GEOMETRIC SERIES

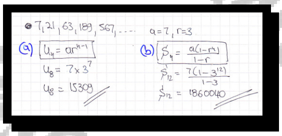
Question 1 (**)

$$7, 21, 63, 189, 567, \ldots$$

a) Find, using algebra, the value of the eighth term of the above sequence.

b) Determine the sum of the first twelve terms of the sequence.

$\boxed{u_8 = 15309}$, $\boxed{S_{12} = 1860040}$



Question 2 (**)

$$59049, 19683, 6561, 2187, 729, \ldots$$

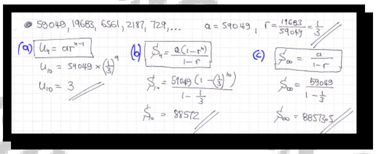
For above sequence, …

a) … calculate, using algebra, the value of the tenth term.

b) … determine the sum of the first ten terms.

c) … find the sum to infinity.

$\boxed{u_{10} = 3}$, $\boxed{S_{10} = 88572}$, $\boxed{S_\infty = 88573.5}$



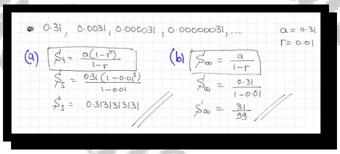
Question 3 (**)

$$0.31,\ 0.0031,\ 0.000031,\ 0.00000031,\ \ldots$$

a) Calculate, using algebra, the sum of the first five terms of the above sequence.

b) Find, as an exact fraction, the sum to infinity of the series.

$\boxed{S_5 = 0.3131313131}$, $\boxed{S_\infty = \frac{31}{99}}$



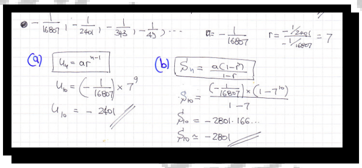
Question 4 (**)

$$-\frac{1}{16807},\ -\frac{1}{2401},\ -\frac{1}{343},\ -\frac{1}{49},\ \ldots$$

a) Determine, using algebra, the value of the tenth term of the above sequence.

b) Find, to the nearest integer, the sum of the first ten terms of the above sequence.

$\boxed{u_{10} = -2401}$, $\boxed{S_{10} \approx -2801}$



Question 5 **(**)**

The first and second term of a geometric series is 90 and 15, respectively.

a) State the common ratio of the series.

b) Calculate the sum to infinity of the series.

$\boxed{}$, $\boxed{r = \frac{1}{6}}$, $\boxed{S_\infty = 108}$

(a) $r = \frac{u_2}{u_1} = \frac{15}{90} = \frac{1}{6}$

(b) $S_\infty = \frac{a}{1-r} = \frac{90}{1-1/6} = \frac{90}{5/6} = 108$

Question 6 **(**)**

The third and fourth term of a geometric progression is 144 and 108, respectively.

Find …

a) … the common ratio of the progression.

b) … the fifth term of the progression.

c) … the sum to infinity of the progression.

$\boxed{}$, $\boxed{r = \frac{3}{4}}$, $\boxed{u_5 = 81}$, $\boxed{S_\infty = 1024}$

(a) $u_3 = 144$
$u_4 = 108$
$r = \frac{u_4}{u_3} = \frac{108}{144} = \frac{3}{4}$

(b) $u_5 = u_4 \times r$
$u_5 = 108 \times \frac{3}{4}$
$u_5 = 81$

(c) $u = ar^{n-1}$
$144 = a \times \left(\frac{3}{4}\right)^2$
$144 = \frac{9}{16}a$
$\boxed{a = 256}$
$\therefore S_\infty = \frac{a}{1-r}$
$S_\infty = \frac{256}{1-3/4}$
$S_\infty = 1024$

Question 7 **(**)**

Evaluate the sum

$$\sum_{r=1}^{15}\left(3\times 2^{r}\right),$$

showing clearly all the relevant workings.

$\boxed{196602}$

$$\sum_{r=1}^{15}(3\times 2^{r}) = (3\times 2^{1})+(3\times 2^{2})+(3\times 2^{3})+(3\times 2^{4})+\dots+(3\times 2^{15})$$
$$= \underbrace{6+12+24+48+\dots}_{\text{This is a G.P.}}$$

$a=6$

$r=2$

$n=15$

USING $S_n = \dfrac{a(r^n-1)}{r-1}$

$$S_{15} = \frac{6(2^{15}-1)}{2-1} = 196602$$

Question 8 **(**+)**

A geometric series has common ratio $\frac{1}{3}$.

a) Find the first term of the series, given that the sum of its first four terms is 36.

b) Determine the sum to infinity of the series.

$\boxed{a=24.3}$, $\boxed{S_\infty = 36.45}$

(a) $S_4 = \dfrac{a(1-r^4)}{1-r}$

$36 = \dfrac{a\left(1-\left(\frac{1}{3}\right)^4\right)}{1-\frac{1}{3}}$

$36 = \dfrac{a\left(1-\frac{1}{81}\right)}{\frac{2}{3}}$

$24 = a \times \frac{80}{81}$

$a = 24.3$

(b) $S_\infty = \dfrac{a}{1-r}$

$S_\infty = \dfrac{24.3}{1-\frac{1}{3}}$

$S_\infty = \dfrac{24.3}{\frac{2}{3}}$

$S_\infty = 36.45$

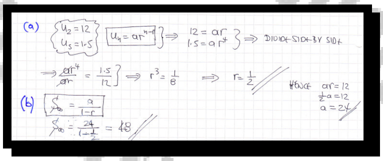
Question 9 **(**+)**

The second and the fifth term of a geometric series is 12 and 1.5, respectively.

a) Find the first term and the common ratio of the series.

b) Calculate the sum to infinity of the series.

$\boxed{a = 24}$, $\boxed{r = 0.5}$, $\boxed{S_\infty = 48}$



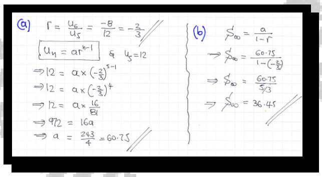
Question 10 **(**+)**

The fifth and the sixth term of a geometric series is 12 and -8, respectively.

a) Find the first term and the common ratio of the series.

b) Calculate the sum to infinity of the series.

$\boxed{a = 60.75}$, $\boxed{r = -\frac{2}{3}}$, $\boxed{S_\infty = 36.45}$



Question 11 (**+)

The common ratio of a geometric series is twice as large as its first term.

a) Given that the sum to infinity of the series is 1, find the exact value of the first term of the series.

b) Determine, as an exact fraction, the value of the fifth term of the series.

$\boxed{}$, $\boxed{a = \frac{1}{3}}$, $\boxed{u_5 = \frac{16}{243}}$

(a) $\boxed{r = 2a}$ & $S_\infty = 1$

$\frac{a}{1-r} = 1$

$\boxed{a = 1 - r}$

So $a = 1 - r$

$a = 1 - 2a$

$3a = 1$

$a = \frac{1}{3}$

(b) NEED THE COMMON RATIO

$r = 2a = \frac{2}{3}$

$u_n = ar^{n-1}$

$u_5 = \frac{1}{3} \times \left(\frac{2}{3}\right)^4$

$u_5 = \frac{16}{243}$

Question 12 (**+)

The common ratio of a geometric series is $\frac{1}{2}$ and the sum of its first three terms is 98.

a) Find the first term of the series.

b) Determine the sum to infinity of the series.

$\boxed{a = 56}$, $\boxed{u_\infty = 112}$

(a) $S_3 = 98$

$\Rightarrow a + ar + ar^2 = 98$

$\Rightarrow a + \frac{1}{2}a + a\left(\frac{1}{2}\right)^2 = 98$

$\Rightarrow a + \frac{1}{2}a + \frac{1}{4}a = 98$ $(\times 4)$

$\Rightarrow 4a + 2a + a = 392$

$\Rightarrow 7a = 392$

$\Rightarrow a = 56$

(b) $S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{56}{1-\frac{1}{2}}$

$S_\infty = \frac{56}{\frac{1}{2}}$

$S_\infty = 112$

Question 13 (**+)

The second and the fourth term of a geometric series are 3 and 1.08, respectively.

Calculate the sum to infinity of the series, given that all the terms of the series are positive.

$$\boxed{S_\infty = 12.5}$$

$u_2 = 3$, $u_4 = 1.08$ $\quad \boxed{u_n = ar^{n-1}} \Rightarrow \left.\begin{array}{l} 3 = ar \\ 1.08 = ar^3 \end{array}\right\} \Rightarrow$ DIVIDE SIDE BY SIDE

$\Rightarrow \frac{\cancel{a}r^3}{\cancel{a}r} = \frac{1.08}{3} \Rightarrow r^2 = 0.36 \Rightarrow r = 0.6$ $(r>0)$ // THUS $ar = 3$, $0.6a = 3$, $a = 5$ //

$\therefore \boxed{S_\infty = \frac{a}{1-r}} \Rightarrow S_\infty = \frac{5}{1-0.6} = 12.5$ //

Question 14 (**+)

Miss Velibright started working as an accountant in a large law firm in the year 2001.

Her starting salary was £$22,000$ and her contract promised that she will be receiving a pay rise of 5% every year thereafter. Miss Velibright plans to retire in 2030.

Find to the nearest £, …

a) …her salary in the year 2030.

b) … her total earnings in employment for the years 2001 to 2030, inclusive.

$\boxed{\text{£}90,555}$, $\boxed{\text{£}1,461,655}$

(a) $a = 22000$, $r = 1.05$

$\boxed{u_n = ar^{n-1}}$

$u_{30} = 22000 \times 1.05^{29}$

$u_{30} = 90554.98\ldots$

$\therefore$ £90,555 //

(b) $S_n = \frac{a(r^n - 1)}{r-1}$ $\quad r>1$

$S_{30} = \frac{22000(1.05^{30} - 1)}{1.05 - 1}$

$S_{30} = 1461654.645\ldots$

$\therefore$ £1,461,655 //

Question 15 (**+)

The k^{th} term of a geometric progression is given by

$$u_k = 15625 \times 1.25^{-k}.$$

a) Find the first three terms of the progression.

b) Find the sum to infinity of the progression.

c) Evaluate the sum

$$\sum_{k=1}^{10} u_k ,$$

giving the answer to the nearest integer.

$\boxed{}$, $\boxed{u_1 = 12500, \quad u_2 = 10000, \quad u_3 = 8000}$, $\boxed{S_\infty = 62500}$, $\boxed{S_{10} = 55789}$

a) USING THE FORMULA GIVEN $u_k = 15625 \times 1.25^{-k}$

$u_1 = 15625 \times 1.25^{-1} = 12500$
$u_2 = 15625 \times 1.25^{-2} = 10000$
$u_3 = 15625 \times 1.25^{-3} = 8000$

b) USING THE STANDARD FORMULA WITH $a = 12500$, $r = 0.8$

$S_\infty = \frac{a}{1-r} = \frac{12500}{1-0.8} = \frac{12500}{0.2} = 62500$

c) USING $S_n = \frac{a(1-r^n)}{1-r}$ WITH $n = 10$

$$\sum_{k=1}^{10} u_k = u_1 + u_2 + u_3 + \cdots + u_{10} = S_{10} = \frac{12500(1-0.8^{10})}{1-0.8} = 55789.1136\ldots \simeq 55789$$

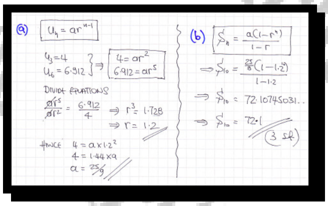
Question 16 (**+)

The third and the sixth term of a geometric series is 4 and 6.912, respectively.

a) Find the exact value of the first term and the common ratio of the series.

b) Calculate, to three significant figures, the sum of the first ten terms of the series.

$\boxed{\text{C2K}}$, $\boxed{a = \frac{25}{9}}$, $\boxed{r = \frac{6}{5}}$, $\boxed{S_{10} \approx 72.1}$



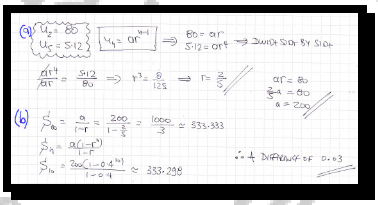
Question 17 (**+)

The second and the fifth term of a geometric progression is 80 and 5.12, respectively.

a) Find the value of the first term and the common ratio of the progression.

b) Calculate, correct to two decimal places, the difference between the sum of the first ten terms of the progression and its sum to infinity.

$\boxed{a = 200}$, $\boxed{r = \frac{2}{5}}$, $\boxed{S_{\infty} - S_{10} = 0.03}$



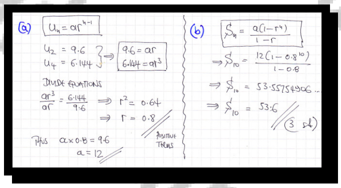
Question 18 (**+)

All the terms of a geometric progression are positive. The second and the fourth term of the progression is 9.6 and 6.144, respectively.

a) Find the first term and the common ratio of the progression.

b) Calculate, correct to three significant figures, the sum of the first ten terms of the progression.

$\boxed{a=12}$, $\boxed{r=0.8}$, $\boxed{S_{10} \approx 53.6}$



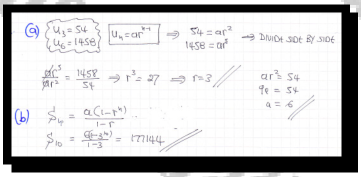
Question 19 (**+)

The third and the sixth term of a geometric series is 54 and 1458, respectively.

a) Find the first term and the common ratio of the series.

b) Determine the sum of the first ten terms of the series.

$\boxed{}$, $\boxed{a=6}$, $\boxed{r=3}$, $\boxed{S_{10}=177144}$



Question 20 (**+)

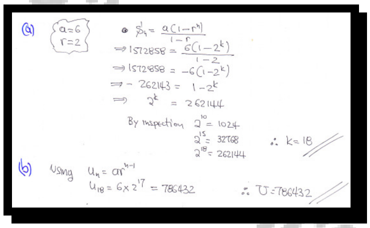
$$6,\ 12,\ 24,\ 48,\ 96,\ \ldots,\ U.$$

The geometric sequence above has k terms and its last term is U.

a) Given that the sum of its terms is $1\,572\,858$, find the value of k.

b) Determine the value of U.

$\boxed{k=18}$, $\boxed{U=786432}$


(a) $a=6$, $r=2$

$S_n = \frac{a(1-r^n)}{1-r}$

$\Rightarrow 1572858 = \frac{6(1-2^k)}{1-2}$

$\Rightarrow 1572858 = -6(1-2^k)$

$\Rightarrow -262143 = 1-2^k$

$\Rightarrow 2^k = 262144$

By inspection $2^{10}=1024$, $2^{15}=32768$, $2^{18}=262144$ $\therefore k=18$

(b) Using $u_n = ar^{n-1}$

$u_{18} = 6\times 2^{17} = 786432$ $\therefore U=786432$

Question 21 (**+)

The first few terms of a geometric sequence are given below

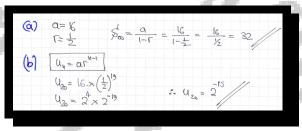
$$16, 8, 4, 2, \ldots$$

Find …

a) … the sum to infinity of the series.

b) … the exact value of the twentieth term, giving the answer as a power of 2.

$\boxed{S_\infty = 32}$, $\boxed{u_{20} = 2^{-15}}$


(a) $a=16$, $r=\frac{1}{2}$ $\quad S_\infty = \frac{a}{1-r} = \frac{16}{1-\frac{1}{2}} = \frac{16}{\frac{1}{2}} = 32$

(b) $\boxed{u_n = ar^{n-1}}$

$u_{20} = 16\times\left(\frac{1}{2}\right)^{19}$

$u_{20} = 2^4\times 2^{-19}$ $\therefore u_{20} = 2^{-15}$

Question 22 (***)

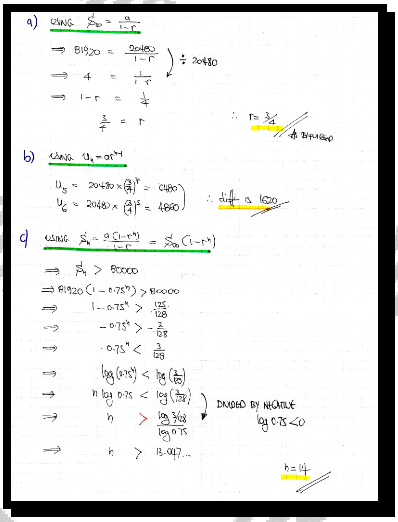
A geometric series has first term 20480 and its sum to infinity is 81920.

a) Show that the common ratio of the series is $\frac{3}{4}$.

b) Calculate the difference between the fifth and the sixth term of the series.

c) Determine the smallest number of terms that should be added so that their total **exceeds** 80000.

$\boxed{}$, $\boxed{1620}$, $\boxed{14}$



a) USING $S_\infty = \frac{a}{1-r}$

$\Rightarrow 81920 = \frac{20480}{1-r}$) $\div 20480$

$\Rightarrow 4 = \frac{1}{1-r}$

$\Rightarrow 1-r = \frac{1}{4}$

$\frac{3}{4} = r$ $\quad \therefore r = \frac{3}{4}$ AS REQUIRED

b) USING $u_n = ar^{n-1}$

$u_5 = 20480 \times \left(\frac{3}{4}\right)^4 = 6480$

$u_6 = 20480 \times \left(\frac{3}{4}\right)^5 = 4860$

$\therefore$ diff is 1620

c) USING $S_n = \frac{a(1-r^n)}{1-r} = S_\infty(1-r^n)$

$\Rightarrow S_n > 80000$

$\Rightarrow 81920(1-0.75^n) > 80000$

$\Rightarrow 1 - 0.75^n > \frac{125}{128}$

$\Rightarrow -0.75^n > -\frac{3}{128}$

$\Rightarrow 0.75^n < \frac{3}{128}$

$\Rightarrow \log(0.75^n) < \log\left(\frac{3}{128}\right)$

$\Rightarrow n \log 0.75 < \log\left(\frac{3}{128}\right)$) DIVIDED BY NEGATIVE $\log 0.75 < 0$

$\Rightarrow n > \frac{\log 3/128}{\log 0.75}$

$\Rightarrow n > 13.047...$

$n = 14$

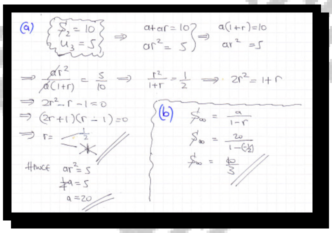
Question 23 (***)

The sum of the first two terms of a geometric series is 10 and the third term is 5.

a) Find the first term and the common ratio of the series.

b) Determine the sum to infinity of the series.

$\boxed{a = 20}$, $\boxed{r = -0.5}$, $\boxed{S_\infty = \frac{40}{3}}$

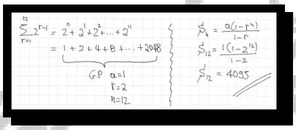


Question 24 (***)

Evaluate the following sum

$$\sum_{r=1}^{12} 2^{r-1} .$$

$\boxed{4095}$



Question 25 (***)

The maximum speed, in mph, that can be achieved in each of the five gears of a sports car form a geometric progression.

The maximum speed obtained in first gear is 32 mph while the car can achieve a maximum speed of 162 mph in fifth gear.

Find the maximum speed that can be achieved in third gear.

$\boxed{}$, $\boxed{72 \text{ mph}}$

$u_1 = 32$
$u_5 = 162$

$\boxed{u_n = ar^{n-1}}$

$\Rightarrow 162 = 32 \times r^4$

$\Rightarrow r^4 = \frac{81}{16}$

$\Rightarrow r = \frac{3}{2}$

$u_3 = 32 \times \left(\frac{3}{2}\right)^2$

$u_3 = 72$ mph

Question 26 (***)

The sum to infinity of a geometric progression of positive terms is 270 and the sum of its first two terms is 240.

Find the first term and the common ratio of the progression.

$\boxed{a = 180}$, $\boxed{r = \frac{1}{3}}$

$S_\infty = 270$

$\frac{a}{1-r} = 270$

$\boxed{a = 270(1-r)}$

$S_2 = 240$

$a + ar = 240$

$\boxed{a(1+r) = 240}$

$270(1-r)(1+r) = 240$

$(1-r)(1+r) = \frac{8}{9}$

$1 - r^2 = \frac{8}{9}$

$\frac{1}{9} = r^2$

$r = +\frac{1}{3}$ (POSITIVE TERMS)

& $a = 270(1-r)$

$a = 270\left(1 - \frac{1}{3}\right)$

$a = 180$

Question 27 (***)

The sum to infinity of a geometric series is four times as large as its first term.

The sum of its first two terms is 2240.

Find the sum of the first five terms of the series.

$$\boxed{S_5 = 3500}$$

- $S_\infty = 4a$

$\Rightarrow \frac{a}{1-r} = 4a$

$\Rightarrow \cancel{a} = 4\cancel{a}(1-r)$

$\Rightarrow 1 = 4(1-r)$

$\Rightarrow \frac{1}{4} = 1-r$

$\Rightarrow \boxed{r = \frac{3}{4}}$

- $S_2 = 2240$

$\Rightarrow a + ar = 2240$

$\Rightarrow a + \frac{3}{4}a = 2240$

$\Rightarrow 4a + 3a = 8960$

$\Rightarrow 7a = 8960$

$\Rightarrow \boxed{a = 1280}$

NOW $S_n = \frac{a(1-r^n)}{1-r}$

$S_5 = \frac{1280\left(1-\left(\frac{3}{4}\right)^5\right)}{1-\frac{3}{4}} = 3905$

Question 28 (***)

Grandad gave Kevin £10 on his first birthday and he increased the amount by 20% on each subsequent birthday.

a) Calculate the amount of money that Kevin received from his grandad on his 10^{th} birthday

Kevin received the last birthday amount of money from his grandad on his n^{th} birthday and on that birthday the amount he received exceeded £1000 for the first time.

b) Show clearly that

$$n > \frac{2}{\log_{10}(1.2)} + 1.$$

c) State the value of n.

$\boxed{}$, $\boxed{£51.60}$, $\boxed{n = 27}$

(a) $a = 10$, $r = 1.2$

$\Rightarrow u_n = ar^{n-1}$

$\Rightarrow u_{10} = 10 \times 1.2^9$

$\Rightarrow u_{10} = 51.597...$

$\therefore$ £51.60

(b) $u_n > 1000$

$\Rightarrow ar^{n-1} > 1000$

$\Rightarrow 10 \times 1.2^{n-1} > 100$

$\Rightarrow 1.2^{n-1} > 100$

$\Rightarrow \log_{10}(1.2)^{n-1} > \log_{10} 100$

$\Rightarrow (n-1)\log_{10}(1.2) > 2$

$\Rightarrow (n-1) > \frac{2}{\log_{10}(1.2)}$

$\Rightarrow n > \frac{2}{\log_{10}(1.2)} + 1$

AS REQUIRED

(c) [illegible]

$n > 26.258...$

$\therefore$ 27th BIRTHDAY

Question 29 (***+)

The first three terms of a geometric series are

$$k\,,\ 6 \quad \text{and} \quad 5k+8 \ \text{respectively,}$$

where k is a constant.

a) Show that one of the possible values of k is 2, and find the other.

b) Given that $k=2$, find the sum of the first 10 terms of the series.

$$\boxed{k=-\frac{18}{5}}\,,\ \boxed{S_{10}=59048}$$

(a) $\left.\begin{array}{l} u_1 = k \\ u_2 = 6 \\ u_3 = 5k+8 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \frac{6}{k} = r \\ \frac{5k+8}{6} = r \end{array}\right\} \Rightarrow \frac{6}{k} = \frac{5k+8}{6}$

$\Rightarrow 5k^2+8k=36$

$\Rightarrow 5k^2+8k-36=0$

$\Rightarrow (k-2)(5k+18)=0$

$\Rightarrow k = \left\langle \begin{array}{l} 2 \\ -\frac{18}{5} \end{array}\right.$

(b) $k=2$

$\left.\begin{array}{l} u_1 = 2 \\ u_2 = 6 \\ u_3 = 18 \end{array}\right. \begin{array}{l} \times 3 \\ \times 3 \end{array}$ $\quad \therefore a=2,\ r=3$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_{10} = \frac{2(1-3^{10})}{1-3}$

$S_{10} = 59048$

Question 30 (***+)

$$1+(1+x)+(1+x)^2+(1+x)^3+(1+x)^4+\ldots$$

It is given that the above series is convergent.

Determine the range of values of x, and its sum to infinity in terms of x.

$$\boxed{-2<x<0}\,,\ \boxed{S_\infty=-\frac{1}{x}}$$

$1+(1+x)+(1+x)^2+(1+x)^3+\ldots$ IS A G.P. WITH $a=1$, $r=1+x$

IF CONVERGENT $-1<r<1$

$-1<1+x<1$

$-2<x<0$

& $S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{1}{1-(1+x)}$

$S_\infty = -\frac{1}{x}$

Question 31 (***+)

The first term of a geometric progression is 1200 and its sum to infinity is 1600.

a) Find the sum of the first five terms of the progression.

The n^{th} term of the progression is denoted by u_n.

b) Evaluate the sum

$$\sum_{r=6}^{\infty} u_r .$$

$\boxed{}$, $\boxed{S_5 = 1598.4375}$, $\boxed{1.5625}$

a) USING THE INFORMATION GIVEN

$$S_\infty = \frac{a}{1-r} \implies 1600 = \frac{1200}{1-r}$$
$$\implies 1600(1-r) = 1200$$
$$\implies 1-r = \frac{3}{4}$$
$$\implies \frac{1}{4} = r$$

SUMMING THE FIRST FIVE TERMS

$$S_n = \frac{a(1-r^n)}{1-r} \implies S_5 = \frac{1200(1-0.25^5)}{1-0.25}$$
$$\implies S_5 = 1600 \times \frac{1023}{1024}$$
$$\implies S_5 = 1598.4375 \approx 1598$$

b) PROCEED AS FOLLOWS

$$\sum_{r=6}^{\infty} u_r = u_6 + u_7 + u_8 + \dots$$
$$= (u_1+u_2+u_3+u_4+\dots) - (u_1+u_2+u_3+u_4+u_5)$$
$$= S_\infty - S_5$$
$$= 1600 - 1598.4375$$
$$= \frac{25}{16}$$
$$= 1.5625$$

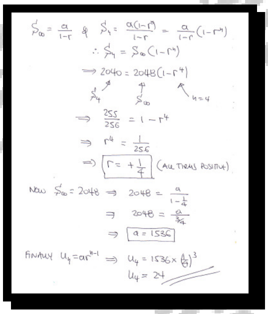
Question 32 **(***+)**

The sum of the first four terms of a geometric series is 2040 and the sum to infinity of the series is 2048.

All the terms of the series are positive.

Find the fourth term of the series.

$\boxed{u_4 = 24}$



Question 33 **(***+)**

The second term of an geometric progression is 2 and the common ratio is $\frac{1}{2}\sqrt{2}$.

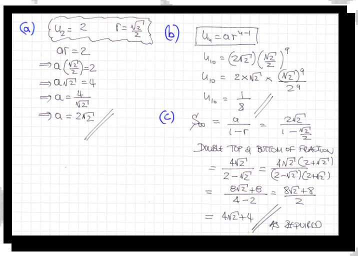
a) Find the first term of the progression.

b) Determine the value of the tenth term.

c) Show, by detailing all steps in the calculation, that the sum to infinity of the progression is

$$4+4\sqrt{2}.$$

$\boxed{a=\sqrt{8}}$, $\boxed{u_{10}=\frac{1}{8}}$



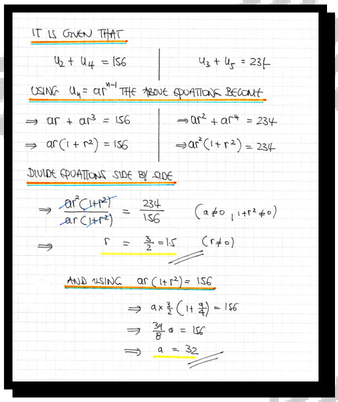
Question 34 (***+)

In a geometric series the sum of the second and fourth term is 156.

In the same geometric series the sum of the third and the fifth term is 234.

Find the first term and the common ratio of the series.

$\boxed{}$, $\boxed{a = 32}$, $\boxed{r = 1.5}$



IT IS GIVEN THAT

$u_2 + u_4 = 156$ | $u_3 + u_5 = 234$

USING $u_n = ar^{n-1}$ THE ABOVE EQUATIONS BECOME

$\Rightarrow ar + ar^3 = 156$ | $\Rightarrow ar^2 + ar^4 = 234$

$\Rightarrow ar(1+r^2) = 156$ | $\Rightarrow ar^2(1+r^2) = 234$

DIVIDE EQUATIONS SIDE BY SIDE

$$\Rightarrow \frac{\cancel{a}r^2\cancel{(1+r^2)}}{\cancel{a}r\cancel{(1+r^2)}} = \frac{234}{156} \qquad (a \neq 0,\ 1+r^2 \neq 0)$$

$$\Rightarrow r = \frac{3}{2} = 1.5 \qquad (r \neq 0)$$

AND USING $ar(1+r^2) = 156$

$$\Rightarrow a \times \frac{3}{2}\left(1 + \frac{9}{4}\right) = 156$$

$$\Rightarrow \frac{39}{8}a = 156$$

$$\Rightarrow a = 32$$

Question 35 (***+)

The second term of a geometric series is 4 and its sum to infinity is 18.

a) Show that the common ratio r of the series is a solution of the equation

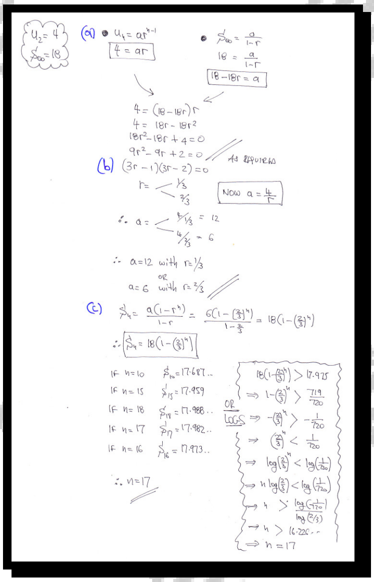
$$9r^2 - 9r + 2 = 0.$$

b) Find the two possible values of r and the corresponding values of the first term of the series.

The sum of the first n terms of the series is denoted by S_n.

c) Given that r takes the larger of the two values found in part (b) determine the smallest value of n for which S_n exceeds 17.975.

$\boxed{r = \frac{1}{3}, a = 12}$, $\boxed{r = \frac{2}{3}, a = 6}$, $\boxed{n = 17}$



Question 36 (***+)

The first three terms of a geometric series are given below as functions of x.

$$x^2, \quad (x+12) \quad \text{and} \quad (2x-3).$$

a) Show that x is a solution of the equation

$$x^3 - 2x^2 - 12x - 72 = 0.$$

b) Show clearly that $x = 6$ is the only solution of the above equation.

c) Find the sum to infinity of the series.

$\boxed{}$, $\boxed{S_\infty = 72}$

(a) $u_1 = x^2$
$u_2 = x+12$
$u_3 = 2x-3$

IF GEOMETRIC $\frac{u_2}{u_1} = \frac{u_3}{u_2}$

$\Rightarrow \frac{x+12}{x^2} = \frac{2x-3}{x+12}$

$\Rightarrow (x+12)^2 = (2x-3)x^2$

$\Rightarrow x^2 + 24x + 144 = 2x^3 - 3x^2$

$\Rightarrow 0 = 2x^3 - 4x^2 - 24x + 144$

$\Rightarrow x^3 - 2x^2 - 12x + 72 = 0$

AS REQUIRED

(b) $(x-6)(x^2 + Ax + 12)$ BY INSPECTION.

$-6Ax$

$12x$

LOOKING AT COEFFICIENT OF x

$12x - 6Ax = -12x$

$12 - 6A = -12$

$24 = 6A$

$A = 4$

$\therefore (x-6)(x^2 + 4x + 12) = 0$

$b^2 - 4ac = 4^2 - 4 \times 1 \times 12 = 16 - 48 = -32 < 0$

NO MORE SOLUTIONS, ONLY SOLUTION $x = 6$

(c) $u_1 = x^2 = 36$
$u_2 = x + 12 = 18$
$u_3 = 2x - 3 = 9$

$\therefore \begin{cases} a = 36 \\ r = \frac{1}{2} \end{cases}$

HENCE $S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{36}{1-\frac{1}{2}}$

$S_\infty = 72$

Question 37 (***+)

A geometric series consists of positive terms only.

The first and the second term of the series add up to 270.

Its sum to infinity is 288.

Determine the first term and the common ratio of the series.

$\boxed{a = 216}$, $\boxed{r = \frac{1}{4}}$

- $u_1 + u_2 = 270$

$a + ar = 270$

$a(1+r) = 270$

$\boxed{a = \frac{270}{1+r}}$

- $S_\infty = 288$

$\frac{a}{1-r} = 288$

$\boxed{a = 288(1-r)}$

$\frac{270}{1+r} = 288(1-r)$

$270 = 288(1-r)(1+r)$

$\frac{15}{16} = (1-r)(1+r)$

$\frac{15}{16} = 1 - r^2$

$r^2 = \frac{1}{16}$

$r = \frac{1}{4}$

USING $a = 288(1-r)$

$a = 288(1-\frac{1}{4})$

$a = 216$

Question 38 **(***+)**

The first and the second term of a geometric series add up to 240.

The first and the third term of the same geometric series add up to 200.

Determine the two possible values of the sum to infinity of the series.

$\boxed{}$, $\boxed{S_\infty = 270 \text{ or } 320}$

- $u_1 + u_2 = 240$
$\Rightarrow a + ar = 240$
$\Rightarrow a(1+r) = 240$
$\Rightarrow a = \dfrac{240}{1+r}$

- $u_1 + u_3 = 200$
$\Rightarrow a + ar^2 = 200$
$\Rightarrow a(1+r^2) = 200$
$\Rightarrow a = \dfrac{200}{1+r^2}$

$$\Rightarrow \frac{240}{1+r} = \frac{200}{1+r^2}$$
$$\Rightarrow 240(1+r^2) = 200(1+r)$$
$$\Rightarrow 6(1+r^2) = 5(1+r)$$
$$\Rightarrow 6 + 6r^2 = 5 + 5r$$
$$\Rightarrow 6r^2 - 5r + 1 = 0$$
$$\Rightarrow (3r-1)(2r-1)$$
$$\Rightarrow r = \frac{1}{2} \text{ or } \frac{1}{3}$$
$$\Rightarrow a = \frac{240}{1+\frac{1}{2}} = 160 \quad \text{or} \quad a = \frac{240}{1+\frac{1}{3}} = 180$$

USING $a = 160,\ r = \frac{1}{2}$ $\quad S_\infty = \dfrac{160}{1-\frac{1}{2}} = 320$ //

$a = 180,\ r = \frac{1}{3}$ $\quad S_\infty = \dfrac{180}{1-\frac{1}{3}} = 270$ //

Question 39 (***+)

The manufacturer of a certain brand of washing machine is to replace an old model with a new model. There will be a "phase out" period for the old model and a "phase in" period for the new model, both lasting 24 months and starting at the same time.

On the first month of the phase out period 5000 old washing machines will be produced and each month thereafter, this figure will reduce by 20%.

a) Show that on the fifth month of the "phase out" period 2048 old washing machines will be produced.

b) Find how many old washing machines will be produced during the "phase out" period.

On the first month of the "phase in" period 1000 new washing machines will be produced and each month thereafter, this figure will increase by 5%.

c) Calculate how many new washing machines will be produced on the last month of the "phase in" period.

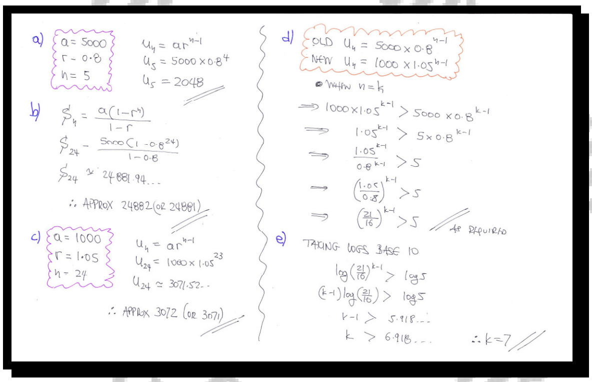
On the k^{th} month of the "phase in/phase out" period, for the first time more new washing machines will be produced than old washing machines.

d) Show that k satisfies

$$\left(\frac{21}{16}\right)^{k-1} > 5.$$

e) Use logarithms to determine the value of k.

$\boxed{}$, $\boxed{24881 \text{ or } 24882}$, $\boxed{3071 \text{ or } 3072}$, $\boxed{k=7}$


a) $a = 5000$, $r = 0.8$, $n = 5$

$u_n = ar^{n-1}$

$u_5 = 5000 \times 0.8^4$

$u_5 = 2048$

b) $S_n = \frac{a(1-r^n)}{1-r}$

$S_{24} = \frac{5000(1-0.8^{24})}{1-0.8}$

$S_{24} \approx 24881.94\ldots$

$\therefore$ APPROX 24882 (OR 24881)

c) $a = 1000$, $r = 1.05$, $n = 24$

$u_n = ar^{n-1}$

$u_{24} = 1000 \times 1.05^{23}$

$u_{24} \simeq 3071.52\ldots$

$\therefore$ APPROX 3072 (OR 3071)

d) OLD $u_n = 5000 \times 0.8^{n-1}$

NEW $u_n = 1000 \times 1.05^{n-1}$

• WHEN $n = k$

$\Rightarrow 1000 \times 1.05^{k-1} > 5000 \times 0.8^{k-1}$

$\Rightarrow 1.05^{k-1} > 5 \times 0.8^{k-1}$

$\Rightarrow \frac{1.05^{k-1}}{0.8^{k-1}} > 5$

$\Rightarrow \left(\frac{1.05}{0.8}\right)^{k-1} > 5$

$\Rightarrow \left(\frac{21}{16}\right)^{k-1} > 5$ AS REQUIRED

e) TAKING LOGS BASE 10

$\log\left(\frac{21}{16}\right)^{k-1} > \log 5$

$(k-1)\log\left(\frac{21}{16}\right) > \log 5$

$k-1 > 5.918\ldots$

$k > 6.918\ldots$

$\therefore k = 7$

Question 40 (***+)

The third and fourth term of a geometric series is

$$(x+1) \text{ and } (x^3+x^2-4x-4),$$

respectively.

a) Determine the common ratio of the series.
Give the answer as simplified quadratic expression, in terms of x.

b) Write the fourth term of the series as a product of three linear factors.

c) Given that $x=3$, find the first term of the series.

$\boxed{r=x^2-4}$, $\boxed{u_4=(x-2)(x+2)(x+1)}$, $\boxed{a=\frac{4}{25}}$

(a) $u_3 = x+1$
$u_4 = x^3+x^2-4x-4$

• $r = \frac{u_4}{u_3} = \frac{x^3+x^2-4x-4}{x+1}$

BY LONG DIVISION OR INSPECTION IN THIS CASE

$= \frac{x^2\cancel{(x+1)}-4\cancel{(x+1)}}{\cancel{x+1}}$

$= x^2-4$

(b) $u_4 = x^3+x^2-4x-4 = (x+1)(x^2-4) = (x+1)(x-2)(x+2)$

(c) $u_3=4$
$u_4=20$
$r=5$

$u_n = ar^{n-1}$

$4 = a\times 5^2 \leftarrow$ USE $u_3=4$

$25a=4$

$a=\frac{4}{25}$

Question 41 (***+)

In a certain quiz game, contestants answer questions consecutively until they get a question wrong.

They win £10 for answering the first question correctly, £20 for answering the second question correctly, £40 for answering the third question correctly, and so on so that the amounts won for each successive question is a term of a geometric series.

When contestants answer a question wrong their game is over and they get to keep $\frac{1}{10}$ of their **total** winnings up to that point.

Connor answers 5 questions correctly.

a) Show that Connor won £31.

The highest prize won in this game, by a contestant called Ray, was £2,097,151.

b) Use algebra to find the number of questions that Ray answered correctly.

Full workings, justifying every step in the calculations, must be shown in this part of the question.

$\boxed{}$, $\boxed{21}$

(a) Q1 Q2 Q3 Q4 Q5 ~~Q6~~

10 20 (30) 40 (70) 80 (150) 160 (310)

BUT HE KEEPS $\frac{1}{10}$ OF TOTAL WINNINGS

$\therefore \frac{1}{10} \times 310 = £31$

(b) RAY WON £2097151

RAY'S TOTAL UNTIL HIS WRONG ANSWER WAS 20971510

$a = 10$
$r = 2$
$S_n = 20971510$
$n = ?$

$S_n = \frac{a(1-r^n)}{1-r}$

$\Rightarrow 20971510 = \frac{10(1-2^n)}{1-2}$

$\Rightarrow 20971510 = \frac{10(1-2^n)}{-1}$

$\Rightarrow -2097151 = 1-2^n$

$\Rightarrow 2^n = 2097152$

By logarithms (or inspection)

$\Rightarrow \log 2^n = \log(2097152)$

$\Rightarrow n\log 2 = \log(2097152)$

$\Rightarrow n = \frac{\log(2097152)}{\log 2}$

$\Rightarrow n = 21$

Question 42 (***+)

The first three terms of a geometric series are

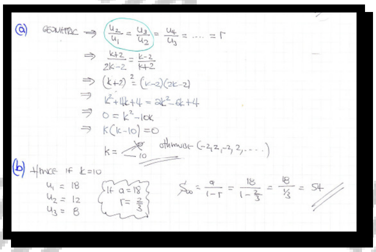
$$(2k-2), \quad (k+2) \quad \text{and} \quad (k-2) \quad \text{respectively,}$$

where k is a non zero constant.

a) Show clearly that $k = 10$.

b) Find the sum to infinity of the series.

$\boxed{S_\infty = 54}$

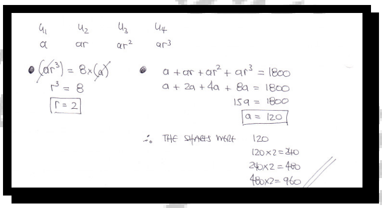


Question 43 (***+)

Four brothers shared £1800 so that their shares formed the terms of a geometric progression.

Given that the largest share was 8 times as large as the smallest share, determine the individual amounts each brother got.

$\boxed{\text{£120, £240, £480, £960}}$



Question 44 (***+)

The second and third term of a geometric progression are 9.6 and 9.216, respectively.

a) Show that the sum to infinity of the progression is 250.

The sum of the first k terms of the progression is greater than 249.

b) Show clearly that

$$0.96^k < 0.004.$$

c) Hence determine the smallest value of k.

$\boxed{k = 136}$

(a) $r = \frac{u_3}{u_2} = \frac{9.216}{9.6} = \frac{24}{25} = 0.96$

$\therefore u_n = ar^{n-1}$

$9.6 = a \times 0.96^1$

$\boxed{a = 10}$

$\therefore S_\infty = \frac{a}{1-r} = \frac{10}{1-0.96} = 250$ // AS REQUIRED

(b) $S_n = \frac{a(1-r^n)}{1-r}$

$\Rightarrow \frac{10(1-0.96^k)}{1-0.96} > 249$

$\Rightarrow \frac{10(1-0.96^k)}{0.04} > 249$

$\Rightarrow 1 - 0.96^k > 0.996$

$\Rightarrow -0.96^k > -0.004$

$\Rightarrow 0.96^k < 0.004$ // AS REQUIRED

(c) BY LOGS OR TRIAL & IMPROVEMENT

$\Rightarrow 0.96^k < 0.004$

$\Rightarrow \log(0.96^k) < \log(0.004)$

$\Rightarrow k\log(0.96) < \log 0.004$

$\Rightarrow k > \frac{\log(0.004)}{\log(0.96)}$

$\log(0.96)$ IS NEGATIVE

$\Rightarrow k > 135.257$

$\therefore k = 136$ //

Question 45 (***+)

The second term of a geometric series is -12 and its sum to infinity is 16.

Show clearly that the eleventh term of the series is $\frac{3}{128}$.

$\boxed{\text{proof}}$

• $u_2 = -12$

$\boxed{ar = -12}$ SINCE $u_n = ar^{n-1}$

• $S_\infty = 16$

$\boxed{16 = \frac{a}{1-r}}$ SINCE $S_\infty = \frac{a}{1-r}$

$\left\{ \begin{array}{l} a = \frac{-12}{r} \\ a = 16(1-r) \end{array} \right\}$

THUS

$\Rightarrow -\frac{12}{r} = 16(1-r)$

$\Rightarrow -12 = 16r(1-r)$

$\Rightarrow -12 = 16r - 16r^2$

$\Rightarrow 16r^2 - 16r - 12 = 0$

$\Rightarrow 4r^2 - 4r - 3 = 0$

$\Rightarrow (2r-3)(2r+1) = 0$

$\Rightarrow r = \cancel{\frac{3}{2}}$ ← OTHERWISE NO SUM TO INFINITY, $-\frac{1}{2}$

$\therefore a = \frac{-12}{-\frac{1}{2}} = 24$

HENCE

$u_n = ar^{n-1}$

$u_{11} = 24 \times \left(-\frac{1}{2}\right)^{10}$

$u_{11} = \frac{3}{128}$ // AS REQUIRED

Question 46 (***+)

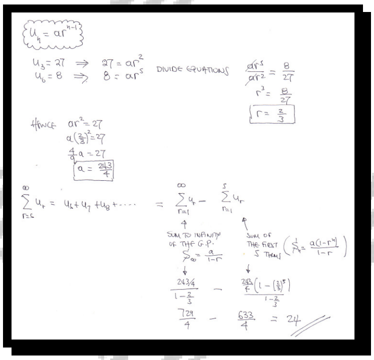
The third and the sixth term of a geometric progression is 27 and 8, respectively.

Show clearly that

$$\sum_{r=6}^{\infty} u_r = 24,$$

where u_r is the r^{th} term of the progression.

$\boxed{\text{proof}}$

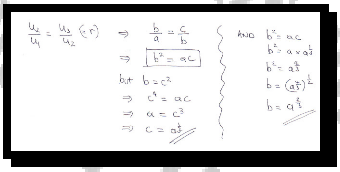


Question 47 (***+)

The first three terms u_1, u_2, u_3 of a geometric series are a, b and c respectively.

Given that $b = c^2$ express u_2 and u_3 in terms of a.

$\boxed{b = a^{\frac{2}{3}}}$, $\boxed{c = a^{\frac{1}{3}}}$



Question 48 (***+)

A steamboat uses 5 tonnes of coal to cover a standard journey designed for tourists.

Due to the engines becoming less efficient, the steamboat requires in each journey 2% more coal than the previous journey.

a) Calculate, in tonnes correct to three decimal places, …

i. … the amount of coal the steamboat will use on the tenth journey.

ii. … the total amount of coal the steamboat will use in the first ten journeys.

The company that owns the steamboat has stocked up with 360 tonnes of coal and plans to use all the coal during a single tourist season.

b) Assuming that in the first journey the steamboat used 5 tonnes of coal, and the consumption of coal increased by 2% in each subsequent journey, show clearly that

$$1.02^n \leq 2.44,$$

where n is the total number of journeys during a single tourist season.

c) Hence, or otherwise, determine the maximum number of journeys that the steamboat can make a single tourist season.

$\boxed{}$, $\boxed{5.975}$, $\boxed{54.749}$, $\boxed{45}$

(a) $a = 5$ ← TONNES

$r = 1.02$ ← 2% INCREASE

(I) $u_n = ar^{n-1}$

$u_{10} = 5 \times 1.02^9$

$u_{10} = 5.975$ (3 d.p)

(II) $S_n = \dfrac{a(1-r^n)}{1-r}$

$S_{10} = \dfrac{5(1-1.02^{10})}{1-1.02}$

$S_{10} = 54.749$ (3 d.p)

(b) $S_n < 360$

$\Rightarrow \dfrac{5(1-1.02^n)}{1-1.02} \leq 360$

$\Rightarrow \dfrac{5(1-1.02^n)}{-0.02} \leq 360$

$\Rightarrow -250(1-1.02^n) \leq 360$

$\Rightarrow -250 + 250 \times 1.02^n \leq 360$

$\Rightarrow 250 \times 1.02^n \leq 610$

$\Rightarrow 1.02^n \leq 2.44$

AS REQUIRED

(c) $1.02^n \leq 2.44$

$\Rightarrow \log(1.02^n) \leq \log(2.44)$

$\Rightarrow n \log(1.02) \leq \log(2.44)$

$\Rightarrow n \leq \dfrac{\log(2.44)}{\log(1.02)}$

$\Rightarrow n \leq 45.044\ldots$

$\therefore$ 45 JOURNEYS

Question 49 (***+)

The first, second and third term of a geometric series are

$$(2k-5), \qquad k \qquad \text{and} \qquad (k-6),$$

respectively, where k is a non zero constant.

a) Show that k is a solution of the equation

$$k^2 - 17k + 30 = 0.$$

b) Given that the series converges, find its sum to infinity.

c) Given instead that series does not converge, find the sum of its first ten terms.

$\boxed{}$, $\boxed{S_\infty = \frac{125}{2} = 62.5}$, $\boxed{S_{10} = 341}$

(a) $\frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \dots = r$

$\frac{k}{2k-5} = \frac{k-6}{k} \Rightarrow k^2 = (k-6)(2k-5)$

$\Rightarrow k^2 = 2k^2 - 5k - 12k + 30$

$\Rightarrow k^2 = 2k^2 - 17k + 30$

$\Rightarrow 0 = k^2 - 17k + 30$ // AS REQUIRED

(b) $k^2 - 17k + 30 = 0$

$(k-15)(k-2) = 0$

$k = \begin{cases} 2 \\ 15 \end{cases}$

- IF $k = 2 \Rightarrow -1, 2, -4, \dots$ DIVERGES
- IF $k = 15 \Rightarrow 25, 15, 9, \dots$ CONVERGES

$r = \frac{15}{25} = \frac{9}{15} = \frac{3}{5}$

HENCE $\left.\begin{array}{l} a = 25 \\ r = \frac{3}{5} \end{array}\right\} \Rightarrow S_\infty = \frac{a}{1-r} = \frac{25}{1-\frac{3}{5}} = \frac{125}{2}$ //

(c) HENCE $\left.\begin{array}{l} a = -1 \\ r = -2 \end{array}\right\} \Rightarrow S_n = \frac{a(1-r^n)}{1-r}$

$\frac{2}{-1} = \frac{-4}{2} = \dots$

$S_{10} = \frac{-1(1-(-2)^{10})}{1-(-2)}$

$S_{10} = \frac{-1+(-2)^{10}}{3}$

$S_{10} = 341$ //

Question 50 (***+)

Three consecutive terms in geometric progression are given in sequential order as

$$(1-5p), \quad \frac{1}{2} \quad \text{and} \quad (4p-2),$$

where p is a constant.

Show that one possible value of p is $\frac{1}{4}$ and find the other.

$\boxed{}$, $\boxed{p=\frac{9}{20}}$

$u_n = 1-5p$

$u_{n+1} = \frac{1}{2}$

$u_{n+2} = 4p-2$

FOR A G.P: $\frac{u_{n+1}}{u_n} = \frac{u_{n+2}}{u_{n+1}} = r$

$\Rightarrow \frac{\frac{1}{2}}{1-5p} = \frac{4p-2}{\frac{1}{2}}$

$\Rightarrow \frac{1}{4} = (4p-2)(1-5p)$

$\Rightarrow \frac{1}{4} = 4p - 20p^2 - 2 + 10p$

$\Rightarrow 20p^2 - 14p + 2 + \frac{1}{4} = 0$

$\Rightarrow 80p^2 - 56p + 8 + 1 = 0$

$\Rightarrow 80p^2 - 56p + 9 = 0$

$\Rightarrow (4p-1)(20p-9)$

$\therefore p = \frac{1}{4}, \frac{9}{20}$

Question 51 (***+)

The sum to infinity of a geometric series is 3 times as large as its first term and the third term of the same series is 40.

a) Find the value of the first term of the series.

b) Determine the exact value of the sum of the first four terms of the series.

$\boxed{}$, $\boxed{a=90}$, $\boxed{S_4=\frac{650}{3}}$

a) SETTING UP TWO EQUATIONS

$\Rightarrow S_\infty = 3 \times a$

$\Rightarrow \frac{a}{1-r} = 3a$

$\Rightarrow \frac{1}{1-r} = 3$

$\Rightarrow \frac{1}{3} = 1-r$

$\Rightarrow r = \frac{2}{3}$

$\Rightarrow u_3 = 40$

$\Rightarrow ar^2 = 40$

$\Rightarrow a \times \left(\frac{2}{3}\right)^2 = 40$

$\Rightarrow \frac{4}{9}a = 40$

$\Rightarrow 4a = 360$

$\Rightarrow a = 90$

b) USING THE SUMMATION FORMULA FOR A G.P.

$\Rightarrow S_n = \frac{a(1-r^n)}{1-r}$

$\Rightarrow S_4 = \frac{90\left(1-\left(\frac{2}{3}\right)^4\right)}{1-\frac{2}{3}}$

$\Rightarrow S_4 = \frac{90\left(1-\frac{16}{81}\right)}{\frac{1}{3}}$

$\Rightarrow S_4 = \frac{650}{3} = 216\frac{2}{3}$

Question 52 (***+)

A geometric series, whose terms alternate in sign, has its first term denoted by a and its common ratio denoted by r.

The sum of the first n terms of the series is denoted by S_n.

It is given that

$$S_4 = 5S_2 .$$

a) Find the value of r.

b) Given further that the fifth term of the series is 36, determine the value of a.

$\boxed{}$, $\boxed{r = -2}$, $\boxed{a = \frac{9}{4}}$

a) STARTING WITH THE INFORMATION GIVEN

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

$a \neq 0$, $1-r \neq 0$

$$\Rightarrow S_4 = 5S_2$$

$$\Rightarrow \frac{\cancel{a}(1-r^4)}{\cancel{1-r}} = 5 \times \frac{\cancel{a}(1-r^2)}{\cancel{1-r}}$$

$$\Rightarrow 1-r^4 = 5(1-r^2)$$

$$\Rightarrow 1-r^4 = 5-5r^2$$

$$\Rightarrow 0 = r^4 - 5r^2 + 4$$

$$\Rightarrow 0 = (r^2-4)(r^2-1)$$

$$\Rightarrow r^2 = \begin{cases} 4 \\ 1 \end{cases}$$

$$\Rightarrow r = \begin{cases} \cancel{2} \\ -2 \\ \cancel{1} \\ \cancel{-1} \end{cases}$$

← TERMS ALTERNATE IN SIGN

$r \neq 0, 1, -1$

b)

$u_n = ar^{n-1}$

$u_5 = ar^4$

$36 = a(-2)^4$

$36 = 16a$

$a = \frac{9}{4}$

Question 53 (***+)

The sum to infinity of a geometric series is 675 and its second term is 27 times larger than its fifth term.

Find the value of the first term of the series.

$\boxed{}$, $\boxed{a = 450}$

$S_{\infty} = 675$
$u_2 = 27u_5$

- $\boxed{S_{\infty} = \frac{a}{1-r}}$

$675 = \frac{a}{1-r}$

$\Rightarrow 675 = \frac{a}{1-\frac{1}{3}}$

$\Rightarrow 675 = \frac{a}{\frac{2}{3}}$

$\Rightarrow a = 450$

- $\boxed{u_n = ar^{n-1}}$

$\Rightarrow u_2 = 27u_5$

$\Rightarrow \cancel{a}r = 27 \times \cancel{a}r^4 \quad (a \neq 0)$

$\Rightarrow r = 27r^4$

$\Rightarrow 1 = 27r^3 \quad (r \neq 0)$

$\Rightarrow \frac{1}{27} = r^3$

$\Rightarrow \boxed{r = \frac{1}{3}}$

Question 54 (***+)

The eighth term of a geometric progression is ten times as large as its fourth term.

The common ratio of the progression is positive.

a) Show that the common ratio of the series is approximately 1.778.

The sum of the first eight terms of a **different** geometric progression is ten times as large as the sum of its first four terms.

The common ratio of the progression r, is positive.

b) Show that r is a solution of the equation

$$r^8 - 10r^4 + 9 = 0.$$

c) By reducing the above equation to a suitable quadratic, or otherwise, show that

$$r = \sqrt{3}.$$

$\boxed{}$, $\boxed{\text{proof}}$

(a) $\boxed{u_n = ar^{n-1}}$

$\Rightarrow u_8 = 10u_4$

$\Rightarrow \cancel{a}r^7 = 10\cancel{a}r^3$

$\Rightarrow r^7 = 10r^3$

$\Rightarrow r^4 = 10$

$\Rightarrow r = 1.778279\ldots \quad (r>0)$

$\Rightarrow$ I.E. APPROX 1.778

(b) $\boxed{S_n = \frac{a(1-r^n)}{1-r}}$

$\Rightarrow S_8 = 10S_4$

$\Rightarrow \frac{a(1-r^8)}{1-r} = 10 \times \frac{a(1-r^4)}{1-r}$

$\boxed{a \neq 0 \text{ \& } r \neq 1}$

$\Rightarrow 1-r^8 = 10(1-r^4)$

$\Rightarrow 1-r^8 = 10-10r^4$

$\Rightarrow 0 = r^8 - 10r^4 + 9$

AS REQUIRED

(c) $\Rightarrow (r^4)^2 - 10(r^4) + 9 = 0$

$\Rightarrow (r^4 - 1)(r^4 - 9) = 0$

$\Rightarrow r^4 = 1,\ 9$

$\Rightarrow r^2 = 1,\ \cancel{-1},\ 3,\ \cancel{-3}$

$\Rightarrow r = 1\ (r\neq 1),\ \cancel{-1}\ (r>0),\ \sqrt{3},\ \cancel{-\sqrt{3}}\ (r>0)$

$\Rightarrow r = \sqrt{3}$

Question 55 (****)

The second and fifth term of a geometric progression are 72 and -9 respectively.

a) Find the first term and the common ratio of the progression.

b) Show that the **difference** between the sum to infinity and the sum of the first n terms of the progression is given by

$$3 \times 2^{5-n}.$$

$\boxed{a = -144}$, $\boxed{r = -\frac{1}{2}}$

(a) $\left.\begin{array}{l} u_2 = 72 \\ u_5 = -9 \end{array}\right\}$ USING $u_n = ar^{n-1}$ $\Rightarrow$ $\left.\begin{array}{l} ar = 72 \\ ar^4 = -9 \end{array}\right) \Rightarrow$ DIVIDE

$$\frac{ar^4}{ar} = \frac{-9}{72}$$

$$r^3 = -\frac{1}{8}$$

$$r = -\frac{1}{2}$$

AND SINCE $ar = 72$

$$-\frac{1}{2}a = 72$$

$$a = -144$$

(b) NOW $S_\infty - S_n = \frac{a}{1-r} - \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}\left[1 - (1-r^n)\right]$

$$= \frac{a}{1-r} \times r^n = \frac{ar^n}{1-r} = \frac{-144 \times \left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{-144\left(-2^{-1}\right)^n}{\frac{3}{2}} = -96(-2)^{-n} = \pm 96 \times 2^{-n}$$

DEPENDING ON WHETHER n IS ODD OR EVEN

$\therefore$ DIFFERENCE IS $96 \times 2^{-n} = 3 \times 2^5 \times 2^{-n} = 3 \times 2^{5-n}$ AS REQUIRED

ALTERNATIVE: $S_\infty = \frac{a}{1-r} = \frac{-144}{1-\left(-\frac{1}{2}\right)} = -96$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{-144\left(1-\left(-\frac{1}{2}\right)^n\right)}{1-\left(-\frac{1}{2}\right)} = -96\left(1-\left(\frac{1}{2}\right)^n\right)$$

$\therefore$ DIFFERENCE $= \cancel{-96} - \left[\cancel{-96}\left(1-\left(-\frac{1}{2}\right)^n\right)\right]$

$$= -96\left(-\frac{1}{2}\right)^n = -3 \times 2^5 \times (-2)^{-n}$$

$$= \pm 3 \times 2^5 \times 2^{-n} = 3 \times 2^{5-n}$$

Question 56 (****)

Max is revising for an exam by practicing papers.

He takes 3 hours and 20 minutes to complete the first paper and 3 hours and 15 minutes to complete the second paper.

It is assumed that the times Max takes to complete each successive paper are consecutive terms of a geometric progression.

a) Assuming this model, show that Max will take approximately …

i. … 176 minutes to complete the sixth paper.

ii. … 35 hours to complete the first 12 papers.

Max aims to be able to complete a paper in under two hours.

b) Determine, by using logarithms, the minimum number of papers he needs to practice in order to achieve his target according to this model.

$\boxed{}$, $\boxed{22}$

(a) WORKING IN MINUTES

$u_1 = 200 \leftarrow$ 3 HOURS – 20 MINUTES

$u_2 = 195 \leftarrow$ 3 HOURS – 15 MINUTES

$\therefore$ COMMON RATIO $r = \frac{195}{200} = \frac{39}{40}$

(I) $\boxed{u_n = ar^{n-1}}$

$u_6 = 200 \times \left(\frac{39}{40}\right)^5$

$u_6 = 176.219\ldots$

$\therefore$ Approx 176 min

(II) $S_n = \frac{a(1-r^n)}{1-r}$

$S_{12} = \frac{200\left(1-\left(\frac{39}{40}\right)^{12}\right)}{1-\frac{39}{40}}$

$S_{12} = 2096.01\ldots$ MINUTES

$\downarrow \div 60$

$34.93\ldots$ HOURS

$\therefore$ APPROX 35 HOURS

(b) $u_n < 120$

$\Rightarrow ar^{n-1} < 120$

$\Rightarrow 200 \times \left(\frac{39}{40}\right)^{n-1} < 120$

$\Rightarrow \left(\frac{39}{40}\right)^{n-1} < \frac{3}{5}$

$\Rightarrow \log\left[\left(\frac{39}{40}\right)^{n-1}\right] < \log\left(\frac{3}{5}\right)$

$\Rightarrow (n-1)\log\frac{39}{40} < \log\frac{3}{5}$

$\Rightarrow n-1 > \frac{\log 3/5}{\log \frac{39}{40}}$

THIS IS NEGATIVE!

$\Rightarrow n-1 > 20.176\ldots$

$\Rightarrow n > 21.176\ldots$

$\therefore n = 22$

Question 57 (****)

The sum to infinity of a geometric progression is four times as large as its second term.

a) Show that the common ratio of the series is $\frac{1}{2}$.

It is further given that the sum of the first four terms of the progression is 5760.

b) Find the first term of the progression.

The sum of the first k terms of the progression is **three less** than its sum to infinity.

c) Use algebra to determine the value of k.

$\boxed{}$, $\boxed{a = 3072}$, $\boxed{k = 11}$

a) USING STANDARD FORMULAE

$\Rightarrow S_\infty = 4 \times u_2$

$\Rightarrow \frac{a}{1-r} = 4 \times ar^1$

$\Rightarrow \frac{\cancel{a}}{1-r} = 4\cancel{a}r$

$\Rightarrow \frac{1}{1-r} = 4r$

$\Rightarrow 1 = 4r - 4r^2$

$\Rightarrow 4r^2 - 4r + 1 = 0$

$\Rightarrow (2r-1) = 0$

$\Rightarrow r = \frac{1}{2}$

b) USING $S_4 = \frac{a(1-r^4)}{1-r}$

$\Rightarrow S_4 = 5760$

$\Rightarrow \frac{a(1-0.5^4)}{1-0.5} = 5760$

$\Rightarrow a\left(1 - \frac{1}{16}\right) = 5760 \times \frac{1}{2}$

$\Rightarrow a \times \frac{15}{16} = 2880$

$\Rightarrow 15a = 46080$

$\Rightarrow a = 3072$

c) FINALLY WE HAVE NOTING $S_\infty = 4u_2 = 4ar = 4 \times 3072 \times 0.5 = 6144$

$S_k = S_\infty - 3$

$\frac{3072(1-0.5^k)}{1-0.5} = 6144 - 3$

$3072(1-0.5^k) = 6141 \times 0.5$

$1 - 0.5^k = \frac{2047}{2048}$

$\frac{1}{2048} = \left(\frac{1}{2}\right)^k$

BY INSPECTION OR LOGS

$k = 11$

$\log\left(\frac{1}{2048}\right) = \log\left(\frac{1}{2}\right)^k$

$k\log\left(\frac{1}{2}\right) = \log\left(\frac{1}{2048}\right)$

$k = \frac{\log\left(\frac{1}{2048}\right)}{\log\left(\frac{1}{2}\right)}$

$k = 11$

Question 58 (****)

The sum of the first two terms of a geometric progression is twice as large as the sum of its second and third term.

a) Show that the common ratio of the series is $\frac{1}{2}$

The sum to infinity of the geometric progression is 80.

b) Determine the exact value of the sum of the first six terms of the progression.

$$\boxed{S_6 = \frac{315}{4}}$$

(a) $(u_1+u_2) = 2(u_2+u_3)$

$\Rightarrow u_1+u_2 = 2u_2+2u_3$

$\Rightarrow u_1 = u_2+2u_3$

$\Rightarrow a = ar+2(ar^2)$

$\Rightarrow 1 = r+2r^2 \quad (a \neq 0)$

$\Rightarrow 2r^2+r-1=0$

$\Rightarrow (2r-1)(r+1)=0$

$\Rightarrow r = \cancel{-1}$ (OTHERWISE NO G.P) or $\frac{1}{2}$

(b) $\boxed{S_\infty = \frac{a}{1-r}}$

$80 = \frac{a}{1-\frac{1}{2}}$

$\Rightarrow 80 = \frac{a}{\frac{1}{2}}$

$\Rightarrow \boxed{40 = a}$

Hence

$S_n = \frac{a(1-r^n)}{1-r}$

$S_6 = \frac{40(1-0.5^6)}{1-0.5}$

$S_6 = \frac{315}{4}$

Question 59 **(****)**

The fourth term and the seventh term of a geometric progression is 81 and 24, respectively.

a) Determine the sum to infinity of the progression.

The tenth term of the progression is denoted by u_{10}.

b) Show clearly that

$$\log_2(u_{10}) = 6 - 2\log_2 3.$$

$$\boxed{S_\infty = \frac{6561}{8}}$$

(a) $ar^3 = 81$, $ar^6 = 24$ $\Rightarrow \frac{ar^6}{ar^3} = \frac{24}{81}$ $\Rightarrow r^3 = \frac{8}{27}$ $\Rightarrow \boxed{r = \frac{2}{3}}$

& $ar^3 = 81$

$a\left(\frac{2}{3}\right)^3 = 81$

$\frac{8}{27}a = 81$

$\boxed{a = \frac{2187}{8}}$

Hence $S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{2187/8}{1-\frac{2}{3}}$

$S_\infty = \frac{6561}{8} = 820.125$

(b) $u_n = ar^{n-1}$

$u_{10} = ar^9$

$\log_2(u_{10}) = \log_2(ar^9)$

$\log_2(u_{10}) = \log_2\left(\frac{2187}{8} \times \left(\frac{2}{3}\right)^9\right)$

$\log_2(u_{10}) = \log_2\left(\frac{64}{9}\right)$

$\log_2(u_{10}) = \log_2 64 - \log_2 9$

$\log_2(u_{10}) = \log_2 2^6 - \log_2 3^2$

$\log_2(u_{10}) = 6\log_2 2 - 2\log_2 3$

$\log_2(u_{10}) = 6 - 2\log_2 3$

AS REQUIRED

Question 60 (****)

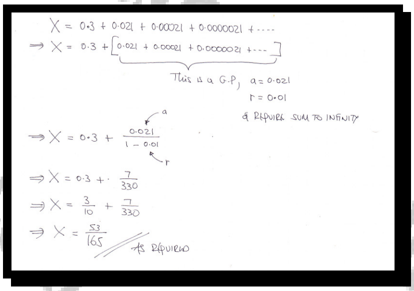
$$X = 0.3\dot{2}\dot{1} = 0.321212121...$$

By writing this decimal as the sum

$$X = 0.3 + 0.021 + 0.00021 + 0.0000021 + ...$$

show that $X = \dfrac{53}{165}$.

$\boxed{\text{proof}}$


$X = 0.3 + 0.021 + 0.00021 + 0.0000021 + \cdots$

$\Rightarrow X = 0.3 + \underbrace{\left[0.021 + 0.00021 + 0.0000021 + \cdots\right]}$

This is a G.P, $a = 0.021$, $r = 0.01$

& REQUIRE SUM TO INFINITY

$\Rightarrow X = 0.3 + \dfrac{0.021}{1 - 0.01}$

$\Rightarrow X = 0.3 + \dfrac{7}{330}$

$\Rightarrow X = \dfrac{3}{10} + \dfrac{7}{330}$

$\Rightarrow X = \dfrac{53}{165}$ // AS REQUIRED

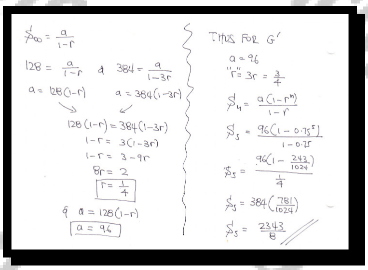
Question 61 (****)

A geometric series G, whose first term is a and common ratio is r, has a sum to infinity of 128.

Another geometric series G', with first term also a and common ratio $3r$ has a sum to infinity of 384.

Determine the exact value of the sum of the first five terms of G'.

$$\boxed{S_5 = \frac{2343}{8}}$$


$S_\infty = \dfrac{a}{1-r}$

$128 = \dfrac{a}{1-r}$ & $384 = \dfrac{a}{1-3r}$

$a = 128(1-r)$ $\quad a = 384(1-3r)$

$128(1-r) = 384(1-3r)$

$1-r = 3(1-3r)$

$1-r = 3-9r$

$8r = 2$

$\boxed{r = \frac{1}{4}}$

& $a = 128(1-r)$

$\boxed{a = 96}$

THUS FOR G'

$a = 96$

"r" $= 3r = \frac{3}{4}$

$S_n = \dfrac{a(1-r^n)}{1-r}$

$S_5 = \dfrac{96(1-0.75^5)}{1-0.75}$

$S_5 = \dfrac{96\left(1 - \frac{243}{1024}\right)}{\frac{1}{4}}$

$S_5 = 384\left(\dfrac{781}{1024}\right)$

$S_5 = \dfrac{2343}{8}$ //

Question 62 (****)

The third term of a geometric series is 4 and its sum to infinity is 27.

a) Show that

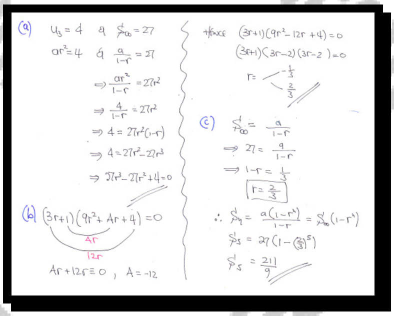
$$27r^3 - 27r^2 + 4 = 0,$$

where r is the common ratio of the series.

b) Given that one possible value of r is $-\frac{1}{3}$ find the other.

c) Given further that the first term of the series is 9, find in exact form the sum of the first five terms of the series.

$\boxed{r = \frac{2}{3}}$, $\boxed{S_5 = \frac{211}{9}}$



(a) $u_3 = 4$ & $S_\infty = 27$

$ar^2 = 4$ & $\frac{a}{1-r} = 27$

$\Rightarrow \frac{ar^2}{1-r} = 27r^2$

$\Rightarrow \frac{4}{1-r} = 27r^2$

$\Rightarrow 4 = 27r^2(1-r)$

$\Rightarrow 4 = 27r^2 - 27r^3$

$\Rightarrow 27r^3 - 27r^2 + 4 = 0$

(b) $(3r+1)(9r^2 + Ar + 4) = 0$

Ar

$12r$

$Ar + 12r \equiv 0$, $A = -12$

Hence $(3r+1)(9r^2 - 12r + 4) = 0$

$(3r+1)(3r-2)(3r-2) = 0$

$r = -\frac{1}{3}$, $\frac{2}{3}$

(c) $S_\infty = \frac{a}{1-r}$

$\Rightarrow 27 = \frac{9}{1-r}$

$\Rightarrow 1 - r = \frac{1}{3}$

$\boxed{r = \frac{2}{3}}$

$\therefore S_n = \frac{a(1-r^n)}{1-r} = S_\infty(1-r^n)$

$S_5 = 27\left(1 - \left(\frac{2}{3}\right)^5\right)$

$S_5 = \frac{211}{9}$

Question 63 (****)

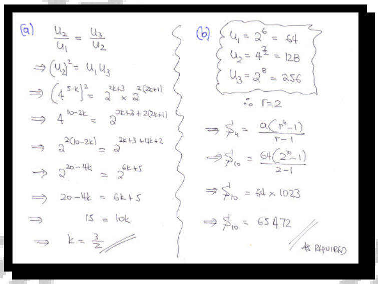
The first three terms of a geometric series are

$$u_1 = 2^{2k+3}, \ u_2 = 4^{5-k} \text{ and } u_3 = 2^{2(2k+1)}.$$

a) Find the value of k.

b) Show that the sum of the first ten terms of the series is 65472.

$$\boxed{k = \frac{3}{2}}$$

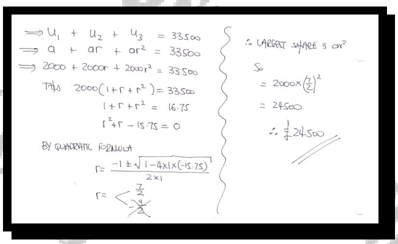


Question 64 (****)

The amount of £33500 is to be divided into three shares, so that the three shares form the terms of a geometric progression.

Given that the value of the smallest share is £2000, find the value of the largest share.

$\boxed{}$, $\boxed{£24500}$



Question 65 (****)

The first term of a geometric series is 24 and the sum of its first four terms is 45.

a) Show that

$$8r^3 + 8r^2 + 8r - 7 = 0$$

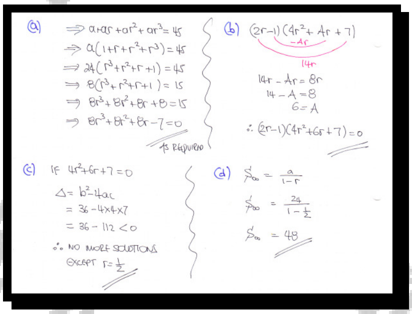
where r is the common ratio of the progression.

b) Given that $r = \frac{1}{2}$ is a solution of the above equation, factorize the equation into a linear and a quadratic factor.

c) Show that $r = \frac{1}{2}$ is the only real solution of the above equation.

d) Determine the sum to infinity of the progression.

$\boxed{(2r-1)\left(4r^2+6r+7\right)}$, $\boxed{S_\infty = 48}$


(a) $\Rightarrow a + ar + ar^2 + ar^3 = 45$
$\Rightarrow a(1 + r + r^2 + r^3) = 45$
$\Rightarrow 24(r^3 + r^2 + r + 1) = 45$
$\Rightarrow 8(r^3 + r^2 + r + 1) = 15$
$\Rightarrow 8r^3 + 8r^2 + 8r + 8 = 15$
$\Rightarrow 8r^3 + 8r^2 + 8r - 7 = 0$
AS REQUIRED

(b) $(2r-1)(4r^2 + Ar + 7)$
$-Ar$
$14r$
$14r - Ar = 8r$
$14 - A = 8$
$6 = A$
$\therefore (2r-1)(4r^2+6r+7) = 0$

(c) IF $4r^2 + 6r + 7 = 0$
$\Delta = b^2 - 4ac$
$= 36 - 4 \times 4 \times 7$
$= 36 - 112 < 0$
$\therefore$ NO MORE SOLUTIONS EXCEPT $r = \frac{1}{2}$

(d) $S_\infty = \frac{a}{1-r}$
$S_\infty = \frac{24}{1-\frac{1}{2}}$
$S_\infty = 48$

Question 66 (****)

The first three terms of a geometric series are

$$(2x+4),\ (3x+2) \text{ and } \left(x^2-11\right),$$

where x is a constant.

a) Show that x is a solution of the cubic equation

$$2x^3-5x^2-34x-48=0.$$

b) Show that $x=6$ is the only real solution of the above equation.

c) Determine the sum of the first eight terms of the geometric series.

$\boxed{}$, $\boxed{S_8=\frac{325089}{1024}\approx 317.47}$

(a) $u_1=2x+4$
$u_2=3x+2$
$u_3=x^2-11$

$\Rightarrow \frac{u_3}{u_2}=\frac{u_2}{u_1}$

$\Rightarrow u_3u_1=(u_2)^2$

$\Rightarrow (x^2-11)(2x+4)=(3x+2)^2$

$\Rightarrow 2x^3+4x^2-22x-44=9x^2+12x+4$

$\Rightarrow 2x^3-5x^2-34x-48=0$

(b) IF $x=6$ $\quad 2\times6^3-5\times6^2-34\times6-48$
$=432-180-204-48=0$

$\therefore x=6$ IS A SOLUTION

NOW

$$\begin{array}{r|l} & 2x^2+7x+8 \\ \hline x-6 & 2x^3-5x^2-34x-48 \\ & \underline{-2x^3+12x^2} \\ & 7x^2-34x-48 \\ & \underline{-7x^2+42x} \\ & 8x-48 \\ & \underline{-8x+48} \\ & 0 \end{array}$$

THUS

$2x^3-5x^2-34x-48=(x-6)(2x^2+7x+8)$

b^2-4ac
$7^2-4\times2\times8<0$

ONLY SOLUTION IS $x=6$

(c) $u_1=16$
$u_2=20$ $\quad \therefore r=\frac{20}{16}=\frac{5}{4}$
$u_3=25$

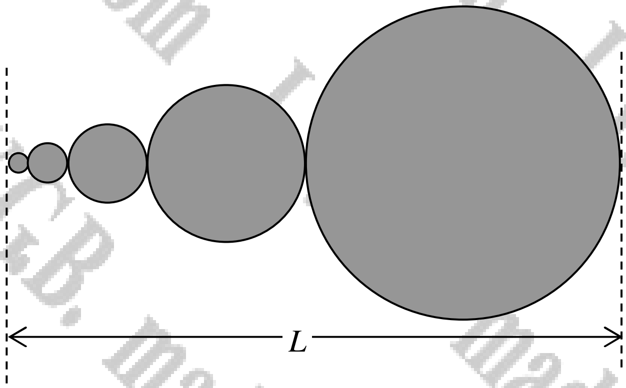
HENCE $\quad S_n=\frac{a(r^n-1)}{r-1}$

$S_8=\frac{16(1.25^8-1)}{1.25-1}$

$S_8=\frac{325089}{1024}$

$S_8\simeq 317.47$

Question 67 (****)



The figure above shows a pattern of 5 circles, touching externally, whose centres lie on a straight line of length L units.

The radii of these circles form a geometric progression, where the radius of the smaller circle is 3 units and that of the fifth (larger) circle is 48 units.

a) Find the common ratio of the geometric progression.

The pattern is extended by 5 more circles to 10 circles.

b) Determine the new value of L.

c) Calculate, in terms of π, the total area of the 10 circles of the new pattern.

$\boxed{r=2}$, $\boxed{L=6138}$, $\boxed{\text{area}=3{,}145{,}725\pi}$

a) USING $u_n = ar^{n-1}$

$\Rightarrow 48 = 3 \times r^4$

$\Rightarrow 16 = r^4$

$\Rightarrow r = +2$ (ALL TERMS ARE POSITIVE)

b) USING THE SUM FORMULA FOR $n=10$ & SUM, NOTING THAT IT NEEDS TO BE DOUBLED (DIAMETER SUM IS NEEDED)

$L_{NEW} = 2 \times \frac{a(r^n - 1)}{r-1}$ $\quad r=2,\ a=3,\ n=10$

$L_{NEW} = 2 \times \frac{3(2^{10}-1)}{2-1}$

$L_{NEW} = 6138$

c) FORM AN EXPRESSION TO SEE THE PATTERN

$\Rightarrow \text{AREA} = \underset{\text{1ST}}{\pi \times 3^2} + \underset{\text{2ND}}{\pi \times (2\times 3)^2} + \underset{\text{3RD}}{\pi \times (2^2 \times 3)^2} + \underset{\text{4th}}{\pi(2^3\times 3)^2} + \dots + \pi(2^9 \times 3)^2$

$\Rightarrow \text{AREA} = \pi \times 3^2 \left[\underbrace{1 + 2^2 + 2^4 + 2^6 + \dots + 2^{18}}_{\text{10 terms}}\right]$

$\Rightarrow \text{AREA} = 9\pi \times [1 + 4 + 16 + 64 + \dots + 262144]$

$\Rightarrow \text{AREA} = 9\pi \times \frac{1(4^{10}-1)}{4-1}$

$\Rightarrow \text{AREA} = 3145725\pi$

Question 68 (****)

Showing clearly your method, determine the value of n, given that

$$\sum_{r=1}^{n} 2^{2r-1} = 43690.$$

$\boxed{n = 8}$

$\sum_{r=1}^{n} 2^{2r-1} = 43690$

$2^1 + 2^3 + 2^5 + \dots + 2^{2n-1} = 43690$

$\underbrace{2 + 8 + 32 + \dots + 2^{2n-1}}_{\text{G.P WITH } a=2,\ r=4} = 43690$

USING $S_n = \frac{a(r^n - 1)}{r-1}$

$\Rightarrow 43690 = \frac{2(4^n - 1)}{4-1}$

$\Rightarrow 65535 = 4^n - 1$

$\Rightarrow 4^n = 65536$

BY TRIAL AND IMPROVEMENT

$n = 8$

Question 69 (****)

The n^{th} term of a geometric series is denoted by u_n.

It is further given that $u_1 = 1458$ and $u_6 = 6$.

Evaluate showing clearly your method

$$\sum_{n=7}^{\infty} u_n.$$

$\boxed{\sum_{n=7}^{\infty} u_n = 3}$

$\left.\begin{array}{l} a = 1458 \\ u_6 = 6 \end{array}\right\} \Rightarrow \quad u_n = ar^{n-1}$

$6 = 1458 \times r^5$

$\frac{1}{243} = r^5$

$r = \sqrt[5]{\frac{1}{243}}$

$\boxed{r = \frac{1}{3}}$

Now

$\sum_{n=}^{\infty} u_n = \sum_{n=1}^{\infty} u_n - \sum_{n=1}^{6} u_n = S_\infty - S_6$

$= \frac{a}{1-r} - \frac{a(1-r^6)}{1-r} = \frac{a}{1-r}\left[1-(1-r^6)\right]$

$= \frac{a}{1-r} \times r^6 = \frac{ar^6}{1-r} = \frac{1458 \times \left(\frac{1}{3}\right)^6}{1-\frac{1}{3}} = 3$

Question 70 (****)

The sum of the first n terms of a geometric series is denoted by S_n.

The common ratio of the series, r, is greater than 1.

a) If $S_4 = 5S_2$ find the value of r.

b) Given further that $S_3 = 21$ determine the value of S_{10}.

$\boxed{\text{SYN}}$, $\boxed{r=2}$, $\boxed{S_{10}=3069}$

a) USING THE INFO GIVEN, TO SET AN EQUATION

$\boxed{S_4 = \frac{a(r^4-1)}{r-1}}$

$\Rightarrow S_4 = 5S_2$

$\Rightarrow \frac{a(r^4-1)}{r-1} = 5 \times \frac{a(r^2-1)}{r-1}$

$\Rightarrow \cancel{\frac{a}{r-1}}(r^4-1) = \cancel{\frac{a}{r-1}} \times 5(r^2-1)$

AS $a \neq 0$, $r>1$ WE MAY DIVIDE WITHOUT LOSING SOLUTIONS

$\Rightarrow r^4 - 1 = 5r^2 - 5$

$\Rightarrow r^4 - 5r^2 + 4 = 0$

$\Rightarrow (r^2-4)(r^2-1) = 0$

$\Rightarrow r^2 = \begin{cases} 4 \\ 1 \end{cases} \Rightarrow r = \begin{cases} 2 \\ \cancel{-2} \\ \cancel{1} \\ \cancel{-1} \end{cases}$ $r>1$

b) START BY FINDING THE VALUE OF a

$\Rightarrow S_3 = 21$

$\Rightarrow \frac{a(2^3-1)}{2-1} = 21$

$\Rightarrow 7a = 21$

$\Rightarrow a = 3$

$\therefore S_{10} = \frac{3(2^{10}-1)}{2-1}$

$S_{10} = 3 \times 1023$

$S_{10} = 3069$

Question 71 (****)

The first three terms of a geometric series are

$$u_1 = q(4p+1), \qquad u_2 = q(2p+3) \qquad \text{and} \qquad u_3 = q(2p-3).$$

a) Find the possible values of p.

The sum to infinity of the series is 250.

b) Find the value of q.

$\boxed{}$, $\boxed{p = -\frac{1}{2} \;\cup\; p = 6}$, $\boxed{q = 4}$

a) LOOKING AT THE PATTERN

$$u_1 \qquad u_2 \qquad u_3$$

$$q(4p+1) \xrightarrow{\times r} q(2p+3) \xrightarrow{\times r} q(2p-3)$$

FORMING TWO EQUATIONS

$$\begin{aligned} q(4p+1)\times r &= q(2p+3) \\ q(2p+3)\times r &= q(2p-3) \end{aligned} \implies \begin{aligned} qr(4p+1) &= q(2p+3) \\ qr(2p+3) &= q(2p-3) \end{aligned}$$

DIVIDING SIDE BY SIDE

$$\frac{\cancel{qr}(4p+1)}{\cancel{qr}(2p+3)} = \frac{\cancel{q}(2p+3)}{\cancel{q}(2p-3)} \implies (4p+1)(2p-3) = (2p+3)(2p+3)$$

$$\implies 8p^2 - 10p - 3 = 4p^2 + 12p + 9$$

$$\implies 4p^2 - 22p - 12 = 0$$

$$\implies 2p^2 - 11p - 6 = 0$$

$$\implies (2p+1)(p-6)$$

$$\implies p = \begin{cases} 6 \\ -\frac{1}{2} \end{cases}$$

b) LOOKING AGAIN AT THE FIRST 3 TERMS

- IF $p = -\frac{1}{2}$ $\quad u_1 = -q,\ u_2 = 2q,\ u_3 = -4q \implies r = -2$
- IF $p = 6$ $\quad u_1 = 25q,\ u_2 = 15q,\ u_3 = 9q \implies r = \frac{3}{5}$

AS THERE EXISTS A SUM TO INFINITY $-1 < r < 1$, IE $r = \frac{3}{5}$

$$\implies S_\infty = \frac{a}{1-r}$$

$$\implies 250 = \frac{25q}{1-\frac{3}{5}}$$

$$\implies 250 = \frac{25q}{2/5}$$

$$\implies 100 = 25q$$

$$\implies q = 4$$

Question 72 (****)

The terms of a geometric progression are $u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ \ldots$

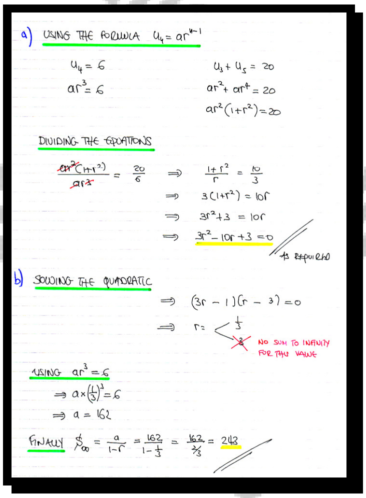
a) Given that $u_4 = 6$ and $u_3 + u_5 = 20$, show that

$$3r^2 - 10r + 3 = 0,$$

where r is the common ratio of the progression.

b) Given further that the progression has a sum to infinity determine its value.

$\boxed{}$, $\boxed{S_\infty = 243}$



a) USING THE FORMULA $u_n = ar^{n-1}$

$u_4 = 6$ $ar^3 = 6$

$u_3 + u_5 = 20$ $ar^2 + ar^4 = 20$ $ar^2(1+r^2) = 20$

DIVIDING THE EQUATIONS

$\frac{\cancel{ar^2}(1+r^2)}{\cancel{ar^3}} = \frac{20}{6} \implies \frac{1+r^2}{r} = \frac{10}{3}$

$\implies 3(1+r^2) = 10r$

$\implies 3r^2 + 3 = 10r$

$\implies 3r^2 - 10r + 3 = 0$ AS REQUIRED

b) SOLVING THE QUADRATIC

$\implies (3r-1)(r-3) = 0$

$\implies r = \frac{1}{3}$ or $\cancel{3}$ NO SUM TO INFINITY FOR THIS VALUE

USING $ar^3 = 6$

$\implies a \times \left(\frac{1}{3}\right)^3 = 6$

$\implies a = 162$

FINALLY $S_\infty = \frac{a}{1-r} = \frac{162}{1-\frac{1}{3}} = \frac{162}{\frac{2}{3}} = 243$

Question 73 (****)

Three consecutive terms of a geometric series are given in terms of a constant x.

$$U_3 = (x+5),\ U_4 = (4x-1) \quad \text{and} \quad U_5 = (2x+3).$$

Find the sum to infinity of the series.

$\boxed{}$, $\boxed{S_\infty = \frac{243}{40} = 6.075}$

LOOKING AT THE "GEOMETRIC PATTERN"

$$\begin{array}{ccc} u_3 & u_4 & u_5 \\ x+5 & 4x-1 & 2x+3 \end{array}$$

$\times r \qquad \times r$

FORMING TWO EQUATIONS

$(x+5)r = 4x-1$
$(4x-1)r = 2x+3$

DIVIDE EQUATIONS TO ELIMINATE r

$$\frac{(x+5)\cancel{r}}{(4x-1)\cancel{r}} = \frac{4x-1}{2x+3}$$
$\Rightarrow (4x-1)^2 = (2x+3)(x+5)$
$\Rightarrow 16x^2 - 8x + 1 = 2x^2 + 13x + 15$
$\Rightarrow 14x^2 - 21x - 14 = 0$
$\Rightarrow 2x^2 - 3x - 2 = 0$
$\Rightarrow (2x+1)(x-2)$
$\Rightarrow x = \begin{cases} -\frac{1}{2} \\ 2 \end{cases}$

USING EACH OF THE VALUES OF x FOUND

- IF $x = -\frac{1}{2}$
  $u_3 = \frac{9}{2}$ $\times -\frac{2}{3}$
  $u_4 = -3$ $\times -\frac{2}{3}$
  $u_5 = 2$
- IF $x = 2$
  $u_3 = 7$
  $u_4 = 7$
  $u_5 = 7$
  NO G.P.

IF $x = -\frac{1}{2}$ WE GET $r = -\frac{2}{3}$, SO USING $u_n = ar^{n-1}$

$\Rightarrow u_3 = \frac{9}{2}$
$\Rightarrow ar^2 = \frac{9}{2}$
$\Rightarrow a\left(-\frac{2}{3}\right)^2 = \frac{9}{2}$
$\Rightarrow \frac{4}{9}a = \frac{9}{2}$
$\Rightarrow a = \frac{81}{8}$

FINALLY THE SUM TO INFINITY CAN BE FOUND

$\Rightarrow S_\infty = \frac{a}{1-r}$
$\Rightarrow S_\infty = \frac{81/8}{1-\left(-\frac{2}{3}\right)}$
$\Rightarrow S_\infty = \frac{81/8}{5/3}$
$\Rightarrow S_\infty = \frac{243}{40}$

Question 74 (****)

Liquid is kept in containers, which due to evaporation and ongoing chemical reactions, at the end of each month the volume of the liquid in these containers reduces by 10% compared with the volume at the start of the same month.

One such container is filled up with 250 litres of liquid.

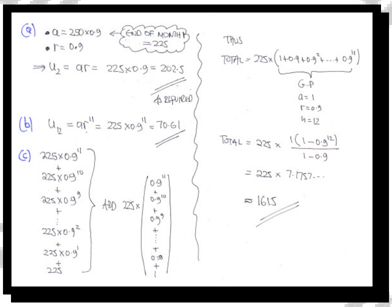
a) Show that the volume of the liquid in the container at the end of the second month is 202.5 litres.

b) Find the volume of the liquid in the container at the end of the twelfth month.

At the start of each month a new container is filled up with 250 litres of liquid, so that at the end of twelve months there are 12 containers with liquid.

c) Use an algebraic method to calculate the total amount of liquid in the 12 containers at the end of 12 months.

$\boxed{}$, $\boxed{\approx 70.6}$, $\boxed{\approx 1615}$



Question 75 (****)

It is given that

$$\sum_{r=1}^{n} u_r = 128 - 2^{7-n},$$

where u_r is the r^{th} term of a geometric progression.

a) Find the sum of the first 8 terms of the progression.

b) Determine the value of u_8.

c) Find the common ratio of the progression.

$\boxed{}$, $\boxed{\sum_{r=1}^{8} u_r = 127.5}$, $\boxed{u_8 = 0.5}$, $\boxed{r = 0.5}$

a) USING THE RESULT GIVEN

$$S_8 = \sum_{r=1}^{8} u_r = 128 - 2^{7-8} = 128 - 2^{-1} = 127.5$$

b) USING THE SUMMATION

$\Rightarrow u_8 = S_8 - S_7$

$\Rightarrow u_8 = 127.5 - \left[128 - 2^{7-7}\right]$

$\Rightarrow u_8 = 127.5 - [128 - 1]$

$\Rightarrow u_8 = 0.5$

c) FIND THE FIRST TERM

$\Rightarrow a = u_1 = S_1$

$\Rightarrow a = 128 - 2^{7-1}$

$\Rightarrow a = 64$

EITHER

$\Rightarrow u_8 = ar^7$

$\Rightarrow \frac{1}{2} = 64 \times r^7$

$\Rightarrow r^7 = \frac{1}{128}$

$\Rightarrow r = \sqrt[7]{\frac{1}{128}}$

$\Rightarrow r = \frac{1}{2}$

OR

$\Rightarrow S_2 = u_1 + u_2 = 128 - 2^{7-2}$

$\Rightarrow u_1 + u_2 = 128 - 32$

$\Rightarrow 64 + u_2 = 96$

$\Rightarrow u_2 = 32$

$\therefore r = \frac{u_2}{u_1} = \frac{32}{64} = \frac{1}{2}$

Question 76 (****)

A certain type of plastic sheet blocks 7% of the sunlight.

It is required to block at least 95% of the sunlight by placing N of these plastic sheets on top of each other.

Use algebra, to determine the least value of N.

$\boxed{}$, $\boxed{N=42}$

MODEL AS FOLLOWS

1 SHEET CUTS 7% OF THE LIGHT $\Rightarrow$ IT ALLOWS $93\% = 0.93$

2 SHEETS $\Rightarrow$ ALLOW $0.93 \times 0.93 = 0.93^2$

3 SHEETS $\Rightarrow$ ALLOW $0.93 \times 0.93^2 = 0.93^3$

ETC

WE NEED TO CUT OUT AT LEAST 95% OF THE LIGHT, I.E. ALLOW AT MOST 5%

$$u_n = ar^{n-1}$$
$$\Rightarrow 0.93 \times 0.93^{n-1} \leqslant 0.05$$
$$\Rightarrow 0.93^n \leqslant 0.05$$
$$\Rightarrow \log(0.93^n) \leqslant \log(0.05)$$
$$\Rightarrow n\log(0.93) \leqslant \log(0.05)$$
$$\Rightarrow n \geqslant \frac{\log(0.05)}{\log(0.93)} \qquad \left[\log(0.93) < 0\right]$$
$$\Rightarrow n \geqslant 41.2801\ldots$$
$$\therefore n = 42$$

Question 77 (****+)

A geometric has positive terms and positive common ratio r.

The difference between the first and the fourth term of a geometric progression is five times as large as the difference between its second and its third term.

a) Show that the common ratio r of the progression is a solution of the equation

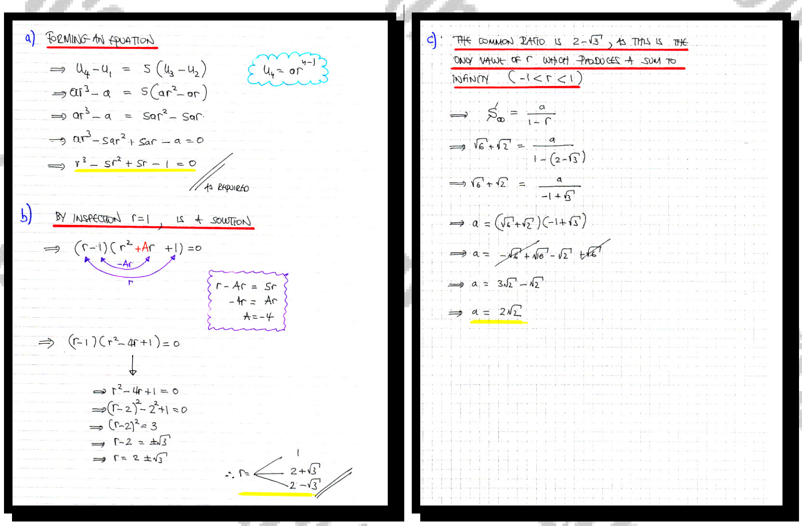
$$r^3 - 5r^2 + 5r - 1 = 0.$$

b) Find, in exact surd form where appropriate, the solutions of the above equation.

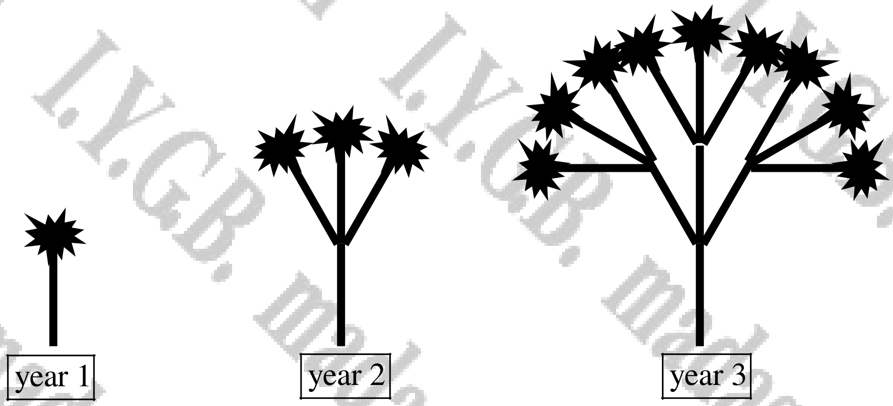
The sum to infinity of the progression is $\sqrt{6}+\sqrt{2}$.

c) Determine, in exact surd form, the first term of the progression.

$\boxed{}$, $\boxed{r = 1,\ 2-\sqrt{3},\ 2+\sqrt{3}}$, $\boxed{a = 2\sqrt{2}}$



Question 78 (****+)



The figure above shows a flowering plant. In year 1 it produces a single stem with a flower at the end.

In year 2, the flower withers and in its place three more stems are produced, with each new stem having a new flower at its end, i.e. 4 stems in total.

In year 3, the flowers wither again and in each of their places a new stems is produced, with each new stem having a new flower at its end, i.e. 13 stems in total.

This flowering pattern continues every year.

a) Find an expression for …

i. … the number of flowers in the n^{th} year.

ii. … the number of stems in the n^{th} year.

One such plant has 1093 stems.

b) Determine the number of flowers of this plant.

[continues overleaf]

[continued from overleaf]

A different plant of the above variety has over 750 flowers.

c) Determine the **least** number of stems of this plant.

$\boxed{}$, $\boxed{f_n = 3^{n-1}}$, $\boxed{S_n = \dfrac{3^n - 1}{2}}$, $\boxed{729}$, $\boxed{3280}$

(a)

YEAR	FLOWERS	STEMS
1	1	1
2	3	1+3=4
3	9	4+9=13
4	27	13+27=40
⋮	⋮	⋮
	GP $a=1$ $r=3$	SUM OF ITS TERMS

$\therefore$ Flowers $\boxed{f_n = 3^{n-1}}$

$S_n = \dfrac{a(r^n-1)}{r-1} = \dfrac{1(3^n-1)}{3-1}$

$\boxed{S_n = \dfrac{3^n-1}{2}}$

(b) $S_n = 1093$

$\dfrac{3^n-1}{2} = 1093$

$3^n = 2187$

BY TRIAL & IMPROVEMENT

$n=7$

$\therefore f_7 = 3^6 = 729$

(c) $f_n > 750$

$\Rightarrow 3^{n-1} > 750$

$\Rightarrow \log 3^{n-1} > \log 750$

$\Rightarrow (n-1)\log 3 > \log 750$

$\Rightarrow n-1 > 6.02\ldots$

$\Rightarrow n > 7.02\ldots$

$\Rightarrow n = 8$

Thus

$S_8 = \dfrac{3^8-1}{2}$

$S_8 = 3280$

Question 79 (****+)

The sum of the first 2 terms of a geometric progression is 40.

The sum of the first 4 terms of the same geometric progression is 130.

Determine the two possible values of the sum of the first 5 terms of the geometric progression.

$\boxed{}$, $\boxed{S_5 = 211 \quad \text{or} \quad S_5 = -275}$

$$\boxed{S_n = \frac{a(r^n - 1)}{r - 1}}$$

$$\left.\begin{array}{l} S_2 = 40 \Rightarrow a + ar = 40 \Rightarrow a(1+r) = 40 \\ S_4 = 130 \Rightarrow \dfrac{a(r^4-1)}{r-1} = 130 \Rightarrow \dfrac{a(r^2+1)(r^2-1)}{r-1} = 130 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} a(1+r) = 40 \\ \dfrac{a(r^2+1)\cancel{(r-1)}(r+1)}{\cancel{r-1}} = 130 \\ (r \neq 1) \end{array}\right\} \Rightarrow \begin{array}{l} a(r+1) \times (r^2+1) = 130 \\ 40(r^2+1) = 130 \\ r^2 + 1 = \frac{13}{4} \\ r^2 = \frac{9}{4} \\ r = \pm\frac{3}{2} \end{array}$$

IF $r = \frac{3}{2}$ $\quad a = \dfrac{40}{1+\frac{3}{2}} = 16$

IF $r = -\frac{3}{2}$ $\quad a = \dfrac{40}{1-\frac{3}{2}} = -80$

THUS $S_5 = \dfrac{16(1.5^5 - 1)}{1.5 - 1} = 211$

OR

$S_5 = \dfrac{-80((-1.5)^5 - 1)}{-1.5 - 1} = -275$

Question 80 (****+)

A geometric series has first term a and common ratio r.

The ratio of the sum of the first 5 terms of the series, to the sum of the reciprocals of the first 5 terms of the series, is 49.

Given further that the sum of the first and third term of the series is 35, determine the value of a and the two possible values of r.

$\boxed{}$, $\boxed{a=28}$, $\boxed{r=\pm\frac{1}{2}}$

FORM TWO EQUATIONS

$u_1 + u_3 = 35$

$a + ar^2 = 35$

$a(1+r^2) = 35$

$\boxed{a = \frac{35}{1+r^2}}$

$$\frac{a + ar + ar^2 + ar^3 + ar^4}{\frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \frac{1}{ar^3} + \frac{1}{ar^4}} = 49$$

$$\Rightarrow \frac{a(1+r+r^2+r^3+r^4)}{\frac{1}{a}\left(1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}\right)} = 49$$

$$\Rightarrow \frac{a^2(1+r+r^2+r^3+r^4)}{1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}} = 49$$

MULTIPLY TOP & BOTTOM OF THE FRACTION IN THE L.H.S BY r^4

$$\Rightarrow \frac{a^2(1+r+r^2+r^3+r^4)\,r^4}{\left(1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}\right)r^4} = 49$$

$$\Rightarrow \frac{a^2r^4\cancel{(1+r+r^2+r^3+r^4)}}{\cancel{r^4+r^3+r^2+r+1}} = 49$$

$$\Rightarrow \boxed{a^2r^4 = 49}$$

$$\Rightarrow ar^2 = \begin{cases} 7 \\ -7 \end{cases}$$

$$\Rightarrow \frac{35}{1+r^2} \times r^2 = \begin{cases} 7 \\ \cancel{-7} \end{cases}$$

$$\Rightarrow \frac{35r^2}{1+r^2} = 7$$

$$\Rightarrow 35r^2 = 7 + 7r^2$$

$$\Rightarrow 28r^2 = 7$$

$$\Rightarrow r^2 = \frac{1}{4}$$

$$\Rightarrow r = \pm\frac{1}{2} \quad \& \quad a = 28$$

Question 81 (****+)

Anton is planning to save for a house purchase deposit over a period of 5 years.

He opens an account known as a "Homesaver" and plans to pay into this account £200 at the start of every month, and continue to do so for 5 years.

The account pays 0.5% compound interest **per month**, with the interest credited to the account at the end of every month.

a) Show clearly that at the **end** of the third month the balance of the account will be £606.02 .

b) Calculate the total amount in Anton's "Homesaver" account after 5 years.

£14023.78

a) FORMING A TABLE

START OF MONTH	£	END OF MONTH
1	200	$200 \times 1.005 = 201$
2	200+201	$401 \times 1.005 = 403.005$
3	200+403.005	$603.005 \times 1.005 = 606.020025$

$\therefore$ £606.02 // AS REQUIRED

b) MONTH END

1 $\quad 200 \times 1.005$

2 $\quad 200 \times 1.005^2 + 200 \times 1.005^1$

3 $\quad 200 \times 1.005^3 + 200 \times 1.005^2 + 200 \times 1.005^1$

⋮

60 $\quad 200 \times 1.005^{60} + 200 \times 1.005^{59} + 200 \times 1.005^{58} + \dots + 200 \times 1.005^1$

HENCE THE REQUIRED TOTAL IS

$$\Rightarrow \text{TOTAL} = 200 \times 1.005^1 + 200 \times 1.005^2 + 200 \times 1.005^3 + \dots + 200 \times 1.005^{60}$$

$$\Rightarrow \text{TOTAL} = 200\Big[\underbrace{1.005^1 + 1.005^2 + 1.005^3 + \dots + 1.005^{60}}_{\text{This is a G.P. with } a = 1.005,\ r = 1.005,\ n = 60}\Big]$$

$$\Rightarrow \text{TOTAL} = 200 \times \frac{1.005\,(1.005^{60} - 1)}{1.005 - 1} = £14023.78$$

Question 82 (****+)

The second term of a geometric series is -12 and its sum to infinity is 16.

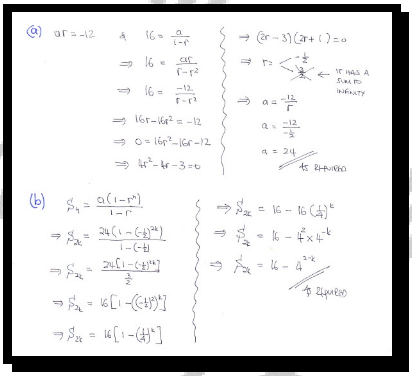
a) Show that the first term of the series is 24.

The sum of the first n term of the series is denoted by S_n.

b) Show clearly that

$$S_{2k} = 16 - 4^{2-k}.$$

proof



Question 83 (****+)

A pension contribution scheme is scheduled as follows.

A £1250 contribution is made at the **start** of every year.

The total money in the scheme at the end of every year is re-invested at a constant compound interest rate of 6% per annum.

a) Show that at the start of the third year, after the annual contribution has been made, the amount in the pension scheme is £3979.50 .

b) Calculate the amount in the pension scheme at the start of the fortieth year, after the annual contribution is made.

$\boxed{}$, $\boxed{£193452.46}$

(a)
START 1 : 1250
END 1 : 1250×1.06

START 2 : $1250 + (1250 \times 1.06)$
END 2 : $[1250 + (1250 \times 1.06)] \times 1.06 = 1250 \times 1.06 + 1250 \times 1.06^2$

START 3 : $1250 + 1250 \times 1.06 + 1250 \times 1.06^2 = 3979.50$ // AS REQUIRED

(b)
END 3 : $(1250 + 1250 \times 1.06 + 1250 \times 1.06^2) \times 1.06 = 1250 \times 1.06 + 1250 \times 1.06^2 + 1250 \times 1.06^3$

START 4 : $1250 + 1250 \times 1.06^1 + 1250 \times 1.06^2 + 1250 \times 1.06^3$

So .....

START 40 : $1250 + 1250 \times 1.06^1 + 1250 \times 1.06^2 + 1250 \times 1.06^3 + \dots + 1250 \times 1.06^{39}$

$$\text{TOTAL} = 1250\Big[\underbrace{1 + 1.06 + 1.06^2 + 1.06^3 + \dots + 1.06^{39}}_{\text{G.P WITH } a=1,\ r=1.06,\ n=40}\Big]$$

$$= 1250 \times \frac{1(1.06^{40} - 1)}{1.06 - 1}$$

$$= 193452.457\dots$$

$\therefore$ £193,452.46 //

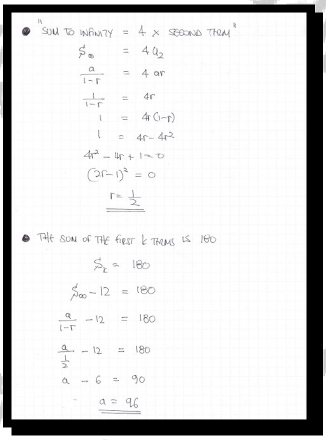
Question 84 (****+)

The sum of the first k terms of a geometric progression is 180.

It is further given that the sum of the first k terms of this geometric progression is **twelve less** than its sum to infinity.

If the sum to infinity of the geometric progression is four times as large as its second term, use algebra to determine the value of k.

$\boxed{}$, $\boxed{k = 4}$

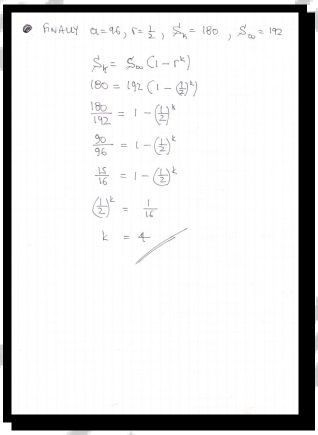


• "SUM TO INFINITY = 4 × SECOND TERM"

$$S_\infty = 4u_2$$
$$\frac{a}{1-r} = 4ar$$
$$\frac{1}{1-r} = 4r$$
$$1 = 4r(1-r)$$
$$1 = 4r - 4r^2$$
$$4r^2 - 4r + 1 = 0$$
$$(2r-1)^2 = 0$$
$$r = \frac{1}{2}$$

• THE SUM OF THE FIRST k TERMS IS 180

$$S_k = 180$$
$$S_\infty - 12 = 180$$
$$\frac{a}{1-r} - 12 = 180$$
$$\frac{a}{\frac{1}{2}} - 12 = 180$$
$$a - 6 = 90$$
$$a = 96$$



• FINALLY $a = 96$, $r = \frac{1}{2}$, $S_k = 180$, $S_\infty = 192$

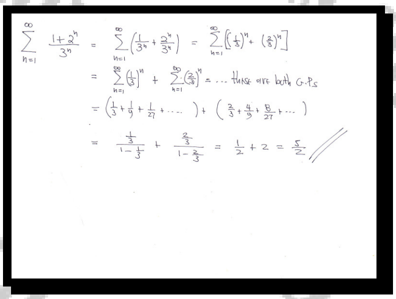
$$S_k = S_\infty(1 - r^k)$$
$$180 = 192\left(1 - \left(\tfrac{1}{2}\right)^k\right)$$
$$\frac{180}{192} = 1 - \left(\frac{1}{2}\right)^k$$
$$\frac{90}{96} = 1 - \left(\frac{1}{2}\right)^k$$
$$\frac{15}{16} = 1 - \left(\frac{1}{2}\right)^k$$
$$\left(\frac{1}{2}\right)^k = \frac{1}{16}$$
$$k = 4$$

Question 85 (*****)

Evaluate showing clearly your method

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}.$$

$\boxed{\frac{5}{2}}$



$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{1}{3^n} + \frac{2^n}{3^n}\right) = \sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n\right]$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \ldots \text{ these are both G.Ps}$$
$$= \left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots\right) + \left(\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \ldots\right)$$
$$= \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{1}{2} + 2 = \frac{5}{2}$$

Question 86 (****+)

The sum to infinity of a geometric series is 2187.

The $(k-1)^{\text{th}}$ and k^{th} term of the same series are 96 and 64, respectively.

Determine the value of

$$\sum_{n=k+1}^{\infty} u_n \, ,$$

where u_n is the n^{th} term of the series.

$$\boxed{} \, , \ \boxed{\sum_{n=k+1}^{\infty} u_n = 128}$$

$$\boxed{\begin{array}{lll} u_{k-1} = 96 & u_k = 64 & S_\infty = 2187 \\ ar^{k-2} = 96 & ar^{k-1} = 64 & \frac{a}{1-r} = 2187 \end{array}}$$

- FROM THE FIRST TWO RELATIONSHIPS

$$r = \frac{u_k}{u_{k-1}} = \frac{64}{96} = \frac{2}{3}$$

- FROM THE THIRD RELATIONSHIP

$$\frac{a}{1-\frac{2}{3}} = 2187$$

$$\frac{a}{\frac{1}{3}} = 2187$$

$$a = 729$$

- NEXT WE HAVE

$$u_k = 64$$

$$ar^{k-1} = 64$$

$$729 \times \left(\frac{2}{3}\right)^{k-1} = 64$$

$$\left(\frac{2}{3}\right)^{k-1} = \frac{64}{729}$$

BY INSPECTION, TRIAL & IMPROVEMENT (AS k IS A POSITIVE INTEGER) OR LOGARITHMS

$$\left(\frac{2}{3}\right)^{k-1} = \left(\frac{2}{3}\right)^{6}$$

$$k = 7$$

- FINALLY WE HAVE

$$\sum_{n=k+1}^{\infty} u_n = \sum_{n=1}^{\infty} u_n - \sum_{n=1}^{k} u_n$$

$$= S_\infty - S_k$$

$$= 2187 - \frac{a(1-r^k)}{1-r}$$

$$= 2187 - \frac{729\left(1-\left(\frac{2}{3}\right)^7\right)}{1-\frac{2}{3}}$$

$$= 2187 - 2059$$

$$= 128$$

Question 87 (****+)

Liquid is kept in containers, which due to evaporation and ongoing chemical reactions, at the end of each month the volume of the liquid in these containers reduces by 4% compared with the volume at the start of the same month.

At the start of each month a new container is filled up with 200 litres of liquid, so that at the end of thirty months there are 30 containers with liquid.

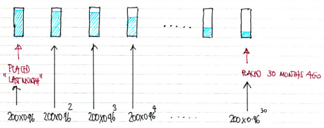
Calculate the total amount of liquid in the 30 containers at the end of 30 months.

$\boxed{}$, $\boxed{\approx 3389}$

STARTING THE MODELLING BY LOOKING AT A SINGLE CONTAINER

MONTH	START	END
1	200	200×0.96
2	200×0.96	200×0.96^2
3	200×0.96^2	200×0.96^3
4	200×0.96^3	200×0.96^4
⋮		
k	$200 \times 0.96^{k-1}$	200×0.96^k

LOOKING AT ALL THE 30 CONTAINERS, PLACED OVER THE ENTIRE 30 MONTH PERIOD



ADDING ALL THESE AMOUNTS

$\Rightarrow \text{TOTAL} = 200 \times 0.96 + 200 \times 0.96^2 + 200 \times 0.96^3 + \dots + 200 \times 0.96^{30}$

$\Rightarrow \text{TOTAL} = 200\left[\underbrace{0.96^1 + 0.96^2 + 0.96^3 + \dots + 0.96^{30}}_{\boxed{\text{THIS IS A G.P } a = 0.96,\ r = 0.96,\ n = 30}}\right]$

$\Rightarrow \text{TOTAL} = 200 \times \dfrac{0.96\left[1 - 0.96^{30}\right]}{1 - 0.96}$ $\qquad \boxed{S_n = \dfrac{a(1-r^n)}{1-r}}$

$\Rightarrow \text{TOTAL} = 4800\left(1 - 0.96^{30}\right)$

$\Rightarrow \text{TOTAL} = 3389$ // (NEAREST LITRE)

Question 88 (*****)

The r^{th} term of a progression is given by

$$u_r = ak^{r-1},$$

where a and k are non zero constants with $k \neq \pm 1$.

Show that

$$\sum_{r=1}^{n} \left(u_r \times u_{r+1}\right) = \frac{a^2 k \left(1-k^{2n}\right)}{1-k^2}.$$

$\boxed{}$, $\boxed{\text{proof}}$

$$u_r = ak^{r-1} \Rightarrow \left\{ \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & \dots & u_n \\ a & ak & ak^2 & ak^3 & & ak^{n-1} \end{array} \right\}$$

HENCE

$$\sum_{r=1}^{n}\left(u_r u_{r+1}\right) = u_1u_2 + u_2u_3 + u_3u_4 + \dots + u_nu_{n+1}$$

$$= a(ak) + ak(ak^2) + ak^2(ak^3) + \dots + ak^{n-1}(ak^n)$$

$$= a^2k + a^2k^3 + a^2k^5 + \dots + a^2k^{2n-1}$$

$$= a^2k\,\underbrace{\left[1 + k^2 + k^4 + \dots + k^{2n-2}\right]}_{\substack{\text{G.P WITH } a=1 \\ r=k^2 \\ n \text{ TERMS}}}$$

$$= a^2k \times \frac{1\left(1-(k^2)^n\right)}{1-k^2}$$

$$= \frac{a^2k\left(1-k^{2n}\right)}{1-k^2}$$

AS REQUIRED

Question 89 (*****)

An elastic ball is dropped from a height of 20 metres, and bounces repeatedly.

The ball bounces off the ground to a height which is $\frac{1}{2}$ the height from which it was last dropped.

a) Show that after the n^{th} bounce the ball reaches a height of $5\times 2^{2-n}$ metres.

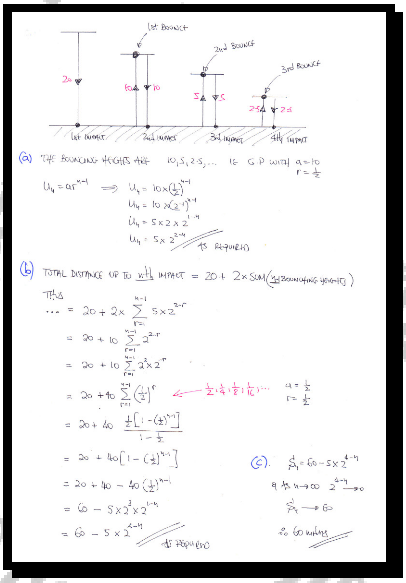
b) Show clearly that the total distance covered by the ball **up and including the n^{th} impact** is given by

$$60-5\times 2^{4-n}.$$

The ball keeps bouncing off the ground in this fashion until it comes to rest.

c) Determine the total distance covered by the ball until it comes to rest.

$\boxed{60 \text{ metres}}$



(a) THE BOUNCING HEIGHTS ARE $10, 5, 2.5, \ldots$ IE G.P WITH $a=10$, $r=\frac{1}{2}$

$u_n = ar^{n-1} \Rightarrow u_n = 10\times\left(\frac{1}{2}\right)^{n-1}$

$u_n = 10\times(2^{-1})^{n-1}$

$u_n = 5\times 2\times 2^{1-n}$

$u_n = 5\times 2^{2-n}$ AS REQUIRED

(b) TOTAL DISTANCE UP TO nth IMPACT $= 20 + 2\times \text{SUM}$ ($n-1$ BOUNCING HEIGHTS)

THUS

$\ldots = 20 + 2\times\sum_{r=1}^{n-1} 5\times 2^{2-r}$

$= 20 + 10\sum_{r=1}^{n-1} 2^{2-r}$

$= 20 + 10\sum_{r=1}^{n-1} 2^2\times 2^{-r}$

$= 20 + 40\sum_{r=1}^{n-1}\left(\frac{1}{2}\right)^r \leftarrow \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ $a=\frac{1}{2}$, $r=\frac{1}{2}$

$= 20 + 40\,\dfrac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^{n-1}\right]}{1-\frac{1}{2}}$

$= 20 + 40\left[1-\left(\frac{1}{2}\right)^{n-1}\right]$

$= 20 + 40 - 40\left(\frac{1}{2}\right)^{n-1}$

$= 60 - 5\times 2^3\times 2^{1-n}$

$= 60 - 5\times 2^{4-n}$ AS REQUIRED

(c) $S_n = 60 - 5\times 2^{4-n}$

& AS $n\to\infty$ $2^{4-n}\to 0$

$S_n \to 60$

$\therefore$ 60 metres

Question 90 (*****)

The n^{th} term of a geometric progression is denoted by u_n.

Find the possible values of the common ratio given that

$$u_{n+2} = 5u_{n+1} - 6u_n .$$

$\boxed{r = 2,\ 3}$

$u_n = ar^{n-1}$

Now $u_{n+2} = 5u_{n+1} - 6u_n$

$\Rightarrow ar^{n+1} = 5ar^n - 6ar^{n-1}$ $\quad a \neq 0$

$\Rightarrow r^{n+1} = 5r^n - 6r^{n-1}$

$\Rightarrow r^n \times r = 5 \times r^n - 6r^n \times r^{-1}$

$\Rightarrow r = 5 - 6r^{-1}$ $\quad r \neq 0$

$\Rightarrow r = 5 - \frac{6}{r}$

$\Rightarrow r^2 = 5r - 6$

$\Rightarrow r^2 - 5r + 6 = 0$

$\Rightarrow (r-2)(r-3) = 0$

$\Rightarrow r = \begin{cases} 2 \\ 3 \end{cases}$

Question 91 (*****)

The trapezium rule with n equally spaced intervals is to be used to estimate the value of the following integral

$$\int_0^1 2^x \, dx .$$

Show that the value of this estimate is given by

$$\frac{1}{2n}\left[\frac{2^{\frac{1}{n}}+1}{2^{\frac{1}{n}}-1}\right].$$

$\boxed{}$, $\boxed{\text{proof}}$

x	0	$\frac{1}{n}$	$\frac{2}{n}$	$\frac{3}{n}$	$\dots\dots$	$\frac{n-1}{n}$	$\frac{n}{n}=1$
y	2^0	$2^{\frac{1}{n}}$	$2^{\frac{2}{n}}$	$2^{\frac{3}{n}}$	$\dots\dots$	$2^{\frac{n-1}{n}}$	2^1

$$\int_0^1 2^x \, dx \approx \frac{\text{"THICKNESS"}}{2}\left[\text{FIRST} + \text{LAST} + 2\times\text{REST}\right]$$

$$\approx \frac{\frac{1}{n}}{2}\left[2^0+2^1+2\big(\underbrace{2^{\frac{1}{n}}+2^{\frac{2}{n}}+2^{\frac{3}{n}}+\dots+2^{\frac{n-1}{n}}}_{n-1 \text{ TERMS}}\big)\right]$$

$$\approx \frac{1}{2n}\left[1+2+2\left[\left(2^{\frac{1}{n}}\right)+\left(2^{\frac{1}{n}}\right)^2+\left(2^{\frac{1}{n}}\right)^3+\dots+\left(2^{\frac{1}{n}}\right)^{n-1}\right]\right]$$

$$\approx \frac{1}{2n}\left[3+2\times\frac{2^{\frac{1}{n}}\left[\left(2^{\frac{1}{n}}\right)^{n-1}-1\right]}{2^{\frac{1}{n}}-1}\right]$$

SUMMATION OF A G.P.

$$S_N = \frac{a\left(r^N-1\right)}{r-1}$$

$$\approx \frac{1}{2n}\left[3+2\times\frac{\left(2^{\frac{1}{n}}\right)^n-2^{\frac{1}{n}}}{2^{\frac{1}{n}}-1}\right]$$

$$\approx \frac{1}{2n}\left[3+2\times\frac{2-2^{\frac{1}{n}}}{2^{\frac{1}{n}}-1}\right]$$

$$\approx \frac{1}{2n}\left[\frac{3\times2^{\frac{1}{n}}-3+4-2\times2^{\frac{1}{n}}}{2^{\frac{1}{n}}-1}\right]$$

$$\approx \frac{1}{2n}\left[\frac{2^{\frac{1}{n}}+1}{2^{\frac{1}{n}}-1}\right]$$

Question 92 (*****)

It is given that $0<r<1$, $0<R<1$ and $r<2R$.

It is further given that

$$\sum_{n=0}^{\infty} R^n = \left(\sum_{n=0}^{\infty} r^n \right)^2 .$$

Show clearly that

$$\sum_{n=0}^{\infty} \left(\frac{r}{2R} \right)^n = \frac{2(2-r)}{3-2r} .$$

$\boxed{\text{proof}}$

AS r & R ARE NUMERICALLY LESS THAN 1, THESE TWO REPRESENT THE SUMS TO INFINITY

$$\Rightarrow \sum_{n=0}^{\infty} R^n = \left[\sum_{n=0}^{\infty} r^n \right]^2$$

$$S_{\infty} = \frac{a}{1-r}$$

$$\Rightarrow \frac{1}{1-R} = \left(\frac{1}{1-r} \right)^2$$

$$\Rightarrow \frac{1}{1-R} = \frac{1}{1-2r+r^2}$$

$$\Rightarrow 1-R = 1-2r+r^2$$

$$\Rightarrow 2r-r^2 = R$$

NOW ANOTHER SUM TO INFINITY

$$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{r}{2R} \right)^n = 1 + \frac{r}{2R} + \left(\frac{r}{2R} \right)^2 + \left(\frac{r}{2R} \right)^3 + \cdots \qquad \frac{r}{2R}<1$$

$$= \frac{1}{1-\frac{r}{2R}} = \frac{2R}{2R-r}$$

$$= \frac{2(2r-r^2)}{2(2r-r^2)-r}$$

DIVIDE "TOP & BOTTOM" BY r, $r \neq 0$

$$= \frac{2(2-r)}{2(2-r)-1}$$

$$= \frac{2(2-r)}{4-2r-1}$$

$$= \frac{2(2-r)}{3-2r}$$

AS REQUIRED

Question 93 (*****)

Evaluate showing clearly your method

$$\sum_{n=1}^{\infty} \frac{3^n - 2}{4^{n+1}} .$$

$\boxed{\text{SYN}}$, $\boxed{\frac{7}{12}}$

SPLIT THE SUMMATION AS FOLLOWS

$$\sum_{n=1}^{\infty}\left[\frac{3^n-2}{4^{n+1}}\right] = \sum_{n=1}^{\infty}\left[\frac{3^n}{4^{n+1}} - \frac{2}{4^{n+1}}\right] = \sum_{n=1}^{\infty}\left[\frac{3^n}{4^{n+1}}\right] - \sum_{n=1}^{\infty}\left[\frac{2}{4^{n+1}}\right]$$

$$= \sum_{n=1}^{\infty}\left[\frac{3^n}{4\times 4^n}\right] - \sum_{n=1}^{\infty}\left[\frac{2}{4\times 4^n}\right]$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$$

THIS IS A G.P.

$a = \frac{3}{4}$

$r = \frac{3}{4}$

$S_\infty = \frac{\frac{3}{4}}{1-\frac{3}{4}}$

$S_\infty = 3$

THIS IS A G.P

$a = \frac{1}{4}$

$r = \frac{1}{4}$

$S_\infty = \frac{\frac{1}{4}}{1-\frac{1}{4}}$

$S_\infty = \frac{1}{3}$

PUTTING ALL THE RESULTS TOGETHER

$$\sum_{n=1}^{\infty}\left[\frac{3^n-2}{4^{n+1}}\right] = \frac{1}{4}\times 3 - \frac{1}{2}\times\frac{1}{3} = \frac{3}{4} - \frac{1}{6} = \frac{7}{12}$$

Question 94 (*****)

An elastic ball is dropped from a height of h metres.

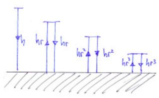
The ball bounces off the ground to a height which is r times the height from which it was dropped, where $0 < r < 1$.

The ball keeps bouncing off the ground in this fashion until it comes to rest.

Given the ball covers a total distance d show that

$$r = \frac{d-h}{d+h}.$$

$\boxed{}$, $\boxed{\text{proof}}$



$d = h + 2hr + 2hr^2 + 2hr^3 + \dots$

$d = h[1 + 2r + 2r^2 + 2r^3 + \dots]$

$d = h[1 + 2\underbrace{(r + r^2 + r^3 + \dots)}_{\text{This is a G.P with } a=r,\ r=r}]$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{r}{1-r}$

Thus

$\Rightarrow d = h\left[1 + 2\left(\frac{r}{1-r}\right)\right]$

$\Rightarrow d = h\left(1 + \frac{2r}{1-r}\right)$

$\Rightarrow \frac{d}{h} = 1 + \frac{2r}{1-r}$

$\Rightarrow \frac{d}{h} - 1 = \frac{2r}{1-r}$

$\Rightarrow \frac{d-h}{h} = \frac{2r}{1-r}$

$\Rightarrow (d-h)(1-r) = 2rh$

$\Rightarrow (d-h) - r(d-h) = 2rh$

$\Rightarrow d-h = 2rh + r(d-h)$

$\Rightarrow d-h = r[2h + (d-h)]$

$\Rightarrow d-h = r[h+d]$

$\Rightarrow \frac{d-h}{h+d} = r$

$\Rightarrow r = \frac{d-h}{d+h}$ // AS REQUIRED

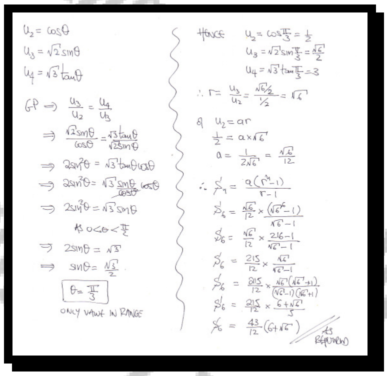
Question 95 (*****)

The 2^{nd}, 3^{rd} and 4^{th} terms of a geometric progression are $\cos\theta$, $\sqrt{2}\sin\theta$ and $\sqrt{3}\tan\theta$, respectively, where $0 < \theta < \frac{\pi}{2}$.

Show clearly that the sum of the first 6 terms of the progression is

$$\frac{43}{12}\left(6+\sqrt{6}\right).$$

$\boxed{\text{proof}}$



$u_2 = \cos\theta$
$u_3 = \sqrt{2}\sin\theta$
$u_4 = \sqrt{3}\tan\theta$

GP $\Rightarrow \frac{u_3}{u_2} = \frac{u_4}{u_3}$
$\Rightarrow \frac{\sqrt{2}\sin\theta}{\cos\theta} = \frac{\sqrt{3}\tan\theta}{\sqrt{2}\sin\theta}$
$\Rightarrow 2\sin^2\theta = \sqrt{3}\tan\theta\cos\theta$
$\Rightarrow 2\sin^2\theta = \sqrt{3}\frac{\sin\theta}{\cancel{\cos\theta}}\cancel{\cos\theta}$
$\Rightarrow 2\sin^2\theta = \sqrt{3}\sin\theta$
As $0<\theta<\frac{\pi}{2}$
$\Rightarrow 2\sin\theta = \sqrt{3}$
$\Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$
$\boxed{\theta = \frac{\pi}{3}}$
ONLY VALUE IN RANGE

HENCE $u_2 = \cos\frac{\pi}{3} = \frac{1}{2}$
$u_3 = \sqrt{2}\sin\frac{\pi}{3} = \frac{\sqrt{6}}{2}$
$u_4 = \sqrt{3}\tan\frac{\pi}{3} = 3$

$\therefore r = \frac{u_3}{u_2} = \frac{\sqrt{6}/2}{1/2} = \sqrt{6}$

& $u_2 = ar$
$\frac{1}{2} = a \times \sqrt{6}$
$a = \frac{1}{2\sqrt{6}} = \frac{\sqrt{6}}{12}$

$\therefore S_n = \frac{a(r^n-1)}{r-1}$
$S_6 = \frac{\sqrt{6}}{12} \times \frac{(\sqrt{6}^6-1)}{\sqrt{6}-1}$
$S_6 = \frac{\sqrt{6}}{12} \times \frac{216-1}{\sqrt{6}-1}$
$S_6 = \frac{215}{12} \times \frac{\sqrt{6}}{\sqrt{6}-1}$
$S_6 = \frac{215}{12} \times \frac{\sqrt{6}(\sqrt{6}+1)}{(\sqrt{6}-1)(\sqrt{6}+1)}$
$S_6 = \frac{215}{12} \times \frac{6+\sqrt{6}}{5}$
$S_6 = \frac{43}{12}(6+\sqrt{6})$
AS REQUIRED

Question 96 (*****)

It is given that

$$S_n = \sum_{k=1}^{n}\left[\sum_{r=1}^{k}\left(2^r\right)\right].$$

Show that

$$S_n = 2^{n+2} - 2n - 4.$$

$\boxed{}$, $\boxed{\text{proof}}$

THE FIRST SUMMATION (INNER NEST) IS GEOMETRIC

e.g. $\sum_{r=1}^{k} 2^r = 2^1 + 2^2 + 2^3 + \dots + 2^k$ SUM OF THE FIRST k TERMS

HENCE WE HAVE

$$\sum_{k=1}^{n}\left[\sum_{r=1}^{k} 2^r\right] = \sum_{k=1}^{n}\left[\frac{2(2^k-1)}{2-1}\right]$$

$$= \sum_{k=1}^{n}\left[2(2^k-1)\right]$$

$$= 2\sum_{k=1}^{n}\left[2^k - 1\right]$$

$$= 2\sum_{k=1}^{n}\left[2^k\right] - 2\sum_{k=1}^{n} 1$$

REPEATING AS THE FIRST IS THE VERY SAME G.P AS ABOVE

$$= 2\left[\frac{2(2^n-1)}{2-1}\right] - 2 \times n$$

$$= 2\left[2(2^n-1)\right] - 2n$$

$$= 4(2^n-1) - 2n$$

$$= 4 \times 2^n - 4 - 2n$$

$$= 2^{n+2} - 2n - 4$$

// AS REQUIRED

Question 97 (*****)

The first two terms of a geometric series are 10 and $(10-x)$.

Given that the series is convergent determine …

a) … the range of values of x.

b) … the range of the sum to infinity of the series.

$\boxed{\text{S1}}$, $\boxed{0 < x < 20}$, $\boxed{S_\infty > 5}$

a) u_1 u_2

10 $10-x$

$$10 \xrightarrow{r} 10-x$$

$$\boxed{r = \frac{10-x}{10}}$$

NOW IF THE G.P IS CONVERGENT THEN $-1 < r < 1$

$$-1 < \frac{10-x}{10} < 1$$
$$-10 < 10-x < 10$$
$$-20 < -x < 0$$
$$0 < x < 20$$

b) $$S_\infty = \frac{a}{1-r} = \frac{10}{1-\frac{10-x}{10}} = \frac{100}{10-(10-x)} = \frac{100}{x}.$$

BUT FROM PART (a)

$$0 < x < 20$$
$$\frac{1}{20} < \frac{1}{x} < \infty$$
$$\frac{1}{x} > \frac{1}{20}$$
$$\frac{100}{x} > 5$$
$$S_\infty > 5$$

Question 98 (*****)

$$1+\frac{1}{1+x}+\frac{1}{(1+x)^2}+\frac{1}{(1+x)^3}+\frac{1}{(1+x)^4}+\ldots$$

It is given that the above series is convergent.

Determine its sum to infinity in terms of x, and the range of the possible values of x.

$$\boxed{S_\infty=\frac{1+x}{x}}, \quad \boxed{x<-2 \text{ or } x>0}$$

$1+\frac{1}{1+x}+\frac{1}{(1+x)^2}+\frac{1}{(1+x)^3}+\ldots$ IS A G.P. WITH $a=1$, $r=\frac{1}{1+x}$

$S_\infty=\frac{a}{1-r}=\frac{1}{1-\frac{1}{1+x}}=\frac{1+x}{(1+x)-1}=\frac{1+x}{x}$

FOR CONVERGENCE

$\Rightarrow -1<r<1$

$\Rightarrow -1<\frac{1}{1+x}<1$

$\Rightarrow -1<\frac{1+x}{(1+x)^2}<1$

$\Rightarrow -(1+x)^2<1+x<(1+x)^2$

SOLVE SEPARATELY

$-(1+x)^2<1+x$	$1+x<(1+x)^2$
$-x^2-2x-1<1+x$	$1+x<1+2x+x^2$
$-x^2-3x-2<0$	$0<x^2+x$
$x^2+3x+2>0$	$x^2+x>0$
$(x+2)(x+1)>0$	$x(x+1)>0$
C.V. $=-2$	C.V. $=0$
$x<-2$	$x>0$

HENCE

$x<-2$ OR $x>0$

Question 99 (*****)

The $(k-1)^{\text{th}}$ and k^{th} term of a convergent geometric progression are 108 and 81, respectively.

Determine the value of

$$\sum_{n=k+1}^{\infty} u_n ,$$

where u_n is the n^{th} term of the series.

$\boxed{}$, $\boxed{\sum_{n=k+1}^{\infty} u_n = 243}$

$$u_{k-1} = 108 \quad \cdot \quad u_k = 81 \quad \cdot \quad \sum_{n=k+1}^{\infty} u_n = ?$$

- FROM THE FIRST TWO RELATIONSHIPS

$$r = \frac{u_k}{u_{k-1}} = \frac{81}{108} = \frac{3}{4}$$

- THEN USING $u_k = 81$

$$ar^{k-1} = 81$$
$$a\left(\tfrac{3}{4}\right)^{k-1} = 81$$
$$a\left(\tfrac{3}{4}\right)^{k} = 81 \times \tfrac{3}{4} \qquad \text{OR} \qquad ar^k = \frac{81 \times 3}{4}$$

- FINALLY WE HAVE

$$\begin{aligned}
\sum_{n=k+1}^{\infty} u_n &= \sum_{n=1}^{\infty} u_n - \sum_{n=1}^{k} u_n \\
&= S_{\infty} - S_k \\
&= S_{\infty} - \frac{a(1-r^k)}{1-r} \\
&= S_{\infty} - S_{\infty}(1-r^k) \\
&= S_{\infty}\left(1-(1-r^k)\right) \\
&= S_{\infty} r^k \\
&= \frac{a}{1-r} \times r^k \\
&= \frac{ar^k}{1-r} \\
&= \frac{\frac{81 \times 3}{4}}{1-\frac{3}{4}} \\
&= \frac{\frac{81 \times 3}{4}}{\frac{1}{4}} \\
&= 81 \times 3 \\
&= 243
\end{aligned}$$

Question 100 (*****)

The first two terms of a geometric series are 2 and x.

Given the series is convergent determine the range of the sum to infinity of the series.

$\boxed{S_\infty > 1}$

$\left.\begin{array}{l} u_1 = 2 \\ u_2 = x \end{array}\right\} \Rightarrow r = \frac{x}{2}$

IF CONVERGENT $-1 < r < 1$

$-1 < \frac{x}{2} < 1$

$\boxed{-2 < x < 2}$

NOW $S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{2}{1-\frac{x}{2}}$

$\boxed{S_\infty = \frac{4}{2-x}}$

THUS $-2 < x < 2$

$-2 < -x < 2$ ← THEY SWAP BUT INEQUALITY DOES NOT CHANGE

$0 < 2-x < 4$

$\frac{1}{4} < \frac{1}{2-x} < \infty$

$1 < \frac{4}{2-x} < \infty$

$\therefore S_\infty > 1$ //

Question 101 (*****)

A convergent geometric progression has positive first term and positive common ratio.

Show that the sum to infinity of the progression is at least four times as large as its second term.

V , proof

STARTING WITH THE "GIVENS"

$S_\infty = \frac{a}{1-r}$ $\quad 0 < r < 1$, $a > 0$ $\quad u_2 = ar$

NOW CONSIDER THE "RATIO" BELOW

$\frac{S_\infty}{u_2} = \frac{\frac{a}{1-r}}{ar} = \frac{a}{ar(1-r)} = \frac{1}{r(1-r)}$

PARTIAL FRACTIONS (BY COVER UP)

$\frac{S_\infty}{u_2} = \frac{1}{r} + \frac{1}{1-r} = \frac{1}{r} - \frac{1}{r-1}$

NOW CONSIDER A NEW FUNCTION & CALCULUS

$\frac{S_\infty}{u_2} = f(r) = \frac{1}{r} - \frac{1}{r-1}$

$f'(r) = -\frac{1}{r^2} + \frac{1}{(r-1)^2}$

$f''(r) = \frac{2}{r^3} - \frac{2}{(r-1)^3}$

LOOK FOR STATIONARY POINTS

$\Rightarrow -\frac{1}{r^2} + \frac{1}{(r-1)^2} = 0$

$\Rightarrow \frac{1}{(r-1)^2} = \frac{1}{r^2}$

$\Rightarrow (r-1)^2 = r^2$

$\Rightarrow r^2 - 2r \quad 1 \quad r^2$

$\Rightarrow 1 = 2r$

$\Rightarrow r = \frac{1}{2}$

HENCE WE HAVE

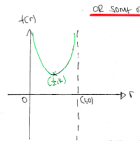
$f(\frac{1}{2}) = \frac{S_\infty}{u_2} = \frac{1}{\frac{1}{2}(1-\frac{1}{2})} = \frac{1}{\frac{1}{2}\times\frac{1}{2}} = 4$

$f''(\frac{1}{2}) = \frac{2}{\frac{1}{8}} - \frac{2}{-\frac{1}{8}} = 16+16 = 32 > 0$

IE LOCAL MINIMUM

NEED A SKETCH OF $f(r)$ FOR $0 < 0 < 1$

OR SOME OTHER ARGUMENT



$f(0) = f(1) = +\infty$

$\therefore \quad f(r) \geq 4 \quad 0 < r < 1$

$\frac{S_\infty}{u_2} \geq 4$

$S_\infty \geq 4u_2$ //

AS REQUIRED

Question 102 (*****)

By showing a detailed method, sum the following series.

$$\frac{2}{1}+\frac{3}{2}+\frac{4}{4}+\frac{5}{8}+\frac{6}{16}+\frac{7}{32}\ldots$$

$\boxed{}$, $\boxed{6}$

- LET $S = \frac{2}{1}+\frac{3}{2}+\frac{4}{4}+\frac{5}{8}+\frac{6}{16}+\frac{7}{32}+\cdots+\frac{n+1}{2^{n-1}}+\cdots$

…(MULTIPLY THROUGH BY $\frac{1}{2}$)

$$\frac{1}{2}S = \frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\frac{6}{32}+\cdots$$

- "LINE UP" THE TWO EXPRESSIONS AS FOLLOWS

$$S = \frac{2}{1}+\frac{3}{2}+\frac{4}{4}+\frac{5}{8}+\frac{6}{16}+\frac{7}{32}+\cdots$$
$$-\frac{1}{2}S = \quad -\frac{2}{2}-\frac{3}{4}-\frac{4}{8}-\frac{5}{16}-\frac{6}{32}-\cdots$$

$$\Rightarrow \frac{1}{2}S = 2+\underbrace{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots}_{\text{THIS IS A GEOMETRIC PROGRESSION WITH } a=\frac{1}{2},\ r=\frac{1}{2}}$$

$$\therefore S_{\infty} = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$\Rightarrow \frac{1}{2}S = 2+1$$

$$\Rightarrow \frac{1}{2}S = 3$$

$$\Rightarrow S = 6$$

Question 103 (*****)

Solve the following simultaneous equations.

$$2\sum_{r=0}^{\infty}\left[\log_2 a\right]^r = \sum_{k=1}^{\infty}(1+b)^{-k} \quad \text{and} \quad \sum_{k=1}^{1}(1+b)^{-k} - \sum_{r=0}^{1}\left[\log_2 a\right]^r = \frac{7}{5}.$$

You may leave the answers as indices in their simplest form, where appropriate.

$$\boxed{}, \quad \boxed{[a,b]=\left[\frac{3}{2},\frac{1}{4}\right]=\left[-\frac{4}{5},2^{\frac{13}{5}}\right]}$$

$$\boxed{2\sum_{r=0}^{\infty}(\log_2 a)^r = \sum_{k=1}^{\infty}\frac{1}{(1+b)^k}} \quad \boxed{\sum_{k=1}^{1}\frac{1}{(1+b)^k} - \sum_{r=0}^{1}(\log_2 a)^r = \frac{7}{5}}$$

- BOTH ARE GEOMETRIC PROGRESSIONS

$$\boxed{\begin{array}{ll} 1+(\log_2 a)+(\log_2 a)^2+(\log_2 a)^3+\dots & \text{FIRST TERM } 1,\ r=\log_2 a \\ \frac{1}{1+b}+\frac{1}{(1+b)^2}+\frac{1}{(1+b)^3}+\frac{1}{(1+b)^4}+\dots & \text{FIRST TERM } \frac{1}{1+b},\ r=\frac{1}{1+b} \end{array}}$$

- FROM THE FIRST EQUATION, "$S_\infty = \frac{a}{1-r}$"

$$\Rightarrow 2\times\frac{1}{1-\log_2 a} = \frac{\frac{1}{1+b}}{1-\frac{1}{1+b}}$$

$$\Rightarrow \frac{2}{1-\log_2 a} = \frac{1}{(1+b)-1}$$

$$\Rightarrow \frac{2}{1-\log_2 a} = \frac{1}{b}$$

$$\Rightarrow 2b = 1-\log_2 a$$

$$\Rightarrow \boxed{\log_2 a = 1-2b}$$

- FROM THE SECOND EQUATION WE OBTAIN

$$\Rightarrow \frac{1}{1+b} - (1+\log_2 a) = \frac{7}{5}$$

$$\Rightarrow \frac{1}{1+b} - 1 - \log_2 a = \frac{7}{5}$$

$$\Rightarrow \frac{1}{1+b} - \log_2 a = \frac{12}{5}$$

$$\Rightarrow \frac{1}{1+b} - (1-2b) = \frac{12}{5}$$

$$\Rightarrow \frac{1}{1+b} - 1 + 2b = \frac{12}{5}$$

$$\Rightarrow \frac{1}{1+b} + 2b = \frac{17}{5}$$

$$\Rightarrow \frac{5}{1+b} + 10b = 17$$

$$\Rightarrow 5 + 10b(1+b) = 17(1+b)$$

$$\Rightarrow 5 + 10b + 10b^2 = 17 + 17b$$

$$\Rightarrow 10b^2 - 7b - 12 = 0$$

$$\Rightarrow (2b-3)(5b+4) = 0$$

$$\Rightarrow b = \begin{cases} 3/2 \\ -4/5 \end{cases}$$

- USING $\log_2 a = 1-2b$

$$a = 2^{1-2b}$$

$$\Rightarrow a = \begin{cases} 2^{-2} = \frac{1}{4} \\ 2^{13/5} \end{cases}$$

$$\therefore (a,b) = \left(\frac{1}{4},\frac{3}{2}\right) \text{ OR } \left(2^{\frac{13}{5}},-\frac{4}{5}\right)$$

Question 104 (*****)

Sum the following series of infinite terms.

$$\frac{1}{3}+\frac{3}{9}+\frac{7}{27}+\frac{15}{81}+\frac{31}{243}+\frac{63}{729}+\ldots$$

$\boxed{}$, $\boxed{\frac{3}{2}}$

$$\boxed{\frac{1}{3}+\frac{3}{9}+\frac{7}{27}+\frac{15}{81}+\frac{31}{243}+\frac{63}{729}+\ldots}$$

- EVIDENTLY THE DENOMINATOR IS "POWERS" OF 3

THE NUMERATORS DO NOT FORM A LINEAR, NOR A QUADRATIC — IF NOT YET SPOTTED WHAT IT IS ATTEMPT DIFFERENCING

1 3 7 15 31 63
2 4 8 16 32
2 4 8 16
2 4 8

"WHEN WE GET THE SAME SEQUENCE BACK FROM DIFFERENCING THEN WE ARE LOOKING AT AN EXPONENTIAL SEQUENCE (GEOMETRIC)"

- WE CAN SEE THAT THE NUMERATOR FORM 2^n-1

- THUS WE REQUIRE

$$\sum_{r=1}^{\infty}\frac{2^r-1}{3^r}=\sum_{r=1}^{\infty}\left[\left(\frac{2}{3}\right)^r-\left(\frac{1}{3}\right)^r\right]=\sum_{r=1}^{\infty}\left(\frac{2}{3}\right)^r-\sum_{r=1}^{\infty}\left(\frac{1}{3}\right)^r$$

G.P $a=\frac{2}{3}$ $r=\frac{2}{3}$

G.P. $a=\frac{1}{3}$ $r=\frac{1}{3}$

- USING THE STANDARD FORMULA FOR THE SUM TO INFINITY OF A G.P.

I.E $S_{\infty}=\frac{a}{1-r}$

$$\ldots=\frac{\frac{2}{3}}{1-\frac{2}{3}}-\frac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{\frac{2}{3}}{\frac{1}{3}}-\frac{\frac{1}{3}}{\frac{2}{3}}=2-\frac{1}{2}=\frac{3}{2}$$

Question 105 (*****)

Sum the following series of infinite terms.

$$\frac{1}{2}+\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{5}{32}+\frac{8}{64}+\frac{13}{128}+\ldots$$

$\boxed{}$, $\boxed{2}$

$$\boxed{\frac{1}{2}+\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{5}{32}+\frac{8}{64}+\frac{13}{128}+\ldots}$$

THE NUMERATOR IS THE FIBONACCI SERIES, THE DENOMINATOR IS A G.P WITH COMMON RATIO 2, OR $\frac{1}{2}$ FOR THE WHOLE FRACTIONAL TERM

THUS LET THE REQUIRED SUM BE S

- $S=\frac{1}{2}+\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{5}{32}+\frac{8}{64}+\frac{13}{128}+\ldots$
- $-\frac{1}{2}S=-\frac{1}{4}-\frac{1}{8}-\frac{2}{16}-\frac{3}{32}-\frac{5}{64}-\frac{8}{128}-\ldots$
- $-\frac{1}{4}S=-\frac{1}{8}-\frac{1}{16}-\frac{2}{32}-\frac{3}{64}-\frac{5}{128}-\ldots$

ADDING $\frac{1}{4}S=\frac{1}{2}$

$S=2$

Question 106 (*****)

It is given that the following series converges to a limit L.

$$\sum_{r=1}^{\infty}\left[\frac{2x-1}{x+2}\right]^{r}.$$

Determine with full justification the range of possible values of L.

$\boxed{}$, $\boxed{L > -\frac{1}{2}}$

$$\sum_{r=1}^{\infty}\left(\frac{2x-1}{x+2}\right)^{r} = L$$

- FIRSTLY THIS IS A GEOMETRIC PROGRESSION, WHICH CONVERGES TO L

$$a = r = \frac{2x-1}{x+2}$$

$$S_{\infty} = \frac{a}{1-r} = \frac{\frac{2x-1}{x+2}}{1-\frac{2x-1}{x+2}} = \frac{2x-1}{(x+2)-(2x-1)} = \frac{2x-1}{3-x}$$

- NEXT WE REQUIRE THE RANGE OF VALUES OF x, FOR WHICH THE SUM TO INFINITY EXISTS

$$|r| < 1$$

$$-1 < \frac{2x-1}{x+2} < 1$$

$$\Rightarrow \frac{2x-1}{x+2} < 1 \qquad \Rightarrow \frac{2x-1}{x+2} > -1$$

$$\Rightarrow \frac{2x-1}{x+2} - 1 < 0 \qquad \Rightarrow \frac{2x-1}{x+2} + 1 > 0$$

$$\Rightarrow \frac{2x-1-x-2}{x+2} < 0 \qquad \Rightarrow \frac{2x-1+x+2}{x+2} > 0$$

$$\Rightarrow \frac{x-3}{x+2} < 0 \qquad \Rightarrow \frac{3x+1}{x+2} > 0$$

$$-2 < x < 3 \qquad x < -2 \text{ OR } x > -\tfrac{1}{3}$$

$$\therefore -\tfrac{1}{3} < x < 3$$

- THUS WE HAVE

$$L(x) = \frac{2x-1}{3-x}, \quad -\tfrac{1}{3} < x < 3$$

$$L'(x) = \frac{(3-x)\times 2 - (2x-1)\times(-1)}{(3-x)^2} = \frac{2(3-x)+(2x-1)}{(3-x)^2}$$

$$L'(x) = \frac{5}{(3-x)^2} > 0 \quad \text{FOR THE ABOVE DOMAIN}$$

- AS $L(x)$ IS AN INCREASING FUNCTION, THE MINIMUM & MAXIMUM CAN BE EASILY FOUND

$$L\left(-\tfrac{1}{3}\right) = \frac{2\left(-\frac{1}{3}\right)-1}{3-\left(-\frac{1}{3}\right)} = \frac{-\frac{5}{3}}{\frac{10}{3}} = -\frac{1}{2}$$

$$L(3) = +\infty$$

$$\therefore L > -\tfrac{1}{2}$$

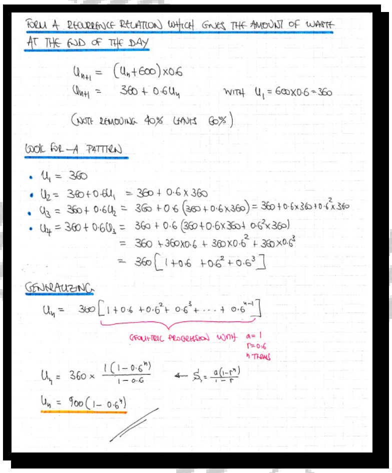
Question 107 (*****)

A factory gets permission to dispose, at the start of every day, 600 kg of waste into a stream of water.

The running stream removes 40% of the any waste present, by the end of the day.

Determine a simplified expression for the amount of waste present in the stream at the end of the n^{th} day.

$\boxed{\text{MP}}$, $\boxed{u_n = 900\left[1-\left(\frac{3}{5}\right)^n\right]}$



FORM A RECURRENCE RELATION WHICH GIVES THE AMOUNT OF WASTE AT THE END OF THE DAY

$u_{n+1} = (u_n + 600) \times 0.6$

$u_{n+1} = 360 + 0.6u_n$ WITH $u_1 = 600 \times 0.6 = 360$

(NOTE REMOVING 40% LEAVES 60%)

LOOK FOR A PATTERN

- $u_1 = 360$
- $u_2 = 360 + 0.6u_1 = 360 + 0.6 \times 360$
- $u_3 = 360 + 0.6u_2 = 360 + 0.6(360 + 0.6 \times 360) = 360 + 0.6 \times 360 + 0.6^2 \times 360$
- $u_4 = 360 + 0.6u_3 = 360 + 0.6(360 + 0.6 \times 360 + 0.6^2 \times 360)$
  $= 360 + 360 \times 0.6 + 360 \times 0.6^2 + 360 \times 0.6^3$
  $= 360[1 + 0.6 + 0.6^2 + 0.6^3]$

GENERALIZING

$u_n = 360[\underbrace{1 + 0.6 + 0.6^2 + 0.6^3 + \dots + 0.6^{n-1}}_{\text{GEOMETRIC PROGRESSION WITH } a=1,\ r=0.6,\ n \text{ TERMS}}]$

$u_n = 360 \times \frac{1(1-0.6^n)}{1-0.6} \leftarrow S_n = \frac{a(1-r^n)}{1-r}$

$u_n = 900(1-0.6^n)$

Question 108 (*****)

Evaluate the following expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{1}{3^{m+n}}\right].$$

Detailed workings must be shown.

V, $\boxed{}$, $\boxed{\frac{9}{4}}$

WORK AS FOLLOWS

$$\sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\left[\frac{1}{3^{m+n}}\right]\right] = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\left[\frac{1}{3^m}\times\frac{1}{3^n}\right]\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{3^n}\sum_{m=0}^{\infty}\left[\frac{1}{3^m}\right]\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{3^n}\underbrace{\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots\right)}_{\text{G.P.}}\right]$$

THIS IS A GEOMETRIC PROGRESSION WITH $a=1$, $r=\frac{1}{3}$ & $S_\infty = \frac{a}{1-r}$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{3^n}\times\frac{1}{1-\frac{1}{3}}\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{3}{2}\times\frac{1}{3^n}\right]$$

$$= \frac{3}{2}\sum_{n=0}^{\infty}\left[\frac{1}{3^n}\right]$$

$$= \frac{3}{2}\underbrace{\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots\right)}_{\text{SAME G.P. AS ABOVE WITH } S_\infty = \frac{1}{1-\frac{1}{3}}}$$

$$= \frac{3}{2}\times\frac{3}{2}$$

$$= \frac{9}{4}$$

Question 109 (*****)

Evaluate the following expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\frac{1}{2^{m+n}}\right].$$

Detailed workings must be shown.

V, $\boxed{}$, $\boxed{\frac{8}{3}}$

WORK AS FOLLOWS

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\frac{1}{2^{m+n}}\right] = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\left[\frac{1}{2^m}\times\frac{1}{2^n}\right]\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\sum_{m=0}^{n}\left(\frac{1}{2^m}\right)\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\underbrace{\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}\right)}_{\substack{\text{G.P WITH } a=1\\ r=\frac{1}{2}\\ n+1 \text{ TERMS}}}\right]$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{n+1} = \frac{a(1-r^{n+1})}{1-r}$$

THUS WE SIMPLIFY TO

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\times\frac{1\left(1-\left(\frac{1}{2}\right)^{n+1}\right)}{1-\frac{1}{2}}\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2^n}\times 2\times\left(1-\left(\frac{1}{2}\right)^{n+1}\right)\right]$$

$$= 2\sum_{n=0}^{\infty}\left[\frac{1}{2^n}\left(1-\frac{1}{2^{n+1}}\right)\right]$$

$$= 2\sum_{n=0}^{\infty}\left[\frac{1}{2^n}-\frac{1}{2^{2n+1}}\right]$$

WRITE THE GEOMETRIC PROGRESSIONS EXPLICITLY

$$\dots = 2\sum_{n=0}^{\infty}\frac{1}{2^n} - 2\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}$$

$$= 2\left[1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right] - 2\left[\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}+\cdots\right]$$

$a=1$, $r=\frac{1}{2}$ $\qquad$ $a=\frac{1}{2}$, $r=\frac{1}{4}$

USING $S_{\infty}=\frac{a}{1-r}$ IN EACH CASE

$$= 2\times\frac{1}{1-\frac{1}{2}} - 2\times\frac{\frac{1}{2}}{1-\frac{1}{4}}$$

$$= 2\times\frac{1}{\frac{1}{2}} - 2\times\frac{2}{4-1}$$

$$= 2\times 2 - \frac{4}{3}$$

$$= 4 - \frac{4}{3}$$

$$= \frac{8}{3}$$

Question 110 (*****)

Show that the following equation has only one real solution.

$$27n = 4\sum_{r=2}^{\infty}(1+n)^{-r} .$$

$\boxed{}$, $\boxed{n = \frac{1}{3},\ n \neq -\frac{2}{3}}$

THIS IS THE SUM TO INFINITY OF SOME G.P — THE EQUATION IS IN "n" AS r IS A DUMMY VARIABLE

$$\Rightarrow 27n = 4\sum_{r=2}^{\infty}(1+n)^{-r}$$

$$\Rightarrow 27n = 4\sum_{r=2}^{\infty}\frac{1}{(1+n)^r}$$

$$\Rightarrow 27n = 4\left[\frac{1}{(1+n)^2} + \frac{1}{(1+n)^3} + \frac{1}{(1+n)^4} + \frac{1}{(1+n)^5} + \dots\right]$$

THE R.H.S IS A G.P WITH $a = \frac{1}{(1+n)^2}$ & $r = \frac{1}{1+n}$

USING $S_{\infty} = \frac{a}{1-r}$

$$\Rightarrow 27n = \frac{\frac{4}{(n+1)^2}}{1 - \frac{1}{n+1}}$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION IN THE R.H.S BY $(n+1)^2$

$$\Rightarrow 27n = \frac{4}{(n+1)^2 - (n+1)}$$

$$\Rightarrow 27n = \frac{4}{n^2+2n+1-n-1}$$

$$\Rightarrow 27n = \frac{4}{n^2+n}$$

$$\Rightarrow 27n^3 + 27n^2 = 4$$

$$\Rightarrow 27n^3 + 27n^2 - 4 = 0$$

LOOK FOR ROOTS BY INSPECTION — RATIONAL ZEROS THEOREM

$$\downarrow$$

$$\frac{\pm 1, \pm 2, \pm 4}{\pm 1, \pm 3, \pm 9, \pm 27}$$

IF $n = \frac{1}{3}$ $\Rightarrow$ $27 \times \frac{1}{27} + 27 \times \frac{1}{9} - 4 = 1 + 3 - 4 = 0$

$\Rightarrow$ $(3n-1)$ IS A FACTOR

BY LONG DIVISION OR MANIPULATION

$$\Rightarrow 27n^3 + 27n^2 - 4 = 0$$

$$\Rightarrow 9n^2(3n-1) + 12n(3n-1) + 4(3n-1) = 0$$

$$\Rightarrow (3n-1)(9n^2+12n+4) = 0$$

$$\Rightarrow (3n-1)(3n+2)^2 = 0$$

$$\Rightarrow n = \begin{cases} \frac{1}{3} \\ -\frac{2}{3} \end{cases}$$

$$\begin{array}{r|l} & 9n^2 + 12n + 4 \\ \hline 3n-1 & 27n^3 + 27n^2 \quad - 4 \\ & -27n^3 + 9n^2 \\ \hline & \quad 36n^2 \quad - 4 \\ & -36n^2 + 12n \\ \hline & \quad 12n - 4 \\ & -12n + 4 \\ \hline \end{array}$$

NOW BOTH VALUES SATISFY THE QUADRATIC

HOWEVER WE NEED TO CHECK S_{∞}, $-1 < r < 1$

- IF $n = \frac{1}{3}$

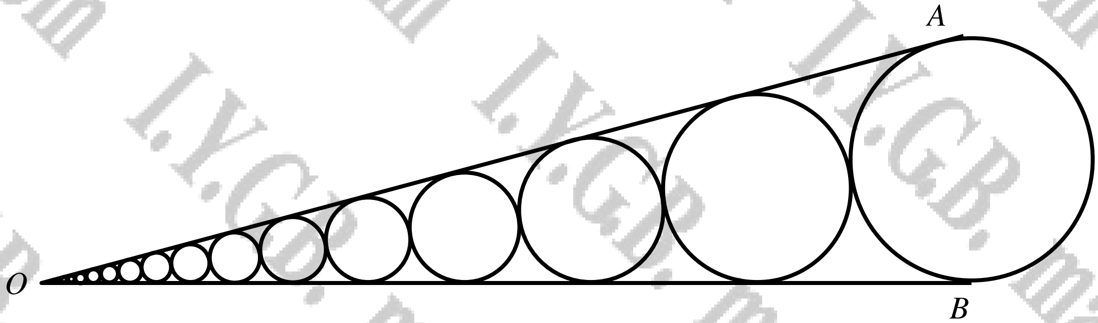
$$r = \frac{1}{n+1} = \frac{1}{\frac{1}{3}+1} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

- IF $n = -\frac{2}{3}$

$$r = \frac{1}{n+1} = \frac{1}{-\frac{2}{3}+1} = \frac{1}{\frac{1}{3}} = 3 > 1$$

ONLY $n = \frac{1}{3}$ IS A SOLUTION

Question 111 (*****)



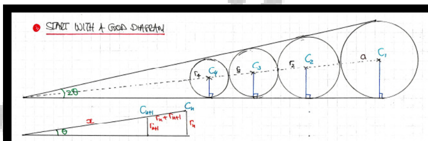
The figure above shows a infinite sequence of circles of decreasing radius, the radius of the larger circle being $\frac{4}{3}$.

The centres of these circles lie on a straight line. The straight lines OA and OB are tangents to every circle in the sequence, the angle AOB denoted by 2θ.

Given that the total area of these circles is 2π, determine the value of θ.

$\boxed{}$, $\boxed{\theta = \frac{1}{6}\pi}$

- START WITH A GOOD DIAGRAM



- LOOKING AT THE FIGURE ABOVE WE GET

$\sin\theta = \frac{r_{n+1}}{x}$ & $\sin\theta = \frac{r_n}{x + r_n + r_{n+1}}$

- ELIMINATE x

$x = \frac{r_{n+1}}{\sin\theta} \implies \sin\theta = \frac{r_n}{\frac{r_{n+1}}{\sin\theta} + r_n + r_{n+1}}$

$\implies r_{n+1} + r_n\sin\theta + r_{n+1}\sin\theta = r_n$

$\implies \frac{r_{n+1}}{r_n} + \sin\theta + \frac{r_{n+1}}{r_n}\sin\theta = 1$

- LET "R" DENOTE $\frac{r_{n+1}}{r_n}$

$\implies R + \sin\theta + R\sin\theta = 1$

$\implies R + R\sin\theta = 1 - \sin\theta$

$\implies R(1+\sin\theta) = 1 - \sin\theta$

$\implies R = \frac{1-\sin\theta}{1+\sin\theta}$

$\implies \frac{r_{n+1}}{r_n} = \frac{1-\sin\theta}{1+\sin\theta} = \text{CONSTANT}$

$\therefore$ A GEOMETRIC PROGRESSION WHICH CONVERGES

- AS THE RADII FORM A G.P. WHICH CONVERGES SO WILL THE AREAS OF THE CIRCLES

$\implies \text{AREA} = \pi a^2 + \pi r_1^2 + \pi r_2^2 + \pi r_3 + \pi r_4^2 + \dots$

$\implies \text{AREA} = \pi\left[a^2 + (aR)^2 + (aR^2)^2 + (aR^3)^2 + (aR^4)^2 + \dots\right]$

$\implies \text{AREA} = \pi\left[a^2 + a^2R^2 + a^2R^4 + a^2R^6 + a^2R^8 + \dots\right]$

$\implies \text{AREA} = \pi a^2\left[1 + R^2 + R^4 + R^6 + R^8 + \dots\right]$

$\implies 2\pi = \pi\left(\frac{4}{3}\right)^2\left(\frac{1}{1-R^2}\right) \leftarrow S_\infty = \text{"}\frac{a}{1-r}\text{"}$ (OR STANDARD BINOMIAL)

$\implies 2 = \frac{16}{9}\left(\frac{1}{1-R^2}\right)$

$\implies 9 = 8\left(\frac{1}{1-R^2}\right)$

$\implies 9(1-R^2) = 8$

$\implies 9\left[1 - \left(\frac{1-\sin\theta}{1+\sin\theta}\right)^2\right] = 8$

$\implies 9\left[\frac{(1+\sin\theta)^2 - (1-\sin\theta)^2}{(1+\sin\theta)^2}\right] = 8$

$\implies 9\left[\frac{\cancel{1} + 2\sin\theta + \cancel{\sin^2\theta} - \cancel{1} + 2\sin\theta - \cancel{\sin^2\theta}}{(1+\sin\theta)^2}\right] = 8$

$\implies \frac{36\sin\theta}{(1+\sin\theta)^2} = 8$

$\implies \frac{9\sin\theta}{1 + 2\sin\theta + \sin^2\theta} = 2$

$\implies 9\sin\theta = 2 + 4\sin\theta + 2\sin^2\theta$

$\implies 2\sin^2\theta - 5\sin\theta + 2 = 0$

$\implies (2\sin\theta - 1)(\sin\theta - 2) = 0$

$\implies \sin\theta = \cancel{2}, \ \frac{1}{2}$

$\implies \theta = \frac{\pi}{6}$

Question 112 (*****)

By showing a detailed method, sum the following series.

$$\sum_{r=0}^{9}\left[(r+1)\times 11^{r}\times 10^{9-r}\right].$$

You may leave the answer in index form.

$\boxed{}$, $\boxed{\displaystyle\sum_{r=0}^{9}\left[(r+1)\times 11^{r}\times 10^{9-r}\right]=10^{11}}$

$$\boxed{\sum_{r=0}^{9}\left[(r+1)\times 11^{r}\times 10^{9-r}\right]=\,?}$$

- WRITE A FEW TERMS OUT AND LOOK FOR A PATTERN

$$\Rightarrow S = 1(11)^0(10)^9+2(11)^1(10^8)+3(11)^2(10)^7+\dots+9(11)^8(10)^1+10(11)^9(10)^0$$

$$\Rightarrow S = 1(11)^0(10)^9+2\left(\tfrac{11}{10}\right)^1(10)^9+3\left(\tfrac{11}{10}\right)^2(10)^9+\dots+9\left(\tfrac{11}{10}\right)^8(10)^9+10\left(\tfrac{11}{10}\right)^9 10^9$$

$$\Rightarrow S = 10^9\left[1+2\left(\tfrac{11}{10}\right)+3\left(\tfrac{11}{10}\right)^2+\dots+9\left(\tfrac{11}{10}\right)^8+10\left(\tfrac{11}{10}\right)^9\right]$$

- THE SERIES ARITHMETIC & GEOMETRIC FEATURES — MULTIPLY BY $-\frac{11}{10}$

$$\Rightarrow -\tfrac{11}{10}S = 10^9\left[\quad -\left(\tfrac{11}{10}\right)-2\left(\tfrac{11}{10}\right)^2-\dots-8\left(\tfrac{11}{10}\right)^8-9\left(\tfrac{11}{10}\right)^9-10\left(\tfrac{11}{10}\right)^{10}\right]$$

- ADD THE LAST TWO LINES ABOVE

$$\Rightarrow \left(1-\tfrac{11}{10}\right)S = 10^9\Big[\underbrace{1+\left(\tfrac{11}{10}\right)^1+\left(\tfrac{11}{10}\right)^2+\left(\tfrac{11}{10}\right)^3+\dots+\left(\tfrac{11}{10}\right)^8+\left(\tfrac{11}{10}\right)^9}_{\text{THIS IS A G.P.}}-10\left(\tfrac{11}{10}\right)^{10}\Big]$$

$a=1$, $r=\frac{11}{10}$, $n=10$ $\quad S_n=\dfrac{a(r^n-1)}{r-1}$

$$\Rightarrow -\tfrac{1}{10}S = 10^9\times\frac{\left(\frac{11}{10}\right)^{10}-1}{\frac{11}{10}-1}+10^9\left(-10\times\left(\tfrac{11}{10}\right)^{10}\right)$$

$$\Rightarrow -\tfrac{1}{10}S = 10^9\times\frac{\left(\frac{11}{10}\right)^{10}-1}{\frac{1}{10}}-10^{10}\times\left(\tfrac{11}{10}\right)^{10}$$

$$\Rightarrow -\tfrac{1}{10}S = 10^{10}\left(\left(\tfrac{11}{10}\right)^{10}-1\right)-10^{10}\times\left(\tfrac{11}{10}\right)^{10}$$

$$\Rightarrow -\tfrac{1}{10}S = \cancel{10^{10}\times\left(\tfrac{11}{10}\right)^{10}}-10^{10}-\cancel{10^{10}\times\left(\tfrac{11}{10}\right)^{10}}$$

$$\Rightarrow S = 10^{11}$$

Question 113 (*****)

A family of infinite geometric series S_k, has first term $\frac{k-1}{k!}$ and common ratio $\frac{1}{k}$, where $k = 3,4,5,6,\ldots,99,100$.

Find the value of

$$\frac{10^4}{100!} + \sum_{k=3}^{100}\left[\left[(k-1)(k-2)-1\right]S_k\right].$$

$\boxed{}$, $\boxed{2}$

WRITE THE FIRST FEW TERMS OF THIS GENERAL GEOMETRIC PROGRESSION

$$\Rightarrow S_k = \frac{k-1}{k!} + \frac{k-1}{k!}\frac{1}{k} + \frac{k-1}{k!}\frac{1}{k^2} + \frac{k-1}{k!}\frac{1}{k^3} + \ldots$$

$$\Rightarrow S_k = \frac{k-1}{k!}\Big[\underbrace{1 + \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \ldots}_{\text{SUMMING TO INFINITY USING } S_\infty = \frac{a}{1-r}}\Big]$$

$$\Rightarrow S_k = \frac{k-1}{k!} \times \frac{1}{1-\frac{1}{k}}$$

$$\Rightarrow S_k = \frac{k-1}{k!} \times \frac{k}{k-1}$$

$$\Rightarrow S_k = \frac{k}{k!}$$

$$\Rightarrow \boxed{S_k = \frac{1}{(k-1)!}}$$

NEXT CONSIDER THE SUMMATION WITH THE GENERAL S_k FOUND

$$\sum_{k=3}^{100}\left[S_k\left[(k-1)(k-2)-1\right]\right]$$

$$= \sum_{k=3}^{100}\left[\frac{(k-1)(k-2)-1}{(k-1)!}\right]$$

$$= \sum_{k=3}^{100}\left[\frac{(k-1)(k-2)}{(k-1)!} - \frac{1}{(k-1)!}\right]$$

$$= \sum_{k=3}^{100}\left[\frac{1}{(k-3)!} - \frac{1}{(k-1)!}\right]$$

WRITING THE SUM EXPLICITLY IN A "TABLE FORM"

$$\sum_{k=3}^{100}\left[\frac{1}{(k-3)!} - \frac{1}{(k-1)!}\right] = \begin{array}{ccc} \frac{1}{0!} & - & \cancel{\frac{1}{2!}} \\ \frac{1}{1!} & - & \cancel{\frac{1}{3!}} \\ \cancel{\frac{1}{2!}} & - & \cancel{\frac{1}{4!}} \\ \cancel{\frac{1}{3!}} & - & \cancel{\frac{1}{5!}} \\ \vdots & & \vdots \\ \cancel{\frac{1}{96!}} & - & \frac{1}{98!} \\ \cancel{\frac{1}{97!}} & - & \frac{1}{99!} \end{array}$$

$$= \frac{1}{0!} + \frac{1}{1!} - \left(\frac{1}{98!} + \frac{1}{99!}\right)$$

$$= 2 - \left(\frac{99+1}{99!}\right)$$

$$= 2 - \frac{100}{99!}$$

FINALLY ADDING THE TERM AT THE FRONT OF THE SUMMATION

$$\frac{10^4}{100!} + \sum_{k=3}^{100}\left[\left[(k-1)(k-2)-1\right]S_k\right] = \frac{10^4}{100!} + 2 - \frac{100}{99!}$$

$$= \frac{100^2}{100!} + 2 - \frac{100}{99!}$$

$$= \cancel{\frac{100}{99!}} + 2 - \cancel{\frac{100}{99!}}$$

$$= 2$$