directly towards the hiker. When the rescuers are half-way into their journey, the hiker rings again.

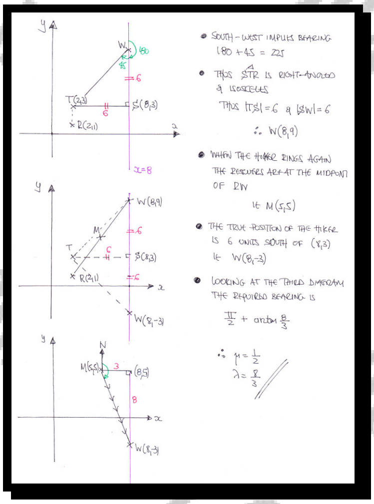
He says that he made a mistake in reading his compass and the ruined tower is in fact located North-West from his position.

The rescuers turn and head directly towards the true location of the hiker.

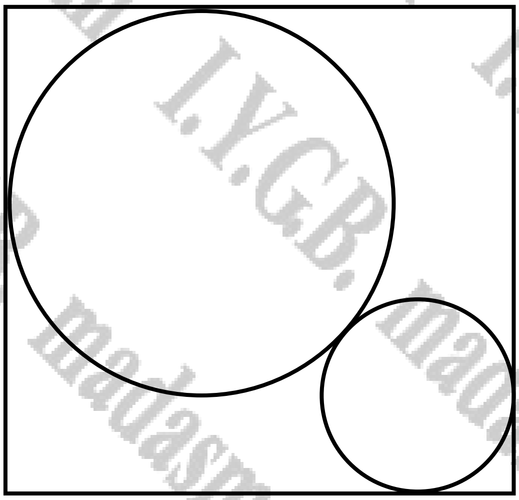
Calculate the angle, as a bearing, at which the rescuers are heading after the hikers second phone call.

Give the answer in the form $\mu\pi + \arctan\lambda$, where μ and λ are constants to be found.

$\boxed{}$, $\boxed{\frac{1}{2}\pi + \arctan\frac{8}{3}}$



Question 119 (*****)

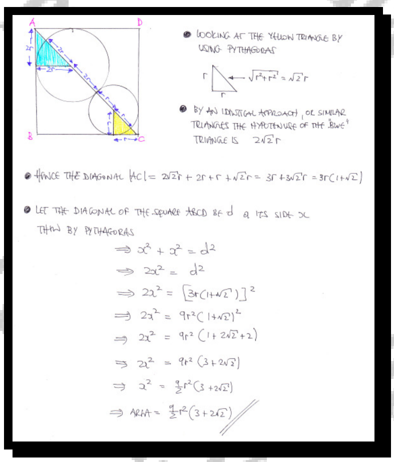


Two circles of different radii are touching each other externally.

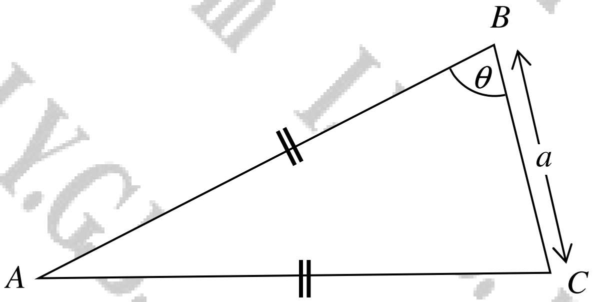
The two circles are enclosed by a square so that all 4 sides of the squares are tangents to the circles, as shown in the figure above.

Given that the radius of the smaller circle is r and the radius of the larger circle is $2r$, determine the exact area of the square in terms of r.

$\boxed{}$, $\boxed{\text{area} = \frac{9}{2}r^2\left(3+2\sqrt{2}\right)}$



Question 120 (*****)



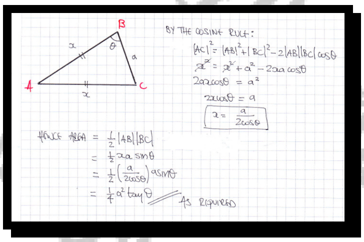
The figure above shows an isosceles triangle ABC, where $AB = AC$.

The side BC has length a and the angle ABC is θ.

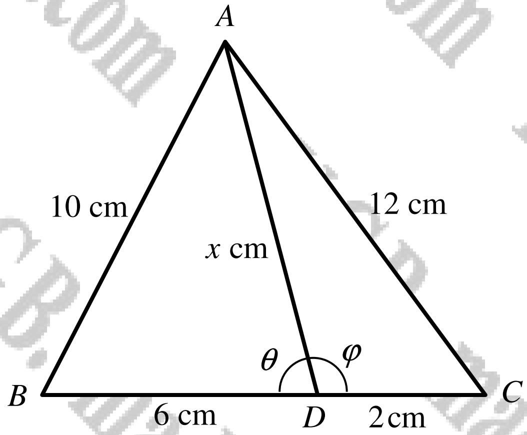
Show that the area of the triangle is

$$\frac{1}{4}a^2 \tan\theta .$$

[ ], [proof]



Question 121 (*****)



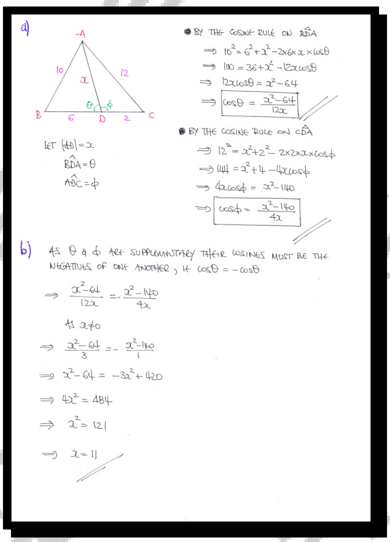
The figure above shows the triangle ABC, where $|AB| = 10$ cm, $|AC| = 12$ cm and $|AB| = 8$ cm. The point D lies on BC so that $|AD| = 6$ cm, $|DC| = 2$ cm and $|AD| = x$ cm. The angle BDA is denoted by θ and the angle CDA is denoted by φ.

a) Express $\cos\theta$ and $\cos\varphi$ in terms of x.

b) Use part **(a)** to find the length of AD.

c) Hence show that the area of the triangle ABD is exactly $\frac{45}{4}\sqrt{7}$ cm^2.

[ ] , [proof]



a)

LET $|AD| = x$, $B\hat{D}A = \theta$, $A\hat{D}C = \phi$

● BY THE COSINE RULE ON $B\hat{D}A$

$\Rightarrow 10^2 = 6^2 + x^2 - 2 \times 6 \times x \times \cos\theta$

$\Rightarrow 100 = 36 + x^2 - 12x\cos\theta$

$\Rightarrow 12x\cos\theta = x^2 - 64$

$\Rightarrow \boxed{\cos\theta = \frac{x^2 - 64}{12x}}$

● BY THE COSINE RULE ON $C\hat{D}A$

$\Rightarrow 12^2 = x^2 + 2^2 - 2 \times 2 \times x \times \cos\phi$

$\Rightarrow 144 = x^2 + 4 - 4x\cos\phi$

$\Rightarrow 4x\cos\phi = x^2 - 140$

$\Rightarrow \boxed{\cos\phi = \frac{x^2 - 140}{4x}}$

b) AS θ & ϕ ARE SUPPLEMENTARY THEIR COSINES MUST BE THE NEGATIVES OF ONE ANOTHER, IE $\cos\theta = -\cos\theta$

$\Rightarrow \frac{x^2 - 64}{12x} = -\frac{x^2 - 140}{4x}$

AS $x \neq 0$

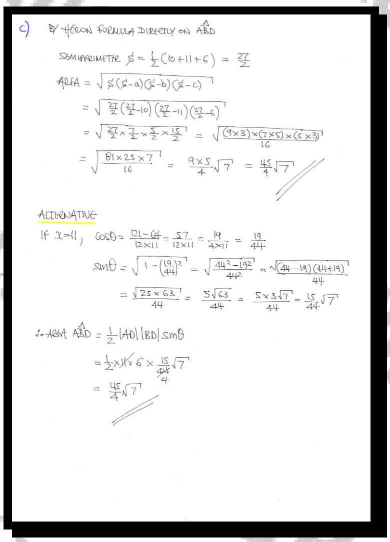
$\Rightarrow \frac{x^2 - 64}{3} = -\frac{x^2 - 140}{1}$

$\Rightarrow x^2 - 64 = -3x^2 + 420$

$\Rightarrow 4x^2 = 484$

$\Rightarrow x^2 = 121$

$\Rightarrow x = 11$



c) BY HERON FORMULA DIRECTLY ON $A\hat{B}D$

SEMIPERIMETER $s = \frac{1}{2}(10 + 11 + 6) = \frac{27}{2}$

AREA $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{\frac{27}{2}\left(\frac{27}{2} - 10\right)\left(\frac{27}{2} - 11\right)\left(\frac{27}{2} - 6\right)}$

$= \sqrt{\frac{27}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{15}{2}} = \sqrt{\frac{(9 \times 3) \times (7 \times 5) \times (5 \times 3)}{16}}$

$= \sqrt{\frac{81 \times 25 \times 7}{16}} = \frac{9 \times 5}{4}\sqrt{7} = \frac{45}{4}\sqrt{7}$

ALTERNATIVE

IF $x = 11$, $\cos\theta = \frac{121 - 64}{12 \times 11} = \frac{57}{12 \times 11} = \frac{19}{4 \times 11} = \frac{19}{44}$

$\sin\theta = \sqrt{1 - \left(\frac{19}{44}\right)^2} = \sqrt{\frac{44^2 - 19^2}{44^2}} = \frac{\sqrt{(44-19)(44+19)}}{44}$

$= \frac{\sqrt{25 \times 63}}{44} = \frac{5\sqrt{63}}{44} = \frac{5 \times 3\sqrt{7}}{44} = \frac{15}{44}\sqrt{7}$

$\therefore$ AREA $A\hat{B}D = \frac{1}{2}|AD||BD|\sin\theta$

$= \frac{1}{2} \times 11 \times 6 \times \frac{15}{44}\sqrt{7}$

$= \frac{45}{4}\sqrt{7}$

Question 122 (*****)

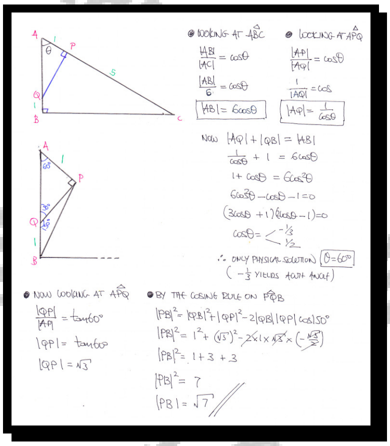
The triangle ABC is such so that $\measuredangle ABC = 90°$ and $|AC| = 6$ cm.

The point P lies on AC and the point Q lies on AB in such a way so that

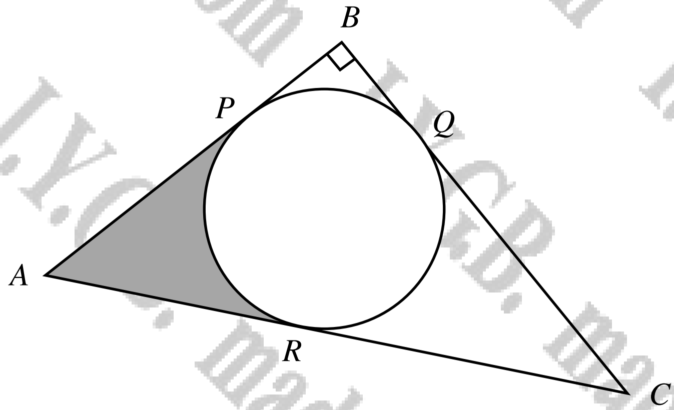
$$\measuredangle APQ = 90° \text{ and } |AP| = |QB| = 1 \text{ cm}.$$

Show that the straight line segment PB is exactly $\sqrt{7}$ cm.

[ ], [proof]



Question 123 (*****)



The triangle ABC has a right angle at B, with $|AB| = 21$ cm and $|BC| = 20$ cm.

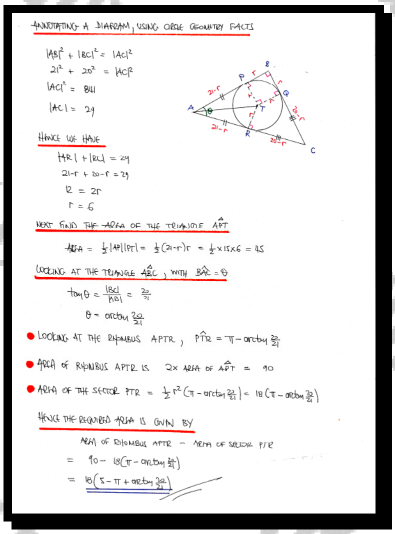
A circle is drawn inside the triangle so that the three sides of the triangle are tangents to the circle.

The points P, Q and R are the respective points of tangency with AB, BC and AC.

Show that the area of the finite region bounded by AP, AR and the circular arc PR, shown shaded in the figure above, is

$$18\left[5 - \pi + \arctan\left(\frac{20}{21}\right)\right].$$

[ ], proof



ANNOTATING A DIAGRAM, USING CIRCLE GEOMETRY FACTS

$|AB|^2 + |BC|^2 = |AC|^2$

$21^2 + 20^2 = |AC|^2$

$|AC|^2 = 841$

$|AC| = 29$

HENCE WE HAVE

$|AR| + |RC| = 29$

$21 - r + 20 - r = 29$

$12 = 2r$

$r = 6$

NEXT FIND THE AREA OF THE TRIANGLE $A\hat{P}T$

$\text{AREA} = \frac{1}{2}|AP||PT| = \frac{1}{2}(21-r)r = \frac{1}{2} \times 15 \times 6 = 45$

LOOKING AT THE TRIANGLE $A\hat{B}C$, WITH $B\hat{A}C = \theta$

$\tan\theta = \frac{|BC|}{|AB|} = \frac{20}{21}$

$\theta = \arctan\frac{20}{21}$

- LOOKING AT THE RHOMBUS APTR, $P\hat{T}R = \pi - \arctan\frac{20}{21}$
- AREA OF RHOMBUS APTR IS $2\times$ AREA OF $A\hat{P}T = 90$
- AREA OF THE SECTOR PTR $= \frac{1}{2}r^2\left(\pi - \arctan\frac{20}{21}\right) = 18\left(\pi - \arctan\frac{20}{21}\right)$

HENCE THE REQUIRED AREA IS GIVEN BY

AREA OF RHOMBUS APTR − AREA OF SECTOR PTR

$= 90 - 18\left(\pi - \arctan\frac{20}{21}\right)$

$= 18\left(5 - \pi + \arctan\frac{20}{21}\right)$

Question 124 (*****)

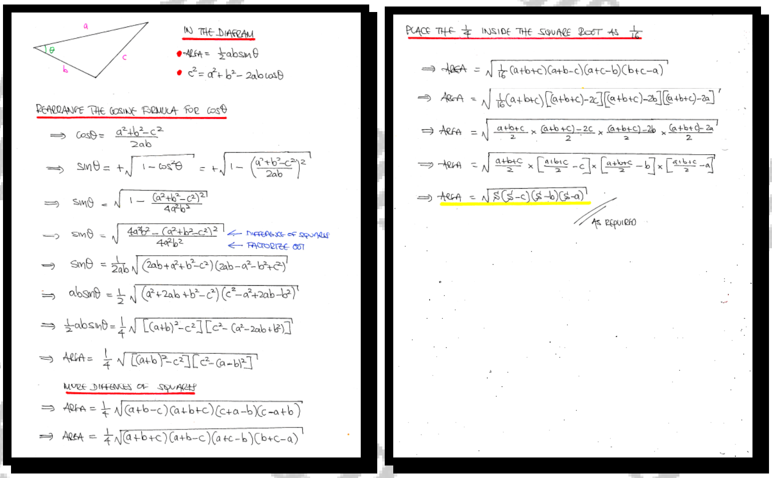
Heron's method for determining the area of any triangle asserts that, if a triangle has side lengths a, b and c, then its area is given by

$$\sqrt{s(s-a)(s-b)(s-c)},$$

where $s=\frac{1}{2}(a+b+c)$, the semi-perimeter of the triangle.

[*you may find the cosine rule and the trigonometric form for the area of a triangle useful in this question*]

V, [ ], proof



IN THE DIAGRAM

- AREA $=\frac{1}{2}ab\sin\theta$
- $c^2=a^2+b^2-2ab\cos\theta$

REARRANGE THE COSINE FORMULA FOR $\cos\theta$

$\Rightarrow \cos\theta=\frac{a^2+b^2-c^2}{2ab}$

$\Rightarrow \sin\theta=+\sqrt{1-\cos^2\theta}=+\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2}$

$\Rightarrow \sin\theta=\sqrt{1-\frac{(a^2+b^2-c^2)^2}{4a^2b^2}}$

$\Rightarrow \sin\theta=\sqrt{\frac{4a^2b^2-(a^2+b^2-c^2)^2}{4a^2b^2}}$ ← DIFFERENCE OF SQUARES ← FACTORIZE OUT

$\Rightarrow \sin\theta=\frac{1}{2ab}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)}$

$\Rightarrow ab\sin\theta=\frac{1}{2}\sqrt{(a^2+2ab+b^2-c^2)(c^2-a^2+2ab-b^2)}$

$\Rightarrow \frac{1}{2}ab\sin\theta=\frac{1}{4}\sqrt{[(a+b)^2-c^2][c^2-(a^2-2ab+b^2)]}$

$\Rightarrow$ AREA $=\frac{1}{4}\sqrt{[(a+b)^2-c^2][c^2-(a-b)^2]}$

MORE DIFFERENCES OF SQUARES

$\Rightarrow$ AREA $=\frac{1}{4}\sqrt{(a+b-c)(a+b+c)(c+a-b)(c-a+b)}$

$\Rightarrow$ AREA $=\frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$

PLACE THE $\frac{1}{4}$ INSIDE THE SQUARE ROOT AS $\frac{1}{16}$

$\Rightarrow$ AREA $=\sqrt{\frac{1}{16}(a+b+c)(a+b-c)(a+c-b)(b+c-a)}$

$\Rightarrow$ AREA $=\sqrt{\frac{1}{16}(a+b+c)[(a+b+c)-2c][(a+b+c)-2b][(a+b+c)-2a]}$

$\Rightarrow$ AREA $=\sqrt{\frac{a+b+c}{2}\times\frac{(a+b+c)-2c}{2}\times\frac{(a+b+c)-2b}{2}\times\frac{(a+b+c)-2a}{2}}$

$\Rightarrow$ AREA $=\sqrt{\frac{a+b+c}{2}\times\left[\frac{a+b+c}{2}-c\right]\times\left[\frac{a+b+c}{2}-b\right]\times\left[\frac{a+b+c}{2}-a\right]}$

$\Rightarrow$ AREA $=\sqrt{s(s-c)(s-b)(s-a)}$

// AS REQUIRED

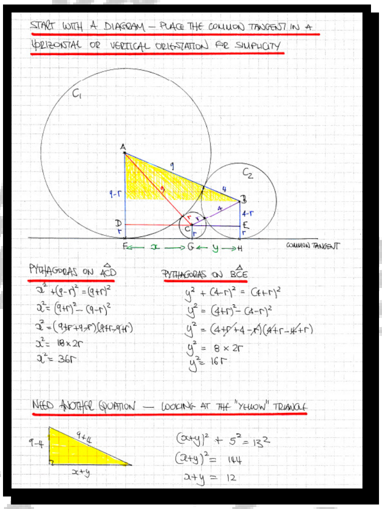
Question 125 (*****)

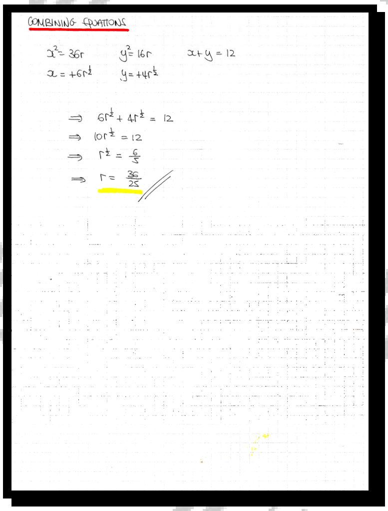
Two circles, C_1 and C_2, are touching each other **externally**, and have respective radii of 9 and 4 units.

A third circle C_3, of radius r, touches C_1 **and** C_2 **externally**.

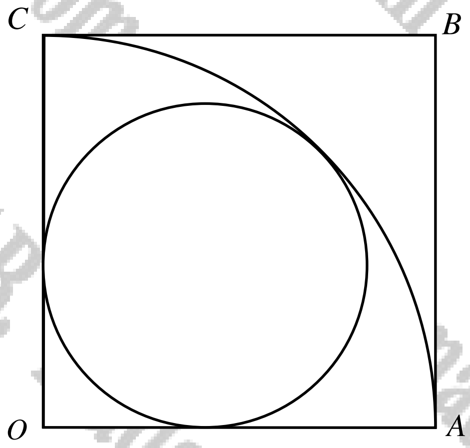
Given further that all three circles have a common tangent, determine the value of r.

$\boxed{}$, $\boxed{r = \frac{36}{25} = 1.44}$





Question 126 (*****)



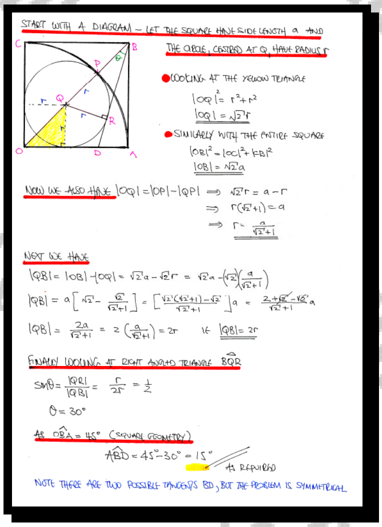
A quarter circular arc AC is inscribed inside a square $OABC$.

The centre of the arc is located at O and the radius of the arc is the same as the side length of the square.

A circle is drawn inside the square so that it touches the quarter circle AC **internally**, and the sides of the square, OA and OC, are tangents to this circle, as shown in the figure above.

If the straight line AD is a tangent to this circle, show that $\measuredangle ABD = 15°$.

[ ], proof



START WITH A DIAGRAM – LET THE SQUARE HAVE SIDE LENGTH a AND THE CIRCLE, CENTRED AT Q, HAVE RADIUS r

- LOOKING AT THE YELLOW TRIANGLE

$|OQ|^2 = r^2 + r^2$

$|OQ| = \sqrt{2}r$

- SIMILARLY WITH THE ENTIRE SQUARE

$|OB|^2 = |OC|^2 + |CB|^2$

$|OB| = \sqrt{2}a$

NOW WE ALSO HAVE $|OQ| = |OP| - |QP| \implies \sqrt{2}r = a - r$

$\implies r(\sqrt{2}+1) = a$

$\implies r = \frac{a}{\sqrt{2}+1}$

NEXT WE HAVE

$|QB| = |OB| - |OQ| = \sqrt{2}a - \sqrt{2}r = \sqrt{2}a - (\sqrt{2})\left(\frac{a}{\sqrt{2}+1}\right)$

$|QB| = a\left[\sqrt{2} - \frac{\sqrt{2}}{\sqrt{2}+1}\right] = \left[\frac{\sqrt{2}(\sqrt{2}+1) - \sqrt{2}}{\sqrt{2}+1}\right]a = \frac{2+\sqrt{2}-\sqrt{2}}{\sqrt{2}+1}a$

$|QB| = \frac{2a}{\sqrt{2}+1} = 2\left(\frac{a}{\sqrt{2}+1}\right) = 2r$ ie $|QB| = 2r$

FINALLY LOOKING AT RIGHT ANGLED TRIANGLE $B\hat{Q}R$

$\sin\theta = \frac{|QR|}{|QB|} = \frac{r}{2r} = \frac{1}{2}$

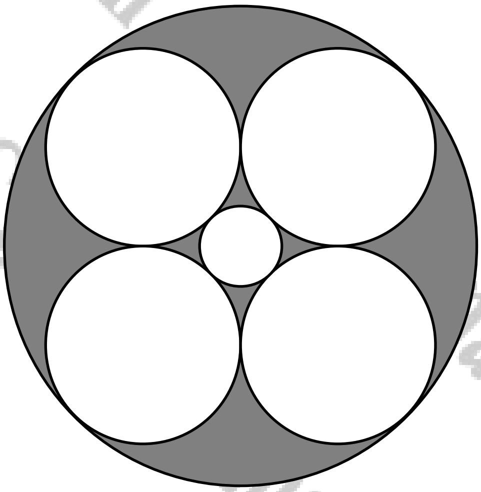
$\theta = 30°$

AS $O\hat{B}A = 45°$ (SQUARE GEOMETRY)

$A\hat{B}D = 45° - 30° = 15°$ AS REQUIRED

NOTE THERE ARE TWO POSSIBLE TANGENTS BD, BUT THE PROBLEM IS SYMMETRICAL

Question 127 **(*****)** **non calculator**



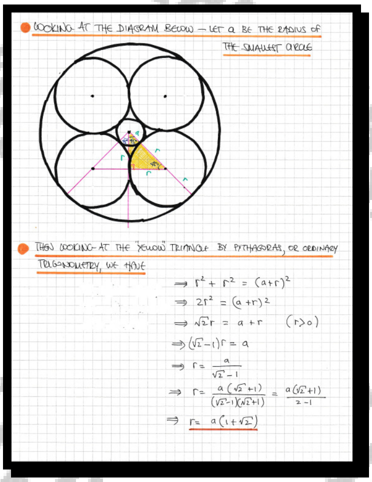
The figure above shows 4 identical circles touching each other so that their centres form a square.

A smaller circle is touching all 4 of the identical circles externally, and all 4 of the identical circles are touching internally a larger circle.

Determine, correct to 1 decimal place, the fraction of the larger circle not occupied, by the other 5 circles, shown shaded in the figure.

You may assume that $\sqrt{2} \approx 1.4142$.

$\boxed{}$, $\boxed{0.284 \approx 28.4}$



LOOKING AT THE DIAGRAM BELOW – LET a BE THE RADIUS OF THE SMALLEST CIRCLE

THEN LOOKING AT THE "YELLOW" TRIANGLE BY PYTHAGORAS, OR ORDINARY TRIGONOMETRY, WE HAVE

$\Rightarrow r^2 + r^2 = (a+r)^2$

$\Rightarrow 2r^2 = (a+r)^2$

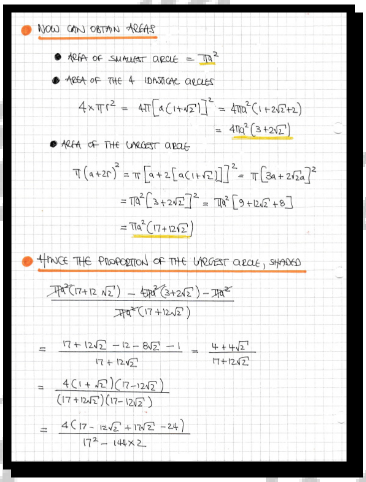
$\Rightarrow \sqrt{2}r = a + r \quad (r>0)$

$\Rightarrow (\sqrt{2}-1)r = a$

$\Rightarrow r = \frac{a}{\sqrt{2}-1}$

$\Rightarrow r = \frac{a(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{a(\sqrt{2}+1)}{2-1}$

$\Rightarrow r = a(1+\sqrt{2})$



NOW CAN OBTAIN AREAS

- AREA OF SMALLEST CIRCLE $= \pi a^2$
- AREA OF THE 4 IDENTICAL CIRCLES

$4 \times \pi r^2 = 4\pi\left[a(1+\sqrt{2})\right]^2 = 4\pi a^2(1+2\sqrt{2}+2)$
$= 4\pi a^2(3+2\sqrt{2})$

- AREA OF THE LARGEST CIRCLE

$\pi(a+2r)^2 = \pi\left[a+2\left[a(1+\sqrt{2})\right]\right]^2 = \pi\left[3a+2\sqrt{2}a\right]^2$
$= \pi a^2\left[3+2\sqrt{2}\right]^2 = \pi a^2\left[9+12\sqrt{2}+8\right]$
$= \pi a^2(17+12\sqrt{2})$

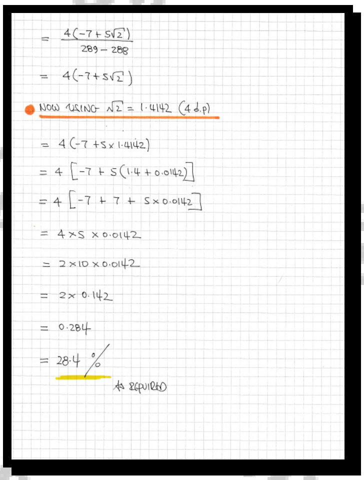
HENCE THE PROPORTION OF THE LARGEST CIRCLE, SHADED

$\frac{\pi a^2(17+12\sqrt{2}) - 4\pi a^2(3+2\sqrt{2}) - \pi a^2}{\pi a^2(17+12\sqrt{2})}$

$= \frac{17+12\sqrt{2}-12-8\sqrt{2}-1}{17+12\sqrt{2}} = \frac{4+4\sqrt{2}}{17+12\sqrt{2}}$

$= \frac{4(1+\sqrt{2})(17-12\sqrt{2})}{(17+12\sqrt{2})(17-12\sqrt{2})}$

$= \frac{4(17-12\sqrt{2}+17\sqrt{2}-24)}{17^2-144\times 2}$



$= \frac{4(-7+5\sqrt{2})}{289-288}$

$= 4(-7+5\sqrt{2})$

NOW USING $\sqrt{2} = 1.4142$ (4 d.p)

$= 4(-7+5\times 1.4142)$

$= 4\left[-7+5(1.4+0.0142)\right]$

$= 4\left[-7+7+5\times 0.0142\right]$

$= 4\times 5\times 0.0142$

$= 2\times 10\times 0.0142$

$= 2\times 0.142$

$= 0.284$

$= 28.4\%$

AS REQUIRED